NORMALIZED SOLUTIONS FOR SCHRÖDINGER EQUATIONS WITH CRITICAL SOBOLEV EXPONENT AND MIXED NONLINEARITIES

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Abstract. In this paper, we consider the following nonlinear Schrödinger equations with mixed nonlinearities:

$$\begin{cases} -\Delta u = \lambda u + \mu |u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} u^2 = a^2, \end{cases}$$

where $N \geq 3$, $\mu > 0$, $\lambda \in \mathbb{R}$ and $2 < q < 2^* = \frac{2N}{N-2}$. We prove in this paper

(1) Existence of solutions of mountain-pass type for $N = 3$ and $2 < q < 2 + \frac{4}{N}$;
(2) Existence and nonexistence of ground states for $2 + \frac{4}{N} \leq q < 2^*$ with $\mu > 0$ large;
(3) Precisely asymptotic behaviors of ground states and mountain-pass solutions as $\mu \to 0$ and μ goes to its upper bound.

Our studies answer some questions proposed by Soave in [49, Remarks 1.1, 1.2 and 8.1].

Keywords: Normalized solution; Ground state; Variational method; Critical nonlinearity; Mixed nonlinearities.

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1. Introduction

In this paper, we consider the following nonlinear scalar field equation:

$$\begin{cases} -\Delta u = \lambda u + \mu |u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 3$, $\mu > 0$, $\lambda \in \mathbb{R}$ and $2 < q < 2^* = \frac{2N}{N-2}$.

(1.1) is a special case of the following model,

$$\begin{cases} -\Delta u = \lambda u + f(u) \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

which is related to finding the stationary waves of nonlinear Schrödinger equations:

$$i\psi_t + \Delta\psi + g(|\psi|^2)\psi = 0 \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Indeed, a stationary wave of (1.3) is of the form $\psi(t, x) = e^{i\lambda t}u(x)$ where $\lambda \in \mathbb{R}$ is a constant and $u(x)$ is a time-independent function, then it is well-known that ψ is a solution of (1.3) if and only if u is a solution of (1.2) with $f(u) = g(|u|^2)u$. As pointed out in [49, 50], in general, the function u is complex valued and thus, (1.2) can be regarded as a complex valued system coupled by the nonlinearities $f(u) = g(|u|^2)u$.

As pointed out in [29,36], the studies on (1.2) can be traced back to the semi-classical papers [12,13,42,43,52]. In these studies, there are two different methods to study (1.2). The first one is to fix the number $\lambda < 0$ in (1.2) and restrict the unknown u to be real valued. In this case, (1.2) is a single equation and under some mild assumptions on $f(u)$, the solutions of (1.2) are critical points of the functional

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 - \lambda|u|^2)dx - \int_{\mathbb{R}^N} F(u)dx$$

in the usual Sobolev space $H^1(\mathbb{R}^N)$, where $F(u) = \int_0^u f(t)dt$. In this case, particular attention is devoted to least action solutions, namely solutions minimizing $\mathcal{J}(u)$ among all non-trivial solutions. We also refer the readers to [2–5] and the references therein for the recent results on the special case (1.1) in this direction. Another one is to fix the L^2 norm of the unknown u, that is, to find solutions of (1.2) with prescribed mass. In this case, (1.2) is always rewritten as follows

$$\begin{cases} -\Delta u = \lambda u + f(u) \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} u^2 = a^2, \end{cases} \tag{1.4}$$

where $\int_{\mathbb{R}^N} u^2 = a^2$ is the prescribed mass, and in this case, $\lambda \in \mathbb{R}$ is a part of the unknown which appears in (1.4) as a Lagrange multiplier. In particular, (1.1) is rewritten as

$$\begin{cases} -\Delta u = \lambda u + \mu|u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} u^2 = a^2. \end{cases} \tag{1.5}$$

Similar to the first case, under some mild assumptions on $f(u)$, the solutions of (1.4) are critical points of the functional

$$\mathcal{F}_\mu(u) = \int_{\mathbb{R}^N}\big(\frac{1}{2}|\nabla u|^2 - F(u)\big)dx$$

on the smooth manifold

$$\mathcal{S}_a = \{u \in H^1(\mathbb{R}^N) \mid \|u\|_2^2 = a^2\}.$$

In particular, the solutions of (1.5) are critical points of the C^2-functional

$$\mathcal{E}_\mu(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu}{q}\|u\|_q^q - \frac{1}{2^*}\|u\|_{2^*}^{2^*}$$

on $\mathcal{S}_a$, where we denote the usual norm in $L^p(\mathbb{R}^N)$ by $\|\cdot\|_p$. In this case, solutions of (1.4) are always called normalized solutions which are particularly relevant for the nonlinear Schrödinger equation (1.3) since the mass is preserved along the time evolution in (1.3). Thus, normalized solutions of (1.4) seems to be particularly meaningful from the physical viewpoint, moreover, these solutions often offer a good insight of the dynamical properties of the stationary solutions for the nonlinear Schrödinger equation (1.3), such as stability or instability (cf. [11,18]). In this case, particular attention is also devoted to least action solutions which are also called ground states for normalized solutions, namely solutions minimizing $\mathcal{F}_\mu(u)$ among all non-trivial solutions. The studies on normalized solutions of (1.4) is a hot topic in the community of nonlinear PDEs nowadays, thus, it is impossible for us to provide a complete references. We just refer the readers to [1, 8–10, 15, 17, 23, 33, 35–37, 40, 46–51] and the references therein. In these references, we would like to highlight [36, 49, 50] to the readers for their detail introductions and references on normalized solutions of (1.4) and new directions on the study of normalized solutions of autonomous problems. We also would like to point out [26–32] and the references therein for the studies on normalized solutions of problems with trapping potentials.

It is well-known that the number $p = 2 + \frac{4}{N}$ plays an important role in studying normalized solutions which is called the L^2 critical exponent or mass critical exponent in the literature. Since $2^* = \frac{2N}{N-2} > 2 + \frac{4}{N}$, the nonlinearity of (1.5) grows faster than $|u|^{p-2}u$ at infinity and thus, it is well-known that $\mathcal{E}_\mu(u)$ is unbounded from below on $\mathcal{S}_a$, which makes one to find new constraints to prove the existence of ground states of $\mathcal{E}_\mu(u)$ on $\mathcal{S}_a$. The new constraint, which is introduced by Bartsch and Soave in [9] for the general problem (1.4) and is widely used nowadays in studying normalized solutions, is the following L^2-Pohozaev manifold:

$$\mathcal{P}_{a,\mu} = \{u \in \mathcal{S}_a \mid \|\nabla u\|_2^2 = \mu\gamma_q\|u\|_q^q + \|u\|_{2^*}^{2^*}\},$$

where

$$\gamma_q = \frac{N(q-2)}{2q}. \tag{1.6}$$

By the Pohozaev identity of (1.5), $\mathcal{P}_{a,\mu}$ contains all nontrivial solutions of (1.5). Thus, we have the following definition of ground states of (1.5).

Definition 1.1. *We say (u_0, λ_0) is a ground state of (1.5) if u_0 is a critical point of $\mathcal{E}_\mu|_{\mathcal{S}_a}(u)$ with $\mathcal{E}_\mu|_{\mathcal{S}_a}(u_0) = \inf_{u\in\mathcal{P}_{a,\mu}} \mathcal{E}_\mu(u)$.*

The L^2-Pohozaev manifold $\mathcal{P}_{a,\mu}$ is quite related to the fibering maps

$$\Psi(u, s) = \frac{e^{2s}}{2}\|\nabla u\|_2^2 - \frac{\mu e^{q\gamma_q s}}{q}\|u\|_q^q - \frac{e^{2^* s}}{2^*}\|u\|_{2^*}^{2^*},$$

which is introduced by Jeanjean in [33] for the general problem (1.4) and is well studied by Soave in [49]. According to the fibering maps $\Psi(u, s)$, $\mathcal{P}_{a,\mu}$ can be naturally divided into the following three parts:

$$\mathcal{P}_+^{a,\mu} = \{u \in \mathcal{S}_a \cap \mathcal{P}_{a,\mu} \mid 2\|\nabla u\|_2^2 > \mu q\gamma_q^2\|u\|_q^q + 2^*\|u\|_{2^*}^{2^*}\},$$
$$\mathcal{P}_0^{a,\mu} = \{u \in \mathcal{S}_a \cap \mathcal{P}_{a,\mu} \mid 2\|\nabla u\|_2^2 = \mu q\gamma_q^2\|u\|_q^q + 2^*\|u\|_{2^*}^{2^*}\},$$
$$\mathcal{P}_-^{a,\mu} = \{u \in \mathcal{S}_a \cap \mathcal{P}_{a,\mu} \mid 2\|\nabla u\|_2^2 < \mu q\gamma_q^2\|u\|_q^q + 2^*\|u\|_{2^*}^{2^*}\}.$$

Let

$$m_{a,\mu}^{\pm} = \inf_{u\in\mathcal{P}_\pm^{a,\mu}} \mathcal{E}_\mu(u), \tag{1.7}$$

then Soave proved the following results in [49, Theorems 1.1 and 1.4]:

(1) For $2 < q < 2 + \frac{4}{N}$, there exists $\alpha_{N,q} > 0$ such that if $\mu a^{q-q\gamma_q} < \alpha_{N,q}$ then $m_{a,\mu}^+ = \inf_{u\in\mathcal{P}_+^{a,\mu}} \mathcal{E}_\mu(u) = \inf_{u\in\mathcal{P}_{a,\mu}} \mathcal{E}_\mu(u) < 0$ and it can be attained by some $u_{a,\mu,+}$ which is real valued, positive, radially symmetric and radially decreasing. Moreover, (1.5) has a ground state $(u_{a,\mu,+}, \lambda_{a,\mu,+})$ with $\lambda_{a,\mu,+} < 0$, and $m_{a,\mu}^+ \to 0$ and $\|\nabla u_{a,\mu,+}\|_2 \to 0$ as $\mu \to 0$.

(2) For $2+\frac{4}{N}\leq q<2^*$, there exists $\alpha_{N,q}>0$ such that if $\mu a^{q-q\gamma_q}<\alpha_{N,q}$ then $m_{a,\mu}^-=\inf_{u\in\mathcal{P}_-^{a,\mu}}\mathcal{E}_\mu(u)=\inf_{u\in\mathcal{P}_{a,\mu}}\mathcal{E}_\mu(u)\in(0,\frac{1}{N}S^{\frac{N}{2}})$ and it can be attained by some $u_{a,\mu,-}$ which is real valued, positive, radially symmetric and radially decreasing, where S is the optimal constant in the Sobolev embedding, that is,

$$\|u\|_{2^*}^2\leq S^{-1}\|\nabla u\|_2^2 \quad \text{for all } u\in D^{1,2}(\mathbb{R}^N). \tag{1.8}$$

Moreover, (1.5) has a ground state $(u_{a,\mu,-},\lambda_{a,\mu,-})$ with $\lambda_{a,\mu,-}<0$, and $m_{a,\mu}^-\to\frac{1}{N}S^{\frac{N}{2}}$ and $\|\nabla u_{a,\mu,-}\|_2\to S^{\frac{N}{2}}$ as $\mu\to0$.

In the L^2-subcritical case $2<q<2+\frac{4}{N}$, since $\mathcal{E}_\mu(u)|_{\mathcal{S}_a}$ is unbounded from below, it could be naturally to expect that $\mathcal{E}_\mu(u)|_{\mathcal{S}_a}$ has a second critical point of mountain-pass type, which is also positive, real valued and radially symmetric. This natural expectation has been pointed out by Soave in [49, Remark 1.1] which can be summarized to be the following question:

(Q_1) **Does $\mathcal{E}_\mu(u)|_{\mathcal{S}_a}$ has a critical point of mountain-pass type in the L^2-subcritical case $2<q<2+\frac{4}{N}$?**

Remark 1.1. *In preparing this paper, we notice that in the very recent work [34], the question (Q_1) has been solved for $N\geq4$. Thus, it only need to consider the case $N=3$ for the question (Q_1).*

Besides, since Soave only considered the case that $\mu a^{q-q\gamma_q}>0$ small in [49, Theorem 1.1], it is also natural to ask what will happen if $\mu>0$ and $\mu a^{q-q\gamma_q}>0$ is large. This natural question has been proposed by Soave in [49] as an open problem, which can be summarized to be the following one:

(Q_2) **Does $\mathcal{E}_\mu(u)|_{\mathcal{S}_a}$ have a ground state if $\mu>0$ and $\mu a^{q-q\gamma_q}>0$ large?**

In [49], Soave conjectures that the answer of (Q_2) is *negative* in general.

Finally, in these results, the asymptotic behavior is only for $\|\nabla u_{a,\mu,-}\|_2$ in the cases of $2+\frac{4}{N}\leq q<2^*$. Thus, it is also natural to ask if it is possible to characterize the asymptotic behavior of $u_{a,\mu,-}$, and not only of $\|\nabla u_{a,\mu,-}\|_2$. In [49, Remark 8.1], Soave pointed out that in dimensions $N=3,4$, it could be proved that $\|\nabla u_{a,\mu,-}\|_2\to S^{\frac{N}{2}}$, but $u_{a,\mu,-}\rightharpoonup0$ in H^1 while, in dimensions $H\geq5$, both $u_{a,\mu,-}\rightharpoonup0$ and $u_{a,\mu,-}\rightharpoonup\widetilde{u}\neq0$ could happen. He also *conjectures* that the weak limit of $\{u_{a,\mu,-}\}$ will be the Aubin-Talanti babbles in the higher dimensions $N\geq5$. Soave's conjecture can be slightly generalized to the following question:

(Q_3) **Can we capture the precisely asymptotic behavior of $u_{a,\mu,-}$ as $\mu\to0$?**

In this paper, we are interested in these questions and we shall give some answers to them, which will give more information on the ground states of (1.5). Our first result, which is devoted to the existence and nonexistence of ground states, can be stated as follows.

Theorem 1.1. *Let $N\geq3$, $2<q<2^*$ and $a,\mu>0$.*

(1) *If $N=3$ and $2<q<2+\frac{4}{N}$, then for $\mu a^{q-q\gamma_q}<\alpha_{N,q}$, $m_{a,\mu}^-$ can be attained by some $u_{a,\mu,-}$ which is real valued, positive, radially symmetric and radially decreasing, and thus, (1.5) has a second solution $u_{a,\mu,-}$ with some $\lambda_{a,\mu,-}<0$.*

(2) *If* $q = 2+\frac{4}{N}$, *then* $m^-_{a,\mu}$ *can not be attained for* $\mu a^{q-q\gamma_q} \geq \alpha_{N,q}$ *and thus,* (1.5) *has no ground states for* $\mu a^{q-q\gamma_q} \geq \alpha_{N,q}$.

(3) *If* $2+\frac{4}{N} < q < 2^*$, *then for all* $\mu > 0$, *and* $m^-_{a,\mu}$ *can be attained by some* $u_{a,\mu,-}$ *which is real valued, positive, radially symmetric and radially decreasing, and thus,* (1.5) *has a ground state* $u_{a,\mu,-}$ *with some* $\lambda_{a,\mu,-} < 0$ *for all* $\mu > 0$.

Remark 1.2. (a) (1) *of Theorem 1.1, which together the results of [34], gives a completely positive answer to the question* (Q_1).

(b) *As pointed out in the very recent work [34], the crucial point in studying* (Q_1) *is to obtain a good energy estimate of* $m^-_{a,\mu}$ *for* $2 < q < 2+\frac{4}{N}$ *such that the compactness of minimizing sequence or* (PS) *sequence at the energy level* $m^-_{a,\mu}$ *still holds. As for other concave-convex problems (cf. [6]) and observed in [34], the threshold of such compactness should be* $m^+_{a,\mu}+\frac{1}{N}S^{\frac{N}{2}}$. *Since* $m^-_{a,\mu}$ *is a mountain-pass level, the classical idea, which can be traced back to [16], is to use the ground state* $u_{a,\mu,+}$ *and the Aubin-Talanti babbles to construct a good path, whose energy can be well controlled from above to make sure that it is smaller than the threshold* $m^+_{a,\mu}+\frac{1}{N}S^{\frac{N}{2}}$. *This strategy is already used in [34] to prove the existence of critical points of* $\mathcal{E}_\mu(u)|_{\mathcal{S}_a}$ *of mountain-pass type for* $N \geq 4$ *and* $2 < q < 2+\frac{4}{N}$. *Unlike [34] in which nonradial test function composing of* $u_{a,\mu,+}$ *and a bubble at* ∞ *is used, here we directly use the radial superposition of* $u_{a,\mu,+}$ *and the Aubin-Talenti bubble. This test function seems to be more natural and it works for all dimensions.*

(c) (2) *and* (3) *of Theorem 1.1 give partial answers to* (Q_2) *and they are proved by observing the non-increasing of* $m^-_{a,\mu}$ *and suitable choices of test functions. These two conclusions imply that the* L^2*-critical and supercritical perturbations have quite different influence on* (1.5). *Moreover, it seems that the critical mass of ground states also exists for* (1.5) *in the* L^2*-critical case.*

Our next result will be devoted to the precisely asymptotic behaviors of the solutions found in [49, Theorem 1.1], [34, Theorem 1.6] and Theorem 1.1 as $\mu \to 0$. To state this result, let us first introduce some necessary notations. By [55, Theorem B], the Gagliardo-Nirenberg inequality,

$$\|u\|_q \leq C_{N,q}\|u\|_2^{1-\gamma_q}\|\nabla u\|_2^{\gamma_q} \quad \text{for all } u \in H^1(\mathbb{R}^N), \tag{1.9}$$

has a minimizer ϕ_0, which satisfies

$$\begin{cases} -\Delta\phi_0+\nu_0\phi_0 = \sigma_0\phi_0^{q-1} \quad \text{in } \mathbb{R}^N, \\ \phi_0(0) = \max_{x\in\mathbb{R}^N}\phi_0(x), \\ \phi_0(x) > 0 \quad \text{in } \mathbb{R}^N, \\ \phi_0(x) \to 0 \quad \text{as } |x| \to +\infty, \end{cases} \tag{1.10}$$

where $\nu_0 = \frac{4}{N(q-2)}(1-\frac{(q-2)(N-2)}{4})$, $\sigma_0 = \frac{4}{N(q-2)}$ and $C_{N,q}$ is the best constant in the Gagliardo-Nirenberg inequality. On the other hand, the Aubin-Talanti babbles,

$$U_\varepsilon(x) = [N(N-2)]^{\frac{N-2}{4}}\left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)^{\frac{N-2}{2}}, \tag{1.11}$$

is the only solutions to the following equation:

$$\begin{cases} -\Delta u = u^{2^*-1} \quad \text{in } \mathbb{R}^N, \\ u(0) = \max_{x\in\mathbb{R}^N} u(x), \\ u(x) > 0 \quad \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to +\infty. \end{cases}$$

Now, our second result can be stated as follows.

Theorem 1.2. *Let $N \geq 3$, $2 < q < 2^*$ and $a, \mu > 0$ such that $\mu > 0$ is sufficiently small. Let $\widetilde{u}_\mu$ be the minimizer of $\mathcal{E}_\mu(u)|_{\mathcal{S}_a}$ in $\mathcal{P}_+^{a,\mu}$ and $\widehat{u}_\mu$ be the minimizer of $\mathcal{E}_\mu(u)|_{\mathcal{S}_a}$ in $\mathcal{P}_-^{a,\mu}$. Then*

(1) *For $2 < q < 2 + \frac{4}{N}$, $\widetilde{w}_{a,\mu}(x) = s_\mu^{\frac{N}{2}} \widetilde{u}_\mu(s_\mu x) \to \nu_a^{\frac{1}{q-2}} \phi_0(\sqrt{\nu_a} x)$ strongly in $H^1(\mathbb{R}^N)$ as $\mu \to 0$, where ϕ_0 is the unique solution of (1.10),*

$$\nu_a = \left(\frac{a^2}{\|\phi_0\|_2^2}\right)^{\frac{2(q-2)}{4-N(q-2)}} \tag{1.12}$$

and $s_\mu \sim \mu^{\frac{1}{2-q\gamma_q}}$ is the unique solution of the following system:

$$\begin{cases} s_\mu^2 \|\nabla\psi_{\nu_a,1}\|_2^2 - \mu\gamma_q \|\psi_{\nu_a,1}\|_q^q s_\mu^{q\gamma_q} - \|\psi_{\nu_a,1}\|_{2^*}^{2^*} s_\mu^{2^*} = 0, \\ 2s_\mu^2 \|\nabla\psi_{\nu_a,1}\|_2^2 - \mu q\gamma_q^2 \|\psi_{\nu_a,1}\|_q^q s_\mu^{q\gamma_q} - 2^* \|\psi_{\nu_a,1}\|_{2^*}^{2^*} s_\mu^{2^*} > 0, \end{cases} \tag{1.13}$$

where $\psi_{\nu_a,1}(x) = \nu_a^{\frac{1}{q-2}} \phi_0(\sqrt{\nu_a} x)$. Moreover, up to translations and rotations, $\widetilde{u}_\mu$ is the unique ground state of (1.5) for $\mu > 0$ sufficiently small.

