CLUSTERED TRAVELLING VORTE RINGS TO THE AXISYMMETRIC THREE-DIMENSIONAL INCOMPRESSIBLE EULER FLOWS

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ABSTRACT. For the three dimensional axisymmetric Euler flow, we construct a family of solutions with multiple travelling vortex rings, with large speed of order $O(|\ln\varepsilon|)$, where $\varepsilon > 0$ is a small parameter. Our construction is based on the analysis of the following nonlinear elliptic equation:

$$\begin{cases} \partial_{rr}\psi + \frac{3}{r}\partial_r\psi + \partial_{zz}\psi = -F((\psi - \frac{\alpha}{2}|\ln\varepsilon|)r^2), (r,z) \in \mathbb{R}^2, \\ \psi_r(0,z) = 0 \text{ for } r = 0, \end{cases}$$

for some special functions F, where α is a parameter. The location of the vortex rings are governed by some balancing systems, which can be solved by the polynomial method in several special cases. For the non-swirl case, in the core of each vortex ring, our solutions can be regarded as a rescaled finite mass solution of the Liouville equation. The results can be generalized directly to the case with swirl, for which we also construct different types of solutions with multiple vortex rings.

1. INTRODUCTION

The Euler equation for an ideal imcompressible homogeneous fluid in dimension 3 can be written in the form:

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p \text{ in } \mathbb{R}^3 \times (0,T), \\ div\mathbf{u} = 0 \text{ in } \mathbb{R}^3 \times (0,T), \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0 \text{ in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the 3-dimensional velocity vector of the fluid and p is the scalar pressure.

An important quantity associated to the velocity $\mathbf{u}$ in the equation (1.1) is its vorticity, which is defined by

$$\mathbf{w} = curl\mathbf{u}.$$

In terms of this quantity, under suitable conditions, the Euler equation (1.1) is equivalent to its vorticity-stream formulation:

$$\begin{cases} \mathbf{w}_t + (\mathbf{u}\cdot\nabla)\mathbf{w} = (\mathbf{w}\cdot\nabla)\mathbf{u} \text{ in } \mathbb{R}^3 \times (0,T), \\ \mathbf{u} = curl\psi^0,\ -\Delta\psi^0 = \mathbf{w} \text{ in } \mathbb{R}^3 \times (0,T), \\ \mathbf{w}(\cdot,0) = curl\mathbf{u}_0 \text{ in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where ψ^0 is the vector stream function. Note that the velocity can actually be recovered from the vorticity from the Biot-Savart law, at least under some natural conditions. We refer to [28] and the references therein for a detailed introduction to the mathematical aspect of the Euler equation. In particular, a rigorous treatment of the vorticity-stream formulation of 3D Euler flow can be found in Section 2.4 of [28].

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In this paper, we shall use the vorticity-stream formulation to show the existence of solutions with multiple vortex rings and having the form of travelling wave type. Our solution can be regarded as a special family of the so called vortex filament solutions. Roughly speaking, vortex filament solutions to the 3-dimensional Euler equation are those solutions whose vorticities $\mathbf{w}$ are large and uniformly concentrated near an evolving smooth curve $\Gamma(t)$.

The study of vortex filaments traces back to Helmholtz and Kelvin. In 1867, Helmholtz considered the situation where the vorticity is concentrated in a circular vortex filament, he found that the vortex rings have steady form and travel with a large constant velocity along the axis of the ring. In 1970, Fraenkel [15] established the first rigorous construction result of a vortex ring concentrated around a torus with fixed radius of small section $\varepsilon > 0$, travelling with constant speed of $O(|\ln \varepsilon|)$. After this pioneering work, existence of steady vortex rings and their asymptotics have been analyzed in many works. For instance, in [1, 31], variational method for the stream function is used to construct a single vortex ring with vanishing circulation in the limit. In [5, 6], vortex rings are constructed via rearrangement of functions. The paper [16] established the existence of vortex ring by analyzing a kinetic energy associated to the vorticity function. [13] constructed vortex rings in various domains and studied their asymptotic limit as a small parameter tends to 0. In the recent papers [8, 9], single vortex rings are shown to exist for a broad class of nonlinearities. The method used there is also of variational nature, for the corresponding elliptic equations. Finally, let us mention the paper [30], where magnetic relaxation method is used to formally show the existence of steady vortex rings. We would like to point out that most of these work deals essentially with one steady vortex ring.

The dynamics of the Euler equation for vortex filaments have been studied long time ago. Da Rios [10] and Levi-Civita [23] formally found the general law of motion of a vortex filament with a thin section of radius $\varepsilon > 0$, uniformly distributed around an involving curve $\Gamma(t)$, see also the survey paper by Ricca [33]. If $\Gamma(t)$ is parametrized as $x = \gamma(s,t)$, where s designates its arclength parameter, then $\gamma(s,t)$ asymptotically obey a law of the form

$$\gamma_t = 2c|\ln \varepsilon|(\gamma_s \times \gamma_{ss})$$

as $\varepsilon \to 0$, or scaling $t = |\ln \varepsilon|^{-1}\tau$,

$$\gamma_\tau = 2c\kappa \mathbf{b}_{\gamma(\tau)} \tag{1.3}$$

where c corresponds to the circulation of the velocity field on the boundary of sections to the filament. Here for the curve $\Gamma(\tau)$ parametrized as $x = \gamma(\tau,t)$, we denote by $\mathbf{t}_{\Gamma(\tau)}$, $\mathbf{n}_{\Gamma(\tau)}$, $\mathbf{b}_{\Gamma(\tau)}$ the tangent, normal, binormal unit vector, κ be its curvature.

In [21], Jerrad and Seis studied the evolution of the vortex filaments under mild assumption of vorticity concentration, by establishing new estimates for the corresponding Hamiltonian-Possion structure. They showed in a rigorous way that $\Gamma(\tau)$ indeed evolves by the law (1.3), up to some errors which can be precisely controlled.

As explained in [21], solutions of the Euler equation for which the vorticity remains close for a significant period of time to a filament should exist and evolve by the binormal curvature flow may be loosely termed vortex filament conjecture. This conjecture in its full generality is still not completely resolved. The case of steady vortex filaments can be regarded as a special situation of this type. Besides those steady vortex rings mentioned above, we also have the example of knotted vortex filaments with small vorticities, studied in [14]. Recently, Davila, Del Pino, Musso and Wei [12] consider the case when the curve $\Gamma(\tau)$ is given by the travelling helix which satisfies the binormal law (1.3) and construct solutions of (1.2) with concentrating vorticity for this helix. We also point out that in [7], evolution of multiple vortex rings in a short time period is analyzed.

It is worth mentioning that the nearly parallel interacting filaments of the 3D Euler equation have also been studied in [22] and the law for it is the same for the dynamics of almost parallel vortex filaments for the Gross-Pitaevskii equation [20]. Construction of vortex filament with small vorticities around general set has also been studied in [14].

In this paper, we would like to study the existence of vortex filament solutions of travelling wave type, where the concentrating region of the vorticity $\mathbf{w}$ has the shape of multiple travelling rings. More precisely, we will construct solutions of multiple vortex rings, for which m rings have positive circulation and the other n rings have negative circulation. Moreover, the vortex rings will collapse to the same circle of radius of $O(1)$ as the parameter ε goes to zero, and the mutual distances between these rings are of the order $O(\frac{1}{|\ln\varepsilon|})$, which is much smaller than the radius of the rings.

Our construction is based on the Lyapunov-Schmidt reduction. The reduced system we obtain here tells us that the location of the rings(represented by the points $\mathbf{a}_j, \mathbf{b}_k$) are essentially determined by following system (Balancing condition):

$$\begin{cases} \sum\limits_{j=1, j\neq k}^{m} \frac{\tilde{\gamma}_j}{\mathbf{a}_k - \mathbf{a}_j} - \sum\limits_{j=1}^{n} \frac{\tilde{\beta}_j}{\mathbf{a}_k - \mathbf{b}_j} = \tilde{\sigma}_k, \text{ for } k = 1, ..., m, \\ -\sum\limits_{j=1, j\neq k}^{n} \frac{\tilde{\beta}_j}{\mathbf{b}_k - \mathbf{b}_j} + \sum\limits_{j=1}^{m} \frac{\tilde{\gamma}_j}{\mathbf{b}_k - \mathbf{a}_j} = -\tilde{\rho}_k, \text{ for } k = 1, ..., n. \end{cases} \tag{1.4}$$

Here $\mathbf{a}_j, j = 1, ..., m$, $\mathbf{b}_\ell, \ell = 1, ..., n$, are complex numbers in the right half plane Π defined in (2.2), $\tilde{\gamma}_j > 0$, $-\tilde{\beta}_j < 0$ correspond to the circulation of the rings, and $\tilde{\sigma}_j, \tilde{\rho}_j$ are constants related to the radius and speed of the travelling rings. Moreover, the solvability of our original problem is related to the non-degeneracy of the solution to (1.4) which will be explained in Definition 7.1.

We remark that similar reduced system has been obtained when we study the multi vortex ring solution for the 3-dimensional Gross-Pitaevskii equation in our previous work [4], when all the degree of the standard vortex are equal to $+1$ or -1. It has been shown there that the existence and non-degeneracy of symmetric $(\mathbf{a}_j, \mathbf{b}_\ell)$ are related to some generalized Adler-Moser polynomials.

To make our construction possible, the solution $\mathbf{a}_j, \mathbf{b}_\ell$ to the system (1.4) has to satisfy some extra symmetric properties. We therefore introduce the following conditions:

$(\mathcal{M}_1)$. The points $\mathbf{a}_j, \mathbf{b}_\ell$, $j = 1, ..., m$, $\ell = 1, ..., n$ are all distinct. The set of points of $\{\mathbf{a}_1, ..., \mathbf{a}_m\}$ and $\{\mathbf{b}_1, ..., \mathbf{b}_n\}$ are both symmetric with respect to the r axis.

$(\mathcal{M}_2)$. $\mathbf{a}_j, \mathbf{b}_\ell$, $j = 1, ..., m$, $\ell = 1, ..., n$ is nondegenerate solution of (1.4) in the sense of Definition 7.1.

To state our results in a more precise way, let us now introduce some notations. For $i = 1, \cdots, m+n$, given distinct points $(r_i, z_i) \in \Pi$, let $\Gamma_i(\tau)$ be a circle with radius r_i translating with constant speed along its axis parametrized as

$$\gamma_i(s,\tau) = (r_i \cos \frac{s}{r_i},\, r_i \sin \frac{s}{r_i},\, z_i + \frac{c_i}{r_i}\tau)^T. \tag{1.5}$$

Then we have the following existence result:

Theorem 1.1. *Suppose $\mathbf{a}_j, \mathbf{b}_\ell$, $j = 1, ..., m$, $\ell = 1, ..., n$ is a solution of (1.4) satisfying condition $(\mathcal{M}_1)$-$(\mathcal{M}_2)$. For any $r_0 > 0$, there exist distict points $(r_i, z_i) \in \Pi$ for $i = 1, \cdots, m+n$, and a smooth solution $\mathbf{w}_\varepsilon$ to (1.2), defined for $\tau \in (-\infty, +\infty)$ such that for all τ,*

$$\mathbf{w}_\varepsilon(x, \tau |\ln \varepsilon|^{-1}) \sim \sum_{i=1}^{m+n} \kappa_i \delta_{\Gamma_i(\tau)} \mathbf{t}_{\Gamma_i(\tau)},$$

where κ_i is the circulation(see (2.16) and (7.4)), Γ_i is the traveling circle parametrized by (1.5), δ is the Dirac mass, and $\mathbf{t}_{\Gamma(\tau)}$ is unit the tangent vector of Γ_i at the point $\mathbf{t}_{\Gamma(\tau)}$. The corresponding velocity vector is axisymmetric. Moreover, the solutions have the following properties:

1. *The vortex rings satisfy $|(r_i, z_i) - (r_0, 0)| = O(\frac{1}{|\ln \varepsilon|})$ as $\varepsilon \to 0$;*
2. *After suitable translation in the r direction with the order $O(1)$, and a scaling by a factor $|\ln \varepsilon|$, the position of the vortex rings in the (r, z) plane is close to those points*
$$\{\mathbf{a}_1, ..., \mathbf{a}_m, \mathbf{b}_1, ..., \mathbf{b}_n\}.$$
In other words, the locations of the vortex rings are determined by the balancing condition.

Solutions described in our main theorem can be constructed for the cases either with swirl or without swirl. More precise description of the solutions will explained in the following sections. The motion of vortex filaments corresponding to our solutions is the natural generalization of the motion of point vortices for the 2D incompressible Euler equations. In that case, their desingularization has been analyzed in [11, 29, 35] and reference therein.

When $\tilde{\beta}_j = \tilde{\gamma}_i = 1$ in (1.4), it has been shown in [4] that for $(m, n) \in \mathbb{S}$ where

$$\mathbb{S} := \{(2,1), (3,2), (4,3), (5,4), (6,5)\},$$

there exists solution to (1.4) satisfying $(\mathcal{M}_1)$-$(\mathcal{M}_2)$. Hence one can construct solutions to (1.2) with such $m+n$ travelling vortex rings. In Section 8 of this paper, we show that there exist abundance of balancing configurations for many choices of different circulations. In particular the case with two different circulations and three different circulations will also be studied.

We point out that solutions with vortex rings in other PDE settings have been built in [3, 24, 25]. Partial results on spectral stability of a columnar vector for the 3D Euler has been obtain in [17, 18] and nonlinear stability of point vortices in [19].

Let us now sketch the main ideas of the proof. We will use the method of finite dimensional Lyapunov-Schmidt reduction. The first step is to reduce our problem to a 2D problem in $\Pi \times (0, T)$:

$$\begin{cases} r w_t + \nabla^\perp (r^2 \psi) \cdot \nabla w = 0 \text{ in } \Pi \times (0, T), \\ -(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2)\psi = w \text{ in } \Pi \times (0, T), \\ \partial_r \psi(0, z) = 0 \in \partial\Pi \times (0, T). \end{cases} \tag{1.6}$$

After some manipulation, in the case with swirl, we are lead from this system to the equation (2.15). This reduction from 3D to 2D case is done in Section 2.1. We will find travelling wave solutions to (1.6) with multi vortex rings by analyzing the nonlinear elliptic equation (2.15). Our second step is to choose the nonlinear function $F(s)$ in (2.15) to be of the form e^s. We then build suitable approximate solutions by using a combination of solutions of the classical Liouville equation. This is contained in Section 2.2. This approximate solutions can be perturbed into a genuine solution by Lyapunov-Schmidt reduction method. The balancing condition comes from the requirement that the projection of the error of the approximate solution to kernels of the corresponding linearized operator should at the main order be equal to zero. Once we have this balancing condition, then a true solution can be found using implicit function theorem.

This paper is organized as follows. In Section 2, we will reformulate our problem and reduce it to a two dimensional elliptic problem and introduce the approximate solution. In Section 3, we will get the error estimate caused by the approximate solution ψ_0 introduced in (2.22). Section 4-6 are devoted to the inner-outer gluing procedure which solves a nonlinear projected problem. Section 7 is devoted to the reduced problem and the fully solvability of our Theorem 2.1. Theorem 1.1 is a direct consequence of Theorem 2.1. In Section 8, we are devoted to the study of the balancing condition (2.23). In the last section, we study the case with swirl and construct multiple vortex ring solution such that the mutual distance and the radius are both of $O(1)$ due to the effect of the swirl. We emphasize that this type of solutions are in general not available for the case without swirl.

