MATH 516-101-2021 Homework Two
Due Date: by 6pm, October 6, 2021

1. Let $G(x, y)$ be Green's function in $\Omega \subset \mathbb{R}^2$. Show that $\forall x \neq y, x, y \in \Omega$ (a) $G(x, y) = G(y, x)$; $(b) G(x, y) > 0$.
2. This problem is concerned with Perron's method. (a) A function $u \in C^0(\Omega)$ is subharmonic if for every $B_\rho(x_0) \subset \Omega$ and a harmonic function h in $B_\rho(x_0)$ with $u \leq h$ on $\partial B_\rho(x_0)$ then $u \leq h$ in $B_\rho(x_0)$. Show that u is subharmonic if and only if for every $B_\rho(x_0) \subset \Omega$ it holds that $u(x_0) \leq \frac{1}{|\partial B_\rho(x_0)|} \int_{\partial B_\rho(x_0)} u$.

(b) Let $\xi \in \partial\Omega$ and $w(x)$ be a barrier on $\Omega_1 \subset\subset \Omega$: (i) w is superharmonic in Ω_1; (ii) $w > 0$ in $\bar{\Omega}_1 \setminus \{\xi\}$; $w(\xi) = 0$. Show that w can be extended to a barrier in Ω.

(c) Let $\Omega = \{x^2 + y^2 < 1\} \setminus \{-1 \leq x \leq 0, y = 0\}$. Show that the function $w := -Re(\frac{1}{\mathrm{Ln}(z)}) = -\frac{\log r}{(\log r)^2 + \theta^2}$ is a local barrier at $\xi = 0$.

(d) Consider the following Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega; u = g \text{ on } \partial\Omega$$

where $\Omega = B_1(0) \setminus \{0\}$, $g(x) = 0$ for $x \in \partial B_1(0)$ and $g(0) = -1$. Show that 0 is not a regular point. Hint: the function $-\frac{\epsilon}{|x|^{n-2}}$ is a sub-harmonic function.

3. (a) Show that the problem of minimizing energy

$$I[u] = \int_J x^2 |u'(x)|^2 dx,$$

for $u \in C(\bar{J})$ with piecewise continuous derivatives in $J := (-1, 1)$, satisfying the boundary conditions $u(-1) = 0, u(1) = 1$, is not attained. (b) Consider the problem of minimizing the energy

$$I[u] = \int_0^1 (1 + |u'(x)|^2)^{\frac{1}{4}} dx$$

for all $u \in C^1((0, 1)) \cap C([0, 1])$ satisfying $u(0) = 0, u(1) = 1$. Show that the minimum is 1 and is not attained.

4. Discuss Dirichlet Principle for

$$\begin{cases} \Delta u - c(x)u + f = 0 \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} + a(x)u = g \text{ on } \partial\Omega \end{cases}$$

5. Let

$$\Phi(x - y, t) = (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}}$$

(a) Let $n = 1$ and $f(x)$ be a function such that $f(x_0-)$ and $f(x_0+)$ exists. Show that

$$\lim_{t \to 0} \int_R \Phi(x - x_0, t) f(y) dy = \frac{1}{2}(f(x_0-) + f(x_0+))$$

(b) Let u satisfy

$$u_t = \Delta u, x \in \mathbb{R}^n, t > 0; u(x, 0) = f(x)$$

Suppose that f is continuous and has compact support. Show that $\lim_{t \to +\infty} u(x, t) = 0$ for all x.

(c) Find all solutions $u(x, t)$ of the one-dimensional heat equation $u_t = u_{xx}$ of the form $u = \frac{1}{\sqrt{t}} f(\frac{x}{\sqrt{t}})$.

6. Consider the following general parabolic equation

$$L[u] = a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u - u_t$$

where $0 < C_1 < a(x, t) < C_2, |b(x, t)| \leq C_3, c(x, t) \leq C_4$. Prove the uniqueness of the initial value problem

$$\begin{cases} Lu(x, t) = f(x, t), \text{ in} \Omega_T; \\ u(x, 0) = \phi(x), x \in \Omega, \quad u(x, t) = g(x, t), \ x \in \partial\Omega, t \in (0, T] \end{cases}$$

Hint: consider $v = e^{-C_4 t} u$.