(2) *For $N \geq 5$, $\widehat{u}_\mu \to U_{\varepsilon_0}$ strongly in $H^1(\mathbb{R}^N)$ as $\mu \to 0$, where U_{ε_0} is the Aubin-Talanti babble satisfying $\|U_{\varepsilon_0}\|_2^2 = a^2$. Moreover, up to translations and rotations, $\widehat{u}_\mu$ is the unique minimizer of $\mathcal{E}_\mu(u)|_{\mathcal{S}_a}$ in $\mathcal{P}_-^{a,\mu}$ for $\mu > 0$ sufficiently small.*

(3) *For $N = 3, 4$, $\widehat{w}_{a,\mu}(x) = \varepsilon_\mu^{\frac{N-2}{2}} \widehat{u}_\mu(\varepsilon_\mu x) \to U_{\varepsilon_0}$ strongly in $D^{1,2}(\mathbb{R}^N)$ for some $\varepsilon_0 > 0$ as $\mu \to 0$ up to a subsequence, where ε_μ satisfies*

$$\mu \sim \begin{cases} \varepsilon_\mu^{6-q} e^{-2\varepsilon_\mu^{-2}}, \quad N = 4, 2 < q < 4, \\ \varepsilon_\mu^{\frac{q}{2}-1}, \quad N = 3, 3 < q < 6, \\ \dfrac{\varepsilon_\mu^{\frac{1}{2}}}{\ln(\frac{1}{\varepsilon_\mu})}, \quad N = 3, q = 3, \\ \varepsilon_\mu^{5-\frac{3q}{2}}, \quad N = 3, 2 < q < 3. \end{cases}$$

Remark 1.3. (1) *The precise asymptotic behaviors of $\widetilde{u}_\mu$ and $\widehat{u}_\mu$ stated in (1) and (2) of Theorem 1.2 are captured by comparing the energy values and norms by full using the variational formulas of $\widetilde{u}_\mu$ and $\widehat{u}_\mu$, and minimizers of the Gagliardo-Nirenberg inequality and the Aubin-Talanti bubbles. In this argument, the unique determination of minimizers of the Gagliardo–Nirenberg inequality (1.9) for $2 < q < 2 + \frac{4}{N}$ and Aubin-Talanti bubbles for $N \geq 5$ in $\mathcal{S}_a$, respectively, is crucial. Moreover, (2) of Theorem 1.2 also gives a positive answer to Soave's conjecture on (Q_3).*

(2) *For the local uniqueness, the standard strategy is to assume the contrary and obtain a contradiction by full using the non-degeneracy of minimizers of the Gagliardo–Nirenberg inequality and Aubin-Talanti bubbles in passing to the limit (cf. [21, 27]), which is powerful in studying problems with potentials. Since* (1.5) *is autonomous, we can use a different method, based on the precisely asymptotic behaviors of* $\widetilde{u}_\mu$ *and the implicit function theorem, to prove the local uniqueness of* $\widetilde{u}_\mu$ *in a more direct way. It is worth pointing out that our method is also based on the non-degenerate of minimizers of the Gagliardo–Nirenberg inequality. For* $\widehat{u}_\mu$*, we remark that since the linear operator of the limit equation is different from that of* (1.5)*, our direct methods, based on implicit function theorem, is invalid. Thus, we will still use the standard method, that is to assume the contrary and obtain a contradiction by full using the non-degeneracy of Aubin-Talanti bubbles.*

(3) *Since we loss the* L^2*-integrability of the Aubin-Talanti babbles* $\{U_\varepsilon\}$ *for* $N = 3, 4$*, the asymptotic behavior of* $\widehat{u}_\mu$ *as* $\mu \to 0$ *for* $N = 3, 4$ *is much weaker than that of* $N \geq 5$ *in the sense that, the convergence is only for subsequences, which also leads us to loss the local uniqueness of* $\widehat{u}_\mu$ *for* $\mu > 0$ *sufficiently small in these two cases. We also remark that since we loss the* L^2*-integrability of the Aubin-Talanti babbles* $\{U_\varepsilon\}$ *for* $N = 3, 4$*, the asymptotic behavior of* $\widehat{u}_\mu$ *can not be obtained by merely using variational arguments to compare the energy values and norms as that for* (2) *of Theorem 1.2. Thus, to capture the precisely asymptotic behavior of* $\widehat{u}_\mu$*, we drive some uniformly pointwise estimates of* $\widehat{u}_\mu$ *by the maximum principle (cf. [20]) and some ODE technique used in [7] (see also [24, 38]). With these additional estimates, we obtain the precisely asymptotic behavior of* $\widehat{u}_\mu$ *for* $N = 3, 4$*. It is worth pointing out that, in the case* $N = 3$ *and* $2 < q < 3$*, since the nonlinearity decays too slow at infinity, we need to further employ the bootstrapping argument to drive the desired estimates.*

Our final result is devoted to the asymptotic behavior of the minimizers of $\mathcal{E}_\mu(u)|_{\mathcal{S}_a}$ in $\mathcal{P}_-^{a,\mu}$ as μ close to its upper-bound in the cases of $2 + \frac{4}{N} \leq q < 2^*$. It can be stated as follows.

Theorem 1.3. *Assume* $N \geq 3$, $2 + \frac{4}{N} \leq q < 2^*$ *and* $\mu, a > 0$*. Let* $\widehat{u}_\mu$ *be the minimizer of* $\mathcal{E}_\mu(u)|_{\mathcal{S}_a}$ *in* $\mathcal{P}_-^{a,\mu}$*, found in [49, Theorem 1.1] for* $q = 2 + \frac{4}{N}$ *with* $0 < \mu a^{q - q\gamma_q} < \alpha_{N,q}$ *and found in Theorem 1.1 for* $2 + \frac{4}{N} < q < 2^*$ *with all* $\mu > 0$*. Then*

(1) *For* $q = 2 + \frac{4}{N}$, $\widehat{v}_\mu = (\frac{a}{\|\phi_0\|_2})^{\frac{N-2}{2}} s_\mu^{\frac{N}{2}} \widehat{u}_\mu(\frac{a}{\|\phi_0\|_2} s_\mu x) \to (\nu_a')^{\frac{1}{q-2}} \phi_0(\sqrt{\nu_a'} x)$ *strongly in* $H^1(\mathbb{R}^N)$ *as* $\mu \to \alpha_{N,q,a}$ *up to a subsequence, where* $\alpha_{N,q,a} = a^{q\gamma_q - q}\alpha_{N,q}$ *for some* $\nu_a' > 0$ *and* $s_\mu = (1 - \frac{\mu}{\alpha_{N,q,a}})^{-\frac{N-2}{4}}$.

(2) *For* $2 + \frac{4}{N} < q < 2^*$, $\widehat{v}_\mu = s_\mu^{\frac{N}{2}} \widehat{u}_\mu(s_\mu x) \to \nu_a^{\frac{1}{q-2}} \phi_0(\sqrt{\nu_a} x)$ *strongly in* $H^1(\mathbb{R}^N)$ *as* $\mu \to +\infty$*, where* $s_\mu = \mu^{\frac{1}{q\gamma_q - 2}}$*. Moreover, up to translations and rotations,* $\widehat{u}_\mu$ *is also the unique ground state of* (1.5) *for* $\mu > 0$ *sufficiently large.*

Remark 1.4. (1) *The ideas in proving Theorem 1.3 are similar to that of Theorem 1.2. However, in the* L^2*-critical case* $q = 2 + \frac{4}{N}$*, the convergence of* $\widehat{u}_\mu$ *is much weaker than that in the* L^2*-supcritical case* $2 + \frac{4}{N} < q < 2^*$

in the sense that, it only holds for subsequences. The main reason is that in the L^2-critical case $q = 2 + \frac{4}{N}$, we have $\|\varphi\|_2^2 = const.$ for all φ being a minimizer of the Gagliardo–Nirenberg inequality (1.9). Thus, the precise mass $\|\widehat{u}_\mu\|_2^2 = a^2$ is invalid in determining a unique minimizer of the Gagliardo–Nirenberg inequality (1.9) in the case $q = 2 + \frac{4}{N}$. Moreover, unlike the studies for problems with homogeneous nonlinearities (cf. [26, 27]), combining nonlinearities (L^2-critical and L^2-supercritical) of (1.5) makes the asymptotic behavior of $\widehat{u}_\mu$ to be more complicated, which also make us loss the local uniqueness of $\widehat{u}_\mu$ for $\mu > 0$ close to its upper bound in this case. Indeed, as μ goes to its upper bound in the L^2-critical case, comparing with the studies for problems with homogeneous nonlinearities, the Sobolev critical term of (1.5) is an additionally inhomogenous perturbation in passing to the limit, which makes the oscillations occurring.

Notations. Throughout this paper, C and C' are indiscriminately used to denote various absolutely positive constants. $a \sim b$ means that $C'b \le a \le Cb$ and $a \lesssim b$ means that $a \le Cb$.

2. Asymptotic behavior of $u_{a,\mu,+}$

By [49, Theorem 1.1], $m^+_{a,\mu}$ can always be attained by some $u_{a,\mu,+}$ for $2 < q < 2 + \frac{4}{N}$ and $\mu a^{q-q\gamma_q} < \alpha_{N,q}$, where $m^+_{a,\mu}$ is given by (1.7) and $u_{a,\mu,+}$ is real valued, positive, radially symmetric and radially decreasing. Our goal in this section is to give an asymptotic behavior of $u_{a,\mu,+}$ as $\mu \to 0$, which is more precisely than that in [49, Theorem 1.4], and capture the precisely decaying rate of $u_{a,\mu,+}$ as $\mu \to 0$. We recall that by [49, Theorem 1.1], $u_{a,\mu,+}$ is a solution of (1.5) for some $\lambda_{a,\mu,+} < 0$. To simplify the notation, we shall denote $u_{\mu,+} = u_{a,\mu,+}$ and $\lambda_{\mu,+} = \lambda_{a,\mu,+}$, since we will fix $a > 0$ in what follows. Let us begin with

Lemma 2.1. *Let $2 < q < 2 + \frac{4}{N}$. Then $-\lambda_{\mu,+} \sim \|\nabla u_{\mu,+}\|_2^2 \sim \mu^{\frac{2}{2-q\gamma_q}}$ as $\mu \to 0$.*

Proof. Since $u_{\mu,+} \in \mathcal{P}^{a,\mu}_+$, we have

$$\|\nabla u_{\mu,+}\|_2^2 = \mu\gamma_q\|u_{\mu,+}\|_q^q + \|u_{\mu,+}\|_{2^*}^{2^*} \tag{2.1}$$

and

$$2\|\nabla u_{\mu,+}\|_2^2 > \mu q\gamma_q^2\|u_{\mu,+}\|_q^q + 2^*\|u_{\mu,+}\|_{2^*}^{2^*}.$$

It follows from the Gagliardo–Nirenberg inequality that

$$\|\nabla u_{\mu,+}\|_2^2 \lesssim \mu\|u_{\mu,+}\|_q^q \lesssim \mu\|\nabla u_{\mu,+}\|_2^{q\gamma_q},$$

which together with $q\gamma_q < 2$ for $2 < q < 2 + \frac{4}{N}$, implies

$$\|\nabla u_{\mu,+}\|_2^2 \lesssim \mu^{\frac{2}{2-q\gamma_q}}. \tag{2.2}$$

Thus, by (2.1) and (2.2), we also have

$$\mu\|u_{\mu,+}\|_q^q \lesssim \mu^{\frac{2}{2-q\gamma_q}}. \tag{2.3}$$

Let us define

$$V_\varepsilon(x) = U_\varepsilon(x)\varphi(R_\varepsilon^{-1}x) \tag{2.4}$$

where $U_\varepsilon(x)$ is the Aubin-Talanti babbles given by (1.11) and $\varphi \in C_0^\infty(\mathbb{R}^N)$ is a radial cut-off function with $\varphi \equiv 1$ in B_1, $\varphi \equiv 0$ in B_2^c, and R_ε is chosen such that

$V_\varepsilon \in \mathcal{S}_a$. More precisely, for $N \geq 5$, we choose $\varepsilon = \varepsilon_0$ and $R_{\varepsilon_0} = +\infty$ such that $V_{\varepsilon_0} = U_{\varepsilon_0} \in \mathcal{S}_a$ while for $N = 3, 4$, we choose $\varepsilon > 0$ sufficiently small and then in the later two cases, we have

$$a^2 = \int_{\mathbb{R}^N} (U_\varepsilon(x)\varphi(R_\varepsilon^{-1}x))^2 \sim \varepsilon^2 \int_1^{R_\varepsilon \varepsilon^{-1}} r^{3-N} \sim \begin{cases} \varepsilon^2 \ln(R_\varepsilon \varepsilon^{-1}), & \text{for } N = 4, \\ \varepsilon R_\varepsilon, & \text{for } N = 3, \end{cases} \tag{2.5}$$

which implies $R_\varepsilon \varepsilon^{-1} \to +\infty$ as $\varepsilon \to 0$. Then it is well-known (cf. [44, (4.2)–(4.5)] or [53, Chapter III]) that

$$\|\nabla V_\varepsilon\|_2^2 = S^{\frac{N}{2}} + O((R_\varepsilon \varepsilon^{-1})^{2-N}), \quad \|V_\varepsilon\|_{2^*}^{2^*} = S^{\frac{N}{2}} + O((R_\varepsilon \varepsilon^{-1})^{-N}) \tag{2.6}$$

for $\varepsilon > 0$ sufficiently small, which implies

$$\|\nabla V_\varepsilon\|_2^2 \sim S^{\frac{N}{2}} \sim \|V_\varepsilon\|_{2^*}^{2^*} \tag{2.7}$$

for $\varepsilon > 0$ sufficiently small. Now, we fix $\varepsilon = \varepsilon_0$ and $R_{\varepsilon_0} = +\infty$ for $N = 5$, and fix $\varepsilon > 0$ sufficiently small and choose R_ε as that in (2.5) for $N = 3, 4$ such that (2.7) holds for all $N \geq 3$. By [49, Lemma 4.2], there exists $t(\mu) > 0$ such that $(V_\varepsilon)_{t(\mu)} \in \mathcal{P}_+^{a,\mu}$ for $\mu > 0$ sufficiently small, where

$$(V_\varepsilon)_{t(\mu)} = [t(\mu)]^{\frac{N}{2}} V_\varepsilon(t(\mu)x).$$

Then

$$[t(\mu)]^2 \|\nabla V_\varepsilon\|_2^2 = \mu\gamma_q \|V_\varepsilon\|_q^q [t(\mu)]^{q\gamma_q} + \|V_\varepsilon\|_{2^*}^{2^*} [t(\mu)]^{2^*}$$

and

$$2[t(\mu)]^2 \|\nabla V_\varepsilon\|_2^2 > \mu q\gamma_q^2 \|V_\varepsilon\|_q^q [t(\mu)]^{q\gamma_q} + 2^* \|V_\varepsilon\|_{2^*}^{2^*} [t(\mu)]^{2^*}.$$

Since $q\gamma_q < 2$ for $2 < q < 2 + \frac{4}{N}$, by

$$(2^* - 2)[t(\mu)]^2 \|\nabla V_\varepsilon\|_2^2 < \mu(2^* - q\gamma_q)\gamma_q \|V_\varepsilon\|_q^q [t(\mu)]^{q\gamma_q},$$

it is easy to see that $t(\mu) \to 0$ as $\mu \to 0$ for all $N \geq 3$. It follows that

$$[t(\mu)]^2 \sim \mu[t(\mu)]^{q\gamma_q} \quad \text{as } \mu \to 0,$$

which implies $t(\mu) \sim \mu^{\frac{1}{2-q\gamma_q}}$ as $\mu \to 0$. Thus, by $q\gamma_q < 2$ for $2 < q < 2 + \frac{4}{N}$ once more,

$$\mathcal{E}_\mu((V_\varepsilon)_{t(\mu)}) = (\frac{1}{2} - \frac{1}{q\gamma_q})\|\nabla V_\varepsilon\|_2^2 [t(\mu)]^2 + (\frac{1}{q\gamma_q} - \frac{1}{2^*})\|V_\varepsilon\|_{2^*}^{2^*} [t(\mu)]^{2^*} \sim -\mu^{\frac{2}{2-q\gamma_q}}.$$

Therefore, by $\mathcal{E}_\mu((V_\varepsilon)_{t(\mu)}) \geq m_{a,\mu}^+$ and $m_{a,\mu}^+ \gtrsim -\mu\|u_{\mu,+}\|_q^q$, we have

$$\mu\|u_{\mu,+}\|_q^q \gtrsim \mu^{\frac{2}{2-q\gamma_q}},$$

which together with (2.3), implies

$$\mu\|u_{\mu,+}\|_q^q \sim \mu^{\frac{2}{2-q\gamma_q}}.$$

By the regularity of $u_{\mu,+}$ and the Pohozaev identity, $\lambda_{\mu,+} \sim -\mu\|u_{\mu,+}\|_q^q$, and by (2.2) and $u_{\mu,+} \in \mathcal{P}_+^{a,\mu}$, $\|\nabla u_{\mu,+}\|_2^2 \sim \mu\|u_{\mu,+}\|_q^q$. Therefore,

$$-\lambda_{\mu,+} \sim \|\nabla u_{\mu,+}\|_2^2 \sim \mu^{\frac{2}{2-q\gamma_q}}$$

as $\mu \to 0$. It completes the proof. □

By the well-known uniqueness result (cf. [39]) and the scaling invariance of (1.10),

$$\phi_0(x) = \left(\frac{\nu_0}{\sigma_0}\right)^{\frac{1}{q-2}} w(\sqrt{\nu_0}x),$$

where w is the unique solution of the following equation:

$$\begin{cases} -\Delta u + u = u^{q-1} \quad \text{in } \mathbb{R}^N, \\ u(0) = \max_{x\in\mathbb{R}^N} u(x), \\ u(x) > 0 \quad \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to +\infty, \end{cases} \tag{2.8}$$

A direct calculation also shows that

$$\psi_{\nu,\sigma}(x) = (\frac{\nu}{\sigma})^{\frac{1}{q-2}}\phi_0(\sqrt{\frac{\nu}{\sigma}}x) \tag{2.9}$$

for $\nu, \sigma > 0$ are all minimizers of the Gagliardo-Nirenberg inequality (1.9). Let ν_a be given by (1.12), then for $q \neq 2 + \frac{4}{N}$, $\psi_{\nu_a,1} \in \mathcal{S}_a$ and $\psi_{\nu_a,1}$ is a minimizer of the Gagliardo–Nirenberg inequality, that is,

$$\|\psi_{\nu_a,1}\|_q^q = C_{N,q}^q a^{q-q\gamma_q}\|\nabla\psi_{\nu_a,1}\|_2^{q\gamma_q}. \tag{2.10}$$

For the sake of simplicity, we re-denote $\psi_a = \psi_{\nu_a,1}$.