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2. Formulation of the problem

2.1. **Reduction to elliptic equation in 2D.**

As we mentioned in the introduction, we will construct travelling solutions to (1.2) with multi vortex rings. In this paper, we are in particular interested in the axisymmetric Euler flows. In this case, the velocity field $\mathbf{u}$ can be expressed in the following form:

$$\mathbf{u}(r,z) = u^r(r,z)e_r + u^\theta(r,z)e_\theta + u^z(r,z)e_z$$

where $\{e_r, e_\theta, e_z\}$ is the usual cylindrical coordinate frame given by

$$e_r = \frac{1}{r}(x,y,0)^T, e_\theta = \frac{1}{r}(-y,x,0)^T, e_z = (0,0,1)^T.$$

We also denote the corresponding vorticity $\mathbf{w}$ and the stream function vector ψ^0 in (1.2) as

$$\mathbf{w}(r,z) = \omega^r(r,z)e_r + \omega^\theta(r,z)e_\theta + \omega^z(r,z)e_z$$

and

$$\psi^0(r,z) = \psi^r(r,z)e_r + \psi^\theta(r,z)e_\theta + \psi^z(r,z)e_z.$$

Under these notations, the 3-dimensional Euler equation (1.1) becomes

$$\begin{cases} u^\theta_t + u^r u^\theta_r + u^z u^\theta_z = -\frac{1}{r} u^r u^\theta \text{ in } \Pi \times (0,T) \\ \omega^\theta_t + u^r \omega^\theta_r + u^z \omega^\theta_z = \frac{2}{r} u^\theta u^\theta_z + \frac{1}{r} u^r \omega^\theta \text{ in } \Pi \times (0,T) \\ -(\Delta - \frac{1}{r^2})\psi^\theta = \omega^\theta \text{ in } \Pi \times (0,T) \end{cases} \tag{2.1}$$

where u^θ, ω^θ and ψ^θ are the angnular components of the velocity, vorticity, and stream function vectors, respectively and

$$\Pi = \{(r,z) | r > 0,\ z \in \mathbb{R}\} \tag{2.2}$$

is the right half plane.

Now the relation between velocity and stream function is given by

$$u^r = -\psi^\theta_z,\ u^z = \frac{1}{r}(r\psi^\theta)_r \tag{2.3}$$

in which the incompressibility condition

$$\frac{1}{r}(ru^r)_r + u^z_z = 0 \tag{2.4}$$

is automatically satisfied.

The axisymmetric Euler equations have a formal singularity at $r = 0$, which sometimes is inconvenient to work with. To remove the artificial singularity we introduce

$$u_1 = \frac{u^\theta}{r},\ \omega_1 = \frac{\omega^\theta}{r}, \psi = \frac{\psi^\theta}{r}. \tag{2.5}$$

The transformed equation becomes

$$\begin{cases} u_{1,t} + u^r u_{1,r} + u^z u_{1,z} = 2u_1 \psi_z \text{ in } \Pi \times (0,T) \\ \omega_{1,t} + u^r \omega_{1,r} + u^z \omega_{1,z} = (u_1^2)_z \text{ in } \Pi \times (0,T) \\ -[\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2]\psi = \omega_1 \text{ in } \Pi \times (0,T), \\ \psi_r(0,z,t) = 0 \text{ on } \partial\Pi \times (0,T). \end{cases} \tag{2.6}$$

In terms of the new variables, we have

$$u^r = -r\psi_z,\ u^z = 2\psi + r\psi_r. \tag{2.7}$$

We may also write the transformed equation as

$$\begin{cases} (r^3 u_1)_t + (ru^r)(r^2 u_1)_r + (ru^z)(r^2 u_1)_z = 0 \text{ in } \Pi \times (0,T) \\ r\omega_{1,t} + (ru^r)\omega_{1,r} + (ru^z)\omega_{1,z} = r(u_1^2)_z \text{ in } \Pi \times (0,T) \\ -[\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2]\psi = \omega_1 \text{ in } \Pi \times (0,T), \\ \psi_r(0,z,t) = 0 \text{ on } \partial\Pi \times (0,T). \end{cases} \tag{2.8}$$

Note that

$$ru^r = -(r^2\psi_1)_z,\ ru^z = (r^2\psi)_r.$$

one has

$$(ru^r, ru^z) = \nabla^\perp(r^2\psi).$$

Hence

$$(ru^r)_r + (ru^z)_z = 0$$

and

$$ru^r v_r + ru^z v_z = 0$$

are satisfied

As mentioned in the introduction, for solutions with large vorticities and uniformly concentrated along a smooth curve, the travelling speed is of $O(|\ln \varepsilon|)$. We therefore introduce $t = |\ln \varepsilon|^{-1}\tau$. In the new variable (r, z, τ), replacing ω_1 by W, and letting

$$\hat{u} = r^2 u_1,\ \hat{\psi} = r^2 \psi,$$

we find that the equation (2.8) becomes

$$\begin{cases} |\log \epsilon| r \hat{u}_\tau + \nabla^\perp(\hat{\psi}) \nabla \hat{u} = 0 \text{ in } \Pi \times (0, |\ln \varepsilon|^{-1} T) \\ |\log \epsilon| r W_\tau + \nabla^\perp(\hat{\psi}) \nabla W = \frac{(\hat{u}^2)_z}{r^3} \text{ in } \Pi \times (0, |\ln \varepsilon|^{-1} T) \\ -\Delta_\theta(\frac{\hat{\psi}}{r}) = rW \text{ in } \Pi \times (0, |\ln \varepsilon|^{-1} T) \\ \psi_r(0, z, t) = 0 \end{cases} \tag{2.9}$$

where

$$\Delta_\theta = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}.$$

In this way we have reduced the original 3D problem to a 2D problem with Neumann boundary condition. The purpose of this paper is to construct regular solution $W(r, z, \tau)$ which resemble a superposition of point vortices of the form $\sum_{i=1}^N \kappa_i \delta(x - p_i(\tau))$ such that $p_i(\tau)$ does not change form as time evolves. We focus on travelling solutions with constant speed α such that

$$p_i(\tau) = p_i + \alpha \tau e_2.$$

Note that due to the Galilean invariance of the Euler equation, travelling wave solutions can be transformed to a steady state solutions, and they are expected to play important roles in the long time behavior of the full Euler flow.

In our context, a travelling solution of speed α with the form $W = W(r, z - \alpha\tau)$ will be correspond to solutions of the following system:

$$\begin{cases} -\alpha |\log \epsilon| r \hat{u}_z + \nabla^\perp(\hat{\psi}) \nabla \hat{u} = 0 \text{ in } \Pi \\ -\alpha |\log \epsilon| r W_z + \nabla^\perp(\hat{\psi}) \nabla W = \frac{(\hat{u}^2)_z}{r^3} \text{ in } \Pi \\ -\Delta_\theta(\frac{\hat{\psi}}{r}) = rW \text{ in } \Pi \\ \psi_r(0, z) = 0. \end{cases} \tag{2.10}$$

Note that the first equation of (2.10) will be automatically satisfied if there exists some function $H : \mathbb{R} \to \mathbb{R}$ such that

$$\hat{u} = H(\hat{\psi} - \frac{\alpha}{2} |\ln \varepsilon| r^2). \tag{2.11}$$

For function $\hat{u}$ in this form (2.11), the second equation in (2.10) is satisfied if there exists some function $F : \mathbb{R} \to \mathbb{R}$ such that

$$W = F(\hat{\psi} - \frac{\alpha}{2} |\ln \varepsilon| r^2) + \frac{G(\hat{\psi} - \frac{\alpha}{2} |\ln \varepsilon| r^2)}{r^2} \tag{2.12}$$

where

$$G(s) = H(s) H'(s).$$

Combining (2.11) and (2.12) and consider the third equation in (2.10), we are lead to the following equation for $\hat{\psi}$:

$$\begin{cases} \Delta_\theta(\frac{\hat{\psi}}{r}) = -r\Big(F(\hat{\psi} - \frac{\alpha}{2} |\ln \varepsilon| r^2) + \frac{G(\hat{\psi} - \frac{\alpha}{2} |\ln \varepsilon| r^2)}{r^2}\Big) \text{ in } \Pi \\ \psi_r(0, z) = 0. \end{cases} \tag{2.13}$$

Recall that $\hat{\psi} = r^2\psi$. Coming back to ψ, one has

$$\begin{cases} -\Delta_5\psi = F((\psi - \frac{\alpha}{2}|\ln\varepsilon|)r^2) + \frac{G((\psi-\frac{\alpha}{2}|\ln\varepsilon|)r^2)}{r^2}, \text{ in } \Pi \\ \psi_r(0,z) = 0 \end{cases} \tag{2.14}$$

where

$$\Delta_5\psi = \partial_{rr}\psi + \frac{3}{r}\partial_r\psi + \partial_{zz}\psi.$$

This family of equations is sometimes referred as the Long-Squire equation [26, 34] or more generally the Grad-Shafranov equation in plasma physics [2] in the form

$$-\Delta_5\psi = F((\psi - \frac{\alpha}{2}|\ln\varepsilon|)r^2) + \frac{G((\psi - \frac{\alpha}{2}|\ln\varepsilon|)r^2)}{r^2} + r^2K((\psi - \frac{\alpha}{2}|\ln\varepsilon|)r^2),$$

where F, G, K are arbitrary functions.

Once a solution ψ satisfying (2.14) for an arbitrary choice of functions (F, G) is found, one can easily construct travelling solutions for the original 3D Euler equation (2.1).

In the literature, vortex rings are divided into two cases depending on whether the swirl velocity u^θ is zero. This is equivalent to say whether the function H is zero. When $H = 0$, this is the non-swirl case while $H \neq 0$ is the swirl case. There are more results for the non swirl case for different choice of the functions F.

2.2. **The approximate solutions.** In this paper, we will mainly focus on the existence of solutions with multiple vortex rings for the non swirl case. Our result can be extended to some case with swirl which will be discussed in the last section. In the following we will consider the following equation:

$$\begin{cases} -\Delta_5\psi = F((\psi - \frac{\alpha}{2}|\ln\varepsilon|)r^2) := W \text{ in } \Pi \\ \psi_r(0,z) = 0. \end{cases} \tag{2.15}$$

We aim to construct multi vortex points solution to (2.15). We devote the rest of the paper to build such a solution by elliptic singular perturbation and Lyapunov-Schmidt reduction method.

In the following we will mainly introduce the approximate solutions. First we fix $r_0 > 0$, we want to construct multiple vortex ring solution to the 3D Euler equation (2.1) such that

$$\omega^\theta \to 8\pi\sum_{j=1}^{N}\kappa_j\delta_{p_j} \tag{2.16}$$

where N corresponds to the number of rings and p_i are distinct points in Π, and κ_j can be regarded in some sense to be the circulation of the individual vortex ring. We will work in the following configuration space:

$$I := \{p_j = (r_j, z_j) \in \Pi\,,\ |r_j - r_0| = O(\frac{1}{|\ln\varepsilon|}),\ |p_i - p_j| = O(\frac{1}{|\ln\varepsilon|})\}. \tag{2.17}$$

By the relation of ω^θ and W, (2.16) implies that

$$W \to 8\pi\sum_{j=1}^{N}\kappa_j\frac{\delta_{p_j}}{r_j} := W^s. \tag{2.18}$$

For $p \in \Pi$, let $G(x,p)$ be the Green's function of the problem

$$\begin{cases} -\Delta_5 G(x,p) = 8\pi\delta_p, \text{ in } \Pi \\ \partial_r G(x,p) = 0 \text{ on } \partial\Pi. \end{cases} \tag{2.19}$$

From (2.15) and (2.18), we should have the formal limit

$$\psi \to \sum_{i=1}^{N} \kappa_i \frac{G(x,p_i)}{r_i} := \psi^s. \tag{2.20}$$

We will choose generation function $F(s)$ of the form e^s, whose precise form will be given explicitly below. We can achieve the above limit by solving the elliptic equation

$$-\Delta_5\psi = F(r^2\psi - \frac{\alpha}{2}|\log\epsilon|r^2) = W \tag{2.21}$$

with

$$F(s) = \sum_{i=1}^{N} \varepsilon^{2-\frac{\alpha}{2\kappa_i}r_i} \frac{\kappa_i}{r_i} f_i(\frac{s}{\kappa_i r_i})\chi_{\frac{\delta_1}{|\ln\varepsilon|}}(p_i),$$

where $f_i(s) = e^s\eta^i(s)$ and $\eta^i(s)$ is cutoff function to be defined in (3.4)-(3.5) and $\delta_1 > \delta_0$ for the constant δ_0 to be given below.

We look for a solution that at main order looks like

$$W = F(r^2\psi - \frac{\alpha}{2}|\log\epsilon|r^2) \approx \frac{1}{\varepsilon^2}U\Big(\frac{x-p_i}{\varepsilon}\Big),$$

near each vortex point p_i where

$$U(x) = \frac{8}{(1+|x|^2)^2}.$$

Since as $\varepsilon \to 0$,

$$\frac{1}{\varepsilon^2}U\Big(\frac{x-p_i}{\varepsilon}\Big) \to 8\pi\delta_{p_i}.$$

Note that

$$-\Delta\Gamma_0 = e^{\Gamma_0} \text{ in } \mathbb{R}^2,$$

where $\Gamma_0 = \log U$ and $\Delta = \partial_{rr} + \partial_{zz}$.

Roughly speaking, we want to construct solutions such that W has compact support and concentrate near each p_i using the nonlinear function e^s. But this function does not have compact support. So we use cutoff function η^i which is chosen in such a way that it is supported in $B_{\frac{\delta_0}{|\ln\varepsilon|}}(p_i)$ for some δ_0 small. So in this way W defined above has compact support near each p_i and it behaves like $\frac{1}{\varepsilon^2}U\Big(\frac{x-p_i}{\varepsilon}\Big)$ near each point. Since $\delta_1 > \delta_0$, so the solution to (2.21) is a solution to the Euler equation.

To state our main result, we consider a ε-regularization of (W^s, ψ^s) given in (2.16) and (2.18). To achieve this, we need to study the asymptotic behavior of $G(x,p)$ near the singular point p.

Locally around $x = p_i$, the Green's function can be expanded as

$$G(x,p_i) = \log\frac{1}{|x-p_i|^4}\big(1 - \frac{3}{2r_i}(r-r_i) + H_{i,0}(x)\big) + H_{i,1}(x).$$

Here $H_{i,0}$ and $H_{i,1}$ are smooth functions with

$$\Delta_5 H_{i,1}(x) = 0, \quad \text{and} \quad H_{i,0}(x) = O(|x-p_i|^2) \quad \text{as } x \to p_i,$$

and it satisfies

$$\Delta_5\Big(\log\frac{1}{|x-p_i|^4}H_{i,0}(x)\Big) = -30\frac{(r-r_i)^2}{rr_i|x-p_i|^2} + \frac{9}{2rr_i}\log\frac{1}{|x-p_i|^4}.$$

Let us consider the following ε-regularization of $G(x,p_i)$.

$$G_{i,\varepsilon} = \log\frac{1}{(\mu_i^2\varepsilon^2 + |x-p_i|^2)^2}\big(1 - \frac{3}{2r_i}(r-r_i) + H_{i,0}(x)\big) + H_{i,1}(x),$$

and consider our first approximation as

$$\psi_0 = \sum_{i=1}^N \frac{\kappa_i}{r_i}G_{i,\varepsilon}\eta_0(x-(r_0,0)) + (1-\eta_0)\sum_{i=1}^N \kappa_i\frac{G(x,p_i)}{r_i} \tag{2.22}$$

for some cutoff function $\eta_0(s)$ such that $\eta_0(s) = 1$ for $s < \frac{r_0}{4}$ and $\eta_0(s) = 0$ for $s > \frac{r_0}{2}$. Here $\mu_i > 0$ are numbers to be fixed later.

