MATH 516-101, 2021-2022 Homework Five
Due Date: By 6pm on November 19, 2021

10pts each unless otherwise stated.

1. Consider the following convection-diffusion problem

$$-\Delta u + \sum_{j=1}^{n} b_j \partial_j u + cu = f \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

Assume that $f \in L^2(\Omega), b_j \in C^1(\bar{\Omega}), c \in L^\infty$. Show that if $c - \frac{1}{2}\sum_{j=1}^{n} \partial_j(b_j) \geq 0$ then the above problem has a unique weak solution.

2. Let $n = 3$ and Ω be the ball $|x| < \pi$. Show that a solution of $\Delta u + u = f(x), f \in L^2$ with $u = 0$ on $\partial\Omega$ can only exist if

$$\int_\Omega f(x)\frac{\sin|x|}{|x|}dx = 0$$

3. Use the space $W^{1,2}(\Omega)$ to discuss the weak solution formulation of the following boundary value problem

$$-\Delta u + u = f \text{ in } \Omega,$$

$$\frac{\partial u}{\partial \nu} = g \text{ on } \partial\Omega.$$

Show that if $u \in W^{2,2}(\Omega) \cap W^{1,2}(\Omega)$ is a weak solution, then it satisfies the equation in the sense of distributions and the boundary condition in the sense of trace.

4. (20pts) (a) Show that

$$\int_\Omega (\Delta v)^2 = \sum_{i,j=1}^{n} \int_\Omega (\partial_{ij} v)^2 dx = \|\nabla^2 v\|_{L^2}, \forall v \in W_0^{2,2}(\Omega)$$

(b) Discuss the weak solution of the following boundary value problem

$$\Delta^2 u = f \text{ in } \Omega, u \in W_0^{2,2}(\Omega)$$

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

(Introduce the bilinear form on $W_0^{2,2}$ and prove the existence by Lax-Milgram and uniqueness of the weak solution.)

5. (20pts) Let Ω be a bounded domain in R^2 and $f \in L^2$. Consider the following minimization problem

$$c = \inf_{u \in H_0^1(\Omega)} (\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4}\int_\Omega u^4 + \int_\Omega f(x)u)$$

Show that c can be attained and its minimizer is a weak solution

$$\Delta u = u^3 + f(x), \text{ in } \Omega; u = 0 \text{ on } \partial\Omega$$

Show that the weak solution is also unique.

6. Let $u \in H^1(R^n)$ have compact support and be a weak solution of the semilinear PDE

$$\Delta u = u^5 - f \text{ in } R^n$$

where $f \in L^2$. Prove that $\|D^2 u\|_{L^2(R^n)} \leq C\|f\|_{L^2(R^n)}$.

Hint: mimic the proof of H^2-estimates but without the cut-off function.

7. (20pts) Let u be a weak sub-solution of

$$-\sum_{i,j} \partial_{x_j}(a^{ij}\partial_{x_i}u) + c(x)u = f$$

where $\theta \leq (a^{ij}) \leq C_2 < +\infty$. Suppose that $c(x) \in L^{\frac{n}{2}}(B_1)$, $f \in L^q(B_1)$ where $q > \frac{n}{2}$. Show that there exists a generic constant $\epsilon_0 > 0$ such that if $\int_{B_1} |c|^{\frac{n}{2}} dx \leq \epsilon_0$, then

$$\sup_{B_{1/2}} u^+ \leq C(\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)})$$

Hint: following the Moser's iteration procedure.

8. Show that $u = \log|x|$ is in $H^1(B_1)$, where $B_1 = B_1(0) \subset R^3$ and that it is a weak solution of

$$-\Delta u + c(x)u = 0$$

for some $c(x) \in L^{\frac{3}{2}}(B_1)$ but u is not bounded.