Proposition 2.1. *Let $2 < q < 2 + \frac{4}{N}$. Then $w_{\mu,+} \to \psi_a$ strongly in $H^1(\mathbb{R}^N)$ as $\mu \to 0$, where $w_{\mu,+} = s_\mu^{-\frac{N}{2}}u_{\mu,+}(s_\mu^{-1}x)$ with s_μ being the unique solution of (1.13). Moreover, up to translations and rotations, $u_{\mu,+}$ is the unique ground state of (1.5) for $\mu > 0$ sufficiently small.*

Proof. Since $\psi_a \in \mathcal{S}_a$, by [49, Lemma 4.2], there exists a unique $s_\mu > 0$ such that $(\psi_a)_{s_\mu} \in \mathcal{P}_+^{a,\mu}$ for $\mu > 0$ sufficiently small where $(\psi_a)_{s_\mu} = s_\mu^{\frac{N}{2}}\psi_a(s_\mu x)$. That is,

$$s_\mu^2\|\nabla\psi_a\|_2^2 = \mu\gamma_q\|\psi_a\|_q^q s_\mu^{q\gamma_q} + \|\psi_a\|_{2^*}^{2^*}s_\mu^{2^*} \tag{2.11}$$

and

$$2s_\mu^2\|\nabla\psi_a\|_2^2 > \mu q\gamma_q^2\|\psi_a\|_q^q s_\mu^{q\gamma_q} + 2^*\|\psi_a\|_{2^*}^{2^*}s_\mu^{2^*}. \tag{2.12}$$

As that in the proof of Lemma 2.1, we have

$$\|\nabla(\psi_a)_{s_\mu}\|_2^{2-q\gamma_q} < C_{N,q}^q\gamma_q\mu a^{q-q\gamma_q}\frac{2^*-q\gamma_q}{2^*-2}. \tag{2.13}$$

Since $u_{\mu,+} \in \mathcal{P}_+^{a,\mu}$, we also have

$$\|\nabla u_{\mu,+}\|_2^{2-q\gamma_q} < C_{N,q}^q\gamma_q\mu a^{q-q\gamma_q}\frac{2^*-q\gamma_q}{2^*-2}. \tag{2.14}$$

Now, using $(\psi_a)_{s_\mu}$ as a test function of $m_{a,\mu}^+$ and by (2.10),

$$m_{a,\mu}^+ \leq \mathcal{E}_\mu((\psi_a)_{s_\mu}) = \frac{1}{N}\|\nabla(\psi_a)_{s_\mu}\|_2^2 - \frac{\mu a^{q-q\gamma_q}C_{N,q}^q}{q}(1-\frac{q\gamma_q}{2^*})\|\nabla(\psi_a)_{s_\mu}\|_2^{q\gamma_q}.$$

By the Gagliardo–Nirenberg inequality (1.9),

$$m_{a,\mu}^+ = \mathcal{E}_\mu(u_{\mu,+}) \geq \frac{1}{N}\|\nabla u_{\mu,+}\|_2^2 - \frac{\mu a^{q-q\gamma_q}C_{N,q}^q}{q}(1-\frac{q\gamma_q}{2^*})\|\nabla u_{\mu,+}\|_2^{q\gamma_q}.$$

Let us consider the function

$$f(t) = \frac{1}{N}t^2 - \frac{\mu a^{q-q\gamma_q} C_{N,q}^q}{q}(1 - \frac{q\gamma_q}{2^*})t^{q\gamma_q}.$$

A direct calculation shows that $f(t)$ is strictly decreasing in $(0, t_0)$, where

$$t_0 = \left(C_{N,q}^q \gamma_q \mu a^{q-q\gamma_q} \frac{2^* - q\gamma_q}{2^* - 2}\right)^{\frac{1}{2-q\gamma_q}}.$$

Thus, by (2.13) and (2.14),

$$\|\nabla u_{\mu,+}\|_2^2 \geq \|\nabla (\psi_a)_{s_\mu}\|_2^2. \tag{2.15}$$

By (2.11) and (2.12), we can use similar arguments as that used in the proof of Lemma 2.1 to show that $s_\mu \sim \mu^{\frac{1}{2-q\gamma_q}}$ as $\mu \to 0$. It then follows from (2.9) and (2.11) that

$$s_\mu = (1 + o_\mu(1))\left(\frac{\mu\gamma_q\|\psi_a\|_q^q}{\|\nabla\psi_a\|_2^2}\right)^{\frac{1}{2-q\gamma_q}} = (1 + o_\mu(1))\left(\frac{\mu\gamma_q\|\phi_0\|_q^q}{\|\nabla\phi_0\|_2^2}\right)^{\frac{1}{2-q\gamma_q}}.$$

Since by the Pohozaev identity satisfied by ϕ, we have $\frac{1}{N}\|\nabla\phi_0\|_2^2 = \frac{(q-2)\sigma_0}{2q}\|\phi_0\|_q^q$. By (1.6), $s_\mu = [(\frac{1}{\sigma_0} + o_\mu(1))\mu]^{\frac{1}{2-q\gamma_q}}$. Let

$$w_{\mu,+} = s_\mu^{-\frac{N}{2}} u_{\mu,+}(s_\mu^{-1}x).$$

Since $u_{\mu,+}$ satisfies (1.5), $w_{\mu,+}$ satisfies the following equation:

$$-\Delta w_{\mu,+} = \lambda_{\mu,+} s_\mu^{-2} w_{\mu,+} + \mu s_\mu^{-2+\frac{N}{2}(q-2)} w_{\mu,+}^{q-1} + s_\mu^{-2+\frac{N}{2}(2^*-2)} w_{\mu,+}^{2^*-1}. \tag{2.16}$$

By Lemma 2.1 and

$$\int_{\mathbb{R}^N} w_{\mu,+}^2 = \int_{\mathbb{R}^N} u_{\mu,+}^2 \equiv a^2,$$

we have

$$\|\nabla w_{\mu,+}\|_2^2 + \|w_{\mu,+}\|_2^2 = s_\mu^{-2}\|\nabla u_{\mu,+}\|_2^2 + a^2 \sim 1.$$

Therefore, $\{w_{\mu,+}\}$ is bounded in $H^1(\mathbb{R}^N)$. It follows that $w_{\mu,+} \rightharpoonup w_*$ weakly in $H^1(\mathbb{R}^N)$ as $\mu \to 0$ up to a subsequence. Note that $w_{\mu,+}$ is radial, by Struss's radial lemma (cf. [12, Lemma A.IV, Theorem A.I'] or [44, Lemma 3.1]) and the Sobolev embedding theorem, $w_{\mu,+} \to w_*$ strongly in $L^q(\mathbb{R}^N)$ as $\mu \to 0$ up to a subsequence. By Lemma 2.1 once more, $\{\lambda_{\mu,+}\mu^{\frac{-2}{2-q\gamma_q}}\}$ is bounded. Thus, $\lambda_{\mu,+}\mu^{\frac{-2}{2-q\gamma_q}} \to \alpha_*$ as $\mu \to 0$ up to a subsequence. On the other hand, by $q\gamma_q < 2$ for $2 < q < 2 + \frac{4}{N}$,

$$s_\mu^{-2+\frac{N}{2}(2^*-2)} \sim \mu^{\frac{2^*-2}{2-q\gamma_q}} \to 0$$

as $\mu \to 0$. Now, using (2.15) and (2.16), it is standard to show that $w_{\mu,+} \to w_*$ strongly in $H^1(\mathbb{R}^N)$ as $\mu \to 0$ up to a subsequence, where w_* is the unique solution of the following equation:

$$\begin{cases} -\Delta u + \alpha_* u = \sigma_0 u^{q-1} \quad \text{in } \mathbb{R}^N, \\ u(0) = \max_{x\in\mathbb{R}^N} u(x), \\ u(x) > 0 \quad \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to +\infty, \end{cases} \tag{2.17}$$

by the well-known uniqueness result (cf. [39]) and the scaling invariance of (1.10), $w_*(x)=(\frac{\alpha_*}{\sigma_0})^{\frac{1}{q-2}}w(\sqrt{\alpha_*}x)$, where w is the unique solution of (2.8). It follows from $\|w_{\mu,+}\|_2^2=a^2$ and the strong convergence of $\{w_{\mu,+}\}$ in $H^1(\mathbb{R}^N)$ that $\|w_*\|_2^2=a^2$, which implies $\alpha_*=\nu_a\nu_0$ where ν_a is given by (1.12). Thus, $w_*=\psi_a$. Since ψ_a is unique, $w_{\mu,+}\to\psi_a$ strongly in $H^1(\mathbb{R}^N)$ as $\mu\to 0$. The system (1.13) directly comes from (2.11) and (2.12). It remains to prove the local uniqueness of $u_{\mu,+}$ for $\mu>0$ sufficiently small. Let us consider the following system:

$$\begin{cases}\mathcal{F}(w,\alpha,\beta,\gamma)=\Delta w-\alpha\nu_0 w+\beta w^{q-1}+\gamma w^{2^*-1},\\ \mathcal{G}(w,\alpha,\beta,\gamma)=\|w\|_2^2-a^2,\end{cases}\tag{2.18}$$

where $\alpha,\beta,\gamma>0$ are parameters. It is easy to see that $\mathcal{F}(\psi_a,\nu_a,\sigma_0,0)=0$ and $\mathcal{G}(\psi_a,\nu_a,\sigma_0,0)=0$. Let

$$\mathcal{L}(\psi_a,\nu_a,\sigma_0,0)=\begin{pmatrix}\partial_w\mathcal{F}(\psi_a,\nu_a,\sigma_0,0) & \partial_\alpha\mathcal{F}(\psi_a,\nu_a,\sigma_0,0)\\ \partial_w\mathcal{G}(\psi_a,\nu_a,\sigma_0,0) & \partial_\alpha\mathcal{G}(\psi_a,\nu_a,\sigma_0,0)\end{pmatrix}$$

be the linearization of the system (2.18) at $(\psi_a,\nu_a,\sigma_0,0)$ in $H^1(\mathbb{R}^N)\times\mathbb{R}$, that is,

$$\partial_w\mathcal{F}(\psi_a,\nu_a,\sigma_0,0)=\Delta-\nu_a\nu_0+(q-1)\sigma_0\psi_a^{q-2},\quad \partial_\alpha\mathcal{F}(\psi_a,\nu_a,\sigma_0,0)=-\nu_0\psi_a$$

and

$$\partial_w\mathcal{G}(\psi_a,\nu_a,\sigma_0,0)=2\psi_a,\quad \partial_\alpha\mathcal{G}(\psi_a,\nu_a,\sigma_0,0)=0.$$

Then $\mathcal{L}(\psi_a,\nu_a,\sigma_0,0)[(\phi,\tau)]=0$ if and only if

$$\begin{cases}\Delta\phi-\nu_a\nu_0\phi+(q-1)\sigma_0\psi_a^{q-2}\phi-\tau\nu_0\psi_a=0,\\ \displaystyle\int_{\mathbb{R}^N}\psi_a\phi=0.\end{cases}$$

Let us consider the following system:

$$\begin{cases}\Delta\phi-\nu_a\nu_0\phi+(q-1)\sigma_0\psi_a^{q-2}\phi-\tau\nu_0\psi_a=g,\\ \displaystyle\int_{\mathbb{R}^N}\psi_a\phi=b,\end{cases}\tag{2.19}$$

where $(g,b)\in H^1_{rad}(\mathbb{R}^N)\times\mathbb{R}$ with

$$H^1_{rad}(\mathbb{R}^N)=\{u\in H^1(\mathbb{R}^N)\mid u\text{ is radial}\}.$$

Then $\phi=\phi_g+\tau\nu_0\phi_a$, where ϕ_g and ϕ_a satisfies

$$\Delta\phi_g-\nu_a\nu_0\phi_g+(q-1)\sigma_0\psi_a^{q-2}\phi_g=g\tag{2.20}$$

and

$$\Delta\phi_a-\nu_a\nu_0\phi_a+(q-1)\sigma_0\psi_a^{q-2}\phi_a=\psi_a,\tag{2.21}$$

respectively. By [54, (5.2) and (5.3)], $\phi_a=\frac{1}{q-2}\psi_a+\frac{1}{2}(x\cdot\nabla\psi_a)$ and

$$\int_{\mathbb{R}^N}\phi_a\psi_a=(\frac{1}{q-2}-\frac{4}{N})\|\psi_a\|_2^2\neq 0$$

since $q\neq 2+\frac{4}{N}$. Thus, the unique solution of (2.19) is given by $(\phi_g+\tau_{b,g}\nu_0\phi_a,\tau_{b,g})$ where

$$\tau_{b,g}=\frac{b-\int_{\mathbb{R}^N}\phi_g\psi_a}{\nu_0\int_{\mathbb{R}^N}\phi_a\psi_a}.$$

Since $q < 2^*$, it is well-known that ψ_a is nondegenerate (cf. [41, Theorem 2.12] and [45, Lemma 4.2]). Thus, by $q < 2^*$, (2.20) only has zero solution in $H^1_{rad}(\mathbb{R}^N)$ for $g = 0$, which implies the linear operator $\mathcal{L}(\psi_a, \nu_a, \sigma_0, 0) : H^1_{rad}(\mathbb{R}^N) \times \mathbb{R} \to H^1_{rad}(\mathbb{R}^N) \times \mathbb{R}$ is bijective. Moreover, it is standard to show that

$$|\tau_{b,g}| + \|\phi_g + \tau_{b,g}\phi_a\|_{H^1} \lesssim |b| + \|g\|_{H^1}.$$

Now, by the implicit function theorem, there exists a unique C^1-curve $(w^{\beta,\gamma}, \alpha^{\beta,\gamma})$ in $H^1_{rad}(\mathbb{R}^N)\times\mathbb{R}^3$ for $|\beta-\sigma_0| << 1$ and $|\gamma| << 1$ such that $(w^{\sigma_0,0}, \alpha^{\sigma_0,0}) = (\psi_a, \nu_a)$, and

$$\mathcal{F}(w^{\beta,\gamma}, \alpha^{\beta,\gamma}, \beta, \gamma) \equiv 0, \quad \mathcal{G}(w^{\beta,\gamma}, \alpha^{\beta,\gamma}, \beta, \gamma) \equiv 0.$$

We recall that $w_{\mu,+}$ is radial and satisfies (2.16), and $w_{\mu,+} \to \psi_a$ strongly in $H^1(\mathbb{R}^N)$ as $\mu \to 0$ with $\|w_{\sigma,+}\|_2^2 = a^2$, thus, by the uniqueness of s_μ determined by (1.13), we must have $w_{\mu,+} = w^{\beta(\mu),\gamma(\mu)}$ for $\beta(\mu) = \mu s_\mu^{-2+\frac{N}{2}(q-2)}$ and $\gamma(\mu) = s_\mu^{-2+\frac{N}{2}(2^*-2)}$ with $\mu > 0$ sufficiently small. On the other hand, if $\widetilde{u}_{\mu,+}$ is another ground state of (1.5) with some $\widetilde{\lambda}_{\mu,+} \in \mathbb{R}$ for $\mu > 0$ sufficiently small, then by [49, Theorem 1.3], $\widetilde{u}_{\mu,+} = e^{i\theta}\widehat{u}_{\mu,+}$ where θ is a constant and $\widehat{u}_{\mu,+}$ is real valued and positive. Since by the Pohozaev identity, we always have $\widetilde{\lambda}_{\mu,+} < 0$. By applying the well-known Gidas-Ni-Nirenberg theorem (cf. [25]), $\widetilde{u}_{\mu,+}$ must be radially symmetric. Now, by running the arguments as used above once more, we know that $\widehat{w}_{\mu,+} = s_\mu^{-\frac{N}{2}}\widehat{u}_{\mu,+}(s_\mu^{-1}x) \to \psi_a$ strongly in $H^1(\mathbb{R}^N)$ as $\mu \to 0^+$ with $\|w_{\sigma,+}\|_2^2 = a^2$. It follows from the uniqueness of $w^{\beta(\mu),\gamma(\mu)}$ that $\widehat{u}_{\mu,+} = u_{\mu,+}$ for $\mu > 0$ sufficiently small. Thus, $u_{\mu,+}$ is the unique ground state of (1.5) for $\mu > 0$ sufficiently small up to translations and rotations. □

3. Existence and nonexistence of $u_{a,\mu,-}$

In this section, we shall mainly study the question (Q_1). Since in the very recent work [34], the question (Q_1) has been solved for $N \geq 4$. we only consider the case $N = 3$ and prove that $m^-_{a,\mu}$ can also be attained by some $u_{a,\mu,-}$ for $2 < q < 2 + \frac{4}{N}$ in the case $N = 3$ under some additional assumptions, where $m^-_{a,\mu}$ is also given by (1.7) and $u_{a,\mu,-}$ is also real valued, positive, radially symmetric and radially decreasing. The crucial point in this study is the following energy estimates.

Lemma 3.1. *Let $N = 3$, $2 < q < 2 + \frac{4}{N}$ and $\mu, a > 0$. Then for $\mu a^{q-q\gamma_q} < \alpha_{N,q}$,*

$$m^-_{a,\mu} = \inf_{u\in\mathcal{P}^{a,\mu}_-} \mathcal{E}_\mu(u) < m^+_{a,\mu} + \frac{1}{3}S^{\frac{3}{2}}. \tag{3.1}$$

Proof. Since $N = 3$, we have $U_\varepsilon = 3^{\frac{1}{4}}(\frac{\varepsilon}{\varepsilon^2+|x|^2})^{\frac{1}{2}}$. Let $W_\varepsilon = \chi(x)U_\epsilon$ where $\chi(x)$ is a cut-off function such that $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| > 2$. By simple computations, we have that

$$\|\nabla W_\varepsilon\|_2^2 = S^{\frac{3}{2}} + O(\varepsilon), \quad \|W_\varepsilon\|_6^6 = S^{\frac{3}{2}} + O(\varepsilon^3) \tag{3.2}$$

and

$$\|W_\varepsilon\|_p^p \sim \begin{cases} \varepsilon^{3-\frac{p}{2}}, & 3 < p < 6; \\ \varepsilon^{\frac{3}{2}}\ln\frac{1}{\varepsilon}, & p = 3; \\ \varepsilon^{\frac{p}{2}}, & 2 \leq p < 3. \end{cases} \tag{3.3}$$

Now, we define $\widehat{W}_{\varepsilon,t} = u_{\mu,+} + tW_\varepsilon$ and $\overline{W}_{\varepsilon,t} = s^{\frac{1}{2}}\widehat{W}_{\varepsilon,t}(sx)$. Then it is well-known that

$$\|\nabla\overline{W}_{\varepsilon,t}\|_2^2 = \|\nabla\widehat{W}_{\varepsilon,t}\|_2^2, \quad \|\overline{W}_{\varepsilon,t}\|_6^6 = \|\widehat{W}_{\varepsilon,t}\|_6^6, \tag{3.4}$$

and

$$\|\overline{W}_{\varepsilon,t}\|_2^2 = s^{-2}\|\widehat{W}_{\varepsilon,t}\|_2^2, \quad \|\overline{W}_{\varepsilon,t}\|_q^q = s^{q\gamma_q-q}\|\widehat{W}_{\varepsilon,t}\|_q^q. \tag{3.5}$$

We choose $s = \frac{\|\widehat{W}_{\varepsilon,t}\|_2}{a}$, then $\overline{W}_{\varepsilon,t} \in \mathcal{S}_a$. By [49, Lemma 4.2], there exist $\tau_{\varepsilon,t} > 0$ such that $(\overline{W}_{\varepsilon,t})_{\tau_{\varepsilon,t}} \in \mathcal{P}_-^{a,\mu}$, where $(\overline{W}_{\varepsilon,t})_{\tau_{\varepsilon,t}} = \tau_{\varepsilon,t}^{\frac{3}{2}}\overline{W}_{\varepsilon,t}(\tau_{\varepsilon,t}x)$. Thus,

$$\|\nabla\overline{W}_{\varepsilon,t}\|_2^2\tau_{\varepsilon,t}^{2-q\gamma_q} = \mu\gamma_q\|\overline{W}_{\varepsilon,t}\|_q^q + \|\overline{W}_{\varepsilon,t}\|_{2^*}^{2^*}\tau_{\varepsilon,t}^{2^*-q\gamma_q}. \tag{3.6}$$

Since $u_{\mu,+} \in \mathcal{P}_+^{a,\mu}$, by [49, Lemma 4.2], $\tau_{\varepsilon,0} > 1$. By (3.2) and (3.6), we also know that $\tau_{\varepsilon,t} \to 0$ as $t \to +\infty$ uniformly for $\varepsilon > 0$ sufficiently small. Since $\tau_{\varepsilon,t}$ is unique by [49, Lemma 4.2], it is standard to show that $\tau_{\varepsilon,t}$ is continuous for t, which implies that there exists $t_\varepsilon > 0$ such that $\tau_{\varepsilon,t_\varepsilon} = 1$. It follows that

$$m_{\mu,a}^- \le \sup_{t\ge0}\mathcal{E}_\mu(\overline{W}_{\varepsilon,t}). \tag{3.7}$$

Recall that $u_{\mu,+} \in \mathcal{S}_a$ and W_ε are positive, by (3.2), (3.4) and (3.5), there exists $t_0 > 0$ such that

$$\mathcal{E}_\mu(\overline{W}_{\varepsilon,t}) = (\frac{1}{2}\|\nabla\widehat{W}_{\varepsilon,t}\|_2^2 - \frac{\mu}{q}s^{q\gamma_q-q}\|\widehat{W}_{\varepsilon,t}\|_q^q - \frac{1}{6}\|\widehat{W}_{\varepsilon,t}\|_6^6) < m_{\mu,a}^+ + \frac{1}{3}S^{\frac{3}{2}} - \sigma' \tag{3.8}$$

for $t < \frac{1}{t_0}$ and $t > t_0$ with $\sigma' > 0$. Since $u_{\mu,+}$ is radial solution of (1.5) and exponentially decays to zero as $r \to +\infty$,

$$\int_{\mathbb{R}^3} u_{\mu,+}W_\varepsilon \sim \varepsilon^{\frac{5}{2}}\int_1^{\frac{1}{\varepsilon}}(\frac{1}{1+r^2})^{\frac{1}{2}}r^2 \sim \varepsilon^{\frac{1}{2}}$$

and

$$\int_{\mathbb{R}^3} u_{\mu,+}W_\varepsilon^5 \sim \varepsilon^{\frac{1}{2}}\int_1^{\frac{1}{\varepsilon}}(\frac{1}{1+r^2})^{\frac{5}{2}}r^2 \sim \varepsilon^{\frac{1}{2}}. \tag{3.9}$$