We have the following existence result:

Theorem 2.1. *Suppose $\mathbf{a}_j, \mathbf{b}_\ell$, $j = 1, ..., m$, $\ell = 1, ..., n$ is a solution of (1.4) satisfying conditions $(\mathcal{M}_1)$-$(\mathcal{M}_2)$. For any $r_0 > 0$, there exists $\{p_1, \cdots, p_N\}$ satisfying (2.17) and $\alpha > 0$ such that there exists solution ψ of (2.21) of the form*

$$\psi = \psi_0 + o(1)$$

where ψ_0 is defined in (2.22).

We point out that the solution in Theorem 1.1 contains multiple vortex rings such that the mutual distance are much smaller than the radius of the rings. In the last section, we also study the swirl case and construct different type of solution. There the mutual distance and the radius of the vortex are all of $O(1)$. This is due to the effect of the swirl.

Remark 2.2. *In fact, we can have more precise description of the vortex points. If we write*

$$p_i = (r_0, 0) + \frac{\hat{p}_i}{|\ln\varepsilon|}.$$

Then $\hat{p}_i$ will be perturbation from solution of the following system:

$$\begin{cases} \sum\limits_{j=1,j\neq k}^m \frac{\gamma_j}{\mathbf{a}_k - \mathbf{a}_j} - \sum\limits_{j=1}^n \frac{\beta_j}{\mathbf{a}_k - \mathbf{b}_j} = \sigma_k, \text{ for } k = 1, ..., m, \\ -\sum\limits_{j=1,j\neq k}^n \frac{\beta_j}{\mathbf{b}_k - \mathbf{b}_j} + \sum\limits_{j=1}^m \frac{\gamma_j}{\mathbf{b}_k - \mathbf{a}_j} = -\rho_k, \text{ for } k = 1, ..., n. \end{cases} \tag{2.23}$$

where m, n corresponds to the number of positive and negative circulation, γ_j, β_j corresponds to the absolute value of the circulation, σ_j, ρ_j are some constants related to the radius and travelling speed of the ring.

Remark 2.3. *By our study for the balancing condition (2.23), the travelling speed α need to be non-zero in our construction.*

3. Error estimate

In the previous sections, we have introduced the approximate solution as ψ_0 defined in (2.22). In this section, we will estimate the error caused by this approximate solution.

Let us define

$$S_0(\psi) = \Delta_5\psi + \sum_{i=1}^{N} \varepsilon^{2-\frac{\alpha r_i}{2\kappa_i}} \frac{\kappa_i}{r_i} f\Big(\frac{r^2}{\kappa_i r_i}(\psi - \frac{\alpha}{2}|\log\varepsilon|)\Big)\eta^i \chi_{B_{\frac{\delta_1}{|\ln\varepsilon|}}(p_i)}. \tag{3.1}$$

Before proceeding, let us explain how we choose the cutoff function η^i such that it has compact support near each vortex point. For each fixed index i, we shall do the expansion near the vortex point p_i using the rescaled variable $y = \frac{x-p_i}{\varepsilon\mu_i}$ where $\mu_i > 0$ are to be determined in (3.16) :

$$\begin{aligned} G_{i,\varepsilon}(x) &= (\Gamma_0(y) - 4\log\varepsilon - \log(8\mu_i^4))\Big(1 - \frac{3}{2r_i}\varepsilon\mu_i y_1 + H_{i,0}(p_i+\varepsilon\mu_i y)\Big) + H_{i,1}(p_i+\varepsilon\mu_i y)\\ &= \Gamma_0(y) - 4\log\varepsilon - \log(8\mu_i^4) + H_{i,1}(p_i) \qquad (3.2)\\ &+ \varepsilon\mu_i y_1[\partial_r H_{i,1}(p_i) - \frac{3}{2r_i}(\Gamma_0 - 4\log\varepsilon - \log 8\mu_i^4)]\\ &+ \partial_z H_{i,1}(p_i)\varepsilon\mu_i y_2 + O(\varepsilon^2\mu_i^2|y|^2\ln|\varepsilon y|). \end{aligned}$$

Moreover, for $j \neq i$, one has

$$\begin{aligned} G_{j,\varepsilon}(x) &= \log\frac{1}{|p_i-p_j|^4}\Big(1 - \frac{3}{2r_j}(r_i-r_j) + H_{j,0}(p_i)\Big) + H_{j,1}(p_i) \qquad (3.3)\\ &+ \varepsilon\mu_i y_1\Big(\log\frac{1}{|p_i-p_j|^4}(\partial_r H_{j,0}(p_i) - \frac{3}{2r_j}) - \frac{4(p_i-p_j)_1}{|p_i-p_j|^2}(1 - \frac{3}{2r_j}(r_i-r_j) + H_{j,0}(p_i)) + \partial_r H_{j,1}(p_i)\Big)\\ &+ \varepsilon\mu_i y_2\Big(\log\frac{1}{|p_i-p_j|^4}\partial_z H_{j,0}(p_i) - \frac{4(p_i-p_j)_2}{|p_i-p_j|^2}(1 - \frac{3}{2r_j}(r_i-r_j) + H_{j,0}(p_i)) + \partial_z H_{j,1}(p_i)\Big)\\ &+ O(\varepsilon^2\mu_i^2|y|^2). \end{aligned}$$

From the above expansion, we see that for $|y| = \frac{c}{\varepsilon|\ln\varepsilon|}$, there holds

$$\begin{aligned} &\frac{r^2}{\kappa_i r_i}\Big(\psi_0(x) - \frac{\alpha}{2}|\log\varepsilon|\Big)\\ &= \frac{\alpha r_i}{2\kappa_i}\ln\varepsilon + 4\ln|\ln\varepsilon| - \ln\mu_i^2 - 4\ln c - 4c + O(1). \end{aligned}$$

We will choose our cutoff function such that

$$\eta^i(s) = 1, \tag{3.4}$$

for $s \geq \frac{\alpha r_i}{2\kappa_i}\ln\varepsilon + 4\ln|\ln\varepsilon| - \ln\mu_i^2 + 2\mathbf{c}_i$ where $\mathbf{c}_i > 0$ is large enough, and

$$\eta^i(s) = 0, \tag{3.5}$$

for $s \leq \frac{\alpha r_i}{2\kappa_i}\ln\varepsilon + 4\ln|\ln\varepsilon| - \ln\mu_i^2 + \mathbf{c}_i$. Here $\mathbf{c}_i$ is chosen large such that near each vortex point p_i, we have that

$$\eta^i(\frac{r^2}{\kappa_i r_i}(\psi_0(x) - \frac{\alpha}{2}|\log\varepsilon|)) = 1 \tag{3.6}$$

for $|x-p_i|\le \frac{\delta_0^2}{|\ln\varepsilon|}$, and

$$\eta^i\Big(\frac{r^2}{\kappa_i r_i}(\psi_0(x)-\frac{\alpha}{2}|\log\varepsilon|)\Big)=0 \tag{3.7}$$

for $|x-p_i|>\frac{\delta_0}{|\ln\varepsilon|}$ for some δ_0 small enough.

First we compute the error for $\Delta_5\psi_0$. For $|x-p_i|<\frac{\delta_0}{|\ln\varepsilon|}$, set

$$\Delta_5\frac{G_{i,\varepsilon}(x)}{r_i}=(I1)+(I2)$$

where

$$\begin{aligned}(I1)&=\frac{1}{r_i}\Delta_5\Big[(\Gamma_0-4\log\varepsilon-\log(8\mu_i^4))\Big(1-\frac{3}{2r_i}(r-r_i)\Big)\Big],\\(I2)&=\frac{1}{r_i}\Delta_5[(\Gamma_0-4\log\varepsilon-\log(8\mu_i^4))H_{i,0}(x)].\end{aligned}$$

By direct computation, one has for $y=\frac{x-p_i}{\varepsilon\mu_i}$,

$$\begin{aligned}(I1)=&\frac{1}{r_i}\Big[\Big(-\frac{1}{\varepsilon^2\mu_i^2}U(y)+\frac{3}{\varepsilon\mu_i(r_i+\varepsilon\mu_i y_1)}\Gamma_0'(y)\frac{y_1}{|y|}\Big)\Big(1-\frac{3}{2r_i}\varepsilon\mu_i y_1\Big)\\&-\frac{3}{\varepsilon\mu_i r_i}\Gamma_0'(|y|)\frac{y_1}{|y|}-\frac{9}{2rr_i}(\Gamma_0-4\log\varepsilon-\log(8\mu_i^4))\Big]\\=&\frac{1}{r_i}\Big[-\frac{1}{\varepsilon^2\mu_i^2}U(y)+\frac{3}{2\varepsilon\mu_i r_i}U(y)y_1-\frac{15}{2}\frac{\Gamma_0'(|y|)}{|y|}\frac{y_1^2}{rr_i}\\&-\frac{9}{2rr_i}(\Gamma_0-4\log\varepsilon-\log(8\mu_i^4))\Big].\end{aligned} \tag{3.8}$$

Moreover, using the fact that $H_{i,0}$ satisfies

$$\Delta_5\Big(\log\frac{1}{|x-p_i|^4}H_{i,0}(x)\Big)=-30\frac{(r-r_i)^2}{rr_i|x-p_i|^2}+\frac{9}{2rr_i}\log\frac{1}{|x-p_i|^4},$$

we deduce

$$(I2)=-30\frac{(r-r_i)^2}{rr_i|x-p_i|^2}+\frac{9}{2rr_i}\log\frac{1}{|x-p_i|^4}+O(\frac{1}{1+|y|^2}). \tag{3.9}$$

Combining the above estimates in (3.8) and (3.9), we obtain

$$\Delta_5\frac{G_{i,\varepsilon}}{r_i}=\frac{1}{r_i}\Big[-\frac{1}{\varepsilon^2\mu_i^2}U(y)+\frac{3}{2\varepsilon\mu_i r_i}U(y)y_1+O(\frac{1}{1+|y|^2})\Big]. \tag{3.10}$$

We also have, for $j\neq i$,

$$\Delta_5 G_{j,\varepsilon}=O(\varepsilon^2\mu^2|\ln\varepsilon|^4). \tag{3.11}$$

Furthermore, away from the vortex points, one has

$$\Delta_5 G_{i,\varepsilon}(x)=O(\frac{\varepsilon^2\mu_i^2}{1+|x|^{2+\sigma}}).$$

Hence from (3.10) and (3.11), we deduce

$$\frac{1}{\kappa_i}\Delta_5\psi_0=\frac{1}{r_i}\Big[-\frac{1}{\varepsilon^2\mu_i^2}U(y)+\frac{3}{2\varepsilon\mu_i r_i}U(y)y_1+O(\frac{1}{1+|y|^2}+\varepsilon^2\mu^2|\ln\varepsilon|^4)\Big],$$

while in the region where $\eta_0 \in (0,1)$, we have

$$\Delta_5 \eta_0 \Big(\sum_{i=1}^{N} \kappa_i \frac{G_{i,\varepsilon} - G(x,p_i)}{r_i} \Big) + 2\nabla \eta_0 \cdot \nabla \Big(\sum_{i=1}^{N} \kappa_i \frac{(G_{i,\varepsilon} - G(x,p_i))}{r_i} \Big) = O(\varepsilon^2 \mu_i^2) \tag{3.12}$$

and it has compact support.

We now obtain, for $|x-p_i| < \frac{\delta_0^2}{|\ln \varepsilon|}$,

$$\frac{1}{\kappa_i}\Delta_5 \psi_0 = \frac{1}{r_i}\Big[-\frac{1}{\varepsilon^2\mu_i^2}U(y) + \frac{3}{2\varepsilon\mu_i r_i}U(y)y_1 + O(\frac{1}{1+|y|^2} + \varepsilon^2\mu_i^2|\ln\varepsilon|^4)\Big], \tag{3.13}$$

and for $|x-p_i| > \frac{\delta_0^2}{|\ln \varepsilon|}$,

$$\Delta_5 \psi_0 = O(\frac{\varepsilon^2\mu_i^2}{1+|x|^\nu}) \tag{3.14}$$

for some $\nu > 2$ independent of ε.

Now we come to the second nonlinear term. For $|x-p_i| < \frac{\delta_0^2}{|\ln \varepsilon|}$, we write $y = \frac{x-p_i}{\varepsilon\mu_i}$. In this region , $\eta^i = 1$. Near the vortex point p_i, one has the following expansion:

$$\begin{aligned}
&\frac{r^2}{\kappa_i r_i}\Big(\psi_0(x) - \frac{\alpha}{2}|\log\varepsilon|\Big) \\
=&\Gamma_0(y) - (4 - \frac{\alpha}{2\kappa_i}r_i)\log\varepsilon - \log(8\mu_i^4) + H_{i,1}(p_i) + \sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{r_i}{r_j}G(p_i,p_j) \\
+&\frac{\varepsilon\mu_i y_1}{2r_i}\Big(\Gamma_0(y) - (4 - 2\frac{\alpha r_i}{\kappa_i})\log\varepsilon - \log(8\mu_i^4) \\
&+ 4H_{i,1}(p_i) + 4\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{r_i}{r_j}G(p_i,p_j) + 2r_i\Big(\partial_r H_{i,1}(p_i) + \sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{r_i}{r_j}\partial_r G(p_i,p_j)\Big)\Big) \\
+&\varepsilon\mu_i y_2\Big(\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{r_i}{r_j}\partial_z G(p_i,p_j)) + \partial_z H_{i,1}(p_i)\Big) \\
&+ O(\varepsilon^2\mu_i^2|y|^2\log(\varepsilon|y|)).
\end{aligned} \tag{3.15}$$

We will choose μ_i such that

$$\log(8\mu_i^2) = H_{i,1}(p_i) + \sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{r_i}{r_j}G(p_i,p_j). \tag{3.16}$$

Using the expansion of G_j, we have

$$
\begin{aligned}
&\frac{r^2}{\kappa_i r_i}\Big(\psi_0(x)-\frac{\alpha}{2}|\log\varepsilon|\Big)\\
=&\Gamma_0(y)-(4-\frac{\alpha}{2\kappa_i}r_i)\log\varepsilon-\log\mu_i^2\\
+&\frac{\varepsilon\mu_i y_1}{2r_i}\Big(\Gamma_0(y)-(4-2\frac{\alpha r_i}{\kappa_i})\log\varepsilon+4H_{i,1}(p_i)+2r_i\partial_r H_{i,1}(p_i)-\log(8)-2\log(\mu_i^2)\\
&+2\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{r_i^2}{r_j}\Big(\log\frac{1}{|p_i-p_j|^4}(\partial_r H_{j,0}(p_i)-\frac{3}{2r_j})-\frac{4(p_i-p_j)_1}{|p_i-p_j|^2}(1-\frac{3}{2r_j}(r_i-r_j)+H_{j,0}(p_i))+\partial_r H_{j,1}(p_i)\Big)\\
&+4\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{r_i}{r_j}\Big(\log\frac{1}{|p_i-p_j|^4}\Big(1-\frac{3}{2r_j}(r_i-r_j)+H_{j,0}(p_i)\Big)+H_{j,1}(p_i)\Big)\Big)\\
+&\varepsilon\mu_i y_2\Big(\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{r_i}{r_j}\Big(\log\frac{1}{|p_i-p_j|^4}\partial_z H_{j,0}(p_i)-\frac{4(p_i-p_j)_2}{|p_i-p_j|^2}(1-\frac{3}{2r_j}(r_i-r_j)+H_{j,0}(p_i))+\partial_z H_{j,1}(p_i)\Big)\Big)\\
&+\partial_z H_{i,1}(p_i)+O(\varepsilon^2\mu_i^2|y|^2\log(\varepsilon|y|)).
\end{aligned}
$$

For p_i in the configuration space I defined in (2.17), we know that $\log\mu_i^2=O(\ln|\ln\varepsilon|)$. By the choice of μ_i in (3.16), one has

$$
\begin{aligned}
&\varepsilon^{2-\frac{\alpha}{2\kappa_i}r_i}\frac{1}{r_i}f\Big(\frac{r^2}{\kappa_i r_i}(\psi_0(x)-\frac{\alpha}{2}|\log\varepsilon|)\Big)\\
=&\frac{1}{\varepsilon^2\mu_i^2 r_i}U(y) \qquad (3.17)\\
\times&\exp\Big[\frac{\varepsilon\mu_i y_1}{2r_i}\Big(\Gamma_0(y)-(4-2\frac{\alpha r_i}{\kappa_i})\log\varepsilon+2\log\mu_i^2-\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{2r_i^2}{r_j}\frac{4(p_i-p_j)_1}{|p_i-p_j|^2}+A_i(\mathbf{p})\Big)\Big)\\
-&\varepsilon\mu_i y_2\Big(\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{r_i}{r_j}\frac{4(p_i-p_j)_2}{|p_i-p_j|^2}+B_i(\mathbf{p})\Big)+O(\varepsilon^2\mu_i^2|y|^2\log(\varepsilon|y|))\Big]
\end{aligned}
$$

where $A_i(\mathbf{p})$, $B_i(\mathbf{p})$ are constants depending on $\mathbf{p}$ such that they are of $O(1)$.

