Energy Methods

$$\begin{cases} u_t = \Delta u + f & \Omega_T \\ u = g \text{ on } & \partial\Omega \end{cases}$$

Thm $\exists 1$ sol'n

Proof: $e(t) = \int_\Omega w^2(x,t)\,dx, \qquad 0 \le t \le T, \qquad w = u_1 - u_2$

$$\dot{e}(t) = 2\int_\Omega w\, w_t\, dx = 2\int_\Omega w\,\Delta w\, dx$$

$$= -2\int_\Omega |\nabla w|^2 dx \le 0$$

$e(t) \le e(0) = 0, \qquad 0 \le t \le T$

So $u_1 \equiv u_2$ in Ω_T

Thm (Backward Uniqueness). Suppose $u_1, u_2 \in C^2(\overline{\Omega_T})$ and $u_1(x,T) = u_2(x,T)$

Proof: $\dot{e}(t) = -2\int_\Omega |\nabla w|^2$

$$\ddot{e} = -4\int_\Omega \nabla w \cdot \nabla w_t = 4\int_\Omega (\Delta w)^2$$

$$\int_\Omega |\nabla w|^2 = -\int_\Omega w\,\Delta w \le \left(\int_\Omega w^2\right)^{\frac{1}{2}} \left(\int_\Omega (\Delta w)^2\right)^{\frac{1}{2}}$$

$$\left(\dot{e}(t)\right)^2 \le e(t)\,\ddot{e}(t)$$

Let $f(t) = \log e(t)$

Then $\ddot{f}(t) = \dfrac{\ddot{e}(t)}{e(t)} - \dfrac{\dot{e}(t)^2}{e(t)^2} \ge 0$, f is convex

$$f((1-\tau)t_1 + \tau t) \le (1-\tau) f(t_1) + \tau f(t)$$

$$e((1-\tau)t_1 + \tau t) \le e(t_1)^{1-\tau} e(t)^{\tau} \Rightarrow$$

$t_1 \le t \le t_2, \; e(t_2) = 0$

$\Rightarrow e(t) = 0,$

Part I.4 Wave Equation

$$\begin{cases} u_{tt} - \Delta u = f(x,t) \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) \end{cases}$$

$\boxed{n=1}$ $(\partial_t + \partial_x)(\partial_t - \partial_x)u = 0$

$\partial_t u - \partial_x u = v \quad \Rightarrow \quad v = a(x-t)$

$\partial_t v + \partial_x v = 0$

Part I.1 $\begin{cases} u_t + b \cdot Du = f \\ u = g \end{cases}$

$u = g(x-tb) + \int_0^t f(x+(s-t)b, s)\,ds$

$b = -1$ $\quad f = a(x-t)$.

$$u(x,t) = \int_0^t a(x + \overset{''y}{(t-s)} - s)\,ds + g(x+t)$$

$$= \frac{1}{2}\int_{x-t}^{x+t} a(y)\,dy + b(x+t)$$

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy$$

d'Alembert's Solution : Need $g \in C^2$, $h \in C^1$

$\boxed{n=3}$: Spherical mean method

$(x_0, \cancel{t_0}) = 0.$

$$U(x_0; r, t) = \frac{1}{|\partial B(x_0,r)|}\int_{\partial B(x_0,r)} u(y)\,dS(y)$$

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0, \quad r > 0 \\ U = G, \quad U_t = H \end{cases}$$

$\tilde{U} := rU, \quad \tilde{G} = rG, \quad \tilde{H} = rH$

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 \\ \tilde{U} = G, \quad \tilde{U}_t = \tilde{H} \\ U = 0 \text{ on } r = 0 \end{cases}$$