Thus, by (3.3),

$$s^2 = \frac{\|\widehat{W}_{\varepsilon,t}\|_2^2}{a^2} = 1 + \frac{2t}{a^2}\int_{\mathbb{R}^3} u_{\mu,+}W_\varepsilon + t^2\|W_\varepsilon\|_2^2 = 1 + O(\varepsilon^{\frac{1}{2}})$$

for $t_0^{-1} \le t \le t_0$. Since it is easy to see that $f(t) = (1+t)^q - 1 - t^q - qt - qt^{q-1} \ge 0$ for all $t \ge 0$ in the case of $q \ge 3$, by (3.4), (3.5) and the fact that $u_{\mu,+}$ is a solution of (1.5) for some $\lambda_{\mu,+} < 0$,

$$\begin{aligned}
\mathcal{E}_\mu(\overline{W}_{\varepsilon,t}) &= \frac{1}{2}\|\nabla\widehat{W}_{\varepsilon,t}\|_2^2 - \frac{\mu}{q}s^{q\gamma_q-q}\|\widehat{W}_{\varepsilon,t}\|_q^q - \frac{1}{6}\|\widehat{W}_{\varepsilon,t}\|_6^6 \\
&\le m_{\mu,a}^+ + \mathcal{E}_\mu(tW_\varepsilon) - \int_{\mathbb{R}^3}(tW_\varepsilon)^5u_{\mu,+} \\
&\quad + t(\lambda_{\mu,+}\int_{\mathbb{R}^3} u_{\mu,+}W_\varepsilon + \frac{\mu}{a^2}(\gamma_q - 1)\|\widehat{W}_{\varepsilon,t}\|_q^q\int_{\mathbb{R}^3} u_{\mu,+}W_\varepsilon) + O(\varepsilon) \\
&= m_{\mu,a}^+ + \mathcal{E}_\mu(tW_\varepsilon) - \int_{\mathbb{R}^3}(tW_\varepsilon)^5u_{\mu,+} + O(\varepsilon)
\end{aligned}$$

for $t_0^{-1} \leq t \leq t_0$, where we have used the fact that $\lambda_{\mu,+}a^2 = \lambda_{\mu,+}\|u_{\mu,+}\|_2^2 = \mu(\gamma_q - 1)\|u_{\mu,+}\|_q^q$ which comes from the Pohozaev identity satisfied by $u_{\mu,+}$. Now, for $t_0^{-1} \leq t \leq t_0$, by (3.2), (3.3) and (3.9),

$$\mathcal{E}_\mu(\overline{W}_{\varepsilon,t}) \leq m^+_{\mu,a} + \frac{1}{3}S^{\frac{3}{2}} + O(\varepsilon) - C\varepsilon^{\frac{1}{2}} < m^+_{\mu,a} + \frac{1}{3}S^{\frac{3}{2}}$$

by taking $\varepsilon > 0$ sufficiently small. It follows from (3.8) that

$$\sup_{t\geq 0} \mathcal{E}_\mu(\overline{W}_{\varepsilon,t}) < m^+_{\mu,a} + \frac{1}{3}S^{\frac{3}{2}}. \tag{3.10}$$

The conclusion then follows from (3.7). □

Remark 3.1. *It is worth pointing our that the above argument also works for $N \geq 4$. In these cases, we have*

$$\|\nabla W_\varepsilon\|_2^2 = S^{\frac{N}{2}} + O(\varepsilon^{N-2}), \quad \|W_\varepsilon\|_6^6 = S^{\frac{N}{2}} + O(\varepsilon^N)$$

and

$$\|W_\varepsilon\|_q^q \sim \varepsilon^{N-\frac{(N-2)q}{2}}, \quad \|W_\varepsilon\|_2^2 \sim \begin{cases} \varepsilon^2 \ln\frac{1}{\varepsilon}, & N = 4, \\ \varepsilon^2, & N \geq 5. \end{cases}$$

Moreover, similar to (3.9),

$$\int_{\mathbb{R}^N} u^p_{\mu,+}W_\varepsilon \sim \varepsilon^{\frac{N-2}{2}} \quad \text{for all } p \geq 1.$$

It follows that

$$\begin{aligned}
\mathcal{E}_\mu(\overline{W}_{\varepsilon,t}) &= \frac{1}{2}\|\nabla\widehat{W}_{\varepsilon,t}\|_2^2 - \frac{\mu}{q}s^{q\gamma_q - q}\|\widehat{W}_{\varepsilon,t}\|_q^q - \frac{1}{6}\|\widehat{W}_{\varepsilon,t}\|_6^6 \\
&\leq m^+_{\mu,a} + \mathcal{E}_\mu(tW_\varepsilon) \\
&\quad + t(\lambda_{\mu,+}\int_{\mathbb{R}^N} u_{\mu,+}W_\varepsilon + \frac{\mu}{a^2}(\gamma_q - 1)\|\widehat{W}_{\varepsilon,t}\|_q^q \int_{\mathbb{R}^N} u_{\mu,+}W_\varepsilon) + O(\varepsilon^2 \ln\frac{1}{\varepsilon}) \\
&= m^+_{\mu,a} + \mathcal{E}_\mu(tW_\varepsilon) + O(\varepsilon^{N-2}) \\
&\leq m^+_{\mu,a} + \frac{1}{N}S^{\frac{N}{2}} - C\varepsilon^{N-\frac{(N-2)q}{2}} + O(\varepsilon^2 \ln\frac{1}{\varepsilon}) \\
&< m^+_{\mu,a} + \frac{1}{N}S^{\frac{N}{2}}
\end{aligned}$$

for $t_0^{-1} \leq t \leq t_0$ by taking $\varepsilon > 0$ sufficiently small since $N \geq 4$ and $q > 2$. Our proof is slightly simpler than that of [34] since our test function is radial and we do not need other variational formulas of $m^-_{\mu,a}$.

For every $c > 0$ such that $\mu c^{q-q\gamma_q} < \alpha_{N,q}$, let $u \in \mathcal{P}^{c,\mu}_\pm$, then $v_b = \frac{b}{c}u \in \mathcal{S}_b$ for all $b > 0$. By [49, Lemma 4.2], there exists $\tau_\pm(b) > 0$ such that

$$(v_b)_{\tau_\pm(b)} = (\tau_\pm(b))^{\frac{N}{2}}v_b(\tau_\pm(b)x) \in \mathcal{P}^{b,\mu}_\pm,$$

where $b > 0$ such that $\mu b^{q-q\gamma_q} < \alpha_{N,q}$. Clearly, $\tau_\pm(c) = 1$.

Lemma 3.2. *Let $2 < q < 2 + \frac{4}{N}$. For every $c > 0$ such that $\mu c^{q-q\gamma_q} < \alpha_{N,q}$, $\tau'_\pm(c)$ exist and*

$$\tau'_\pm(c) = \frac{\mu q\gamma_q\|u\|_q^q + 2^*\|u\|_{2^*}^{2^*} - 2\|\nabla u\|_2^2}{c(2\|\nabla u\|_2^2 - \mu q\gamma_q^2\|u\|_q^q - 2^*\|u\|_{2^*}^{2^*})}. \tag{3.11}$$

Moreover, $\mathcal{E}_\mu((v_b)_{\tau_\pm(b)}) < \mathcal{E}_\mu(u)$ *for all* $b > c$ *such that* $\mu b^{q-q\gamma_q} < \alpha_{N,q}$.

Proof. The proof is mainly inspired by [19]. Since $(v_b)_{\tau_\pm(b)} \in \mathcal{P}_\pm^{b,\mu}$, we have

$$(\frac{b}{c}\tau(b))^2\|\nabla u\|_2^2 = (\frac{b}{c})^q(\tau(b))^{q\gamma_q}\mu\gamma_q\|u\|_q^q + (\frac{b}{c}\tau(b))^{2^*}\|u\|_{2^*}^{2^*}.$$

Now, if we define the function

$$\Phi(b,\tau) = (\frac{b\tau}{c})^2\|\nabla u\|_2^2 - (\frac{b}{c})^q\tau^{q\gamma_q}\mu\gamma_q\|u\|_q^q - (\frac{b\tau}{c})^{2^*}\|u\|_{2^*}^{2^*},$$

then $\Phi(b,\tau(b)) \equiv 0$ for $b > 0$ such that $\mu b^{q-q\gamma_q} < \alpha_{N,q}$. Since $u \in \mathcal{P}_\pm^{c,\mu}$,

$$\partial_\tau\Phi(c,1) = 2\|\nabla u\|_2^2 - \mu q\gamma_q^2\|u\|_q^q - 2^*\|u\|_{2^*}^{2^*} \neq 0.$$

It follows from the implicit function theorem that $\tau'_\pm(c)$ exist and (3.11) holds. By (1.6) and $q < 2^*$, $1-\gamma_q > 0$. Thus, by $u \in \mathcal{P}_\pm^{c,\mu}$,

$$\begin{aligned} 1 + c\tau'(c) &= 1 + \frac{\mu q\gamma_q\|u\|_q^q + 2^*\|u\|_{2^*}^{2^*} - 2\|\nabla u\|_2^2}{2\|\nabla u\|_2^2 - \mu q\gamma_q^2\|u\|_q^q - 2^*\|u\|_{2^*}^{2^*}} \\ &= \frac{\mu q\gamma_q(1-\gamma_q)\|u\|_q^q}{2\|\nabla u\|_2^2 - \mu q\gamma_q^2\|u\|_q^q - 2^*\|u\|_{2^*}^{2^*}}. \end{aligned}$$

Since $(v_b)_{\tau_\pm(b)} \in \mathcal{P}_\pm^{b,\mu}$ and $u \in \mathcal{P}_\pm^{c,\mu}$,

$$\begin{aligned} \mathcal{E}_\mu((v_b)_{\tau_\pm(b)}) &= (\frac{1}{2} - \frac{1}{q\gamma_q})\|\nabla(v_b)_{\tau(b)}\|_2^2 + (\frac{1}{q\gamma_q} - \frac{1}{2^*})\|(v_b)_{\tau(b)}\|_{2^*}^{2^*} \\ &= (\frac{b}{c}\tau(b))^2(\frac{1}{2} - \frac{1}{q\gamma_q})\|\nabla u\|_2^2 + (\frac{b}{c}\tau(b))^{2^*}(\frac{1}{q\gamma_q} - \frac{1}{2^*})\|u\|_{2^*}^{2^*} \\ &= (\frac{1}{2} - \frac{1}{q\gamma_q})\|\nabla u\|_2^2 + (\frac{1}{q\gamma_q} - \frac{1}{2^*})\|u\|_{2^*}^{2^*} + o(b-c) \\ &\quad + \frac{1+c\tau'(c)}{c}(2(\frac{1}{2} - \frac{1}{q\gamma_q})\|\nabla u\|_2^2 + 2^*(\frac{1}{q\gamma_q} - \frac{1}{2^*})\|u\|_{2^*}^{2^*})(b-c) \\ &= \mathcal{E}_\mu(u) - \frac{\mu(1-\gamma_q)\|u\|_q^q}{c}(b-c) \\ &\quad + o(b-c). \end{aligned}$$

Therefore,

$$\frac{d\mathcal{E}_\mu((v_b)_{\tau_\pm(b)})}{db}|_{b=c} = -\frac{\mu(1-\gamma_q)\|u\|_q^q}{c} < 0.$$

Since $c > 0$, which satisfies $\mu c^{q-q\gamma_q} < \alpha_{N,q}$, is arbitrary and $(v_b)_{\tau_\pm(b)} \in \mathcal{P}_\pm^{b,\mu}$, we have $\mathcal{E}_\mu((v_b)_{\tau_\pm(b)}) < \mathcal{E}_\mu(u)$ for all $b > c$ such that $\mu b^{q-q\gamma_q} < \alpha_{N,q}$. □

With Lemma 3.2 in hands, we can obtain the following.

Proposition 3.1. *Let* $2 < q < 2 + \frac{4}{N}$ *and* $\mu a^{q-q\gamma_q} < \alpha_{N,q}$. *If* $m_{a,\mu}^- < m_{a,\mu}^+ + \frac{1}{N}S^{\frac{N}{2}}$ *then*

$$m_{a,\mu}^- = \inf_{u\in\mathcal{P}_-^{a,\mu}} \mathcal{E}_\mu(u)$$

can be attained by some $u_{a,\mu,-}$ *which is real valued, positive, radially symmetric and decreasing in* $r = |x|$. *Moreover,* (1.5) *has a second solution* $u_{a,\mu,-}$ *with some* $\lambda_{a,\mu,-} < 0$ *which is real valued, positive, radially symmetric and radially decreasing.*

Proof. Let $\{u_n\} \subset \mathcal{P}_-^{a,\mu}$ be a minimizing sequence. Then by taking $|u_n|$ and adapting the Schwarz symmetrization to $|u_n|$ if necessary, we can obtain a new minimizing sequence, say $\{u_n\}$ again, such that u_n are all real valued, nonnegative, radially symmetric and decreasing in $r = |x|$. Since $\{u_n\} \subset \mathcal{P}_-^{a,\mu}$, we have

$$\mathcal{E}_\mu(u_n) = \frac{\mu}{q}(\frac{q\gamma_q}{2} - 1)\|u_n\|_q^q + \frac{1}{N}\|u_n\|_{2^*}^{2^*}. \tag{3.12}$$

Thus, by the Hölder and Young inequalities and $\{u_n\} \subset \mathcal{P}_-^{a,\mu}$ again, we known that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and thus, $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^N)$ as $n \to \infty$ up to a subsequence. Since u_n are all radial, by Struss's radial lemma (cf. [12, Lemma A.IV, Theorem A.I'] or [44, Lemma 3.1]) and the Sobolev embedding theorem, $u_n \to u_0$ strongly in $L^q(\mathbb{R}^N)$ as $n \to \infty$ up to a subsequence. Without loss of generality, we assume that $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^N)$ and $u_n \to u_0$ strongly in $L^q(\mathbb{R}^N)$ as $n \to \infty$. We claim that $u_0 \neq 0$. If not, then $u_n \to 0$ strongly in $L^q(\mathbb{R}^N)$ as $n \to \infty$. It follows from $\{u_n\} \subset \mathcal{P}_-^{a,\mu}$ that

$$\|\nabla u_n\|_2^2 = \|u_n\|_{2^*}^{2^*} + o_n(1),$$

which together with the Sobolev inequality (1.8), implies that either $u_n \to 0$ strongly in $D^{1,2}(\mathbb{R}^N)$ as $n \to \infty$ or $\|\nabla u_n\|_2^2 = \|u_n\|_{2^*}^{2^*} + o_n(1) \geq S^{\frac{N}{2}} + o_n(1)$. Hence, by (3.12), either $m_{a,\mu}^- = 0$ or $m_{a,\mu}^- \geq \frac{1}{N}S^{\frac{N}{2}}$, which contradicts $\mathcal{E}_\mu(u) \gtrsim 1$ for $u \in \mathcal{P}_-^{c,\mu}$ and Lemma 3.1. We remark that $\mathcal{E}_\mu(u) \gtrsim 1$ for $u \in \mathcal{P}_-^{c,\mu}$ comes from similar arguments as used for [50, Lemma 5.7]. Therefore, we must have $u_0 \neq 0$. Let $v_n = u_n - u_0$. Then there are two cases:

- (i) $v_n \to 0$ strongly in $H^1(\mathbb{R}^N)$ as $n \to \infty$ up to a subsequence.
- (ii) $\|\nabla v_n\|_2^2 + \|v_n\|_2^2 \gtrsim 1$.

In the case (i), $u_0 \in \mathcal{P}_-^{a,\mu}$ and $m_{a,\mu}^-$ is attained by u_0 which is real valued, radially symmetric, nonnegative and decreasing in $r = |x|$. By [49, Proposition 1.5], u_0 is a solution of (1.5) with some $\lambda_0 \in \mathbb{R}$ which appears as a Lagrange multiplier. By multiplying (1.5) with u_0 and integrating by parts, and using $u_0 \in \mathcal{P}_-^{a,\mu}$, we have

$$\lambda_0 a^2 = \mu(\gamma_q - 1)\|u_0\|_q^q < 0,$$

which implies $\lambda_0 < 0$. Now, by the maximum principle and classical elliptic estimates, we know that u_0 is positive. It remains to consider the case (ii). Let $\|u_0\|_2^2 = t_0^2$, then by the Fatou lemma, $0 < t_0 \leq a$. There are two subcases:

- (ii_1) $\|v_n\|_{2^*} \to 0$ as $n \to \infty$ up to a subsequence.
- (ii_2) $\|v_n\|_{2^*}^{2^*} \gtrsim 1$.

In the subcase (ii_1), by [49, Lemma 4.2], there exists $s_0 > 0$ such that $(u_0)_{s_0} \in \mathcal{P}_-^{t_0,\mu}$. By [49, Lemma 4.2] once more, $\{u_n\} \subset \mathcal{P}_-^{a,\mu}$ and $u_n \to u_0$ strongly in $L^{2^*}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ as $n \to \infty$ up to a subsequence,

$$m_{a,\mu}^- + o_n(1) = \mathcal{E}_\mu(u_n) \geq \mathcal{E}_\mu((u_n)_{s_0}) \geq \mathcal{E}_\mu((u_0)_{s_0}) + o_n(1).$$

By Lemma 3.2, we have $m_{t_0,\mu}^- \geq m_{a,\mu}^-$. Thus, $\mathcal{E}_\mu((u_0)_{s_0}) = m_{t_0,\mu}^-$ and $m_{t_0,\mu}^- = m_{a,\mu}^-$. If $t_0 < a$ then by taking $(u_0)_{s_0}$ as the test function in the proof of Lemma 3.2, we know that $m_{t_0,\mu}^- > m_{a,\mu}^-$, which is a contradiction. Thus, in the subcase (ii_1), we must have $t_0 = a$ and so that $m_{a,\mu}^-$ is attained by $(u_0)_{s_0}$ which is real valued, radially symmetric, nonnegative and decreasing in $r = |x|$. As above, we can show that $(u_0)_{s_0}$ is positive and $(u_0)_{s_0}$ is a solution of (1.5) with some $\lambda_0' < 0$. It remains to consider the subcase (ii_2). Let

$$s_n = \left(\frac{\|\nabla v_n\|_2^2}{\|v_n\|_{2^*}^{2^*}} \right)^{\frac{1}{2^*-2}}.$$

Then in the subcase (ii_2), $s_n \lesssim 1$ and by the Sobolev inequality (1.8),

$$\|\nabla (v_n)_{s_n}\|_2^2 = \|(v_n)_{s_n}\|_{2^*}^{2^*} \geq \frac{1}{N} S^{\frac{N}{2}}.$$

Since $0 < t_0 \leq a$, by [49, Lemma 4.2], there exists $\tau_0 > 0$ such that $(u_0)_{\tau_0} \in \mathcal{P}_{-}^{t_0,\mu}$. We claim that $s_n \geq \tau_0$ up to a subsequence. Suppose the contrary that $s_n < \tau_0$ for all n. Then by [49, Lemma 4.2] once more, the Brezis-Lieb lemma (cf. [56, Lemma 1.32]), Lemma 3.2, the fact that $u_n \to u_0$ strongly in $L^q(\mathbb{R}^N)$ as $n \to \infty$ and the boundedness of $\{s_n\}$,

$$\begin{aligned} m_{a,\mu}^- + o_n(1) &= \mathcal{E}_\mu(u_n) \\ &\geq \mathcal{E}_\mu((u_n)_{s_n}) \\ &= \mathcal{E}_\mu((u_0)_{s_n}) + \mathcal{E}_0((v_n)_{s_n}) + o_n(1) \\ &\geq m_{t_0,\mu}^+ + \frac{1}{N} S^{\frac{N}{2}} + o_n(1) \\ &\geq m_{a,\mu}^+ + \frac{1}{N} S^{\frac{N}{2}} + o_n(1), \end{aligned}$$

which is impossible. Thus, we must have $s_n \geq \tau_0$ up to a subsequence. Without loss of generality, we may assume that $s_n \geq \tau_0$ for all $n \in \mathbb{N}$. Again, by [49, Lemma 4.2], the Brezis-Lieb lemma (cf. [56, Lemma 1.32]) and the fact that $u_n \to u_0$ strongly in $L^q(\mathbb{R}^N)$ as $n \to \infty$,

$$m_{a,\mu}^- + o_n(1) = \mathcal{E}_\mu(u_n) \geq \mathcal{E}_\mu((u_n)_{\tau_0}) = \mathcal{E}_\mu((u_0)_{\tau_0}) + \mathcal{E}_0((v_n)_{\tau_0}) + o_n(1).$$

Since $s_n \geq \tau_0$, by [49, Proposition 2.2], $\mathcal{E}_0((v_n)_{\tau_0}) \geq 0$, which, together with Lemma 3.2, implies that $t_0 = a$ and $m_{a,\mu}^-$ is attained by $(u_0)_{\tau_0}$. Clearly, $(u_0)_{\tau_0}$ is real valued, radially symmetric, nonnegative and decreasing in $r = |x|$. As above, we can show that $(u_0)_{\tau_0}$ is positive and $(u_0)_{\tau_0}$ is a solution of (1.5) with some $\lambda_0'' < 0$. Therefore, we have proved that $m_{a,\mu}^-$ can always be attained by some $u_{a,\mu,-}$ which is real valued, radially symmetric, positive and decreasing in $r = |x|$. By [49, Proposition 1.5], (1.5) has a second solution $u_{a,\mu,-}$ which is real valued, radially symmetric, positive and decreasing in $r = |x|$. □

Our next goal in this section is to prove the existence and nonexistence of ground states for $\mu a^{q-q\gamma_q} \geq \alpha_{N,q}$ in the L^2-critical and supercritical cases, which gives partial answers to the question (Q_2). In these two cases, $2 + \frac{4}{N} \leq q < 2^*$, which implies

$$q\gamma_q \geq 2.$$

We recall that the constant $\alpha_{N,q}$ is given by [49, Theorem 1.1]. For $q = 2 + \frac{4}{N}$, by [49, (5,1)],

$$\alpha_{N,q} = C_{N,q}^{-q}(1 + \frac{2}{N}) = \frac{1}{C_{N,q}^q \gamma_q}, \tag{3.13}$$

where $C_{N,q}$ is the optimal constant in the Gagliardo–Nirenberg inequality (1.9).