For $\frac{\delta_0^2}{|\ln\varepsilon|}<|x-p_i|<\frac{\delta_0}{|\ln\varepsilon|}$, similar to (3.17), one has

$$
\varepsilon^{2-\frac{\alpha}{2\kappa_i}r_i}\frac{1}{r_i}f\Big(\frac{r^2}{\kappa_i r_i}(\psi_0(x)-\frac{\alpha}{2}|\log\varepsilon|)\Big)=O(\varepsilon^2|\ln\varepsilon|^4).
$$

Let

$$
\begin{aligned}
S(\psi_0)&=\varepsilon^2\mu_i^2 S_0(\psi_0)\\
&=\varepsilon^2\mu_i^2\Big[\Delta_5\psi_0+\sum_{i=1}^N\varepsilon^{2-\frac{\alpha}{2\kappa_i}r_i}\frac{\kappa_i}{r_i}f\Big(\frac{r^2}{\kappa_i r_i}(\psi_0-\frac{\alpha}{2}|\log\varepsilon|)\Big)\eta^i\chi_{B_{\delta_1}(p_i)}\Big]. \qquad (3.18)
\end{aligned}
$$

We find that for $|x-p_i|<\frac{\delta_0^2}{|\ln\varepsilon|}$,

$$
S(\psi_0)=\frac{\kappa_i}{r_i}[\varepsilon E_{i,0}+O(\frac{\varepsilon^2\mu_i^2\log(\varepsilon|y|)}{1+|y|^2})],
$$

where

$$E_{i,0}(y) = \frac{\mu_i y_1 U(y)}{2r_i}\Big(\Gamma_0(y) - (4-2\frac{\alpha r_i}{\kappa_i})\log\varepsilon - \sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{2r_i^2}{r_j}\frac{4(p_i-p_j)_1}{|p_i-p_j|^2} + A_i(\mathbf{p}) - 2\log\mu_i^2\Big) - \mu_i y_2 U(y)\Big(\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{r_i}{r_j}\frac{4(p_i-p_j)_2}{|p_i-p_j|^2} + B_i(\mathbf{p})\Big). \tag{3.19}$$

As one can see from section 7 , if we define $p_i = (r_0+s)e_1 + \frac{\hat{p}_i}{|\ln\varepsilon|}$, and choose $s=0$ with $\hat{p}_i$ being solution of (7.4), then from the reduction (7.2)-(7.3) to (7.4), one has

$$-(4-2\tfrac{\alpha r_i}{\kappa_i})\log\varepsilon - \textstyle\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{2r_i^2}{r_j}\frac{4(p_i-p_j)_1}{|p_i-p_j|^2} = O(1),$$
$$\textstyle\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{r_i}{r_j}\frac{4(p_i-p_j)_2}{|p_i-p_j|^2} = O(1).$$

Recalling the definition of our configuration space defined in (2.17), for all the points p_j with $O(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|})$ perturbation from the above mentioned points, one has

$$-(4-2\tfrac{\alpha r_i}{\kappa_i})\log\varepsilon - \textstyle\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{2r_i^2}{r_j}\frac{4(p_i-p_j)_1}{|p_i-p_j|^2} = O(\ln|\ln\varepsilon|) = o(|\ln\varepsilon|),$$
$$\textstyle\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\frac{r_i}{r_j}\frac{4(p_i-p_j)_2}{|p_i-p_j|^2} = O(\ln|\ln\varepsilon|) = o(|\ln\varepsilon|).$$

We have that

$$S(\psi_0) = O(\frac{\varepsilon\ln|\ln\varepsilon|}{1+|y|^3} + \frac{\varepsilon\ln(1+|y|)}{1+|y|^3} + \frac{\varepsilon^2\mu_i^2\log(\varepsilon|y|)}{1+|y|^2}) = O(\frac{\varepsilon|\ln|\ln\varepsilon||}{1+|y|^{2+\sigma}}) \tag{3.20}$$

for any $\sigma\in(0,1)$. While for $\frac{\delta_0^2}{|\ln\varepsilon|} < |x-p_i|$, one can easily check that the following estimate holds:

$$S_0(\psi_0) \le \frac{\varepsilon^{1+\sigma}}{1+|x|^\nu}$$

for some $\sigma\in(0,1)$, $\nu>2$.

4. The inner outer gluing system

We have constructed the approximate solution ψ_0 in (2.22) in Section 2 andwe will look for a solution ψ of the equation

$$\begin{cases} \Delta_5\psi + F_1(\psi) = 0 \text{ in } \Pi, \\ \partial_r\psi(0,z) = 0, \end{cases} \tag{4.1}$$

where

$$F_1(\psi) = \sum_{i=1}^N \varepsilon^{2-\frac{\alpha}{2\kappa_i}r_i}\frac{\kappa_i}{r_i} f(\frac{r^2}{\kappa_i r_i}(\psi - \frac{\alpha}{2}|\ln\varepsilon|))\eta^i\chi_{B_{\frac{\delta_1}{|\ln\varepsilon|}}(p_i)}$$

with $f(s)=e^s$.

We look for ψ of the form

$$\psi = \psi_0(x) + \varphi(x)$$

where φ is a perturbation term. We decompose φ as

$$\varphi(x) = \sum_{i=1}^{N} \eta_i(x)\phi_i(y) + \xi(x) \tag{4.2}$$

where $\eta_i(x) = \eta_{\delta_2}(x - p_i)$ for some $\delta_2 < \delta_0^2$ and $y = \frac{x-p_i}{\varepsilon\mu_i}$ be the rescaled variable near each vortex point p_i. Recall that our problem (4.1) can be rewritten as

$$S_0(\psi_0 + \varphi) = \mathcal{L}(\varphi) + N(\varphi) + E = 0 \text{ in } \Pi,$$

where

$$\begin{aligned} E &= S_0(\psi_0), \\ \mathcal{L}(\varphi) &= \Delta_5\varphi - F_1'(\psi_0)\varphi, \\ N(\varphi) &= F_1(\psi_0+\varphi) - F_1(\psi_0) - F_1'(\psi_0)\varphi, \\ F_1'(\psi_0) &= \sum_{i=1}^{N} \varepsilon^{2-\frac{\alpha}{2\kappa_i}r_i}\frac{r^2}{r_i^2} f'\Big(\frac{r^2}{\kappa_i r_i}\big(\psi_0 - \frac{\alpha}{2}|\ln\varepsilon|\big)\Big)\eta^i \chi_{B_{\frac{\delta_1}{|\ln\varepsilon|}}(p_i)}. \end{aligned}$$

We have the following expansion:

$$\begin{aligned} S_0(\psi_0+\varphi) &= \sum_{i=1}^{N}\eta_i\Big[\Delta_5\phi_i + F_1'(\psi_0)(\phi_i+\xi) + E + N\big(\sum_{i=1}^{N}\eta_i\phi_i+\xi\big)\Big] \\ &\quad + \Delta_5\xi + \big(1-\sum_{i=1}^{N}\eta_i\big)\Big[F_1'(\psi_0)\xi + E + N\big(\sum_{i=1}^{N}\eta_i\phi_i+\xi\big)\Big] \\ &\quad + \sum_{i=1}^{N}\Big[\Delta_5\eta_i\phi_i + 2\nabla_x\eta_i\cdot\nabla_x\phi_i\Big]. \end{aligned}$$

It follows that ψ given in (4.2) will solve (4.1), if $(\phi,\xi) = (\phi_1,\cdots,\phi_N,\xi)$ solves the inner and outer problem:

$$\Delta_5\phi_i + F_1'(\psi_0)(\phi_i+\xi) + E + N\big(\sum_{i=1}^{N}\eta_i\phi_i+\xi\big),\ |x-p_i| < 2\delta_2, \tag{4.3}$$

and

$$\begin{cases} \Delta_5\xi + (1-\sum_{i=1}^{N}\eta_i)\Big[F_1'(\psi_0)\xi + E + N(\sum_{i=1}^{N}\eta_i\phi_i+\xi)\Big], \\ \quad + \quad \sum_{i=1}^{N}\Big[\Delta_5\eta_i\phi_i + 2\nabla_x\eta_i\cdot\nabla_x\phi_i\Big] \text{ in } \Pi, \\ \partial_r\xi(0,z) = 0, \end{cases} \tag{4.4}$$

respectively.

Let us write the inner problem (4.3) in terms of the variable $y = \frac{x-p_i}{\varepsilon\mu_i}$. Note that

$$\begin{aligned} \Delta_5\phi &= \Delta_x\phi + \frac{3}{r}\partial_r\phi \\ &= \frac{1}{\varepsilon^2\mu_i^2}\Big[\Delta_y\phi + \frac{\varepsilon\mu_i}{r_i+\varepsilon\mu_i y_1}\partial_{y_1}\phi\Big]. \end{aligned}$$

By the estimate (3.17) in Section 3, one has

$$\varepsilon^2\mu_i^2 F_1'(\psi_0) = e^{\Gamma_0(y)} + b(y)$$

where

$$b(y) = O(\frac{\varepsilon\mu_i \log(2+|y|)}{1+|y|^3} + \frac{o(\varepsilon\mu_i|\ln\varepsilon|)}{1+|y|^3}).$$

Similarly, one can get that

$$\mathcal{N}(\varphi) = \varepsilon^2\mu_i^2 N(\varphi) = (e^{\Gamma_0(y)} + b(y))\varphi^2.$$

Moreover, there holds

$$\tilde{E}_i = \varepsilon^2\mu_i^2 E = S(\psi_0) = O(\frac{\varepsilon\mu_i \log(2+|y|)}{1+|y|^3} + \frac{\varepsilon|\ln\ln\varepsilon|}{1+|y|^3} + \frac{\varepsilon^{1+b}}{1+|y|^{2+a}})$$

for some $a, b \in (0,1)$.

With these estimates at hand, the inner problem (4.3) can be written as

$$\Delta_y\phi_i + f'(\Gamma_0)\phi_i + B_i[\phi_i] + \mathcal{N}_i(\varphi) + \tilde{E}_i + (f'(\Gamma_0) + b(y))\xi = 0 \text{ in } B_R \tag{4.5}$$

where $R = \frac{c}{\varepsilon\mu_i|\ln\varepsilon|}$ and

$$B_i(\phi_i) = b(y)\phi_i + \frac{\varepsilon\mu_i}{r_i + \varepsilon\mu_i y_1}\partial_{y_1}\phi_i. \tag{4.6}$$

We will solve this problem coupled with the outer problem (4.4) such that ϕ_i has the size of error $\tilde{E}_i$ with two powers less of decay in y where ϕ_i is the inner perturbation defined in (4.2) and y is the rescaled variable near each vortex p_i.

$$(1+|y|)|D_y\phi_i| + |\phi_i| \le \frac{c\varepsilon|\ln\ln\varepsilon|}{(1+|y|^a)}$$

for some $a > 0$ independent of ε.

For the outer problem, it can be written as

$$\begin{cases} \Delta_5\xi + G(\xi,\phi) = 0 \text{ in } \Pi, \\ \partial_r\xi(0,z) = 0 \end{cases} \tag{4.7}$$

where

$$G(\xi,\phi) = V(x)\xi + N^o(\varphi) + E^o(x) + \sum_{i=1}^{N} A_i(\phi_i)$$

with

$$\begin{aligned} V(x) &= (1 - \sum_i \eta_i)F_1'(\psi_0), \\ N^o(\varphi) &= (1 - \sum_i \eta_i)N(\varphi), \\ A_i(\phi_i) &= \Big(\Delta_5\eta_i\phi_i + 2\nabla\eta_i\cdot\nabla\phi_i\Big), \\ E^o(x) &= (1 - \sum_i \eta_i)E. \end{aligned}$$

From the previous estimate (3.17) and the estimates there, one has the following:

$$|V(x)| = O(\varepsilon^{1+\sigma}), \tag{4.8}$$

$$|N^o(\varphi)| = O(\varepsilon^{1+\sigma}|\varphi|^2), \tag{4.9}$$

$$|E^o(x)| = O(\varepsilon^{1+\sigma}) \tag{4.10}$$

for some $\sigma > 0$.

In this section, we have done the inner-outer decomposition for the perturbation term φ defined in (4.2), and decompose our problem to the inner problem (4.5) and outer problem (4.7) respectively. In the next two sections, we will study the linear results that are the basic tool to solve the system (4.5)-(4.7) by means of a fixed point argument.

5. Linear theory

In this section, we consider the linear theory related to the inner and outer problem (4.5) and (4.4) in the previous section.