Reflection
$$\tilde{G}(r) = \begin{cases} \hat{G}(r), & r>0 \\ -\hat{G}(-r), & r<0 \end{cases}$$

$$\tilde{H}(r) = \begin{cases} \hat{H}(r), & r>0 \\ -\hat{H}(-r), & r<0 \end{cases}$$

$$\hat{U}(r,t) = \frac{1}{2}\left[\tilde{G}(r+t) + \tilde{G}(r-t)\right] + \frac{1}{2}\int_{r-t}^{r+t} \tilde{H}(y)\,dy$$

$$= \frac{1}{2}\left[\hat{G}(r+t) - \hat{G}(t-r)\right] + \frac{1}{2}\int_{-r+t}^{r+t} \hat{H}(y)\,dy$$

$$\boxed{0<r<t}$$

$$\lim_{r\to 0} \hat{U}(r,t) = u(x,t)$$

$$u(x,t) = \lim_{r\to 0^+} \frac{\hat{U}(x;r,t)}{r}$$

$$= \lim_{r\to 0^+}\left[\frac{\hat{G}(t+r)-\hat{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r}\hat{H}(y)dy\right]$$

$$= \hat{G}'(t) + \hat{H}(t)$$

Recall
$$\hat{G}(r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} r\, g(y)\, dS_y$$

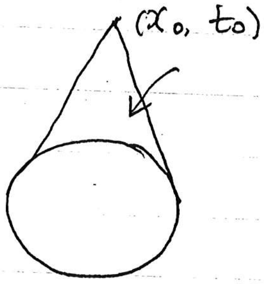
$$u(x,t) = \frac{\partial}{\partial t}\left(t ⨍_{\partial B(x,t)} g\, dS\right) + t ⨍_{\partial B(x,t)} h\, dS_y$$

$$⨍_{\partial B(x,t)} g(y)\,dS = ⨍_{\partial B(0,t)} g(x+tz)\,dS(z)$$

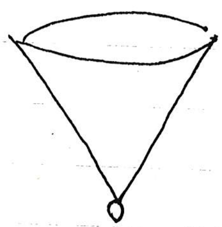
$$\frac{\partial}{\partial t}\left(\cancel{t} ⨍_{\partial B(x,t)} g\,dS\right) = ⨍_{\partial B(0,1)} Dg(x+tz)\cdot z\, dS_z$$

$$= ⨍_{\partial B(x,t)} Dg \cdot \frac{y-x}{t}\, dS_y$$

$$u(x,t) = \fint_{\partial B(x,t)} t h(y) + g(y) + Dg(y)\cdot(y-x)\, dS_y$$



Domain of dependence



Domain of Influence

Kirchoff's formula

$n=2$: Method of Descend

$$\bar{u}(x_1, x_2, x_3) = u(x_1, x_2)$$

$$\bar{x} = (x_1, x_2, 0)$$

$$u(x,t) = \frac{\partial}{\partial t}\left(t \int_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S}\right) + t \int_{\partial \bar{B}(\bar{x},t)} \bar{h}\, d\bar{S}$$

$$\fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} = \frac{1}{4\pi t^2} \int_{\partial \bar{B}(\bar{x},\bar{t})} g\, d\bar{S}$$

$$= \frac{2}{4\pi t^2} \int_{B(x,t)} g(y)\left(1 + |D\gamma(y)|^2\right)^{\frac{1}{2}} dy$$

$\gamma(y) = (t^2 - |y-x|^2)^{\frac{1}{2}}, \quad y \in B(x,t),$ surface area

$$|x_3|^2 + (x - x_0)^2 = t$$

$$x_3 = \pm\sqrt{t^2 - (x-x_0)^2}$$

$$(1+|D\gamma|^2)^{1/2} = t\,(t^2-|y-x|^2)^{-\frac{1}{2}}$$

$$⨍_{\partial\bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} = \frac{1}{2\pi t}\int_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy$$

$$= \frac{t}{2} ⨍_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy$$

$$u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\left(t^2 ⨍_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right)$$

$$+\frac{t^2}{2} ⨍_{B(x,t)} \frac{h(y)}{(t^2-|y-x|^2)^{1/2}}\,dy$$

$$t^2 ⨍_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}} = t ⨍_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{1/2}}\,dz$$