Lemma 3.3. *Let $N \geq 3$ and $2+\frac{4}{N} \leq q < 2^*$. Then $m^-_{a,\mu}$ is strictly decreasing for $0<\mu<a^{q\gamma_q-q}\alpha_{N,q}$ and is nonincreasing for $\mu \geq a^{q\gamma_q-q}\alpha_{N,q}$, where $m^-_{a,\mu}$ is given by (1.7). Moreover, $0<m^-_{a,\mu}<\frac{1}{N}S^{\frac{N}{2}}$ for all $\mu>0$ in the case of $2+\frac{4}{N}<q<2^*$ while, $m^-_{a,\mu}=0$ for $\mu \geq a^{q\gamma_q-q}\alpha_{N,q}$ in the case of $q=2+\frac{4}{N}$.*

Proof. Modified the proof of [49, Lemma 8.2] in a trivial way (or by Lemma 3.2 and [49, Theorem 1.1]), we can show that $m^-_{a,\mu}$ is strictly decreasing for $0<\mu<a^{q\gamma_q-q}\alpha_{N,q}$. For $\mu \geq a^{q\gamma_q-q}\alpha_{N,q}$, let us consider the fibering map

$$\Psi_u(t)=\frac{t^2}{2}\|\nabla u\|_2^2-\frac{\mu t^{q\gamma_q}}{q}\|u\|_q^q-\frac{t^{2^*}}{2^*}\|u\|_{2^*}^{2^*},$$

as that in [49]. For $2+\frac{4}{N}<q<2^*$, it has been proved in [49, Lemma 6.1] that for every $u \in \mathcal{S}_a$, there exists $t_u>0$ such that $\Psi_u(t)$ is strictly increasing in $(0,t_u)$, is strictly decreasing in $(t_u,+\infty)$ and

$$(u)_{t_u}=t_u^{\frac{N}{2}}u(t_u x) \in \mathcal{P}^{a,\mu}_-.$$

Moreover, by [49, Lemma 6.2], we have $m^-_{a,\mu}>0$ for all $\mu>0$ in the L^2-supercritical case $2+\frac{4}{N}<q<2^*$. It follows that we can always choose $v_\varepsilon \in \mathcal{P}^{a,\mu}_-$ such that $\mathcal{E}_\mu(v_\varepsilon)<m^-_{a,\mu}+\varepsilon$ in the L^2-supercritical case $2+\frac{4}{N}<q<2^*$. Then by similar arguments as used for [49, Lemma 8.2] (or by Lemma 3.2), we have

$$m^-_{a,\mu'}<m^-_{a,\mu}+\varepsilon \quad \text{for all } \mu'>\mu.$$

Since $\varepsilon>0$ and $\mu \geq a^{q\gamma_q-q}\alpha_{N,q}$ are arbitrary, $m^-_{a,\mu}$ is nonincreasing for $\mu \geq a^{q\gamma_q-q}\alpha_{N,q}$ in the L^2-supercritical case $2+\frac{4}{N}<q<2^*$. It follows from [49, Lemma 6.4] that $m^-_{a,\mu}<\frac{1}{N}S^{\frac{N}{2}}$ for all $\mu>0$.

In the L^2-critical case $q=2+\frac{4}{N}$, since

$$\sup_{u\in\mathcal{S}_a}\frac{\|\nabla u\|_2}{\|u\|_q}=+\infty.$$

For all $\mu>0$, we can always choose $u \in \mathcal{S}_a$ such that $\frac{\|\nabla u\|_2}{\|u\|_q}>\mu\gamma_q$. Indeed, if $\sup_{u\in\mathcal{S}_a}\frac{\|\nabla u\|_2}{\|u\|_q}\lesssim 1$, then by the Gagliardo-Nirenberg inequality,

$$\|\nabla u\|_2 \lesssim \|u\|_q \lesssim \|\nabla u\|_2^{\gamma_q} \quad \text{for all } u \in \mathcal{S}_a,$$

which implies

$$\sup_{u\in\mathcal{S}_a}\|\nabla u\|_2 \lesssim 1.$$

It is impossible since in any ball $B_R(0)$, the eigenvalue problem $-\Delta u=\lambda u$, with Dirichlet boundary conditions, has a sequence of eigenvalues $\lambda_j \to +\infty$ as $j \to \infty$. We note that $q\gamma_q=2$ in the L^2-critical case $q=2+\frac{4}{N}$. Thus,

$$\Psi_u'(t)=(\|\nabla u\|_2^2-\mu\gamma_q\|u\|_q^q)t-t^{2^*-1}\|u\|_{2^*}^{2^*}=0$$

has a unique solution $t_u>0$ for $u \in \mathcal{S}_a$ such that $\frac{\|\nabla u\|_2}{\|u\|_q}>\mu\gamma_q$. Moreover, $\Psi_u(t)$ is strictly increasing in $(0,t_u)$, is strictly decreasing in $(t_u,+\infty)$ and

$$(u)_{t_u}=t_u^{\frac{N}{2}}u(t_u x) \in \mathcal{P}_{a,\mu}=\mathcal{P}^{a,\mu}_-.$$

Thus, $\mathcal{P}_{a,\mu}=\mathcal{P}^{a,\mu}_- \neq \emptyset$ and $\Psi_u(t_u)=\max_{t\geq 0}\Psi_u(t)$ for all $\mu>0$ and all $u \in \mathcal{S}_a$ such that $\frac{\|\nabla u\|_2}{\|u\|_q}>\mu\gamma_q$. Now, as in the L^2-supercritical case $2+\frac{4}{N}<q<$

2^*, by similar arguments as used for [49, Lemma 8.2], we can show that $m^-_{a,\mu}$ is nonincreasing for $\mu \geq a^{q\gamma_q-q}\alpha_{N,q}$ in the L^2-critical case $q = 2 + \frac{4}{N}$. It remains to prove that $m^-_{a,\mu} = 0$ for $\mu \geq a^{q\gamma_q-q}\alpha_{N,q}$ in the case of $q = 2 + \frac{4}{N}$. Let $\{\varphi_n\}$ be the minimizing sequence of the Gagliardo-Nirenberg inequality (1.9). Then by scaling $\frac{at_n^{\frac{N}{2}}}{\|\varphi_n\|_2}\varphi_n(t_n x)$ if necessary, we may assume that $\|\varphi_n\|_2^2 = a^2$, $\|\varphi_n\|_q^q = 1$ and $\|\nabla\varphi_n\|_2^2 = C_{N,q}^{-\frac{2}{\gamma_q}} a^{\frac{2(\gamma_q-1)}{\gamma_q}} + o_n(1)$. Let us consider the following function:

$$
\begin{aligned}
h_{\varphi_n}(\mu, t) &= t^2(\|\nabla\varphi_n\|_2^2 - \mu\gamma_q\|\varphi_n\|_q^q) - t^{2^*}\|\varphi_n\|_{2^*}^{2^*} \\
&= (C_{N,q}^{-\frac{2}{\gamma_q}} a^{\frac{2(\gamma_q-1)}{\gamma_q}} + o_n(1) - \mu\gamma_q)t^2 - t^{2^*}\|\psi\|_{2^*}^{2^*} \\
&= \gamma_q(\alpha_{N,q}a^{q\gamma_q-q} + o_n(1) - \mu)t^2 - t^{2^*}\|\psi\|_{2^*}^{2^*},
\end{aligned}
$$

where we have used (3.13). By [49, Lemma 5.1], there exists a unique $t_n(\mu) > 0$ such that $h_{\varphi_n}(\mu, t_n(\mu)) = 0$ for $0 < \mu < a^{q\gamma_q-q}\alpha_{N,q}$. Thus, $(\varphi_n)_{t_n(\mu)} \in \mathcal{P}_{a,\mu}$ for $0 < \mu < a^{q\gamma_q-q}\alpha_{N,q}$, where $(\varphi_n)_{t_n(\mu)} = [t_n(\mu)]^{\frac{N}{2}}\varphi_n(t_n(\mu)x)$. Moreover, since $\|\varphi_n\|_q^q = 1$, by the Hölder inequality, $\|\varphi_n\|_{2^*} \gtrsim 1$. It follows that $t(\mu) \to 0$ as $\mu \to a^{q\gamma_q-q}\alpha_{N,q}$, which implies

$$
\mathcal{E}_\mu((\psi)_{t(\mu)}) = \frac{1}{N}\|\varphi_n\|_{2^*}^{2^*}[t_n(\mu)]^{2^*} = o_n(1)
$$

as $\mu \to a^{q\gamma_q-q}\alpha_{N,q}$ in the L^2-critical case $q = 2 + \frac{4}{N}$. Thus, we must have $m^-_{a,\mu} \leq 0$ for $\mu = a^{q\gamma_q-q}\alpha_{N,q}$. By the monotone property of $m^-_{a,\mu}$ stated in Lemma 3.2, $m^-_{a,\mu} \leq 0$ for $\mu \geq a^{q\gamma_q-q}\alpha_{N,q}$. Recall that we always have

$$
\mathcal{E}_\mu(u) = \frac{1}{N}\|u\|_{2^*}^{2^*} \geq 0 \quad \text{for all } u \in \mathcal{P}_{a,\mu}, \tag{3.14}
$$

thus, we must have $m^-_{a,\mu} = 0$ for $\mu \geq a^{q\gamma_q-q}\alpha_{N,q}$. □

With Lemma 3.3 in hands, we can obtain the following.

Proposition 3.2. *Let $N \geq 3$ and $2 + \frac{4}{N} \leq q < 2^*$.*

(1) *If $2 + \frac{4}{N} < q < 2^*$, then $m^-_{a,\mu}$ is attained by same $u_{a,\mu,-}$ which is real valued, positive, radially symmetric and decreasing in $r = |x|$ for all $\mu > 0$, and thus, $u_{a,\mu,-}$ is a solution of (1.5) for all $\mu > 0$ with some $\lambda_{a,\mu,-} < 0$.*

(2) *If $q = 2 + \frac{4}{N}$, then $m^-_{a,\mu}$ can not be attained and (1.5) has no ground states for all $\mu \geq a^{q\gamma_q-q}\alpha_{N,q}$.*

Proof. (1) By Lemma 3.3, $0 < m^-_{a,\mu} < \frac{1}{N}S^{\frac{N}{2}}$ for all $\mu > 0$ in the case of $2 + \frac{4}{N} < q < 2^*$. Now, by following the arguments in [49, Section 6] step by step, we can show that $m^-_{a,\mu}$ is attained by some $u_{a,\mu,-}$ which is real valued, nonnegative, radially symmetric and decreasing in $r = |x|$ for all $\mu > 0$ in the case of $2 + \frac{4}{N} < q < 2^*$. By similar arguments as used for [50, Lemma 6.2], we know that $\mathcal{P}_{a,\mu} = \mathcal{P}_-^{a,\mu} \neq \emptyset$ is a natural constraint in $\mathcal{S}_a$ for all $\mu > 0$ in the case of $2 + \frac{4}{N} < q < 2^*$. Thus, $u_{a,\mu,-}$ is a solution of (1.5) for all $\mu > 0$ with some $\lambda_{a,\mu,-}$ in the case of $2 + \frac{4}{N} < q < 2^*$. As that in the proof of Proposition 3.1, we can show that $\lambda_{a,\mu,-} < 0$ and $u_{a,\mu,-}$ is positive.

(2) Suppose the contrary that $m^-_{a,\mu}$ is attained by some $u_{a,\mu,-}$ for $\mu \geq a^{q\gamma_q-q}\alpha_{N,q}$, then by Lemma 3.3 and (3.14), $\|u_{a,\mu,-}\|_{2^*}^{2^*} = 0$. It is impossible since $u_{a,\mu,-} \in \mathcal{S}_a$.

Thus, $m^-_{a,\mu}$ can not be attained for $\mu \geq a^{q\gamma_q - q}\alpha_{N,q}$. It follows that (1.5) has no ground state for all $\mu \geq a^{q\gamma_q - q}\alpha_{N,q}$. □

4. The asymptotic behavior of $u_{a,\mu,-}$

In this section, we shall mainly study the question (Q_3) and give a precisely description of the asymptotic behavior of $u_{a,\mu,-}$ as $\mu \to 0^+$. Since we consider $\mu \to 0^+$ now, the assumptions of [49, Theorem 1.1], [34, Theorem 1.6] and Proposition 3.1 always hold and thus, $u_{a,\mu,-}$, which is a minimizer of $\mathcal{E}|_{\mathcal{S}_a}(u)$ on $\mathcal{P}^{a,\mu}_-$, exists for all $N \geq 3$, $2 < q < 2^*$ for $\mu > 0$ sufficiently small.

Proposition 4.1. *Let $N \geq 3$, $2 < q < 2^*$ and $u_{a,\mu,-}$ is a critical point of $\mathcal{E}|_{\mathcal{S}_a}(u)$ of mountain pass type. If $N \geq 5$ then $u_{a,\mu,-} \to U_{\varepsilon_0}$ strongly in $H^1(\mathbb{R}^N)$ as $\mu \to 0^+$, where U_{ε_0} is the Aubin-Talanti babble such that $U_{\varepsilon_0} \in \mathcal{S}_a$. Moreover, if $N \geq 9$, then up to translations and rotations, $u_{a,\mu,-}$ is the unique minimizer of $\mathcal{E}|_{\mathcal{S}_a}(u)$ on $\mathcal{P}^{a,\mu}_-$ for $\mu > 0$ sufficiently small.*

Proof. By [49, Theorem 1.4], $m^+_{a,\mu} \to 0$ as $\mu \to 0^+$. Moreover, by [49, Theorem 1.4] again, we know that $m^-_{a,\mu} \to \frac{1}{N}S^{\frac{N}{2}}$ as $\mu \to 0^+$, and

$$\|\nabla u_{a,\mu,-}\|_2^2, \|u_{a,\mu,-}\|_{2^*}^{2^*} \to S^{\frac{N}{2}} \quad \text{as } \mu \to 0^+$$

for $2 + \frac{4}{N} \leq q < 2^*$. On the other hand, by [49, Lemma 4.2] and similar arguments as used for [50, Lemma 5.7], we also have $m^-_{a,\mu} \gtrsim 1$ for $\mu > 0$ sufficiently small in the case of $2 < q < 2 + \frac{4}{N}$. Thus, by adapting similar arguments as used in the proof of [49, Theorem 1.1] for the case of $2 + \frac{4}{N} \leq q < 2^*$ to the case of $2 < q < 2 + \frac{4}{N}$, we can also show that $m^-_{a,\mu} \to \frac{1}{N}S^{\frac{N}{2}}$ as $\mu \to 0^+$, and

$$\|\nabla u_{a,\mu,-}\|_2^2, \|u_{a,\mu,-}\|_{2^*}^{2^*} \to S^{\frac{N}{2}} \quad \text{as } \mu \to 0^+$$

for $2 < q < 2 + \frac{4}{N}$ (see also [34, Theorem 1.7]). It follows that, up to a subsequence, $\{u_{a,\mu,-}\}$ is a minimizing sequence of the following minimizing problem:

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}. \tag{4.1}$$

Since $N \geq 5$, $U_\varepsilon \in L^2(\mathbb{R}^N)$ for all $\varepsilon > 0$. We then choose $\varepsilon_0 > 0$ such that $U_{\varepsilon_0} \in \mathcal{S}_a$. By [49, Lemma 4.2], there exists $t(\mu) > 0$ such that $(U_{\varepsilon_0})_{t(\mu)} \in \mathcal{P}^{a,\mu}_-$ for $\mu > 0$ sufficiently small, that is,

$$[t(\mu)]^2 S^{\frac{N}{2}} = \mu\gamma_q \|U_{\varepsilon_0}\|_q^q [t(\mu)]^{q\gamma_q} + [t(\mu)]^{2^*} S^{\frac{N}{2}}.$$

Clearly, by the implicit function theorem, $t(\mu)$ is of class C^1 for $|\mu| << 1$ such that $t(0) = 1$. It follows from $S^{\frac{N}{2}}(1 - [t(\mu)]^{2^*-2}) = \mu\gamma_q\|U_{\varepsilon_0}\|_q^q [t(\mu)]^{q\gamma_q - 2}$ that

$$t(\mu) = 1 - \frac{\gamma_q \|U_{\varepsilon_0}\|_q^q}{(2^* - 2)S^{\frac{N}{2}}}\mu + o(\mu), \tag{4.2}$$

which implies

$$\begin{aligned} m^-_{a,\mu} &\leq \mathcal{E}_\mu((U_{\varepsilon_0})_{t(\mu)}) \\ &= \frac{1}{N}S^{\frac{N}{2}} - \frac{\mu\gamma_q\|U_{\varepsilon_0}\|_q^q}{2^*} - \frac{\mu}{q}(1 - \frac{q\gamma_q}{2^*})\|U_{\varepsilon_0}\|_q^q + o(\mu) \\ &= \frac{1}{N}S^{\frac{N}{2}} - \frac{\mu\|U_{\varepsilon_0}\|_q^q}{q} + o(\mu) \end{aligned} \tag{4.3}$$

for $N \geq 5$. Since we have $m^-_{a,\mu} = \frac{1}{N}\|\nabla u_{a,\mu,-}\|_2^2 - \frac{\mu}{q}(1-\frac{q\gamma_q}{2^*})\|u_{a,\mu,-}\|_q^q$, by (4.3),

$$\frac{1}{N}\|\nabla u_{a,\mu,-}\|_2^2 - \frac{\mu}{q}(1-\frac{q\gamma_q}{2^*})\|u_{a,\mu,-}\|_q^q \leq \frac{1}{N}S^{\frac{N}{2}} - \frac{\mu\|U_{\varepsilon_0}\|_q^q}{q} + o(\mu). \tag{4.4}$$

On the other hand, by (4.1) and $u_{a,\mu,-} \in \mathcal{P}^{a,\mu}_-$,

$$\begin{aligned} S &\leq \frac{\|\nabla u_{a,\mu,-}\|_2^2}{\|u_{a,\mu,-}\|_{2^*}^2} \\ &= \frac{\|\nabla u_{a,\mu,-}\|_2^2}{(\|\nabla u_{a,\mu,-}\|_2^2 - \mu\gamma_q\|u_{a,\mu,-}\|_q^q)^{\frac{2}{2^*}}} \\ &= (\|\nabla u_{a,\mu,-}\|_2^2 - \mu\gamma_q\|u_{a,\mu,-}\|_q^q)^{\frac{2}{N}} + \frac{\mu\gamma_q\|u_{a,\mu,-}\|_q^q}{S^{\frac{N-2}{2}}} + o(\mu\|u_{a,\mu,-}\|_q^q). \end{aligned}$$

It follows that

$$\|\nabla u_{a,\mu,-}\|_2^2 \geq S^{\frac{N}{2}} - \frac{N-2}{2}\mu\gamma_q\|u_{a,\mu,-}\|_q^q + o(\mu\|u_{a,\mu,-}\|_q^q). \tag{4.5}$$

Combining (4.4) and (4.5), we have

$$\|u_{a,\mu,-}\|_q^q \geq \|U_{\varepsilon_0}\|_q^q + o(1). \tag{4.6}$$

Since $\{u_{a,\mu,-}\}$ is bounded in $H^1(\mathbb{R}^N)$, $u_{a,\mu,-} \rightharpoonup u_{0,-}$ weakly in $H^1(\mathbb{R}^N)$ as $\mu \to 0^+$ up to a subsequence. Since $u_{a,\mu,-}$ is radial and decreasing for $r = |x|$,

$$\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_{a,\mu,-}|^2dx = \int_{B_1(0)}|u_{a,\mu,-}|^2dx.$$

Thus, by (4.6), Lions' lemma [56, Lemma 1.21] and the Sobolev embedding theorem, $u_{0,-} \neq 0$. Note that it is standard to show that $u_{0,-}$ is a weak solution of the following equation,

$$-\Delta U = U^{2^*-1}, \quad \text{in } \mathbb{R}^N,$$

thus, we must have $\|\nabla u_{0,-}\|_2^2 \geq S^{\frac{N}{2}}$. It follows from $\|\nabla u_{a,\mu,-}\|_2^2 \to S^{\frac{N}{2}}$ as $\mu \to 0^+$ that $u_{a,\mu,-} \to u_{0,-}$ strongly in $D^{1,2}(\mathbb{R}^N)$ as $\mu \to 0^+$ up to a subsequence, which implies $u_{0,-} = U_\varepsilon$ for some $\varepsilon > 0$. Since $\|U_{\varepsilon_0}\|_2^2 = a^2$, by the Fatou lemma and (4.6),

$$\|U_\varepsilon\|_q^q \geq \|U_{\varepsilon_0}\|_q^q \quad \text{and} \quad \|U_\varepsilon\|_2 \leq \|U_{\varepsilon_0}\|_2^2.$$

Hence, we must have $\varepsilon = \varepsilon_0$ and thus, $\|u_{0,-}\|_2 = \|U_{\varepsilon_0}\|_2^2 = a^2$, which implies $u_{a,\mu,-} \to U_{\varepsilon_0}$ strongly in $H^1(\mathbb{R}^N)$ as $\mu \to 0^+$ up to a subsequence. Since U_{ε_0} is the unique Aubin-Talanti babble in $\mathcal{S}_a$, we have $u_{a,\mu,-} \to U_{\varepsilon_0}$ strongly in $H^1(\mathbb{R}^N)$ as $\mu \to 0^+$. Moreover, since $u_{a,\mu,-}$ is a solution of (1.5), by the Pohozaev identity and $u_{a,\mu,-} \in \mathcal{P}_{a,\mu}$, we have

$$-\lambda_{a,\mu,-}\|u_{a,\mu,-}\|_2^2 = (1-\gamma_q)\mu\|u_{a,\mu,-}\|_q^q, \tag{4.7}$$

which implies $\lambda_{a,\mu,-} \to 0$ as $\mu \to 0^+$. It remains to prove that $u_{a,\mu,-}$ is the unique minimizer of $\mathcal{E}|_{\mathcal{S}_a}(u)$ on $\mathcal{P}^{a,\mu}_-$ for $\mu > 0$ sufficiently small up to translations and rotations. For this, let us first claim that

$$u_{a,\mu,-} \lesssim \left(\frac{1}{1+r^2}\right)^{\frac{N-2}{2}} \tag{4.8}$$

for all $r \geq 0$ in the case of $\mu > 0$ sufficiently small. Indeed, since $u_{a,\mu,-}$ is a positive and radially decreasing solution of (1.5), by Struss's radial lemma (cf. [12, Lemma

A.IV, Theorem A.I'] or [44, Lemma 3.1]), $u_{a,\mu,-} \lesssim r^{-\frac{N-1}{2}}$ for $r \geq 1$. Thus, by (4.7), $u_{a,\mu,-}$ satisfies

$$-u''_{a,\mu,-} - \frac{N-1}{r}u'_{a,\mu,-} \lesssim u_{a,\mu,-}^{\frac{4}{N-2}}u_{a,\mu,-} \lesssim r^{-(2+\delta)}u_{a,\mu,-} \quad \text{for } r \gtrsim 1 \tag{4.9}$$

for some $\delta > 0$. By bootstrapping we obtain the desired decaying estimate (4.8).