We will consider the following two problems:

$$\Delta_y \phi + e^{\Gamma_0(y)}\phi + B_i(\phi) + h(y) = 0 \text{ in } B_R \tag{5.1}$$

where $B_i(\phi)$ are defined in (4.6), this is the linear problem for the inner problem in a bounded ball. The second one corresponds to the linear problem for the outer problem:

$$\begin{cases} \Delta_5 \xi + g(x) = 0 \text{ in } \Pi, \\ \partial_r \xi(0,z) = 0. \end{cases} \tag{5.2}$$

The following linear theory for (5.1) has been studied in [12]. In order to study this problem, we first study the unperturbed linear problem:

$$\Delta_y \phi + e^{\Gamma_0(y)}\phi + h(y) = 0 \text{ in } \mathbb{R}^2. \tag{5.3}$$

We first introduce the function space we want to work in : for $m > 2$, and $\beta \in (0,1)$, we consider the following norms:

$$\|h\|_m = \sup_{y\in\mathbb{R}^2} (1+|y|^m)|h(y)|,$$
$$\|h\|_{m,\beta} = \|h\|_m + (1+|y|^{m+\beta})[h]_{B_1(y),\beta},$$
$$[h]_{A,\beta} = \sup_{y_1,y_2\in A} \frac{|h(y_1)-h(y_2)|}{|y_1-y_2|^\beta}.$$

We also consider the functions $Z_i(y)$, $i = 0,1,2$ as

$$Z_i(y) = \partial_{y_i}\Gamma_0(y) = -\frac{4y_i}{1+|y|^2} \text{ for } i = 1,2,$$
$$Z_0(y) = 2 + y\cdot\nabla\Gamma_0(y) = 2\frac{1-2|y|^2}{1+2|y|^2}.$$

We have the following estimates:

Lemma 5.1 (Lemma 6.1 in [12]). *Given $m > 2$ and $\beta \in (0,1)$, for any h with $\|h\|_m < \infty$, there exists a constant $C > 0$ and solution $\phi = \mathcal{T}(h)$ of problem (5.3) such that*

$$(1+|y|)|D\phi| + |\phi(y)|$$
$$\leq C\Big(\log(2+|y|)\Big|\int_{\mathbb{R}^2} hZ_0 dy\Big| + (1+|y|)\sum_{i=1}^{2}\Big|\int_{\mathbb{R}^2} hZ_i dy\Big| + (1+|y|)^{2-m}\|h\|_m\Big).$$

Moreover, if $\|h\|_{m,\beta} < \infty$, then the following estimate holds:

$$(1+|y|)^{2+\beta}[D_y^2\phi]_{B_1(y),\beta} + (1+|y|^2)|D_y^2\phi|$$
$$\le C\Big(\log(2+|y|)|\int_{\mathbb{R}^2} hZ_0 dy| + (1+|y|)\sum_{i=1}^{2}|\int_{\mathbb{R}^2} hZ_i dy| + (1+|y|)^{2-m}\|h\|_{m,\beta}\Big).$$

Now we go back to the inner linear problem (5.1). As one can see from the above lemma, in general, even h has sufficient decay, the solution ϕ has logarithmic growth. In order to get better decay estimate of the solution ϕ and get rid of the logarithmic growth, we can have the solvability for the following problem:

$$\Delta\phi + e^{\Gamma_0(y)}\phi + B_i(\phi) + h(y) = \sum_{j=0}^{2} c_{ij} e^{\Gamma_0} Z_i \text{ in } B_R \tag{5.4}$$

for $R > 0$ large. For a function h defined in A, we denote by $\|h\|_{m,\beta,A}$ the norm only taken on A:

$$\|h\|_{m,A} = \sup_{y\in A}(1+|y|^m)|h(y)|,$$
$$\|h\|_{m,\beta,A} = \|h\|_{m,A} + (1+|y|^{m+\beta})[h]_{B_1(y)\cap A,\beta}.$$

Similarly, for a function in $C^{2,\beta}(A)$,

$$\|\phi\|_{*,m-2,A} = \|D^2\phi\|_{m,\beta,A} + \|D\phi\|_{m-1,A} + \|\phi\|_{m-2,A}.$$

Then we have the following solvability result:

Proposition 5.2 (Proposition 6.1 in [12]). *There are number $C > 0$ such that for all R large, problem (5.4) has a solution $\phi = T_i(h)$ for certain $c_{ij} = c_{ij}(h)$ for $j = 0, 1, 2$ and it satisfies*

$$\|\phi\|_{*,m-2,B_R} \le C\|h\|_{m,\beta,B_R}.$$

Moreover, c_{ij} can be estimated as

$$c_{i0}(h) = \gamma_0 \int_{B_R} hZ_0 dy + O(R^{-(m-2)})\|h\|_{m,\beta,B_R},$$
$$c_{ij}(h) = \gamma_j \int_{B_R} hZ_j dy + O(R^{-(m-1)})\|h\|_{m,\beta,B_R}, \text{ for } j = 1, 2,$$

where $\gamma_i^{-1} = \int_{\mathbb{R}^2} e^{\Gamma_0} Z_i^2 dy$.

For the outer linear problem (5.2), we will restrict to the case of functions $g(x)$ that satisfy decay condition

$$\|g\|_\nu = \sup_{x\in\Pi}(1+|x|^\nu)|g(x)| < \infty$$

for some $\nu > 2$. This operator has been studied in [4] and [24].

Consider the barrier function as

$$\mathcal{B}(x) = c_1(1+|x|^2)^{-\frac{\sigma}{2}} \text{ for } \sigma \in (0,1).$$

By direct calculation, one can check that

$$\Delta_5\mathcal{B}(x) \le -Cc_1(1+|x|^2)^{-1-\frac{\sigma}{2}}.$$

Use this as a barrier function, we have the following estimate, whose proof will be omitted here.

Lemma 5.3. *There exists a solution ξ to (5.2) that defines a linear operator $\xi = T^o(g)$ of g and satisfies*

$$|\xi| \leq C\|g\|_\nu (1+|x|)^{-(\nu-2)}.$$

for $\nu > 2$.

6. Solving the inner-outer gluing problem

In this section, we will solve the nonlinear projected inner outer gluing problem as a fixed point problem, and in the next section, we will solve the reduced problem by choosing suitable p_i. More precisely, for any p_i in our configuration space I, we solve the following projected problem:

$$\Delta_y \phi_i + f'(\Gamma_0)\phi_i + B_i[\phi_i] + \mathcal{N}(\varphi) + \tilde{E}_i + (f'(\Gamma_0) + b(y))\xi = \sum_{j=0}^{2} c_{ij} e^{\Gamma_0(y)} Z_j \text{ in } B_R, \tag{6.1}$$

and

$$\begin{cases} \Delta_5 \xi + G(\xi, \phi) = 0 \text{ in } \Pi, \\ \partial_r \xi(0, z) = 0. \end{cases} \tag{6.2}$$

Let X^o be the Banach space of all functions $\xi \in C^{2,\beta}(\Pi)$ such that

$$\|\xi\| < \infty.$$

Then the outer problem (7.1) can be formulated as

$$\xi = T^o(G(\xi, \phi)),\ \xi \in X^o.$$

For the inner problem, we write it as

$$\Delta_y \phi_i + f'(\Gamma_0)\phi_i + B_i(\phi_i) + \mathcal{H}_i(\phi, \xi) = \sum_{j=0}^{2} c_{ij} e^{\Gamma_0(y)} Z_j$$

where

$$\mathcal{H}_i(\phi, \xi) = \mathcal{N}_i(\varphi) + \tilde{E}_i + (f'(\Gamma_0) + b(y))\xi.$$

Let X_* be Banach space of functions $\phi \in C^{2,\beta}(B_R)$ such that

$$\|\phi\|_{*, m-2, B_R} < \infty.$$

We decompose the inner problem as follows: We first introduce constants c_{ij} such that

$$\Delta_y \phi_{i,1} + f'(\Gamma_0)\phi_{i,1} + B_i(\phi_{i,1}) + B_i(\phi_{i,2}) + \mathcal{H}_i(\phi, \xi) = \sum_{j=0}^{2} c_{ij} e^{\Gamma_0(y)} Z_j \text{ in } B_R$$

where $c_{ij} = c_{ij}[\mathcal{H}_i(\phi, \xi) + B_i(\phi_{i,2})]$. This can be solvable by Proposition 5.2, and for $\phi_{i,1} \in X_*$, we have

$$\phi_{i,1} = T_i[\mathcal{H}_i(\phi, \xi) + B_i(\phi_{i,2})].$$

We require that $\phi_{i,2}$ solves the following equation:

$$\Delta_y \phi_{i,2} + e^{\Gamma_0}\phi_{i,2} + c_{i0} e^{\Gamma_0} Z_0 \text{ in } \mathbb{R}^2.$$

By Lemma 5.1, this can be solvable and can be written as

$$\phi_{i,2} = \mathcal{T}[c_{i0}(\mathcal{H}_i(\phi, \xi) + B_i(\phi_{i,2})) e^{\Gamma_0} Z_0].$$

Having in mind the apriori estimate in Lemma 5.1, we require that $\phi_{i,2} \in C^{2,\beta}(\mathbb{R}^2)$ and it satisfies

$$(1+|y|^{2+\beta})[D_y^2\phi]_{B_1(y),\beta} + (1+|y|^2)|D_y^2\phi| + (1+|y||D\phi|) + |\phi| \le C\log(2+|y|).$$

Let $\|\cdot\|_{**,m-2,\beta}$ be the infimum of C such that the above inequality holds, and denote by X_{**} the Banach space of functions $\phi \in C^{2,\beta}(\mathbb{R}^2)$ with $\|\phi\|_{**,m-2,\beta} < \infty$.

Our aim is to find a fix point solution $\xi, \phi_{i,1}, \phi_{i,2}$ for the following problem:

$$(\xi, \phi_{i,1}, \phi_{i,2}) = \mathcal{A}(\xi, \phi_{i,1}, \phi_{i,2})$$

given by

$$\begin{aligned} \xi &= T^o(G(\xi, \phi_{i,1} + \phi_{i,2})),\ \xi \in X^o, \\ \phi_{i,1} &= T_i(\mathcal{H}_i(\phi, \xi) + B_i(\phi_{i,2})),\ \phi_{i,1} \in X_*, \\ \phi_{i,2} &= \mathcal{T}[c_{i0}(\mathcal{H}_i(\phi, \xi) + B_i(\phi_{i,2}))e^{\Gamma_0} Z_0],\ \phi_{i,2} \in X_{**}. \end{aligned}$$

Let $m > 2$ and $a \in (0,1)$ and define

$$\begin{aligned} B_M = \{(\xi, \phi_{i,1}, \phi_{i,2}) \in X^o \times X_*^N \times X_{**}^N :\ & \|\xi\|_\infty \le M\varepsilon^{1+a}, \\ & \|\phi_{i,1}\|_{*,m-2,\beta} \le M\varepsilon|\ln\varepsilon|^{\frac{1}{2}},\ \|\phi_{i,2}\|_{**,m-2,\beta} \le M\varepsilon|\ln\varepsilon|^{\frac{1}{2}}\}. \end{aligned}$$

We will show that $\mathcal{A}$ is a contraction mapping from B_M to itself. We first show that $\mathcal{A}(B_M) \subset B_M$.

First we have by the definition of X_*, X_{**}, we have

$$\begin{aligned} A_i(\phi) &\le \frac{1}{1+|x|^\nu}(|\ln\varepsilon|^2|\phi| + |\frac{|\ln\varepsilon|}{\varepsilon\mu_i}|D_y\phi|) \\ &\le \frac{\varepsilon^\sigma}{1+|x|^\nu}(\|\phi_{i,1}\|_{*,m-2,\beta} + \|\phi_{i,2}\|_{**,m-2,\beta}) \end{aligned}$$

for some $\sigma > 2$. Combining this estimate with the estimates (4.8), (4.9) and (4.10) in the gluing section 4, we have

$$\begin{aligned} |G(\xi, \phi_{i,1}, \phi_{i,2}) \le & \frac{\varepsilon^2|\ln\varepsilon|^4}{1+|x|^\nu}(1+|\xi|+|\xi|^2+|\phi_{i,1}+\phi_{i,2}|^2) \\ & + \frac{\varepsilon^\sigma}{1+|x|^\nu}(\|\phi_{i,1}\|_{*,m-2,\beta} + \|\phi_{i,2}\|_{**,m-2,\beta}). \end{aligned}$$

From Proposition 5.2, we have

$$\|\xi\|_\infty = \|T^o(G(\xi, \phi_{i,1}+\phi_{i,2}))\|_\infty \le C\varepsilon^{1+a}.$$

Next one has, for some $\sigma_0 \in (0,1)$,

$$\mathcal{H}_i(\phi, \xi) \le |\tilde{E}_i| + \frac{C}{1+|y|^{2+\sigma_0}}(\frac{1}{1+|y|^{2-\sigma_0}} + \frac{\varepsilon|\ln\varepsilon|}{1+|y|^{1-\sigma_0}})|\xi| \tag{6.3}$$

$$+ \frac{c}{1+|y|^4}(|\xi|^2 + \sum_{i=1}^N(|\eta_i\phi_{i,1}|^2 + |\eta_i\phi_{i,2}|^2)) \tag{6.4}$$

and recall that $|\tilde{E}_i| \le \frac{c\varepsilon|\ln\ln\varepsilon|}{1+|y|^{2+\sigma_0}}$, we have

$$\|\mathcal{H}_i(\phi, \xi)\|_{m,\beta,B_R} \le C\varepsilon|\ln\varepsilon|^{\frac{1}{2}}.$$

From the expression for B_i in (4.6), we have

$$B_i(\phi_{i,2}) \leq C(\varepsilon\mu_i|D\phi_{i,2}| + \frac{\varepsilon|\ln\varepsilon|}{1+|y|^3}).$$

Thus one has

$$\|B_i(\phi_{i,2})\|_{m,\beta,B_R} \leq C(\frac{1}{|\ln\varepsilon|} + \varepsilon|\ln\varepsilon|)\|\phi_{i,2}\|_{**,m-2,B_R} \leq C\varepsilon|\ln\varepsilon|^{\frac{1}{2}} \tag{6.5}$$

and

$$c_{i0}[\mathcal{H}_i(\phi,\xi) + B_i(\phi_{i,2})] \leq C\|\mathcal{H}_i(\phi,\xi) + B_i(\phi_{i,2})\|_{m,\beta,B_R} \leq C\varepsilon|\ln\varepsilon|^{\frac{1}{2}}.$$

Combining the above estimates, we have

$$\|\phi_{i,1}\|_{*,m-2,B_R} \leq C\varepsilon|\ln\varepsilon|^{\frac{1}{2}}$$

and

$$\|\phi_{i,2}\|_{**,m-2,B_R} \leq C\varepsilon|\ln\varepsilon|^{\frac{1}{2}}.$$

We will show that $\mathcal{A}$ is a contraction mapping. Let $\varphi_j = \sum_{i=1}^N \eta_i(\phi_{i,1}^j + \phi_{i,2}^j) + \xi^j$ for $j = 1,2$ such that

$$(\xi^j, \phi_{i,1}^j, \phi_{i,2}^j) \in B_M.$$

Let $G(\varphi^j) = G(\xi^j, \phi_{i,1}^j, \phi_{i,2}^j)$ and one has

$$\begin{aligned} G(\varphi^1) - G(\varphi^2) \leq\ & |V(x)(\xi^1 - \xi^2)| \\ & + (1 - \sum_i \eta_i)|N(\varphi^1) - N(\varphi^2)| \\ & + \sum_i |A_i(\phi_1^1) - A_i(\phi_1^2)| \\ & + \sum_i |A_i(\phi_2^1) - A_i(\phi_2^2)|. \end{aligned}$$

We will estimate it term by term. We compute

$$\begin{aligned} &|V(x)(\xi^1 - \xi^2)| + (1 - \sum_i \eta_i)|N(\varphi^1) - N(\varphi^2)| \\ &\qquad \leq C\varepsilon^2|\ln\varepsilon|^2(|\xi^1 - \xi^2| + \sum_i \eta_i^2(|\phi_{i,1}^1 - \phi_{i,1}^2|^2 + |\phi_{i,2}^1 - \phi_{i,2}^2|^2)) \end{aligned}$$

and

$$|A_i(\phi_{i,1}^1) - A_i(\phi_{i,1}^2)| \leq \frac{\varepsilon^{\sigma_0}}{1+|x|^\nu}\|\phi_{i,1}^1 - \phi_{i,1}^2\|_{*,m-2,\beta}.$$

In order to estimate $A_i(\phi_{i,2}^1) - A_i(\phi_{i,2}^2)$, we notice that

$$\Delta_y(\phi_{i,2}^1 - \phi_{i,2}^2) + f'(\Gamma_0)(\phi_{i,2}^1 - \phi_{i,2}^2) + c_0^{12}e^{\Gamma_0}Z_0 = 0 \text{ in } \mathbb{R}^2$$

where

$$c_0^{12} = c_{i0}\Big(\mathcal{H}_i(\phi_{i,1} + \phi_{i,2}^1, \xi) + B_i(\phi_{i,2}^1)\Big) - c_0\Big(\mathcal{H}_i(\phi_{i,1} + \phi_{i,2}^2, \xi) + B_i(\phi_{i,2}^2)\Big).$$

By the definition of c_{i0}, we have

$$c_0^{12} = \int_{B_R}\Big[B_i(\phi_{i,2}^1 - \phi_{i,2}^2) + \mathcal{N}(\xi + \eta_i(\phi_{i,1} + \phi_{i,2}^1) + \sum_{j\neq i}\eta_j\phi_j) - \mathcal{N}_i(\xi + \eta_i(\phi_{i,1} + \phi_{i,2}^1) + \sum_{j\neq i}\eta_j\phi_j)\Big]Z_0 dy.$$

Similar to (6.5) and (6.3), we then get

$$c_0^{12} \le \frac{1}{|\ln\varepsilon|}\|\phi_{i,2}^1-\phi_{i,2}^2\|_{**,m-2,\beta}+C\|\phi_{i,2}^1-\phi_{i,2}^2\|_{**,m-2,\beta}^2.$$

Thus,

$$\begin{aligned}A_i(\phi_{i,2}^1-\phi_{i,2}^2) &\le C(\Delta_5\eta_i(\phi_{i,2}^1-\phi_{i,2}^2)+\nabla\eta_i\nabla(\phi_{i,2}^1-\phi_{i,2}^2))\\ &\le C\frac{\varepsilon^{\sigma_0}}{1+|x|^\nu}\|\phi_{i,2}^1-\phi_{i,2}^2\|_{**,m-2,\beta}.\end{aligned}$$

Combining the above estimates, we have

$$\begin{aligned}|G(\varphi^1)-G(\varphi^2)| \le \frac{\varepsilon^\sigma}{1+|x|^\nu}[&\|\xi^1-\xi^2\|_\infty+\|\phi_{i,2}^1-\phi_{i,2}^2\|_{**,m-2,\beta}+\|\phi_{i,1}^1-\phi_{i,1}^2\|_{*,m-2,\beta}\\ &+\|\phi_{i,2}^1-\phi_{i,2}^2\|_{**,m-2,\beta}^2+\|\phi_{i,1}^1-\phi_{i,1}^2\|_{*,m-2,\beta}^2].\end{aligned}$$

We conclude that for $\varphi^i\in B_M$

$$\|T^o(G(\varphi^1)-G(\varphi^2))\|_\infty \le C\varepsilon^\sigma[\|\xi^1-\xi^2\|_\infty+\|\phi_{i,2}^1-\phi_{i,2}^2\|_{**,m-2,\beta}+\|\phi_{i,1}^1-\phi_{i,1}^2\|_{*,m-2,\beta}]$$

Similarly, we can analyze each of the operator in $\mathcal{A}$ and get that $\mathcal{A}$ is a contraction mapping in B_M.