$$\frac{\partial}{\partial t}\left(t^2 ⨍_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right)$$

$$= ⨍_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{1/2}}\,dz + t ⨍_{B(0,1)} \frac{Dg(x+tz)\cdot z}{(1-|z|^2)^{1/2}}\,dz$$

$$= t ⨍_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy + t ⨍_{B(x,t)} \frac{Dg(y)\cdot(y-x)}{(t^2-|y-x|^2)^{1/2}}\,dy$$

$$u(x,t) = \frac{1}{2} ⨍_{B(x,t)} \frac{tg(y)+t^2h(y)+tDg(y)\cdot(y-x)}{(t^2-|y-x|^2)^{1/2}}\,dy$$

Poisson's Formula

Method of descent

$\boxed{n\geq 4}$: $\boxed{n=5}$: $\tilde{U} = \frac{1}{r}\partial_r(r^3U) = 3rU + r^2U_r$

$$\tilde{U}_{tt} - \tilde{U}_{rr} = 0$$

$n = 2k+1$,

$$\tilde{U}(r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}U(x; r,t)\right)$$

Satisfies $\tilde{U}_{tt} - \tilde{U}_{rr} = 0$

n - even, Method of Descent

Nonhomogeneous problem

$$\begin{cases} u_{tt} - \Delta u = f \\ u = 0,\ u_t = 0 \end{cases}$$

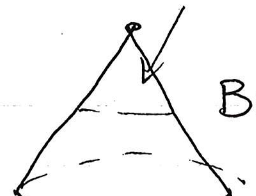
$$\begin{cases} u_{tt}(\cdot; s) - \Delta u(\cdot; s) = 0, & t > s \\ u(\cdot; s) = u_t(\cdot, s) = f(\cdot, s), & t = s \end{cases}$$

$$u(x,t) = \int_0^t u(x, t; s)$$

$\underline{n=1}$, $$u(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y, t-s)\,dy\,ds$$

Some Properties of Wave Equation

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega_T \\ u = g & \text{on } \partial\Omega_T \\ u_t = h & \text{on } \partial\Omega \times \{t=0\} \end{cases}$$



Domain of Dependence

$B(x_0, t_0 - t)$

$\underline{\text{Thm}}$ (Uniqueness):

$$E(t) = \frac{1}{2}\int_\Omega w_t^2 + |\nabla w|^2\,dx \qquad 0 \le t \le T$$

$$\dot{E}(t) = \int_\Omega w_t w_{tt} + Dw \cdot Dw_t\,dx = \int_\Omega w_t (w_{tt} - \Delta w)\,dx = 0.$$

Thm: If $u = u_t \equiv 0$ on $B(x_0, t_0) \times \{t=0\}$
then $u \equiv 0$ within the cone

Proof: $e_t = \frac{1}{2}\int_{B(x_0, t_0-t)} u_t^2(x,t) + |Du|^2 dx$, $0 \le t \le t_0$

$$\dot{e}(t) = \int_{B(x_0, t_0-t)} u_t u_{tt} + Du\, Du_t - \frac{1}{2}\int_{\partial B(x_0, t_0-t)} (u_t^2 + |Du|^2)$$

$$= \int_{B(x_0, t_0-t)} u_t (u_{tt} - \Delta u)$$

$$+ \int_{\partial B(x_0, t_0-t)} \frac{\partial u}{\partial \nu} u_t ds - \frac{1}{2}\int_{\partial B(x_0, t_0-t)} u_t^2 + |Du|^2 ds$$

$$= \int_{\partial B(x_0, t_0-t)} \frac{\partial u}{\partial t} u_t - \frac{1}{2}u_t^2 - \frac{1}{2}|Du|^2 ds$$

$\dot{e}(t) \le 0$, $e(t) \le e(0) = 0$.