MATH 516-101-2022 Homework Two

Due Date: by 6pm, October 5, 2022

1. (10pts) Let u be a harmonic function in $B_1(0)\backslash\{0\} = \{0 < |x| < 1\} \subset \mathbb{R}^2$ be such that $\lim_{x\to 0} \frac{u(x)}{\log|x|} = 0$. Show that u can be extended to be a function $u \in C^2(B_1(0))$.

2. (10pts) Let $G(x, y)$ be Green's function in $\Omega \subset \mathbb{R}^n$. Show that $\forall x \neq y, x, y \in \Omega$, $G(x, y) = G(y, x)$.

3. (20pts) Let $u \in C^0(\Omega)$. Show that the followings are equivalent

(i) for all $x \in \Omega, B_r(x) \subset\subset \Omega$,

$$u(x) \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy$$

(ii) for all $B \subset\subset \Omega$ and for all $h : \bar{B} \to R$ which satisfies

$$-\Delta h = 0, \ \ x \in B, \ \ \ h \geq u \text{ for } x \in \partial B$$

one has $u(x) \leq h(x)$ for all $x \in \bar{B}$. Here $B = B_r(y)$.

(iii) for all $x \in \Omega, B_r(x) \subset\subset \Omega$,

$$u(x) \leq \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(\sigma)d\sigma$$

(iv) for all $x \in \Omega$ and for all $\phi \in C^2$ such that $u - \phi$ has a local maximum at x, then $-\Delta\phi(x) \leq 0$.

4. (30pts) This problem is concerned with Perron's method.

(a) Let $\xi \in \partial\Omega$ and $w(x)$ be a barrier on $\Omega_1 \subset\subset \Omega$: (i) w is superharmonic in Ω_1; (ii) $w > 0$ in $\bar{\Omega}_1\backslash\{\xi\}$; $w(\xi) = 0$. Show that w can be extended to a barrier in Ω.

(b) Let $\Omega = \{x^2 + y^2 < 1\}\backslash\{-1 \leq x \leq 0, y = 0\}$. Show that the function $w := -Re(\frac{1}{\mathrm{Ln}(z)}) = -\frac{\log r}{(\log r)^2+\theta^2}$ is a local barrier at $\xi = 0$.

(c) Consider the following Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega; u = g \text{ on } \partial\Omega$$

where $\Omega = B_1(0)\backslash\{0\}$, $g(x) = 0$ for $x \in \partial B_1(0)$ and $g(0) = -1$. Show that 0 is not a regular point. Hint: the function $-\frac{\epsilon}{|x|^{n-2}}$ is a sub-harmonic function.

5. (10pts) (a) Show that the problem of minimizing energy

$$I[u] = \int_J x^2 |u'(x)|^2 dx,$$

for $u \in C(\bar{J})$ with piecewise continuous derivatives in $J := (-1, 1)$, satisfying the boundary conditions $u(-1) = 0, u(1) = 1$, is not attained. (b) Consider the problem of minimizing the energy

$$I[u] = \int_0^1 (1 + |u'(x)|^2)^{\frac{1}{4}} dx$$

for all $u \in C^1((0, 1)) \cap C([0, 1])$ satisfying $u(0) = 0, u(1) = 1$. Show that the minimum is 1 and is not attained.

6. (10pts) Discuss Dirichlet Principle for

$$\begin{cases} \Delta u - c(x)u + f = 0 \ \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} + a(x)u = g \text{ on } \partial\Omega \end{cases}$$

7. (10pts) Let

$$\Phi(x - y, t) = (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}}$$

(a) Let $n = 1$ and $f(x)$ be a function such that $f(x_0-)$ and $f(x_0+)$ exists. Show that

$$\lim_{t\to 0} \int_R \Phi(x - x_0, t) f(y)dy = \frac{1}{2}(f(x_0-) + f(x_0+))$$

(b) Let u satisfy

$$u_t = \Delta u, x \in \mathbb{R}^n, t > 0; u(x, 0) = f(x)$$

Suppose that f is continuous and has compact support. Show that $\lim_{t\to+\infty} u(x, t) = 0$ for all x.