7. The reduced problem

In the previous section, we have find a solution (ξ,ϕ) such that they solve the following system:

$$\Delta_y\phi_i+f'(\Gamma_0)\phi_i+B_i(\phi_i)+\mathcal{H}_i(\phi,\xi)=\sum_{j=1}^2 c_{ij}e^{\Gamma_0(y)}Z_j \text{ in } B_R,$$

and

$$\begin{cases}\Delta_5\xi+G(\xi,\phi)=0 \text{ in } \Pi,\\ \partial_r\xi(0,z)=0\end{cases}\tag{7.1}$$

Then the full solvability of this problem is reduced to the following:

$$c_{ij}=c_{ij}[B_i(\phi_{i,2})+\mathcal{H}_i(\phi,\xi)]=0 \text{ for } i=1,\cdots,N,\ j=1,2.$$

By the definition of c_{ij}, we know that

$$c_{ij}=\int_{\mathbb{R}^2}[\mathcal{H}_i(\phi_i,\xi)+B_i(\phi_i)]Z_i dy+O(\varepsilon^{1+\sigma}).$$

Hence by the definition of $\mathcal{H}_i$ and the estimates for ϕ_i and ξ, we know that $c_{ij}=0$ can be reduced to the following :

$$\int_{B_R}\tilde{E}_iZ_idy=O(\varepsilon^{1+\sigma}).$$

By the expression of the error $\tilde{E}_i$ in (3.19) in Section 3, one has

$$\int_{B_R}\tilde{E}_iZ_1dy=\frac{\kappa_i}{r_i}\varepsilon\mathcal{F}_1(\mathbf{p},\alpha)+O(\varepsilon|\ln\ln\varepsilon|)$$

where

$$\mathcal{F}_1(\mathbf{p},\alpha)=-\frac{\mu_i}{2r_i}\int_{\mathbb{R}^2}Uy_1Z_1dy\Big[(4-2\frac{\alpha r_i}{\kappa_i})\log\varepsilon+\sum_{j\ne i}\frac{\kappa_j}{\kappa_i}\frac{2r_i^2}{r_j}\frac{4(p_i-p_j)_1}{|p_i-p_j|^2}\Big].$$

While

$$\int_{B_R} \tilde{E}_i Z_2 dy = \frac{\kappa_i}{r_i} \varepsilon \mathcal{F}_2(\mathbf{p}, \alpha) + O(\varepsilon)$$

where

$$\mathcal{F}_2(\mathbf{p}, \alpha) = -\mu_i \int_{\mathbb{R}^2} U y_2 Z_2 dy \sum_{j \neq i} \frac{\kappa_j}{\kappa_i} \frac{r_i}{r_j} \frac{4(p_i - p_j)_2}{|p_i - p_j|^2}.$$

If we put

$$p_j = (r_0 + s) e_1 + \frac{\hat{p}_j}{|\ln \varepsilon|},$$

where s is small perturbation, then the reduced problem becomes

$$\sum_{j \neq i} \frac{\kappa_j}{\kappa_i} \frac{(\hat{p}_i - \hat{p}_j)_1}{|\hat{p}_i - \hat{p}_j|^2} - (\frac{1}{r_i} - \frac{\alpha}{2\kappa_i}) = O(\frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|}) \tag{7.2}$$

and

$$\sum_{j \neq i} \frac{\kappa_j}{\kappa_i} \frac{(\hat{p}_i - \hat{p}_j)_2}{|\hat{p}_i - \hat{p}_j|^2} = O(\frac{1}{|\ln \varepsilon|}). \tag{7.3}$$

Similar reduced problem has been obtained when one study the multi vortex ring solution for the 3D Gross-Pitaevskii equation in [4] when all the κ_j are either 1 or -1.

We relabel these $\hat{p}_j$ such that $\kappa_i > 0$ for $i = 1, \cdots, m$, and $\kappa_i < 0$ for $i = m+1, \cdots, m+n$, and denote by $\mathbf{a}_i = \hat{p}_i$ for $i = 1, \cdots, m$ and $\mathbf{b}_{j-m} = \hat{p}_j$ for $j = m+1, \cdots, m+n$.

Let us set $\tilde{\gamma}_i = \kappa_i$ for $i = 1, \cdots, m$ and $\tilde{\beta}_j = -\kappa_{m+j}$ for $j = 1, \cdots, n$. In this case, we find that at main order, $(\mathbf{a}_1, \cdots, \mathbf{a}_m, \mathbf{b}_1, \cdots, \mathbf{b}_n)$ should satisfy the following system:

$$\begin{cases} \sum_{j=1, j\neq i}^{m} \frac{\tilde{\gamma}_j}{\mathbf{a}_i - \mathbf{a}_j} - \sum_{j=1}^{n} \frac{\tilde{\beta}_j}{\mathbf{a}_i - \mathbf{b}_j} = \frac{\tilde{\gamma}_i}{r_0} - \frac{\alpha}{2}, \\ \sum_{j=1, j\neq i}^{n} \frac{\tilde{\beta}_j}{\mathbf{b}_i - \mathbf{b}_j} - \sum_{j=1}^{m} \frac{\tilde{\gamma}_j}{\mathbf{b}_i - \mathbf{a}_j} = \frac{\tilde{\beta}_i}{r_0} + \frac{\alpha}{2}. \end{cases} \tag{7.4}$$

This can be regarded as a balancing condition between the multiple vortex rings.

We require that the set of points $\{\mathbf{a}_1, \cdots, \mathbf{a}_m, \mathbf{b}_1, \cdots, \mathbf{b}_n\}$ are symmetric with respect to the y_1-axis. From now on, we will work in the space. It can be seen that if $(\mathbf{a}_1, \cdots, \mathbf{a}_m, \mathbf{b}_1, \cdots, \mathbf{b}_n)$ is a solution to (7.4), then any translation is also a solution, so the linearized operator of (7.4) around this solution has at least one dimensional kernel given by $(1, \cdots, 1)$.

Next let us consider the linearized operator around the solution. Let us denote the left hand side of the j-th equation of (7.4) by $\mathbb{F}_j$. Then we can compute the linearization $d\mathbb{F}$ of the map

$$\mathbb{F} : (\mathbf{a}_1, ..., \mathbf{a}_m, \mathbf{b}_1, ..., \mathbf{b}_n) \to (\mathbb{F}_1, ..., \mathbb{F}_{m+n}).$$

$d\mathbb{F}$ evaluated at the point $(\mathbf{a}_1, \cdots, \mathbf{a}_m, \mathbf{b}_1, \cdots, \mathbf{b}_n)$ is a matrix, which can be explicitly computed.

Definition 7.1 (Nondegeneracy). *We call $(\mathbf{a}_1, ..., \mathbf{a}_m, \mathbf{b}_1, ..., \mathbf{b}_n)$ is a non-degenerate solution of (7.4), if the kernel of the linearized operator $d\mathbb{F}$ is one dimensional.*

When $\tilde{\gamma}_i = \tilde{\beta}_j = 1$, the balance problem (7.4) has been studied in [4]. It has been shown in that paper that the problem is related to the roots of some polynomials with rational coefficients, which can be regarded as a generalization of the classical Adler Moser polynomial. Let us recall the results proved there briefly. We have the following:

Lemma 7.2. *For $\tilde{\gamma}_i = \tilde{\beta}_j = 1$, then when the pair $(m, n) \in \mathbb{S}$ where*

$$\mathbb{S} := \{(2,1),(3,2),(4,3),(5,4),(6,5)\},$$

there exists non-degenerate solution to (7.4).

Remark 7.3. *Existence of non-degenerate solutions to the balance equation (7.4) for more general circulation can be found in the next section.*

Let us come back to our reduced problem (7.2)-(7.3). We will see that if there exists a nondegenerate solution $(\mathbf{a}_1^0, \cdots, \mathbf{a}_m^0, \mathbf{b}_1^0, \cdots, \mathbf{b}_n^0)$ of the balance problem (7.4), we can solve the reduced problem by perturbation.

If we define vector $\mathbf{q}$ by

$$\mathbf{a}_j = \mathbf{a}_j^0 + q_j, j = 1, ..., m,$$
$$\mathbf{b}_j = \mathbf{b}_j^0 + q_{j+m}, j = 1, ..., n,$$

then the reduced problem takes the form

$$d\mathbb{F}(\mathbf{q}) = \mathcal{G}(s, \mathbf{q}) - s r_0^{-2} \mathbf{e}_1, \tag{7.5}$$

where $\mathcal{G}(\gamma, \mathbf{q}) = O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)$ as $\varepsilon \to 0$, with higher order dependence on γ, $\mathbf{q}$, and

$$\mathbf{e}_1 = (\tilde{\gamma}_1, \cdots, \tilde{\gamma}_m, \tilde{\beta}_1, \cdots, \tilde{\beta}_n)^T.$$

Since $d\mathbb{F}$ is non-degenerated, the kernel is spanned by $\mathbf{e}_2 := (1, ..., 1)$. We can first project the right hand side $\mathcal{G}$ orthogonal to $\mathbf{e}_1$ and solve this projected problem, and then adjust s such that the projection of $\mathcal{G}$ to $\mathbf{e}_1$ for $\mathcal{G}$ is zero. Moreover we have the following estimates

$$|\mathbf{q}| + |s| = O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right).$$

8. Balancing configuration for more circulations and the polynomial method

In this section, we would like to search balancing configuration for more general circulation κ_j studied in the previous sections.

Let us assume there are m vortex rings with circulation $\tilde{\gamma}_j > 0$ located in the (r, z) plane at the points $\mathbf{a}_j$, $j = 1, ..., m$; and n vortex rings with circulation $-\tilde{\beta}_j < 0$ located at $\mathbf{b}_j$, $j = 1, ..., n$. We would like to point out that these circulations are not necessary integers.

By the computation in the previous sections, the balancing condition of these vortex rings is the following system:

$$\begin{cases} \sum_{j=1, j\neq i}^{m} \frac{\tilde{\gamma}_j}{\mathbf{a}_i - \mathbf{a}_j} - \sum_{j=1}^{n} \frac{\tilde{\beta}_j}{\mathbf{a}_i - \mathbf{b}_j} = \frac{\tilde{\gamma}_i}{r_0} - \alpha_1 := \frac{\sigma_i}{2}, \text{ for } i = 1, ..., m, \\ \sum_{j=1}^{m} \frac{\tilde{\gamma}_j}{\mathbf{b}_i - \mathbf{a}_j} - \sum_{j=1, j\neq i}^{n} \frac{\tilde{\beta}_j}{\mathbf{b}_i - \mathbf{b}_j} = -\frac{\tilde{\beta}_i}{r_0} - \alpha_1 := -\frac{\rho_i}{2}, \text{ for } i = 1, ..., n. \end{cases} \tag{8.1}$$

where $\alpha_1 = \frac{\alpha}{2}$.

In general, it is not easy to solve this system directly, even with numerical methods. On the other hand, for some special cases of $\tilde{\beta}_j, \tilde{\gamma}_j$, the polynomial method turns out to be quite powerful to tackle this type of problem. The case that all $\tilde{\gamma}_j$ and $\tilde{\beta}_j$ are equal to 1 has been treated in [4]. In this section, we consider the case that there are at most three different circulations.

Let $a_1, ..., a_{\tilde{m}}$ be those points with circulation $\tilde{\gamma} > 0$; and $b_1, ..., b_{\tilde{n}}$ with circulation $-\tilde{\beta} < 0$; $c_1, ..., c_{\tilde{t}}$ with circulation $\tilde{k} > 0$. Define generating polynomials

$$P(x) = \prod_{j=1}^{\tilde{m}} \left(x - a_j\right), Q(x) = \prod_{j=1}^{\tilde{n}} \left(x - b_j\right), R(x) := \prod_{j=1}^{\tilde{t}} \left(x - c_j\right).$$

Let $\sigma_1 = \frac{\tilde{\gamma}}{r_0} - \alpha_1, \rho_1 = \frac{\tilde{\beta}}{r_0} + \alpha_1, \delta_1 = \frac{\tilde{k}}{r_0} - \alpha_1$.