Now, let us consider the following system:

$$\begin{cases}\mathcal{F}(w,\alpha,\mu) = \Delta w - \alpha w + \mu w^{q-1} + w^{2^*-1},\\ \mathcal{G}(w,\alpha,\mu) = \|w\|_2^2 - a^2,\end{cases} \tag{4.10}$$

where $\alpha, \mu > 0$ are parameters. It is easy to see that $\mathcal{F}(U_{\varepsilon_0},0,0) = 0$ and $\mathcal{G}(U_{\varepsilon_0},0,0) = 0$. Let

$$\mathcal{L}(U_{\varepsilon_0},0,0) = \begin{pmatrix}\partial_w\mathcal{F}(U_{\varepsilon_0},0,0) & \partial_\alpha\mathcal{F}(U_{\varepsilon_0},0,0)\\ \partial_w\mathcal{G}(U_{\varepsilon_0},0,0) & \partial_\alpha\mathcal{G}(U_{\varepsilon_0},0,0)\end{pmatrix}$$

be the linearization of the system (4.10) at $(U_{\varepsilon_0},0,0)$ in $H^1(\mathbb{R}^N)\times\mathbb{R}$, that is,

$$\partial_w\mathcal{F}(U_{\varepsilon_0},0,0) = \Delta + (2^*-1)U_{\varepsilon_0}^{2^*-2}, \quad \partial_\alpha\mathcal{F}(U_{\varepsilon_0},0,0) = -U_{\varepsilon_0}$$

and

$$\partial_w\mathcal{G}(U_{\varepsilon_0},0,0) = 2U_{\varepsilon_0}, \quad \partial_\alpha\mathcal{G}(U_{\varepsilon_0},0,0) = 0.$$

Then $\mathcal{L}(U_{\varepsilon_0},0,0)[(\phi,\tau)] = 0$ if and only if

$$\begin{cases}\Delta\phi + (2^*-1)U_{\varepsilon_0}^{2^*-2}\phi - \tau U_{\varepsilon_0} = 0,\\ \displaystyle\int_{\mathbb{R}^N} U_{\varepsilon_0}\phi = 0.\end{cases} \tag{4.11}$$

We claim that in $H^1_{rad}(\mathbb{R}^N)\times\mathbb{R}$, $\mathcal{L}(U_{\varepsilon_0},0,0)[(\phi,\tau)] = 0$ if and only if $(\phi,\tau) = (0,0)$. Let $\mathcal{L}(U_{\varepsilon_0},0,0)[(\phi,\tau)] = 0$ for some $(\phi,\tau) \in H^1_{rad}(\mathbb{R}^N)\times\mathbb{R}$. Since it is well-known (cf. [14]) that $W = \frac{N-2}{2}U_{\varepsilon_0} + U'_{\varepsilon_0}r$ is the unique radial solution of the following equation

$$\Delta\phi + (2^*-1)U_{\varepsilon_0}^{2^*-2}\phi = 0$$

in $H^1_{rad}(\mathbb{R}^N)$, by multiplying the first equation of (4.11) with W and integrating by parts, we have

$$0 = \tau\int_{\mathbb{R}^N} WU_{\varepsilon_0} = -\tau\int_{\mathbb{R}^N} U_{\varepsilon_0}^2.$$

It follows that $\tau = 0$ and thus, $\phi = CW$ for some constant $C \in \mathbb{R}$. By the second equation of (4.11),

$$0 = \int_{\mathbb{R}^N} U_{\varepsilon_0}\phi = -C\int_{\mathbb{R}^N} U_{\varepsilon_0}^2,$$

which implies that $\phi = 0$. Thus, the kernel of the linearization of the system (4.10) at $(U_{\varepsilon_0},0,0)$ in $H^1_{rad}(\mathbb{R}^N)\times\mathbb{R}$ is trivial, which implies that the linear operator $\mathcal{L}(U_{\varepsilon_0},0,0) : H^1_{rad}(\mathbb{R}^N)\times\mathbb{R} \to H^1_{rad}(\mathbb{R}^N)\times\mathbb{R}$ is injective. On the other hand, by similar arguments as used for Proposition 2.1, we know that all minimizers of $\mathcal{E}_\mu(u)|_{\mathcal{S}_a}$ on $\mathcal{P}_-^{a,\mu}$ are real valued, positive, radially symmetric and radially decreasing up to translations and rotations. Now, suppose that there are at least two minimizers of $\mathcal{E}_\mu(u)|_{\mathcal{S}_a}$ on $\mathcal{P}_-^{a,\mu}$, say u_μ^* and u_μ^{**}, then without loss of generality, we may assume that they are all real valued, positive, radially symmetric and radially decreasing. The corresponding Lagrange multipliers are λ_μ^* and λ_μ^{**}, respectively. Let

$$w_\mu=\frac{u_\mu^*-u_\mu^{**}}{\|u_\mu^*-u_\mu^{**}\|_{H^1}+|\lambda_\mu^*-\lambda_\mu^{**}|}\quad\text{and}\quad \varsigma_\mu=\frac{\lambda_\mu^*-\lambda_\mu^{**}}{\|u_\mu^*-u_\mu^{**}\|_{H^1}+|\lambda_\mu^*-\lambda_\mu^{**}|},$$

where $\|\cdot\|_{H^1}$ is the usual norm in $H^1(\mathbb{R}^N)$. It is easy to see that $\{w_\mu\}$ is bounded in $H^1(\mathbb{R}^N)$ and $\{\varsigma_\mu\}$ is bounded. Moreover, by (1.5), we also have

$$\begin{aligned}-\Delta w_\mu-\lambda_\mu^* w_\mu-\varsigma_\mu u_\mu^{**} &= \mu(q-1)\Big(u_\mu^*+\theta_\mu(u_\mu^*-u_\mu^{**})\Big)^{q-2}w_\mu\\ &\quad+(2^*-1)\Big(u_\mu^*+\theta_\mu'(u_\mu^*-u_\mu^{**})\Big)^{2^*-2}w_\mu,\end{aligned}\tag{4.12}$$

where $\theta_\mu,\theta_\mu'\in(0,1)$. Since u_μ^* and u_μ^{**} belong to $\mathcal{S}_a$, we also have

$$2\int_{\mathbb{R}^N}u_\mu^* w_\mu=-\|u_\mu^*-u_\mu^{**}\|_2\|w_\mu\|_2.$$

Since the linear operator $\mathcal{L}(U_{\varepsilon_0},0,0):H^1_{rad}(\mathbb{R}^N)\times\mathbb{R}\to H^1_{rad}(\mathbb{R}^N)\times\mathbb{R}$ is injective, it is standard to prove that $(w_\mu,\varsigma_\mu)\rightharpoonup(0,0)$ weakly in $D^{1,2}(\mathbb{R}^N)\times\mathbb{R}$ as $\mu\to0^+$. Now, by multiplying (4.12) with w_μ and integrating by parts, we can use fact that $u_\mu^{**},u_\mu^*\to U_{\varepsilon_0}$ strongly in $H^1(\mathbb{R}^N)$ as $\mu\to+\infty$ to show that $(w_\mu,\varsigma_\mu)\to(0,0)$ strongly in $D^{1,2}(\mathbb{R}^N)\times\mathbb{R}$ as $\mu\to0^+$. Moreover, by (4.7), we also have

$$\varsigma_\mu\sim\mu\int_{\mathbb{R}^N}\Big(u_\mu^*+\theta_\mu(u_\mu^*-u_\mu^{**})\Big)^{q-1}w_\mu=o(\mu).$$

By (4.7), (4.8) and (4.12),

$$-\Delta w_\mu-\frac12\lambda_\mu^* w_\mu\lesssim\frac{\mu}{r^{N-2}}\quad\text{for } r\gtrsim\frac{1}{|\lambda_\mu^*|^{\frac14}}.$$

By (4.7), we also have

$$-\Delta(r^{2-N})-\frac12\lambda_\mu^* r^{2-N}=-\frac12\lambda_\mu^* r^{2-N}\gtrsim\frac{\mu}{r^{N-2}}\quad\text{for } r\gtrsim\frac{1}{|\lambda_\mu^*|^{\frac14}}.$$

Since w_μ is radial and $\{w_\mu\}$ is bounded in $H^1(\mathbb{R}^N)$, by [12, Lemma A.2],

$$|w_\mu|\lesssim r^{-\frac{N-1}{2}}\quad\text{for } r\gtrsim1.\tag{4.13}$$

Thus, by the maximum principle,

$$|w_\mu|\lesssim r^{2-N}\quad\text{for } r\gtrsim\frac{1}{|\lambda_\mu^*|^{\frac14}}.\tag{4.14}$$

For $1\lesssim r\lesssim\frac{1}{|\lambda_\mu^*|^{\frac14}}$, by (4.8), (4.12) and (4.13),

$$-\Delta w_\mu\lesssim\frac{1}{r^4}\Big(\frac{1}{r^{\frac{N-1}{2}}}+\frac{1}{r^{N-2}}\Big)\lesssim r^{-\frac{7+N}{2}}\lesssim r^{-\frac{\alpha+N}{2}}$$

in the case of $N\geq5$, where $\alpha=\frac92$. Recall that for $N\geq5$, $r^{2-\frac{\alpha+N}{2}}$ is also a superharmornic function. Thus, by the maximum principle,

$$|w_\mu|\lesssim r^{2-\frac{\alpha+N}{2}}\quad\text{for } 1\lesssim r\lesssim\frac{1}{|\lambda_\mu^*|^{\frac14}}.\tag{4.15}$$

Note that by $w_\mu \to 0$ strongly in $D^{1,2}(\mathbb{R}^N)$ as $\mu \to 0^+$ and $\|w_\mu\|_{H^1}^2 = 1$, we know that $\|w_\mu\|_2^2 = 1 + o_\mu(1)$. Thus, by $w_\mu \to 0$ strongly in $D^{1,2}(\mathbb{R}^N)$ as $\mu \to 0^+$, the Sobolev embedding theorem and (4.14) and (4.15),

$$1 \sim \int_{\mathbb{R}^N} |w_\mu|^2 \lesssim o_\mu(1) + \int_{r_0}^{\frac{1}{|\lambda_\mu^*|^{\frac{1}{4}}}} r^{3-\alpha} + \int_{\frac{1}{|\lambda_\mu^*|^{\frac{1}{4}}}}^{+\infty} r^{3-N} = o_\mu(1) + \frac{1}{2} r_0^{-\frac{1}{2}},$$

which is a contradiction by taking $r_0 > 0$ sufficiently large. It follows that $u_{a,\mu,-}$ is the unique minimizer of $\mathcal{E}|_{\mathcal{S}_a}(u)$ on $\mathcal{P}_-^{a,\mu}$ for $\mu > 0$ sufficiently small up to translations and rotations if $N \geq 5$. □

For $N = 3, 4$, The Aubin-Talanti babbles $U_\varepsilon \notin L^2(\mathbb{R}^N)$. Thus, we need to modify the arguments for Proposition 4.1 to give a precise description of $u_{a,\mu,-}$ as $\mu \to 0^+$ in these two cases. As in the proof of Proposition 4.1, $\{u_{a,\mu,-}\}$ is also a minimizing sequence of the minimizing problem (4.1) in the cases $N = 3, 4$. Since $u_{a,\mu,-}$ is radially symmetric for $r = |x|$, by Lions' result (cf. [56, Theorem 1.41]), up to subsequence, there exists $\sigma_\mu > 0$ such that for some $\varepsilon_* > 0$,

$$v_{a,\mu,-}(x) = \sigma_\mu^{\frac{N-2}{2}} u_{a,\mu,-}(\sigma_\mu x) \to U_{\varepsilon_*} \text{ strongly in } D^{1,2}(\mathbb{R}^N) \text{ as } \mu \to 0^+. \tag{4.16}$$

We also remark that since $U_{\varepsilon_*} \notin L^2(\mathbb{R}^N)$ for $N = 3, 4$ and $\|v_{a,\mu,-}\|_2^2 = \frac{a^2}{\sigma_\mu^2}$, by the Fatou lemma, we have $\sigma_\mu \to 0$ as $\mu \to 0^+$.

Lemma 4.1. *Let $N = 3, 4$ and $2 < q < 2^*$. Then*

$$1 \sim \begin{cases} \dfrac{\mu\sigma_\mu^{N-\frac{N-2}{2}q}}{-\lambda_{a,\mu,-}}, & \dfrac{N}{N-2} < q < 2^*, \\ \dfrac{\mu\sigma_\mu^{\frac{3}{2}}}{-\lambda_{a,\mu,-}} \ln\left(\dfrac{1}{\sqrt{-\lambda_{a,\mu,-}}\sigma_\mu}\right), & N = 3, q = 3, \\ \dfrac{\mu\sigma_\mu^{3-\frac{q}{2}}\left(\sqrt{-\lambda_{a,\mu,-}}\sigma_\mu\right)^{q-3}}{-\lambda_{a,\mu,-}}, & N = 3, 2 < q < 3. \end{cases}$$

Proof. By the equation (1.5), we know that $v_{a,\mu,-}$ satisfies

$$-\Delta v_{a,\mu,-} - \lambda_{a,\mu,-}\sigma_\mu^2 v_{a,\mu,-} = \mu\sigma_\mu^{N-\frac{N-2}{2}q} v_{a,\mu,-}^{q-1} + v_{a,\mu,-}^{2^*-1} \quad \text{in } \mathbb{R}^N. \tag{4.17}$$

It follows from (4.7) that

$$-\lambda_{a,\mu,-}\sigma_\mu^2 \|v_{a,\mu,-}\|_2^2 = (1-\gamma_q)\mu\sigma_\mu^{N-\frac{N-2}{2}q} \|v_{a,\mu,-}\|_q^q. \tag{4.18}$$

Recall that $\|u_{a,\mu,-}\|_2^2 = \sigma_\mu^2 \|v_{a,\mu,-}\|_2^2 = a^2$ and $\|\nabla u_{a,\mu,-}\|_2^2 \to S^{\frac{N}{2}}$ as $\mu \to 0^+$, by (4.7) and the Hölder inequality, $\lambda_{a,\mu,-} \to 0$ as $\mu \to 0^+$. Clearly,

$$\mu\sigma_\mu^{N-\frac{N-2}{2}q} \to 0 \quad \text{as } \mu \to 0^+. \tag{4.19}$$

By the Hölder inequality once more,

$$|\lambda_{a,\mu,-}|\sigma_\mu^2 \|v_{a,\mu,-}\|_2^2 \lesssim \mu\sigma_\mu^{N-\frac{N-2}{2}q} \|v_{a,\mu,-}\|_2^{N-\frac{N-2}{2}q}.$$

Since $q > 2$, $N - \frac{N-2}{2}q < 2$. It follows from $\|v_{a,\mu,-}\|_2^2 \sim \sigma_\mu^{-2} \to +\infty$ as $\mu \to 0^+$ that

$$|\lambda_{a,\mu,-}|\sigma_\mu^2 = o(\mu\sigma_\mu^{N-\frac{N-2}{2}q}) \quad \text{as } \mu \to 0^+. \tag{4.20}$$

Recall that $v_{a,\mu,-} \to U_{\varepsilon_*}$ strongly in $D^{1,2}(\mathbb{R}^N)$ as $\mu \to 0^+$ up to a subsequence, by (4.19)-(4.20), adapting the Moser iteration (cf. [53, B.3 Lemma]) and the L^p theory of elliptic equations to (4.17) and the Sobolev embedding theorem,

$$v_{a,\mu,-} \to U_{\varepsilon_*} \quad \text{strongly in } L^\infty(\mathbb{R}^N) \text{ as } \mu \to 0^+ \text{ up to a subsequence} \tag{4.21}$$

In what follows, we follow the ideas in [7] (see also [24, 38]) to drive a uniformly upper bound of $v_{a,\mu,-}$. We define

$$\widetilde{v}_{a,\mu,-} = \frac{1}{v_{a,\mu,-}(0)} v_{a,\mu,-}(\sqrt{v_{a,\mu,-}(0)}x).$$

Since $\widetilde{v}_{a,\mu,-}$ is radial, $\widetilde{v}_{a,\mu,-}$ satisfies

$$-\widetilde{v}''_{a,\mu,-} - \frac{N-1}{r}\widetilde{v}'_{a,\mu,-} = f(\widetilde{v}_{a,\mu,-}) \quad \text{in } \mathbb{R}^N. \tag{4.22}$$

where

$$\begin{aligned} f(\widetilde{v}_{a,\mu,-}) &= \lambda_{a,\mu,-}\sigma_\mu^2 v_{a,\mu,-}(0)\widetilde{v}_{a,\mu,-} + \mu\sigma_\mu^{N-\frac{N-2}{2}q}[v_{a,\mu,-}(0)]^{q-1}\widetilde{v}_{a,\mu,-}^{q-1} \\ &\quad +[v_{a,\mu,-}(0)]^{2^*-1}\widetilde{v}_{a,\mu,-}^{2^*-1}. \end{aligned}$$

Let

$$H(r) = r^N(\widetilde{v}'_{a,\mu,-})^2 + (N-2)r^{N-1}\widetilde{v}_{a,\mu,-}\widetilde{v}'_{a,\mu,-} + \frac{N-2}{N}r^N\widetilde{v}_{a,\mu,-}f(\widetilde{v}_{a,\mu,-}).$$

Then by direct calculations and using (4.20)-(4.22),

$$\begin{aligned} H'(r) &= \frac{r^N\widetilde{v}'_{a,\mu,-}}{N}(4|\lambda_{a,\mu,-}|\sigma_\mu^2 - (N-2)(2^*-q)\mu\sigma_\mu^{N-\frac{N-2}{2}q}v_{a,\mu,-}^{q-2})v_{a,\mu,-} \\ &= \mu\sigma_\mu^{N-\frac{N-2}{2}q}\frac{r^N\widetilde{v}'_{a,\mu,-}}{N}(o_\mu(1) - (N-2)(2^*-q)U_{\varepsilon_*}^{q-2})v_{a,\mu,-}. \end{aligned}$$

Since $v_{a,\mu,-} > 0$, $\widetilde{v}'_{a,\mu,-} < 0$ and $v_{a,\mu,-}$ exponentially decays to zero as $r \to +\infty$, there exists $r_\mu > 0$, $H'(r) > 0$ for $0 < r < r_\mu$ and $H'(r) < 0$ for $r > r_\mu$. Thus, $H(r) > H(0) = 0$ for all $r > 0$. Let

$$\Psi(r) = \frac{-\widetilde{v}'_{a,\mu,-}}{r\widetilde{v}_{a,\mu,-}^{\frac{N}{N-2}}}.$$

