MATH 516-101-2021 Homework One

Due Date: By 6pm on September 22nd, 2021

1. (10pts) Find the explicit solution to

$$u_t + b \cdot \nabla u + cu = f(x,t)$$
$$u(x,0) = g(x)$$

Here $c, b = (b_1, ..., b_n)$ are constants.

2. (10pts) Let $f \in C_c^1(R^2)$. Show that the following function

$$v(x) = \frac{1}{2\pi} \int_{R^2} (\log|x-y|) f(y) dy$$

satisfies $-\Delta v(x) = f(x), x \in R^2$. (Extra 5points) Show that $v(x) = \frac{1}{2\pi}(\int_{R^2} f(x)dx) \log|x| + C + O(\frac{1}{|x|})$ as $|x| \to +\infty$.

3. (40pts) This problem concerns the Newtonian potential

$$u(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \frac{1}{(1+|y|^2)^{\frac{l}{2}}} dy \tag{1}$$

where $n \geq 3$.

a) Show that for $l \in (2,n)$, $|u(x)| \leq \frac{C}{|x|^{l-2}}$ for $|x| > 1$

b) Show that for $l = n$, then $|u(x)| \leq \frac{C}{|x|^{n-2}} \log(|x|+2)$ for $|x| > 1$

c) Show that for $l > n$, then $|u(x)| \leq \frac{C}{|x|^{n-2}}$ for $|x| > 1$

d) If $f \in C_c^\infty(R^n), n \geq 3$ and $v = \int_{R^n} \frac{1}{|x-y|^{n-2}} f(y) dy$, show that $\lim_{|x| \to +\infty} |x|^{n-2} v(x) = C$ for some constant C.

Hint: For $|x| = R >> 1$, divide the integral into three parts

$$\int_{R^n} (...)dy = \int_{|y-x|<\frac{|x|}{2}} (...) + \int_{\frac{|x|}{2}<|y-x|<2|x|} (...) + \int_{|y-x|>2|x|} (...)$$

and estimate each parts. For example in the region $|y-x| < \frac{|x|}{2}$ we have $|y| > |x| - |x-y| > \frac{|x|}{2}$ and

$$\int_{|y-x|<\frac{|x|}{2}} \frac{1}{|x-y|^{n-2}} |f(y)| dy \leq \int_0^{\frac{|x|}{2}} \frac{r^{n-1}}{r^{n-2}} dr \frac{C}{|x|^l} \leq \frac{C}{|x|^{l-2}}$$

4. (10pts) For $n > 2$, the Kelvin transform is defined by

$$v(x) = |x|^{2-n} u(\frac{x}{|x|^2})$$

Suppose u satisfies $-\Delta u(x) = u^p$. Find out the new equation satisfied by v. Find out for which exponent p the equation is invariant. Extra points (10pts): Let u satisfy $\Delta^2 u = f(x)$ and we define

$$v(x) = |x|^{4-n} u(\frac{x}{|x|^2})$$

find out the equation satisfied by v. Here $\Delta^2 u = \Delta(\Delta u)$.

5. (10pts) Let $u \in C^2(\Omega)$ satisfy

$$\Delta u \geq 0 \text{ in } \Omega$$

where Ω is a bounded smooth domain in R^n.

Show that

$$u(x) \leq \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y) d\sigma, \ \forall B_r(x) \subset \Omega$$

As a consequence, show that the maximum of u can not be attained in Ω unless u is a constant.

6. (10pts) Let u be a harmonic function in $B_1(0) \backslash \{0\} = \{0 < |x| < 1\}$ be such that $\lim_{x \to 0} |x|^{n-2} u(x) = 0$. Show that $u \in C^2(B_1(0))$. Here $n \geq 3$.

7. (20pts) Determine the Green's function for the annulus $\{1 < |x| < 2\}$ in R^2.