Lemma 8.1. *Suppose the balancing condition* (8.1) *holds. Then* P, Q, R *satisfy*

$$\begin{aligned}
&\frac{\tilde{\gamma}^2 P''}{P} + \frac{\tilde{\beta}^2 Q''}{Q} + \frac{\tilde{k}^2 R''}{R} - 2\tilde{\gamma}\tilde{\beta}\frac{P'Q'}{PQ} - 2\tilde{\beta}\tilde{k}\frac{Q'R'}{QR} + 2\tilde{\gamma}\tilde{k}\frac{P'R'}{PR} \\
&= \sigma_1\tilde{\gamma}\frac{P'}{P} + \rho_1\tilde{\beta}\frac{Q'}{Q} + \delta_1\tilde{k}\frac{R'}{R}.
\end{aligned}$$

Proof. This follows from direct computation, we also refer to [32] for the case of $\sigma_1 = \rho_1 = \delta_1 = 0$.

$$P'(x) = P(x)\sum_{j=1}^{\tilde{m}} \frac{1}{x - a_j}. \tag{8.2}$$

Similar formula holds for Q' and R'. Differentiating (8.2) yields

$$\begin{aligned}
P''(x) &= 2P(x)\sum_{i<j}^{\tilde{m}} \frac{1}{(x - a_i)\left(x - a_j\right)} \\
&= 2P(x)\sum_{i<j}^{\tilde{m}} \left[\left(\frac{1}{x - a_i} - \frac{1}{x - a_j}\right)\frac{1}{a_i - a_j}\right] \\
&= 2P(x)\sum_{i=1}^{\tilde{m}} \left(\frac{1}{x - a_i}\sum_{j\neq i}\frac{1}{a_i - a_j}\right).
\end{aligned}$$

It follows from the balancing condition that

$$\begin{aligned}
\frac{\gamma^2 P''}{P} &= 2\sum_{i=1}^{\tilde{m}} \left[\frac{\tilde{\gamma}}{x - a_i}\left(\sum_{j=1}^{\tilde{n}} \frac{\tilde{\beta}}{a_i - b_j} - \sum_{j=1}^{\tilde{t}} \frac{\tilde{k}}{a_i - c_j} + \frac{\sigma_1}{2}\right)\right], \\
\frac{\tilde{k}^2 R''}{R} &= 2\sum_{i=1}^{\tilde{t}} \left[\frac{\tilde{k}}{x - c_i}\left(\sum_{j=1}^{\tilde{n}} \frac{\tilde{\beta}}{c_i - b_j} - \sum_{j=1}^{\tilde{t}} \frac{\tilde{\gamma}}{c_i - a_j} + \frac{\delta_1}{2}\right)\right], \\
\frac{\tilde{\beta}^2 Q''}{Q} &= 2\sum_{j=1}^{\tilde{n}} \left[\frac{\tilde{\beta}}{x - b_j}\left(\sum_{i=1}^{\tilde{m}} \frac{\tilde{\gamma}}{b_j - a_i} + \sum_{i=1}^{\tilde{t}} \frac{\tilde{k}}{b_j - c_i} + \frac{\rho_1}{2}\right)\right].
\end{aligned}$$

On the other hand, using (8.2), we compute

$$\frac{P'Q'}{PQ} = \sum_{i=1}^{\tilde{m}}\sum_{j=1}^{\tilde{n}} \frac{1}{(x-a_i)\left(x-b_j\right)}$$
$$= \sum_{i=1}^{\tilde{m}}\sum_{j=1}^{\tilde{n}} \left[\left(\frac{1}{x-a_i}-\frac{1}{x-b_j}\right)\frac{1}{a_i-b_j}\right].$$

Hence

$$\frac{\tilde{\gamma}^2 P''}{P} + \frac{\tilde{\beta}^2 Q''}{Q} + \frac{\tilde{k}^2 R''}{R} - 2\tilde{\gamma}\tilde{\beta}\frac{P'Q'}{PQ} - 2\tilde{\beta}\tilde{k}\frac{Q'R'}{QR} + 2\tilde{\gamma}\tilde{k}\frac{P'R'}{PR}$$
$$= \sigma_1\tilde{\gamma}\frac{P'}{P} + \rho_1\tilde{\beta}\frac{Q'}{Q} + \delta_1\tilde{k}\frac{R'}{R}.$$

This completes the proof. □

We emphasize that one can't use the scaling $P(\tilde{\gamma}x), Q(\tilde{\beta}x), R(\tilde{k}x)$ to reduce the equation to the usual(easier) case that all circulations are equal to ± 1.

8.1. **Two different circulations.** In this subsection, let us assume the third generating polynomial $R = 1$, this corresponds to the case of two different circulations. We are then lead to consider the following equation for the unknown polynomials P, Q :

$$\tilde{\gamma}^2 P''Q + \tilde{\beta}^2 PQ'' - 2\tilde{\beta}\tilde{\gamma}P'Q' = \sigma\tilde{\gamma}P'Q + \rho\tilde{\beta}PQ'.$$

Solving this equation in the general case seems to be a nontrivial problem. We would like to find solutions in some special cases. After a scaling, one can always take $\tilde{\gamma} = 1$, that is

$$P''Q + \tilde{\beta}^2 PQ'' - 2\tilde{\beta}P'Q' = \sigma P'Q + \rho\tilde{\beta}PQ'. \tag{8.3}$$

Equation (8.3) is translational invariant. We first consider the case that

$$Q(x) = x.$$

This means that there is only one vortex ring with negative circulation $-\tilde{\beta}$. We seek polynomials P with degree m, whose m (distinct) roots corresponds to the location of m vortex rings of positive circulation 1. We therefore get the following equation for the unknown polynomial P :

$$xP'' - 2\tilde{\beta}P' = \sigma xP' + \rho\tilde{\beta}P. \tag{8.4}$$

Inspecting the highest order term in this equation, we find that a necessary condition for the existence of solution to (8.4) is

$$\sigma m + \rho\tilde{\beta} = 0. \tag{8.5}$$

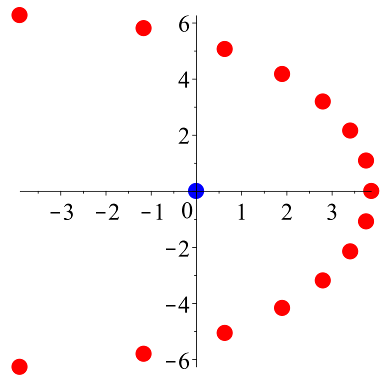
Suppose the polynomial P can be written as

$$P(x) = \sum_{j=0}^{m} c_j x^j, \text{ with } c_m = 1.$$

Then (8.4) has the form

$$\sum_{j=2}^{m} j(j-1)c_j x^{j-1} - 2\tilde{\beta}\sum_{j=1}^{m} jc_j x^{j-1} = \sigma\sum_{j=1}^{m} jc_j x^j + \rho\tilde{\beta}\sum_{j=0}^{m} c_j x^j.$$

FIGURE 1. Roots of P_{15} with $\tilde{\beta} = 8, \rho = 3$



Hence we obtain $-2\tilde{\beta}c_1 = \rho\tilde{\beta}c_0$ and

$$(j+1)\,jc_{j+1} - 2\tilde{\beta}\,(j+1)\,c_{j+1} = \sigma jc_j + \rho\tilde{\beta}c_j, j \geq 1.$$

That is,

$$c_j = \frac{(j+1)\,(j-2\tilde{\beta})}{\sigma j + \rho\tilde{\beta}} c_{j+1}, j = 0, ..., m-1.$$

This uniquely determines c_j.

If we assume that $2\tilde{\beta} \neq j$ for all $j = 0, ..., m-1$, then all the coefficients c_j are nonzero. In particular, 0 is not a root of P and P, Q have no common root.

Let us denote these polynomials by $P_m = P_{m,\tilde{\beta},\sigma}$. Note that ρ is explicitly determined by $m, \tilde{\beta}, \sigma$ through (8.5). To obtain balancing configuration, the condition $\sigma m + \rho\tilde{\beta} = 0$ becomes

$$\left(\frac{1}{r_0} - \alpha_1\right) m + \left(\frac{\tilde{\beta}}{r_0} + \alpha_1\right) \tilde{\beta} = 0.$$

That is,

$$\alpha_1 = \frac{m + \tilde{\beta}^2}{(m - \tilde{\beta})\, r_0}.$$

Note that α_1 can be positive or negative, depending on the value of $m - \tilde{\beta}$. However, α_1 can't be zero. We can find a nondegenerate configuration for generic choice of the circulation $\tilde{\beta}$ whose location is determined by the zeros of P_m.

To proceed, consider the equation

$$P''\xi + \tilde{\beta}^2 P\xi'' - 2\tilde{\beta}P'\xi' = \sigma P'\xi + \rho\tilde{\beta}P\xi'. \tag{8.6}$$

Previous analysis yields abundance of solution pairs (P_m, ξ) with $\xi\,(x) = x$. Recall that if A, B are two functions and η_1 is a solution of the second order ODE of η :

$$\eta'' + A\eta' + B\eta = 0.$$

Variation of parameter formula then tells us that this ODE has another solution of the form

$$\eta_1\,(x) \int \frac{\exp\left(-\int A\,(s)\,ds\right)}{\eta_1^2\,(t)} dt.$$

Hence (8.6) has another solution pair (P_m, ξ_2), where

$$\xi_2(x) = x \int \left(s^{-2} P_m^{\frac{2}{\tilde{\beta}}} e^{\frac{\rho s}{\tilde{\beta}}} \right) ds.$$

Lemma 8.2. *Assume $\frac{2}{\tilde{\beta}} \in \mathbb{N}$. The function $e^{-\frac{\rho x}{\tilde{\beta}}} \xi_2(x)$ is a polynomial of degree $\frac{2m}{\tilde{\beta}} - 1$ for a suitable integration constant.*

Proof. We write P_m as P and expand $P^{\frac{2}{\tilde{\beta}}}$ near $s = 0$. We see that the integrand defining ξ_2 has the form

$$\left(\frac{c_1}{s^2} + \frac{c_2}{s} + c_3 + c_4 s + ... \right) e^{\frac{\rho s}{\tilde{\beta}}}.$$

We compute

$$\left(P^{\frac{2}{\tilde{\beta}}} \right)' + \frac{\rho}{\tilde{\beta}} P^{\frac{2}{\tilde{\beta}}} = \frac{2}{\tilde{\beta}} P^{\frac{2}{\tilde{\beta}}-1} P' + \frac{\rho}{\tilde{\beta}} P^{\frac{2}{\tilde{\beta}}} = \frac{1}{\tilde{\beta}} P^{\frac{2}{\tilde{\beta}}-1} \left(2P' + \rho P \right).$$

On the other hand, at $s = 0$, by (8.6), we have

$$2P'(0) + \rho P(0) = 0.$$

Therefore, $\left(P^{\frac{2}{\tilde{\beta}}} \right)' + \frac{\rho}{\tilde{\beta}} P^{\frac{2}{\tilde{\beta}}} = 0$ at $s = 0$. As a consequence, $e^{-\frac{\rho x}{\tilde{\beta}}} \xi_2(x)$ is polynomial for a suitable integration constant. This completes the proof.

□

To construct more balancing configurations, we need the following

Lemma 8.3. *Suppose p, q satisfy the equation*

$$p'' q + \tilde{\beta}^2 p q'' - 2\tilde{\beta} p' q' = \sigma p' q + \rho \tilde{\beta} p q'.$$

Let $\eta = e^{-\frac{\rho x}{\tilde{\beta}}} q$. Then $p, \tilde{\eta}$ satisfy

$$p'' \eta + \tilde{\beta}^2 p \eta'' - 2\tilde{\beta} p' \eta' = (\sigma + 2\rho) p' \eta - \rho \tilde{\beta} p \eta'. \tag{8.7}$$

Proof. This follows from direct computation. We omit the details.

□

Combining Lemma 8.2 and Lemma 8.3, for $\frac{2}{\tilde{\beta}} \in \mathbb{N}$, we get polynomial $Q_n = Q_{n,\tilde{\beta},\sigma}$, with degree $n = \frac{2m}{\tilde{\beta}} - 1$, which solves the equation

$$P_m'' Q_n + \tilde{\beta}^2 P_m Q_n'' - 2\tilde{\beta} P_m' Q_n' = (\sigma + 2\rho) P_m' Q_n - \rho \tilde{\beta} P_m Q_n'. \tag{8.8}$$

Explicitly,

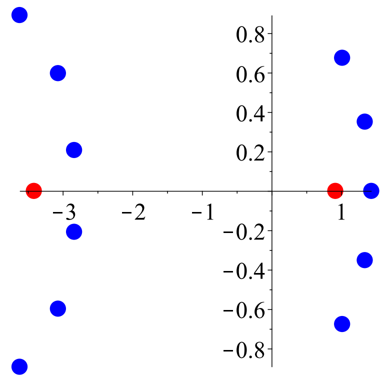
$$Q_n = e^{-\frac{\rho}{\tilde{\beta}} x} x \int s^{-2} P_m^{\frac{2}{\tilde{\beta}}} e^{\frac{\rho s}{\tilde{\beta}}} ds.$$

This provides us with more balancing configurations.

Suppose the corresponding vortex rings are located near $(r_0, 0)$ and with traveling speed α, then there holds

$$\begin{cases} \frac{1}{r_0} - \alpha = \sigma + 2\rho, \\ \frac{\tilde{\beta}}{r_0} + \alpha = -\rho, \\ \sigma m + \rho \tilde{\beta} = 0. \end{cases}$$

FIGURE 2. Roots of P_2 and Q_{11} with $\tilde{\beta} = 1/3, \rho = 8/5$



Therefore, we should have

$$\alpha = \frac{\left(1 + 2\tilde{\beta}\right) m - \tilde{\beta}^2}{\left(\tilde{\beta} - m\right) r_0}.$$

We also point out that due to the condition $\frac{2}{\tilde{\beta}} \in \mathbb{N}$, α can't be zero.

In the special case that $\tilde{\beta} = 2$, the previous technique can be further applied to yield a sequence of solutions. We also refer to [27] for related results.

Proposition 8.4. *There exist sequences of monic polynomials p_j, q_j with* $\deg\left(p_j\right) = 2j + 1$, $\deg\left(q_j\right) = j$, $j \leq 30$, *such that $p_0 = x - 4$, $q_1 = x$, and*

$$p_j'' q_j - 4p_j' q_j' + 4p_j q_j'' = j p_j' q_j - (2j + 1)\, p_j q_j',$$
$$p_j'' q_{j+1} - 4p_j' q_{j+1}' + 4p_j q_{j+1}'' = -\left(j + 1\right) p_j' q_{j+1} + (2j + 1)\, p_j q_{j+1}'.$$

Moreover, p_j, q_j are related through the recurrence relation:

$$q_j' q_{j+1} - q_j q_{j+1}' + \frac{2j + 1}{4} q_j q_{j+1} = \frac{2j + 1}{4} p_j.$$
$$p_j' p_{j+1} - p_j p_{j+1}' + (j + 1)\, p_j p_{j+1} = (j + 1)\, q_{j+1}^4.$$

Before proceeding to the proof, let us remark that with these recurrence relation, we can find the sequence $\left\{p_j, q_j\right\}$ of polynomials in the following order:

$$p_0 \to q_1 \to p_1 \to q_2 \to p_2 \to \ldots$$

We believe the condition $j < 30$ can be dropped. Indeed, if q_j, p_j have no repeated roots, then the recurence relation gives a polynomial q_{j+1}. Similarly, if p_j, q_{j+1} have no common roots, then we get a polynomial p_{j+1}.