Then by direct calculations and using (4.22),

$$\Psi'(r) = \frac{N}{N-2}r^{-(1+N)}\widetilde{v}_{a,\mu,-}^{-\frac{2N-2}{N-2}}H(r) > 0$$

for all $r > 0$. It follows from (4.22) once more that

$$\Psi(r) > \Psi(0) = -\widetilde{v}''_{a,\mu,-}(0) = \frac{d_\mu}{N}$$

where

$$d_\mu = \lambda_{a,\mu,-}\sigma_\mu^2 v_{a,\mu,-}(0) + \mu\sigma_\mu^{N-\frac{N-2}{2}q}[v_{a,\mu,-}(0)]^{q-1} + [v_{a,\mu,-}(0)]^{2^*-1}.$$

Let

$$Z_\mu(r) = \frac{1}{(1+\frac{d_\mu}{N(N-2)}r^2)^{\frac{N-2}{2}}}.$$

Then it is easy to check that $\frac{-Z'_\mu(r)}{[Z_\mu(r)]^{\frac{N}{N-2}}} = \frac{d_\mu}{N}r$. It follows that

$$\frac{\widetilde{v}'_{a,\mu,-}}{\widetilde{v}_{a,\mu,-}^{\frac{N}{N-2}}} \leq \frac{Z'_\mu(r)}{[Z_\mu(r)]^{\frac{N}{N-2}}} \quad \text{for all } r>0,$$

which together with (4.21), implies

$$v_{a,\mu,-} \lesssim \frac{1}{(1+r^2)^{\frac{N-2}{2}}} \quad \text{for all } r>0 \tag{4.23}$$

uniformly for $\mu>0$ sufficiently small. Now, for the cases $\frac{N}{N-2}<q<2^*$,

$$\|v_{a,\mu,-}\|_q^q \lesssim \int_0^{+\infty} \frac{1}{(1+r^2)^{\frac{(N-2)q}{2}}} r^{N-1}dr \lesssim 1.$$

By the Fatou lemma, $\|v_{a,\mu,-}\|_q^q \geq \|U_{\varepsilon_*}\|_q^q + o_\mu(1) \gtrsim 1$. Thus, by (4.18),

$$\frac{\mu\sigma_\mu^{N-\frac{N-2}{2}q}}{-\lambda_{a,\mu,-}} \sim 1 \quad \text{for } \frac{N}{N-2}<q<2^*.$$

For $N=3$ and $q=3$, we need to drive the uniformly exponential decay of $v_{a,\mu,-}$ at infinitely both from below and above to obtain the conclusions. Let

$$\Phi = r^{-1}e^{-\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu r}.$$

Then it is easy to check (cf. [44]) that $-\Delta\Phi - \lambda_{a,\mu,-}\sigma_\mu^2\Phi \leq 0$ for $r\geq 1$ in the case of $N=3$. Since $v_{a,\mu,-}\to U_{\varepsilon_*}$ strongly in $L^\infty(\mathbb{R}^N)$ as $\mu\to 0^+$ up to a subsequence, by the maximum principle,

$$v_{a,\mu,-} \gtrsim r^{-1}e^{-\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu r} \quad \text{for } r\geq 1 \tag{4.24}$$

in the case of $N=3$. On the other hand, let

$$\Upsilon = r^{-1}e^{-\frac{1}{2}\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu r}.$$

Then it is also easy to check that $-\Delta\Upsilon - \frac{1}{2}\lambda_{a,\mu,-}\sigma_\mu^2\Upsilon \geq 0$ for $r\geq 1$. Since $\mu\sigma_\mu^{3-\frac{q}{2}}\to 0$ as $\mu\to 0^+$, by (4.23), for

$$r \gtrsim \frac{1}{|\lambda_{a,\mu,-}|\sigma_\mu^2},$$

we have $-\Delta v_{a,\mu,-} - \frac{1}{2}\lambda_{a,\mu,-}\sigma_\mu^2 v_{a,\mu,-} \leq 0$ in the case of $N=3$. Thus, by the maximum principle and (4.21) once more,

$$v_{a,\mu,-} \lesssim r^{-1}e^{-\frac{1}{2}\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu r} \quad \text{for } r \gtrsim \frac{1}{|\lambda_{a,\mu,-}|\sigma_\mu^2}. \tag{4.25}$$

For $q=3$ and $N=3$, by (4.23) and (4.25),

$$\begin{aligned}\|v_{a,\mu,-}\|_3^3 &\lesssim \int_1^{\frac{1}{|\lambda_{a,\mu,-}|\sigma_\mu^2}} r^{-1} + \int_{\frac{1}{|\lambda_{a,\mu,-}|\sigma_\mu^2}}^{+\infty} e^{-\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu r} \\ &\lesssim \ln\left(\frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu}\right).\end{aligned}$$

By (4.24), for $q=3$ and $N=3$, we also have

$$\|v_{a,\mu,-}\|_3^3 \gtrsim \int_1^{\frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu}} r^{-1} \gtrsim \ln\left(\frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu}\right).$$

Thus, the conclusion for $q=3$ and $N=3$ then follows (4.18). For $N=3$ and $2<q<3$, we need to construct a newly upper bound of $v_{a,\mu,-}$ by adapting ideas in [20]. By (4.23), we have

$$-\Delta v_{a,\mu,-} - \frac{1}{2}\lambda_{a,\mu,-}\sigma_\mu^2 v_{a,\mu,-} \lesssim \mu\sigma_\mu^{3-\frac{q}{2}} r^{1-q} \quad \text{for } r \gtrsim \frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu}.$$

On the other hand, let $\phi_\mu \sim \frac{\mu\sigma_\mu^{3-\frac{q}{2}}}{|\lambda_{a,\mu,-}|\sigma_\mu^2} r^{1-q}$, then by direct calculations,

$$-\Delta\phi_\mu - \frac{1}{2}\lambda_{a,\mu,-}\sigma_\mu^2\phi_\mu \gtrsim \mu\sigma_\mu^{3-\frac{q}{2}} r^{1-q} \quad \text{for } r \gtrsim \frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu}.$$

By (4.23) and the maximum principle,

$$v_{a,\mu,-} \lesssim \frac{\mu\sigma_\mu^{3-\frac{q}{2}}}{|\lambda_{a,\mu,-}|\sigma_\mu^2} r^{1-q} \quad \text{for } r \gtrsim \frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu}. \tag{4.26}$$

Now, using (4.26) as a new barrier and by (4.23), we know that

$$-\Delta v_{a,\mu,-} - \frac{1}{2}\lambda_{a,\mu,-}\sigma_\mu^2 v_{a,\mu,-} \lesssim \mu\sigma_\mu^{3-\frac{q}{2}}\left(\frac{\mu\sigma_\mu^{3-\frac{q}{2}}}{|\lambda_{a,\mu,-}|\sigma_\mu^2} r^{1-q}\right)^{q-1}$$

for $r \gtrsim \frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu}$. Thus, by similar comparisons, we have

$$v_{a,\mu,-} \lesssim \left(\frac{\mu\sigma_\mu^{3-\frac{q}{2}}}{|\lambda_{a,\mu,-}|\sigma_\mu^2}\right)^q r^{-(q-1)^2} \quad \text{for } r \gtrsim \frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu}.$$

By iterating the above arguments n times for a sufficiently large n such that $q(q-1)^n-3>0$, we have

$$v_{a,\mu,-} \lesssim \left(\frac{\mu\sigma_\mu^{3-\frac{q}{2}}}{|\lambda_{a,\mu,-}|\sigma_\mu^2}\right)^{s_n} r^{-(q-1)^n} \quad \text{for } r \gtrsim \frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu}, \tag{4.27}$$

where $s_n = s_{n-1}(q-1)+1$ which implies

$$s_n = \frac{(q-1)^{n+1}-1}{q-2}.$$

By (4.24), we have

$$\|v_{a,\mu,-}\|_q^q \gtrsim \int_1^{\frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu}} r^{2-q} e^{-q\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu r} \gtrsim \left(\frac{1}{|\lambda_{a,\mu,-}|\sigma_\mu^2}\right)^{\frac{3-q}{2}} \tag{4.28}$$

and

$$\|v_{a,\mu,-}\|_2^2 \gtrsim \int_1^{\frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu}} e^{-2\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu r} \gtrsim \left(\frac{1}{|\lambda_{a,\mu,-}|\sigma_\mu^2}\right)^{\frac{1}{2}}.$$

It follows from (4.18) that

$$|\lambda_{a,\mu,-}| \gtrsim \mu\sigma_\mu^{3-\frac{q}{2}}\left(\frac{1}{|\lambda_{a,\mu,-}|\sigma_\mu^2}\right)^{\frac{3-q}{2}} \quad \text{and} \quad \left(\frac{1}{|\lambda_{a,\mu,-}|\sigma_\mu^2}\right)^{\frac{1}{2}}|\lambda_{a,\mu,-}| \lesssim \mu \tag{4.29}$$

which implies

$$\frac{\mu\sigma_\mu^{3-\frac{q}{2}}}{|\lambda_{a,\mu,-}|\sigma_\mu^2} \lesssim |\lambda_{a,\mu,-}|^{\frac{3-q}{2}}\sigma_\mu^{1-q} \lesssim \sigma_\mu^{2(2-q)}.$$

Now, by (4.27) and (4.29),

$$\begin{aligned}
\|v_{a,\mu,-}\|_q^q &\lesssim \int_1^{\frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu}} r^{2-q} + (\sigma_\mu)^{-2((q-1)^{n+1}-1)}\int_{\frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu}}^{+\infty} r^{2-q(q-1)^n} \\
&\lesssim \left(\frac{1}{|\lambda_{a,\mu,-}|\sigma_\mu^2}\right)^{\frac{3-q}{2}} + \sigma_\mu^{2(q(q-1)^n-3)-2((q-1)^{n+1}-1)} \\
&\lesssim \left(\frac{1}{|\lambda_{a,\mu,-}|\sigma_\mu^2}\right)^{\frac{3-q}{2}} + \sigma_\mu^{(q-1)^n(2+o_n(1))} \\
&= \left(\frac{1}{|\lambda_{a,\mu,-}|\sigma_\mu^2}\right)^{\frac{3-q}{2}}. \qquad (4.30)
\end{aligned}$$

The conclusion for $N = 3$ and $2 < q < 3$ follows from (4.18), (4.28) and (4.30). $\square$

With Lemma 4.1 in hands, we can obtain the following.

Proposition 4.2. *Let $N = 3, 4$ and $2 < q < 2^*$. Then*

$$w_{\mu,-} = \varepsilon_\mu^{\frac{N-2}{2}} u_{a,\mu,-}(\varepsilon_\mu x) \to U_{\varepsilon_*} \quad \text{strongly in } D^{1,2}(\mathbb{R}^N) \text{ as } \mu \to 0^+$$

up to a subsequence for some $\varepsilon_* > 0$, where $\varepsilon_\mu > 0$ satisfies

$$\mu \sim \begin{cases} \varepsilon_\mu^{6-q}e^{-2\varepsilon_\mu^{-2}}, & N = 4, 2 < q < 2^*, \\ \varepsilon_\mu^{\frac{q}{2}-1}, & N = 3, 3 < q < 2^*, \\ \frac{\varepsilon_\mu^{\frac{1}{2}}}{\ln(\frac{1}{\varepsilon_\mu})}, & N = 3, q = 3, \\ \varepsilon_\mu^{5-\frac{3q}{2}}, & N = 3, 2 < q < 3. \end{cases} \qquad (4.31)$$

Proof. Let $\{V_\varepsilon\}$ be the family given by (2.4). Since $2 \geq \frac{N}{N-2}$ for $N \geq 4$, By [44, (4.2)–(4.5)],

$$\|V_\varepsilon\|_q^q \sim \begin{cases} \varepsilon^{N-\frac{N-2}{2}q}, & N = 3, 4, \frac{N}{N-2} < q < 2^*, \\ \varepsilon^{\frac{3}{2}}\ln(R_\varepsilon\varepsilon^{-1}), & N = 3, q = 3, \\ \varepsilon^{3-\frac{q}{2}}(R_\varepsilon\varepsilon^{-1})^{3-q}, & N = 3, 2 < q < 3 \end{cases} \qquad (4.32)$$

for $\varepsilon > 0$ sufficiently small. By [49, Lemmas 4.2, 5.1 and 6.1], there exist $t_{\mu,\varepsilon} > 0$ such that

$$\|\nabla V_\varepsilon\|_2^2 = \mu\gamma_q\|V_\varepsilon\|_q^q t_{\mu,\varepsilon}^{q\gamma_q-2} + \|V_\varepsilon\|_{2^*}^{2^*}t_{\mu,\varepsilon}^{2^*-2}$$

and

$$2\|\nabla V_\varepsilon\|_2^2 < \mu q\gamma_q^2\|V_\varepsilon\|_q^q t_{\mu,\varepsilon}^{q\gamma_q-2} + 2^*\|V_\varepsilon\|_{2^*}^{2^*}t_{\mu,\varepsilon}^{2^*-2}.$$

Thus, $\{t_{\mu,\varepsilon}\}$ is uniformly bounded and bounded from below away from 0 for all $\varepsilon, \mu > 0$ sufficiently small. By (2.5) and (2.6),

$$S^{\frac{N}{2}}(1 - t_{\mu,\varepsilon_\mu}^{2^*-2}) = \mu\gamma_q\|V_\varepsilon\|_q^q t_{\mu,\varepsilon}^{q\gamma_q-2} + O((R_\varepsilon\varepsilon^{-1})^{2-N})).$$

Then we can use similar arguments as used for (4.2) to show that

$$t_{\mu,\varepsilon} = 1 - (1+o(1))\frac{\mu\gamma_q\|V_\varepsilon\|_q^q + O((R_\varepsilon\varepsilon^{-1})^{2-N}))}{(2^*-2)S^{\frac{N}{2}}}$$

and thus by similar arguments as used for (4.6),

$$\|u_{a,\mu,-}\|_q^q \geq (1+o(1))(\|V_\varepsilon\|_q^q - C\mu^{-1}(R_\varepsilon\varepsilon^{-1})^{2-N})),$$

which together with (2.5) and (4.32), implies

$$\|u_{a,\mu,-}\|_q^q \gtrsim \begin{cases} \varepsilon^{4-q} - C\mu^{-1}e^{-2\varepsilon^{-2}}, & N=4, 2<q<2^*, \\ \varepsilon^{3-\frac{q}{2}} - C\mu^{-1}\varepsilon^2, & N=3, 3<q<2^*, \\ \varepsilon^{\frac{3}{2}}\ln(\frac{1}{\varepsilon}) - C\mu^{-1}\varepsilon^2, & N=3, q=3, \\ \varepsilon^{\frac{3q}{2}-3} - C\mu^{-1}\varepsilon^2, & N=3, 2<q<3. \end{cases}$$

By choosing ε_μ such that the right hand sides of the above estimate take the maximum, we have (4.31) and

$$\|u_{a,\mu,-}\|_q^q \gtrsim \begin{cases} \varepsilon_\mu^{N-\frac{N-2}{2}q}, & N=3,4, \frac{N}{N-2}<q<2^*, \\ \varepsilon_\mu^{\frac{3}{2}}\ln(\frac{1}{\varepsilon_\mu}), & N=3, q=3, \\ \varepsilon_\mu^{\frac{3q}{2}-3}, & N=3, 2<q<3. \end{cases} \tag{4.33}$$

We define $w_{\mu,-} = \varepsilon_\mu^{\frac{N-2}{2}}u_{a,\mu,-}(\varepsilon_\mu x)$, then $\|w_{\mu,-}\|_{2^*}^{2^*}, \|\nabla w_{\mu,-}\|_2^2 \sim 1$ and by (4.33),

$$\|w_{\mu,-}\|_q^q \gtrsim \begin{cases} 1, & N=3,4, \frac{N}{N-2}<q<2^*, \\ \ln(\frac{1}{\varepsilon_\mu}), & N=3, q=3, \\ \varepsilon_\mu^{2q-6}, & N=3, 2<q<3. \end{cases} \tag{4.34}$$

It is easy to see that

$$\sigma_\mu^{N-\frac{N-2}{2}q}\|v_{a,\mu,-}\|_q^q = \|u_{a,\mu,-}\|_q^q = \varepsilon_\mu^{N-\frac{N-2}{2}q}\|w_{\mu,-}\|_q^q. \tag{4.35}$$

Then by Lemma 4.1, (4.18) and (4.34), we have

$$\sigma_\mu^{N-\frac{N-2}{2}q} \gtrsim \varepsilon_\mu^{N-\frac{N-2}{2}q} \tag{4.36}$$

for $\frac{N}{N-2} < q < 2^*$ and $N=3,4$. On the other hand, we know that

$$w_{\mu,-}(x) = \left(\frac{\varepsilon_\mu}{\sigma_\mu}\right)^{\frac{N-2}{2}} v_{a,\mu,-}\left(\frac{\varepsilon_\mu}{\sigma_\mu}x\right) \tag{4.37}$$

and $\widetilde{w}_{\mu,-}$ satisfies

$$-\Delta\widetilde{w}_{\mu,-} = g(\widetilde{w}_{\mu,-}) \quad \text{in } \mathbb{R}^N.$$

where

$$\widetilde{w}_{\mu,-}=\frac{1}{w_{\mu,-}(0)}w_{\mu,-}([w_{\mu,-}(0)]^s x)$$

with $s\in\mathbb{R}$ and

$$\begin{aligned} g(\widetilde{w}_{\mu,-}) &= \lambda_{a,\mu,-}\varepsilon_\mu^2[w_{\mu,-}(0)]^{2s}\widetilde{w}_{\mu,-}+\mu\varepsilon_\mu^{N-\frac{N-2}{2}q}[w_{\mu,-}(0)]^{2s+q-2}\widetilde{w}_{\mu,-}^{q-1} \\ &\quad +[w_{\mu,-}(0)]^{2s+2^*-2}\widetilde{w}_{\mu,-}^{2^*-1}. \end{aligned}$$

By similar arguments as used for (4.23), we have

$$w_{\mu,-}\lesssim\frac{w_{\mu,-}(0)}{(1+b_\mu r^2)^{\frac{N-2}{2}}}\quad\text{for all } r>0, \tag{4.38}$$

where

$$\begin{aligned} b_\mu &= [w_{\mu,-}(0)]^{2s-1}(\lambda_{a,\mu,-}\varepsilon_\mu^2 w_{\mu,-}(0)+\mu\varepsilon_\mu^{N-\frac{N-2}{2}q}[w_{\mu,-}(0)]^{q-1} \\ &\quad +[w_{\mu,-}(0)]^{2^*-1}). \end{aligned}$$

We recall that $\mu,\sigma_\mu,\lambda_{a,\mu,-}\to 0$ as $\mu\to 0^+$, and by (4.7), we have $\lambda_{a,\mu,-}\lesssim\mu$. Thus, by Lemma 4.1, (4.21) and (4.37),

$$b_\mu \sim \left(\frac{\varepsilon_\mu}{\sigma_\mu}\right)^{(N-2)s+2}. \tag{4.39}$$

Now, take $s=-1$ and by (4.38), we can use similar arguments in the proof of Lemma 4.1 to show that

$$\|w_{\mu,-}\|_q^q\lesssim\left(\frac{\varepsilon_\mu}{\sigma_\mu}\right)^{\frac{N-2}{2}q-\frac{N}{2}(4-N)}$$

for $\frac{N}{N-2}<q<2^*$ and $N=3,4$, which together with (4.35), implies that $\sigma_\mu\lesssim\varepsilon_\mu$ for $\frac{N}{N-2}<q<2^*$ and $N=3,4$. It follows from (4.36) that $\sigma_\mu\sim\varepsilon_\mu$ for $\frac{N}{N-2}<q<2^*$ and $N=3,4$. For the case $N=3$ and $q=3$, by $\|w_{\mu,-}\|_2^2\sim\varepsilon_\mu^{-2}$, Struss's radial lemma (cf. [12, Lemma A.IV, Theorem A.I'] or [44, Lemma 3.1]) and similar arguments as used for (4.25),

$$w_{\mu,-}\lesssim\varepsilon_\mu^{-2}r^{-1}e^{-\frac12\sqrt{|\lambda_{a,\mu,-}|}\varepsilon_\mu r}\quad\text{for } r\gtrsim\frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\varepsilon_\mu}. \tag{4.40}$$

It follows from (4.38) and (4.39) that

$$\|w_{\mu,-}\|_q^q\lesssim\left(\frac{\varepsilon_\mu}{\sigma_\mu}\right)^{\frac32-\frac32(s+2)}\ln\left(\frac{1}{\sqrt{|\lambda_{a,\mu,-}|}\sigma_\mu}\right).$$

By Lemma 4.1, taking $s=-1$ and (4.35), we have $\varepsilon_\mu\gtrsim\sigma_\mu$. By Lemma 4.1, taking $s=2$ and (4.35), we have $\varepsilon_\mu\lesssim\sigma_\mu$. Thus, for $N=3$ and $q=3$, we also have $\varepsilon_\mu\sim\sigma_\mu$. For the case $N=3$ and $2<q<3$, by (4.7), (4.33), Lemma 4.1 and $\mu\sim\varepsilon_\mu^{5-\frac{3q}{2}}$,

$$\sigma_\mu^{\frac{q}{5-q}}\gtrsim\varepsilon_\mu^{\frac{q}{5-q}}\left(\frac{\mu}{\varepsilon_\mu^{5-\frac{3q}{2}}}\right)^{\frac{3-q}{5-q}}\sim\varepsilon_\mu^{\frac{q}{5-q}}$$

which implies $\sigma_\mu \gtrsim \varepsilon_\mu$. Thus, by (4.21), we can adapt the maximum principle as that in the proof of Proposition 4.2 to show that

$$w_{a,\mu,-} \gtrsim r^{-1}e^{-\sqrt{|\lambda_{a,\mu,-}|}\varepsilon_\mu r} \quad \text{for } r \geq 1. \tag{4.41}$$

By (4.41), we can see that the estimates for (4.29) works for ε_μ and thus, we have

$$|\lambda_{a,\mu,-}| \gtrsim \mu\varepsilon_\mu^{3-\frac{q}{2}}\left(\frac{1}{|\lambda_{a,\mu,-}|\varepsilon_\mu^2}\right)^{\frac{3-q}{2}}, \quad \left(\frac{1}{|\lambda_{a,\mu,-}|\varepsilon_\mu^2}\right)^{\frac{1}{2}}|\lambda_{a,\mu,-}| \lesssim \mu$$

and

$$\frac{\mu\varepsilon_\mu^{3-\frac{q}{2}}}{|\lambda_{a,\mu,-}|\varepsilon_\mu^2} \lesssim |\lambda_{a,\mu,-}|^{\frac{3-q}{2}}\varepsilon_\mu^{1-q} \lesssim \varepsilon_\mu^{2(2-q)}.$$

Now, we can follow similar arguments as used in the proof of Lemma 4.1 to show that

$$\|w_{\mu,-}\|_q^q \quad \lesssim \quad \left(\frac{1}{|\lambda_{a,\mu,-}|\varepsilon_\mu^2}\right)^{\frac{3-q}{2}},$$

which, together with Lemma 4.1 and (4.35), implies that $\sigma_\mu \lesssim \varepsilon_\mu$. Thus, we also have $\sigma_\mu \sim \varepsilon_\mu$ as $\mu \to 0^+$ in the case of $N = 3$ and $2 < q < 3$. □

We are ready to give the proofs of Theorem 1.1 and 1.2.