Proof. Let $q_{j+1} = e^{\mu x}\eta$, where μ is a parameter to be determined later on. Let us assume

$$p_j'' q_j - 4p_j' q_j' + 4p_j q_j'' = \frac{j n_j}{2j + 1} p_j' q_j - n_j p_j q_j',$$
$$p_j'' q_{j+1} - 4p_j' q_{j+1}' + 4p_j q_{j+1}'' = -\frac{j + 1}{2j + 1} n_j p_j' q_{j+1} + n_j p_j q_{j+1}'.$$

Denoting $n = n_j$, then η satisfies

$$\begin{aligned} & p_j''\eta - 4p_j'\left(\mu\eta + \eta'\right) + 4p_j\left(\mu^2\eta + 2\mu\eta' + \eta''\right) \\ & = -\frac{j+1}{2j+1} n p_j'\eta + n p_j\left(\mu\eta + \eta'\right). \end{aligned}$$

We then get

$$\begin{aligned} & p_j''\eta - 4p_j'\eta' + 4p_j\eta'' \\ & = \left(-\frac{j+1}{2j+1}n + 4\mu\right) p_j'\eta + (n - 8\mu)\, p_j\eta' + \left(n\mu - 4\mu^2\right) p_j\eta. \end{aligned}$$

Let us choose $\mu = \frac{n}{4}$. Then the equation for η becomes

$$p_j''\eta - 4p_j'\eta' + 4p_j\eta'' = \frac{j}{2j+1} n p_j'\eta - n p_j\eta'.$$

In particular, η solves the same ODE as q_j. Therefore, denoting $W\left(\eta_1, \eta_2\right) = \eta_1'\eta_2 - \eta_1\eta_2'$, we get

$$W\left(q_j, e^{-\mu x} q_{j+1}\right) = \mu p_j e^{-\mu x}.$$

On the other hand,

$$p_{j+1}'' q_{j+1} - 4p_{j+1}' q_{j+1}' + 4p_{j+1} q_{j+1}'' = \frac{(j+1)\, n_{j+1}}{2j+3} p_{j+1}' q_{j+1} - n_{j+1} p_{j+1} q_{j+1}',$$

$$p_j'' q_{j+1} - 4p_j' q_{j+1}' + 4p_j q_{j+1}'' = -\frac{j+1}{2j+1} n_j p_j' q_{j+1} + n_j p_j q_{j+1}'.$$

Let $p_{j+1} = e^{\alpha x}\phi$, then

$$\begin{aligned} & \left(\alpha^2\phi + 2\alpha\phi' + \phi''\right) q_{j+1} - 4\left(\alpha\phi + \phi'\right) q_{j+1}' + 4\phi q_{j+1}'' \\ & = \frac{(j+1)\, n_{j+1}}{2j+3}\left(\alpha\phi + \phi'\right) q_{j+1} - n_{j+1}\phi q_{j+1}'. \end{aligned}$$

That is,

$$\begin{aligned} & \phi'' q_{j+1} - 4\phi' q_{j+1}' + 4\phi q_{j+1}'' \\ & = \left(\frac{(j+1)\, n_{j+1}}{2j+3} - 2\alpha\right)\phi' q_{j+1} + \left(-n_{j+1} + 4\alpha\right)\phi q_{j+1}' + \left(\frac{(j+1)\, n_{j+1}}{2j+3}\alpha - \alpha^2\right)\phi q_{j+1}. \end{aligned}$$

Now we choose α such that

$$\frac{(j+1)\, n_{j+1}}{2j+3}\alpha - \alpha^2 = 0.$$

This implies $\alpha = \frac{(j+1)n_{j+1}}{2j+3}$. Then the equation of ϕ becomes

$$\begin{aligned} & \phi'' q_{j+1} - 4\phi' q_{j+1}' + 4\phi q_{j+1}'' \\ & \quad = -\frac{(j+1)\, n_{j+1}}{2j+3}\phi' q_j + \frac{(2j+1)\, n_{j+1}}{2j+3}\phi q_{j+1}'. \end{aligned}$$

Therefore, if we take n_{j+1} such that

$$n_{j+1} = \frac{2j+3}{2j+1} n_j,$$

then ϕ and p_{j+1} satisfy the same ODE. This means that we can take $n_j = 2j+1$. Hence we obtain

$$p_j'' q_j - 4p_j' q_j' + 4p_j q_j'' = j p_j' q_j - (2j+1)\, p_j q_j',$$
$$p_j'' q_{j+1} - 4p_j' q_{j+1}' + 4p_j q_{j+1}'' = -(j+1)\, p_j' q_{j+1} + (2j+1)\, p_j q_{j+1}'.$$

We also have the following recursive formula

$$W\left(p_j, e^{-\alpha x} p_{j+1}\right) = (j+1)\, e^{-\alpha x} q_{j+1}^4.$$

□

8.2. **Three different circulations.** Let us consider the case of $\tilde{t} = 1$. In this case,

by translation, we may assume without loss of generality that $c_1 = 0$. This means that we have one vortex ring with circulation $\tilde{k}$.

By Lemma 8.1, in the case that the third circulation $\tilde{k}$ is positive, we are lead to consider the following equation:

$$x\left(\tilde{\gamma}^2 P'' Q + \tilde{\beta}^2 P Q''\right) - 2\tilde{\gamma}\tilde{\beta} x P' Q' + 2\tilde{\gamma}\tilde{k} P' Q - 2\tilde{\beta}\tilde{k} P Q' = \sigma\tilde{\gamma} x P' Q + \rho\tilde{\beta} x P Q' + \delta\tilde{k} P Q.$$

After a possible rescaling, we may assume $\tilde{\gamma} = 1$. The above equation becomes

$$x\left(P'' Q - 2\tilde{\beta} P' Q' + \tilde{\beta}^2 P Q''\right) + 2\tilde{k} P' Q - 2\tilde{\beta}\tilde{k} P Q' = \sigma x P' Q + \rho\tilde{\beta} x P Q' + \delta\tilde{k} P Q. \tag{8.9}$$

We consider the case that $Q(x) = x - b_1$, where b_1 is an unknown constant. Substituting this into (8.9), we obtain the following equation for the unknown function P :

$$\begin{aligned} & x(x-b_1) P'' + \left[-2\tilde{\beta} x + 2k(x-b_1) - \sigma x (x-b_1)\right] P' \\ & + \left[-2\tilde{\beta}\tilde{k} - \rho\tilde{\beta} x - \delta\tilde{k}(x-b_1)\right] P = 0. \end{aligned} \tag{8.10}$$

Vanishing of the highest order term x^{m+1} requires

$$\sigma\tilde{m} + \rho\tilde{\beta} + \delta\tilde{k} = 0.$$

For generic parameters $\tilde{m}, \tilde{\beta}, \sigma, \rho$ satisfying this condition, equation (8.10) can be solved, yielding polynomial solution $\tilde{P}_{\tilde{m}} = \tilde{P}_{\tilde{m},\tilde{\beta},\sigma,\rho,\tilde{k}}$. To obtain balancing configuration, we need

$$\left(\frac{1}{r_0} - \alpha\right)\tilde{m} + \left(\frac{\tilde{\beta}}{r_0} + \alpha\right)\tilde{\beta} + \left(\frac{\tilde{k}}{r_0} - \alpha\right)\tilde{k} = 0.$$

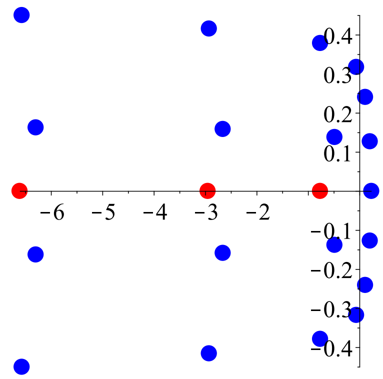
That is,

$$\alpha = \frac{\tilde{m} + \tilde{\beta}^2 + \tilde{k}^2}{\left(\tilde{m} + \tilde{k} - \tilde{\beta}\right) r_0}.$$

Now $(P, Q) = \left(\tilde{P}_{\tilde{m}}, x - b_1\right)$ is a solution pair to equation (8.9). By the variation of parameter formula, $\left(\tilde{P}_{\tilde{m}}, \xi\right)$ is also a solution pair to this equation , where

$$\xi = (x - b_1)\int \frac{\tilde{P}_{\tilde{m}}^{\frac{2}{\tilde{\beta}}}\, t^{\frac{2\tilde{k}}{\tilde{\beta}}}\, e^{\frac{\rho t}{\tilde{\beta}}}}{(t-b_1)^2}\, dt.$$

FIGURE 3. Roots of $\tilde{P}_3$ and $\tilde{Q}_{19}$ with $\tilde{\beta} = 1/2, k = 2, \rho = 4, \delta = 1$



Similar arguments as Lemma 8.2 tells us that the function $\tilde{Q}_n := e^{-\frac{\rho x}{\tilde{\beta}}}\xi$ is a polynomial, with degree $n = \frac{2\tilde{m}+2\tilde{k}}{\tilde{\beta}} - 1$, provided that $\frac{2k}{\tilde{\beta}} \in \mathbb{N}$ and $\frac{2}{\tilde{\beta}} \in \mathbb{N}$. Moreover, the coefficient before its highest order term equals $\frac{\tilde{\beta}}{\rho}$. We remark that ξ satisfies

$$(x - b_1)\,\xi' - \xi = \frac{\rho}{\tilde{\beta}} x^{\frac{2k}{\tilde{\beta}}} e^{\frac{\rho x}{\tilde{\beta}}} \tilde{P}^{\frac{2}{\tilde{\beta}}}.$$

Hence $\tilde{Q}_n$ satisfies

$$(x - b_1)\,\tilde{Q}_n' + \left((x - b_1)\,\frac{\rho}{\tilde{\beta}} - 1\right)\tilde{Q}_n = \frac{\rho}{\tilde{\beta}} x^{\frac{2k}{\tilde{\beta}}} \tilde{P}_{\tilde{m}}^{\frac{2}{\tilde{\beta}}}.$$

Lemma 8.5. *Suppose p, q satisfy the equation*

$$x\left(p''q - 2\tilde{\beta}p'q' + \tilde{\beta}^2 pq''\right) - 2\tilde{k}p'q + 2\tilde{k}\tilde{\beta}pq' = \sigma x p'q + \rho\tilde{\beta}xpq' + \delta\tilde{k}pq.$$

Let $\eta = e^{-\frac{\rho}{\tilde{\beta}}x}q$. Then p, η satisfy

$$\begin{aligned} x\left(p''\eta - 2\tilde{\beta}p'\eta' + \tilde{\beta}^2 p\eta''\right) - 2\tilde{k}p'\eta + 2\tilde{k}\tilde{\beta}p\eta' \\ = (\sigma + 2\rho)\,xp'\eta - \rho\tilde{\beta}xp\eta' + (\delta + 2\rho)\,\tilde{k}p\eta \end{aligned} \tag{8.11}$$

By Lemma 8.5, the pair $(p, \eta) = \left(\tilde{P}_{\tilde{m}}, \tilde{Q}_{\tilde{n}}\right)$ satisfies (8.11). This enables us to find more balancing configuration with three different circulations, similar as the two circulation case. Nondegeneracy can be proved by numerical methods.

9. AXISYMMETRIC FLOW WITH SWIRL

In the previous sections, we have constructed solutions to the axisymmetric Euler flow without swirl, which corresponds to the equation

$$\begin{cases} -\Delta_5\psi = F((\psi - \frac{\alpha}{2}|\ln\varepsilon|)r^2) \text{ in } \Pi, \\ \psi_r(0, z) = 0. \end{cases} \tag{9.1}$$

Recall that in the case with swirl, we have the equation

$$-\Delta_5\psi = F\left(r^2\psi - \frac{\alpha}{2}\left|\ln\varepsilon\right| r^2\right) + \frac{G\left(r^2\psi - \frac{\alpha}{2}\left|\ln\varepsilon\right| r^2\right)}{r^2} := W.$$

The same method as that of the non-swirl case can be applied to this equation, with the choice that $F = G = e^s$ near the vortex rings(also using a cutoff function to make them zero away from the vortex ring). Indeed, in this case, we can compute the error of the approximate solutions following the same lines as before and compute the projection of the error onto the kernel. To analyze the effect of the term $\frac{1}{r^2}$ appearing before the function G, let us expand it near the point $(r_i, 0)$. In the notation adopted in the previous sections, we have

$$\frac{1}{r^2} = \frac{1}{\left(r_i + \varepsilon\mu_i y_1\right)^2} = \frac{1}{r_i^2}\left(1 - \frac{2\varepsilon\mu_i y_1}{r_i} + O\left(\varepsilon^2\right)\right). \tag{9.2}$$

Hence the contribution to the projection of the kernel due to this additional term is of the form $\frac{-2\varepsilon}{r_i}$, compared to the term $\frac{\varepsilon\ln\varepsilon}{r_i}\left(\frac{\alpha r_i}{\kappa_i} - 2\right)$ appeared in (3.17), this is a higher order term. Hence one can construct solutions as the non-swirl case. Moreover in the swirl case, we can also construct solutions different from the one constructed in the previous sections, due to the presence of the term (9.2). Indeed, we would like to contrusct a travelling wave solution with two vortex rings whose distance is of the order $O(1)$, and whose position in the (r, z) plane is close to the points $p_1 := (r_1, 0)$ and $p_2 := (r_2, 0)$, with $r_2 > r_1$ and positive circulation κ_1, κ_2, and $\kappa_2 \geq \kappa_1$.

The nonlinear function W will be chosen such that near $p_i, i = 1, 2$,

$$W = \left(a_i + \frac{b_i}{r^2}\right)\frac{\varepsilon^{2-\frac{r_i\alpha}{2\kappa_i}}\kappa_i}{r_i} e^{\frac{s}{\kappa_i r_i}}.$$

We require

$$a_i + \frac{b_i}{r_i^2} = 1, \text{ for } i = 1, 2. \tag{9.3}$$

Here $a_i, b_i \in \mathbb{R}$. Let $w = (w_1, w_2)$ be point in (r, z) plane. Up to a constant, one can show that

$$G_j(x) = \frac{p_{j,1}^2}{x_1}\int_0^{2\pi}\frac{\cos t}{\sqrt{x_1^2 + p_{j,1}^2 - 2x_1 p_{j,1}\cos t + \left(x_2 - p_{j,2}\right)^2}^2}dt.$$

According to the computation of (3.16), for each $i = 1, 2$, we should choose μ_i such that

$$\ln\left(8\mu_i^2\right) = H_{i,1}(p_i) + \sum_{j\neq i}\frac{\kappa_j r_i}{\kappa_i r_j}G_j(p_i). \tag{9.4}$$

Using (3.17) and (9.2), we see that at the main order the balancing condition for these two rings should have the form

$$\begin{cases} \left(\frac{\alpha}{\kappa_1} - \frac{2}{r_1}\right)\ln\varepsilon + \frac{\kappa_2 r_1}{\kappa_1 r_2}\partial_r G_2(p_1) + \partial_r H_{1,1}(p_1) + \frac{2\ln\mu_1^2}{r_1} + \frac{C_0}{r_1} - \frac{2b_1}{r_1} = 0, \\ \left(\frac{\alpha}{\kappa_2} - \frac{2}{r_2}\right)\ln\varepsilon + \frac{\kappa_1 r_2}{\kappa_2 r_1}\partial_r G_1(p_2) + \partial_r H_{2,1}(p_2) + \frac{2\ln\mu_2^2}{r_2} + \frac{C_0}{r_2} - \frac{2b_2}{r_2} = 0, \end{cases} \tag{9.5}$$

where C_0 is a constant indepenend of i. Therefore, we should choose $\alpha = \alpha_0 + \alpha_\varepsilon$ with $\alpha_\varepsilon = o(|1|)$, such that

$$\frac{\kappa_1}{r_1} = \frac{\kappa_2}{r_2} = \frac{\alpha_0}{2}. \tag{9.6}$$

With this choice, from (9.5), we obtain the condition

$$\begin{aligned} &r_1\partial_r G_2(p_1) + r_1\partial_r H_{1,1}(p_1) + 2\ln\mu_1^2 + C_0 - 2b_1 \\ &= r_2\partial_r G_1(p_2) + r_2\partial_r H_{2,1}(p_2) + 2\ln\mu_2^2 + C_0 - 2b_2. \end{aligned}$$

Observe that (9.4) yields

$$\ln\mu_1^2 - \ln\mu_2^2 = G_2(p_1) - G_1(p_2) + H_{1,1}(p_1) - H_{2,1}(p_2).$$

Hence we arrive at

$$\begin{aligned} 2(b_1 - b_2) = &\, r_1\partial_r G_2(p_1) - r_2\partial_r G_1(p_2) + r_1\partial_r H_{1,1}(p_1) - r_2\partial_r H_{2,1}(p_2) \\ &+ 2(G_2(p_1) - G_1(p_2) + H_{1,1}(p_1) - H_{2,1}(p_2)). \end{aligned}$$

To conclude, we see that once the parameters r_1, r_2, κ_1, b_1 are chosen, we can find $\kappa_2, \alpha, b_2, a_1, a_2$, such that for ε sufficiently small, there is a solution with two vortex rings to the Euler equation with swirl. Details of proof will be omitted, since they will be similar as before.

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