Proof of Theorem 1.1: It follows immediately from Lemma 3.1, Propositions 3.1 and 3.2. □

Proof of Theorem 1.2: It follows immediately from Propositions 2.1, 4.1 and 4.2. □

We close this section by

Proof of Theorem 1.3: (1) Since the proof is similar to that of Proposition 2.1, we only sketch it. In the case of $q = 2+\frac{4}{N}$, we have $\|\varphi\|_2^2 = \|\phi_0\|_2^2$ for all minimizers of the Gagliardo–Nirenberg inequality (1.9), where ϕ is the unique solution of (1.10). Thus, we choose $\varphi = \frac{a}{\|\phi_0\|_2}\phi \in \mathcal{S}_a$ as a test function of $m_{a,\mu}^-$. By using similar arguments as used in the proof of Proposition 2.1 and direct calculations,

$$m_{a,\mu}^- \leq \frac{1}{N}(1-\frac{\mu}{\alpha_{N,q,a}})^{\frac{2^*}{2^*-2}}\left(\frac{\|\nabla\phi_0\|_2}{\|\phi_0\|_{2^*}}\right)^N.$$

It follows from $u_{a,\mu,-} \in \mathcal{P}_{a,\mu}$, the Gagliardo–Nirenberg and Sobolev inequalities that

$$S^{\frac{N}{2}} \leq \frac{\|\nabla u_{a,\mu,-}\|_2^2}{(1-\frac{\mu}{\alpha_{N,q,a}})^{\frac{2}{2^*-2}}} \leq \left(\frac{\|\nabla\phi_0\|_2}{\|\phi_0\|_{2^*}}\right)^N, \tag{4.42}$$

which, together with $u_{a,\mu,-} \in \mathcal{P}_{a,\mu}$ once more and the Pohozaev identity satisfied by $u_{a,\mu,-}$, implies that

$$-\lambda_{\mu,-} = \frac{1-\gamma_q}{a^2}\mu\|u_{a,\mu,-}\|_q^q \geq (1+o_\mu(1))\frac{1-\gamma_q}{a^2\gamma_q}S^{\frac{N}{2}}(1-\frac{\mu}{\alpha_{N,q,a}})^{\frac{2}{2^*-2}}$$

and

$$-\lambda_{\mu,-} = \frac{1-\gamma_q}{a^2}\mu\|u_{a,\mu,-}\|_q^q \leq (1-\frac{\mu}{\alpha_{N,q,a}})^{\frac{2}{2^*-2}}\left(\frac{\|\nabla\phi_0\|_2}{\|\phi_0\|_{2^*}}\right)^N.$$

Thus, $\{v_{a,\mu,-}\}$ is bounded in $H^1(\mathbb{R}^N)$, where

$$v_{a,\mu,-}=(\frac{a}{\|\phi_0\|_2})^{\frac{N-2}{2}}s_\mu^{\frac{N}{2}}u_{a,\mu,-}(\frac{a}{\|\phi_0\|_2}s_\mu x)$$

and $s_\mu=(1-\frac{\mu}{\alpha_{N,q,a}})^{-\frac{N-2}{4}}$. Clearly, $v_{a,\mu,-}$ satisfies

$$-\Delta v_{a,\mu,-}=\lambda_{\mu,-}\frac{a^2}{\|\phi_0\|_2^2}s_\mu^2v_{a,\mu,-}+\mu(\frac{a}{\|\phi_0\|_2})^{\frac{4}{N}}v_{a,\mu,-}^{q-1}+s_\mu^{2-\frac{N}{2}(2^*-2)}v_{a,\mu,-}^{2^*-1}$$

By (3.13) and [55, (I.3)], we know that $\alpha_{N,q,a}(\frac{a}{\|\phi_0\|_2})^{\frac{4}{N}}=1$ for $q=2+\frac{4}{N}$. On the other hand, since $v_{a,\mu,-}$ is radial, it is standard to show that $v_{a,\mu,-}\to\psi_{\nu_a',1}$ strongly in $H^1(\mathbb{R}^N)$ as $\mu\to\alpha_{N,q,a}^-$ up to a subsequence for some $\nu_a'>0$.

(2) In the cases of $2+\frac{4}{N}<q<2^*$, $\frac{2}{q-2}-\frac{N}{2}\neq 0$. Thus, we can choose $\nu_a>0$, as that in (2.10), such that $\|\psi_{\nu_a,1}\|_2^2=a^2$. Again, we use $\psi_{\nu_a,1}\in\mathcal{S}_a$ as a test function of $m_{a,\mu}^-$. By using similar arguments as used in the proof of Proposition 2.1 and direct calculations, $m_{a,\mu}^-\lesssim\mu^{-\frac{2}{q\gamma_q-2}}$ as $\mu\to+\infty$. It follows that $u_{a,\mu,-}\to 0$ strongly in $D^{1,2}(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$ as $\mu\to+\infty$. This, together with $u_{a,\mu,-}\in\mathcal{P}_{a,\mu}$ and the Gagliardo–Nirenberg and Sobolev inequalities, implies

$$\|\nabla u_{a,\mu,-}\|_2^2\geq(1+o_\mu(1))(\mu\gamma_qa^{q-q\gamma_q}C_{N,q}^q)^{-\frac{2}{q\gamma_q-2}}.$$

On the other hand, for the test function $\psi_{\nu_a,1}$, it satisfies

$$\|\nabla\psi_{\nu_a,1}\|_2^2=\mu\gamma_q\|\psi_{\nu_a,1}\|_q^qt_\mu^{q\gamma_q-2}+\|\psi_{\nu_a,1}\|_{2^*}^{2^*}t_\mu^{2^*-2},$$

where $(\psi_{\nu_a,1})_{t_\mu}\in\mathcal{P}_{a,\mu}$. It follows that

$$t_\mu\|\nabla\psi_{\nu_a,1}\|_2\leq\left(\frac{1}{\mu a^{q-q\gamma_q}\gamma_qC_{N,q}^q}\right)^{\frac{1}{q\gamma_q-2}}.$$

Thus,

$$\begin{aligned}\mathcal{E}_\mu((\psi_{\nu_a,1})_{t_\mu}) &= (1+o_\mu(1))(\frac{1}{2}-\frac{1}{q\gamma_q})\|\nabla(\psi_{\nu_a,1})_{t_\mu}\|_2^2\\ &\leq (1+o_\mu(1))(\frac{1}{2}-\frac{1}{q\gamma_q})\left(\frac{1}{\mu a^{q-q\gamma_q}\gamma_qC_{N,q}^q}\right)^{\frac{2}{q\gamma_q-2}}.\end{aligned}$$

Note that $\mathcal{E}_\mu((\psi_{\nu_a,1})_{t_\mu})\geq m_{a,\mu}^-$ and

$$m_{a,\mu}^-=\mathcal{E}_\mu(u_{a,\mu,-})=(1+o_\mu(1))(\frac{1}{2}-\frac{1}{q\gamma_q})\|\nabla u_{a,\mu,-}\|_2^2$$

as $\mu\to+\infty$, we must have

$$\|\nabla u_{a,\mu,-}\|_2^2=(1+o_\mu(1))(\mu\gamma_qa^{q-q\gamma_q}C_{N,q}^q)^{-\frac{2}{q\gamma_q-2}}. \tag{4.43}$$

As in (1), $\{v_{a,\mu,-}\}$ is bounded in $H^1(\mathbb{R}^N)$, where $v_{a,\mu,-}=s_\mu^{\frac{N}{2}}u_{a,\mu,-}(s_\mu x)$ and $s_\mu=\mu^{\frac{1}{q\gamma_q-2}}$. Again, $v_{a,\mu,-}$ satisfies

$$-\Delta v_{a,\mu,-}=\lambda_{\mu,-}s_\mu^2v_{a,\mu,-}+v_{a,\mu,-}^{q-1}+s_\mu^{2-\frac{N}{2}(2^*-2)}v_{a,\mu,-}^{2^*-1}.$$

Using (4.43), the Pohozaev identity satisfied by $u_{a,\mu,-}$ and $u_{a,\mu,-}\in\mathcal{P}_{a,\mu}$ once more, we have

$$-\lambda_{\mu,-}=(1+o_\mu(1))\frac{1-\gamma_q}{a^2}(\mu\gamma_qa^{q-q\gamma_q}C_{N,q}^q)^{-\frac{2}{q\gamma_q-2}}.$$

Now, by similar arguments as used in (1), $v_{a,\mu,-} \to \psi_{\nu'_a,1}$ strongly in $H^1(\mathbb{R}^N)$ as $\mu \to +\infty$ up to a subsequence for some $\nu'_a > 0$. Since in the cases of $2+\frac{4}{N} < q < 2^*$, $\frac{2}{q-2} - \frac{N}{2} \neq 0$. By $\|v_{a,\mu,-}\|_2^2 = a^2$, we must have $\nu'_a = \nu_a$. By the uniqueness of $\psi_{\nu_a,1}$ in $\mathcal{S}_a$, we know that $v_{a,\mu,-} \to \psi_{\nu_a,1}$ strongly in $H^1(\mathbb{R}^N)$ as $\mu \to +\infty$. Using the uniqueness of $\psi_{\nu_a,1}$ in $\mathcal{S}_a$ and the nondegenerate of $\psi_{\nu_a,1}$, we can prove the local uniqueness of $u_{a,\mu,-}$ for $\mu > 0$ sufficiently large by adapting similar arguments as used for $u_{a,\mu,+}$ in the proof of Proposition 2.1. □

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References

[1] N. Ackermann, T. Weth, Unstable normalized standing waves for the space periodic NLS, *Anal. PDE,* **12**(2019), 1177–1213.

[2] T. Akahori, S. Ibrahim, N. Ikoma, H. Kikuchi, Uniqueness and nondegeneracy of ground states to nonlinear scalar field equations involving the Sobolev critical exponent in their nonlinearities for high frequencies, *Calc. Var. PDEs,* **58**(2019), article 120.

[3] T. Akahori, S. Ibrahim, H. Kikuchi, H. Nawa, Existence of a ground state and blow-up problem for a nonlinear Schrödinger equation with critical growth, *Differ. Integral Equ.,* **25**(2012), 383–402.

[4] T. Akahori, S. Ibrahim, H. Kikuchi, H. Nawa, Existence of a ground state and scattering for a nonlinear Schrödinger equation with critical growth, *Sel. Math. New Ser.,* **19**(2013), 545–609.

[5] C.O. Alves, M.A.S. Souto, M. Montenegro, Existence of a ground state solution for a nonlinear scalar field equation with critical growth, *Calc. Var. PDEs,* **43**(2012), 537–554.

[6] A. Ambrosetti, H. Brezis and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.,* **122**(1994), 519–543.

[7] F. Atkinson, L. Peletier, Emden-Fowler equations involving critical exponents, *Nonlinear Anal.,* **10**(1986), 755–776.

[8] T. Bartsch, S. De Valeriola, Normalized solutions of nonlinear schrödinger equations, *Arch. Math.,* **100**(2012), 75–83.

[9] T. Bartsch, N. Soave, A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems, *J. Funct. Anal.,* **272**(2017), 4998–5037.

[10] J. Bellazzini, L. Jeanjean, On dipolar quantum gases in the unstable regime, *SIAM J. Math. Anal.,* **48**(2016), 2028–2058.

[11] H. Berestycki, T. Cazenave, Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires, *C. R. Acad. Sci., Sér. 1 Math.,* **293**(1981), 489–492.

[12] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. I: Existence of a ground state, *Arch. Ration. Mech. Anal.,* **82**(1983), 313–345.

[13] H. Berestycki, P.L. Lions, Nonlinear scalar field equations II: Existence of infinitely many solutions, *Arch. Ration. Mech. Anal.,* **82**(1983), 347–375.

[14] E. Bianchi, H. Egnell, A note on the Sobolev inequality, *J. Funct. Anal.,* **100**(1991), 18–24.

[15] D. Bonheure, J. Casteras, T. Gou, L. Jeanjean, Normalized solutions to the mixed dispersion nonlinear Schrödinger equation in the mass critical and supercritical regime, *Trans. Amer. Math. Soc.,* **372**(2019), 2167–2212.

[16] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Commun. Pure Appl. Math.,* **36**(1983), 437–477.

[17] B. Buffoni, M.J. Esteban, E. Séré, Normalized solutions to strongly indefinite semilinear equations, *Adv. Nonlinear Stud.,* **6**(2006), 323–347.

[18] T. Cazenave, P.-L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, *Commun. Math. Phys.,* **85**(1982), 549–561.

[19] Z. Chen, W. Zou, Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent, *Arch. Rational Mech. Anal.,* **205**(2012), 515-551.
[20] J. Dávila, M. del Pino, I. Guerra, Non-uniqueness of positive ground states of non-linear Schrödinger equations, *Proc. Lond. Math. Soc.,* **106**(2013), 318–344.
[21] Y. Deng, C.-S. Lin, S. Yan, On the prescribed scalar curvature problem in $\mathbb{R}^N$, local uniqueness and periodicity, *J. Math. Pures Appl.,* **104**(2015), 1013–1044.
[22] A. Ferrero, F. Gazzola, Asymptotic behavior of ground states of quasilinear elliptic problems with two vanishing parameters, Part III, *J. Differential Equations,* **198** (2004), 53–90.
[23] G. Fibich, F. Merle, Self-focusing on bounded domains, *Physica D,* **155**(2001), 132–158.
[24] F. Gazzola, J. Serrin, Asymptotic behavior of ground states of quasilinear elliptic problems with two vanishing parameters, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* **19**(2002), 477–504.
[25] B. Gidas, W. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$, in: L. Nachbin (Ed.), Math. Anal. Appl. Part A, Advances in Math. Suppl. Studies, vol. 7A, Academic Press, 1981, pp. 369–402.
[26] Y. Guo, C.-S. Lin, J. Wei, Local uniqueness and refined spike profiles of ground states for two-dimensional attractive Bose-Einstein condensates, *SIAM J. Math. Anal.,* **49**(2017), 3671–3715.
[27] Y. Guo, S. Li, J. Wei, X. Zeng, Ground states of two-component attractive Bose-Einstein condensates I: Existence and uniqueness, *J. Funct. Anal.,* **276**(2019), 183–230.
[28] Y. Guo, S. Li, J. Wei, X. Zeng, Ground states of two-component attractive Bose-Einstein condensates II: Semi-trivial limit behavior, *Trans. Amer. Math. Soc.,* **371**(2019), 6903–6948.
[29] Y. Guo, R. Seiringer, On the mass concentration for Bose-Einstein condensates with attractive interactions, *Lett. Math. Phys.,* **104**(2014), 141–156.
[30] Y. Guo, Z.-Q. Wang, X. Zeng, H.-S. Zhou, Properties for ground states of attractive Gross-Pitaevskii equations with multi-well potentials, *Nonlinearity* **31**(2018), 957–979.
[31] Y. Guo, X. Zeng, H.-S. Zhou, Energy estimates and symmetry breaking in attractive Bose-Einstein condensates with ring-shaped potentials, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire,* **33**(2016), 809–828.
[32] Y. Guo, X. Zeng, The Lieb-Yau conjecture for ground states of pseudo-relativistic Boson stars, *J. Funct. Anal.,* **278**(2020), article 108510.
[33] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal.,* **28**(1997), 1633–1659.
[34] L. Jeanjean, T. Le, Multiple normalized solutions for a Sobolev critical Schrödinger equation, arXiv:2011.02945v1 [math.AP]
[35] L. Jeanjean, T. Luo, Z.-Q. Wang, Multiple normalized solutions for quasi-linear Schrödinger equations, *J. Differ. Equ.,* **259**(2015), 3894–3928.
[36] L. Jeanjean, S.-S. Lu, A mass supercritical problem revisited, *Calc. Var. PDEs,* **59**(2020), article 174.
[37] L. Jeanjean, S.-S. Lu, Nonradial normalized solutions for nonlinear scalar field equations, *Nonlinearity,* **32**(2019), 4942–4966.
[38] M. Knaap, L. Peletier, Quasilinear elliptic equations with nearly critical growth, *Comm. PDEs,* **14**(1989), 1351–1383.
[39] M. Kwong, Uniqueness of positive solution of $\Delta u - u + u^p = 0$ in $\mathbf{R}^N$. *Arch. Rational Mech. Anal.,* **105** (1989), 243–266.
[40] S. Le Coz, A note on Berestycki-Cazenave's classical instability result for nonlinear Schrödinger equations, *Adv. Nonlinear Stud.,* **8**(2008), 455–463.
[41] C.-S. Lin, W.-M. Ni, On the diffusion coefficient of a semilinear Neumann problem. Calculus of variations and partial differential equations (Trento, 1986), Lecture Notes in Math. 1340, Berlin- NewYork: Springer, 1988, pp. 160–174.
[42] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 1, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* **1**(1984), 109–145.
[43] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 2, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1**(1984), 223–283.
[44] V. Moroz, C. B. Muratov, Asymptotic properties of ground states of scalar field equations with a vanishing parameter. *J. Eur. Math. Soc.,* **16**(2014), 1081–1109.

[45] W.-M. Ni, I. Takagi, Locating the peaks of least energy solutions to a semilinear Neumann problem. *Duke Math. J.,* **70**(1993), 247–281.
[46] B. Noris, H. Tavares, G. Verzini, Existence and orbital stability of the ground states with prescribed mass for the L^2-critical and supercritical NLS on bounded domains, *Anal. PDE,* **7**(2014), 1807–1838.
[47] D. Pierotti, G. Verzini, Normalized bound states for the nonlinear Schrödinger equation in bounded domains, *Calc. Var. PDEs,* **56**(2017), article 133.
[48] M. Shibata, A new rearrangement inequality and its application for L^2-constraint minimizing problems, *Math. Z.,* **287**(2017), 341–359.
[49] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities: The Sobolev critical case. *J. Funct. Anal.,* **279**(2020), article 108610.
[50] N. Soave, Normalized ground state for the NLS equations with combined nonlinearities. *J. Differential Equations,* **269**(2020), 6941–6987.
[51] C.A. Stuart, Bifurcation for Dirichlet problems without eigenvalues, *Proc. Lond. Math. Soc.,* **45**(1982), 169–192.
[52] W. Strauss, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.,* **55**(1977), 149–162.
[53] M. Struwe, Variational methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Third edition. Springer-Verlag, Berlin, 2000. xviii+274 pp.
[54] J. Wei, On single interior spike solutions of Gierer-Meinhardt system: uniqueness, spectrum estimates, *Europ. J. Appl. Math.,* **10**(1999), 353–378.
[55] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates. *Commun. Math. Phys.,* **87**(1982/1983), 567–576.
[56] M. Willem, Minimax Theorems. Birkhäuser, Boston, 1996.

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