to a harmonic fcn

Proof: $\{u_n(y)\}$ converge, $\forall \varepsilon>0, \exists N$ s.t $\forall n \geq N$

$$0 \leq u_m(y) - u_n(y) < \varepsilon, \; N < n \leq m$$

By Harnack

$$\sup_{\Omega'} |u_m(x) - u_n(x)| < C\varepsilon,$$

So $u_m(x)$ ~~is a~~ Cauchy sequence, $u_m \to u_0(x)$, M-V-P, harmonic #

Fundamental Solutions and Green's Representation Formula

Let $$\Phi(r) = \begin{cases} \frac{1}{2\pi} \log \frac{1}{r}, & n=2 \\ \frac{1}{(n-2)\omega_n} r^{2-n}, & n \geq 3 \end{cases}$$

Then $\Delta_y \Phi(|x-y|) = 0, \; y \neq x$

Moreover it satisfies

$$\int_{\partial B_r(a)} \frac{\partial \Phi}{\partial \nu_x}(x,a)\, dS_x = -1 \; ; \; \forall r > 0$$

Theorem I.2.10. Suppose Ω is a bounded domain in $\mathbb{R}^n$. and that $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. Then $\forall a \in \Omega$, there holds

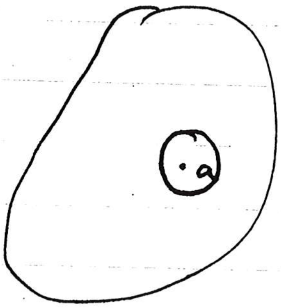
$$u(a) = \int_{\Omega} \Phi(|x-a|)(-\Delta u(x))\, dx + \int_{\partial\Omega} \left(\Phi \frac{\partial u}{\partial \nu_x} - u \frac{\partial \Phi}{\partial \nu_x}\right) dS_x \quad (27)$$

Remark: (i) $\forall a \in \Omega$, $\Phi(|x-a|)$ is integrable in Ω

(ii) For $a \notin \bar{\Omega}$, the expression in the right-hand-side gives zero

(iii) Letting $u \equiv 1$, $\int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu_x} dS_x = -1$

Proof: Applying Green's 2nd Identity formula to u and $\Phi(|x-a|)$ in $\Omega \setminus B_\varepsilon(a)$, $\varepsilon > 0$



$$\int_{\Omega\setminus B_\varepsilon(a)} (\Phi \Delta u - u \Delta \Phi) = \int_{\partial\Omega} \left(\Phi \frac{\partial u}{\partial n} - u \frac{\partial \Phi}{\partial n}\right) ds_x - \int_{\partial B_r(a)} \left(\Phi \frac{\partial u}{\partial n} - u \frac{\partial \Phi}{\partial n}\right) ds_x$$

Note that $\Delta \Phi = 0$ in $\Omega \setminus B_\varepsilon(a)$. Then

$$\int_\Omega \Phi \Delta u = \int_{\partial\Omega} \left(\Phi \frac{\partial u}{\partial n} - u \frac{\partial \Phi}{\partial n}\right) - \lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(a)} \left(\Phi \frac{\partial u}{\partial n} - u \frac{\partial \Phi}{\partial n}\right) ds_x$$

For $n \geq 3$,

$$\left| \int_{\partial B_\varepsilon(a)} \Phi \frac{\partial u}{\partial n} ds \right| = \left| \frac{1}{(n-2)\omega_n} \varepsilon^{2-n} \int_{\partial B_\varepsilon(a)} \frac{\partial u}{\partial n} ds \right|$$

$$\leq \frac{\varepsilon}{n-2} \sup_{\partial B_\varepsilon(a)} |Du| \to 0 \quad \text{as } \varepsilon \to 0$$

$$\int_{\partial B_\varepsilon(a)} u \frac{\partial \Phi}{\partial n} ds = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_\varepsilon(a)} u\, ds \to u(a) \quad \text{as } \varepsilon \to 0.$$

We get the same conclusion for $n=2$ in the same way. #

Remark:

Definition of Green's Function:

x be fixed and

Let $\phi^x(y)$ solve

$$\begin{cases} \Delta \phi^x = 0 & \text{in } \Omega \\ \phi^x = \Phi(y-x) & \text{on } \partial\Omega \end{cases}$$

$$G(x,y) = \Phi(x-y) - \phi^x(y) \quad (x, y \in \Omega,\ x \neq y).$$

Then $\cancel{0 = -\int \Phi(y-x) \, \partial\phi^x}$

$$-\int_\Omega \phi^x(y)\,\Delta u = \int_{\partial\Omega} u \frac{\partial \phi^x}{\partial \nu} - \phi^x \frac{\partial u}{\partial \nu} \qquad (28)$$

Adding (27) & (28) together, we obtain Green's Representation Formula:

$$u(x) = \int_\Omega G(x,y) f(y) - \int_{\partial\Omega} g \frac{\partial G}{\partial \nu}\, ds \qquad (29)$$

for
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Interpreting Green's Function

$$\begin{cases} -\Delta G = \delta_x & \text{in } \Omega \\ G = 0 & \text{on } \partial\Omega \end{cases}$$

Properties of Green's Function

① $G(x,y) = G(y,x)$, $x \neq y$

② $G(x,y) \in C^\infty_y(\Omega \setminus \{x\})$

③ $G > 0$ in $\Omega \setminus \{x\}$

<u>Task</u> (remaining):

① Show existence of ϕ^x

② prove that RHS of (29) Solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

We will prove ① much later. Let us first solve ②.

We consider

$$u_1(x) = \int_\Omega G(x,y) f(y)\,dy$$

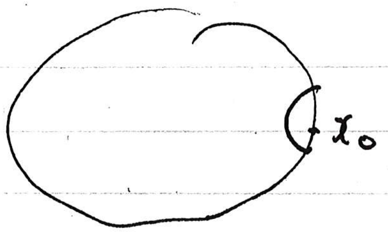
$$u_2(x) = -\int_{\partial\Omega} g \frac{\partial G}{\partial \nu_y}\, dS_y$$

<u>Thm</u>: If $f \in C^1(\Omega) \cap C(\bar{\Omega})$. Then u_1 solves $\begin{cases} -\Delta u_1 = f & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega \end{cases}$

Proof: $\Delta u_1 = \Delta \int_\Omega \underbrace{\Phi(x-y)}_{\Delta \to f(y)} - \phi^x(y) f(y)\,dy$

$= -f(x)$

It remains to prove $\lim_{x \to x_0, x \in \Omega} u(x) = 0$



$$I_1 = \int_{\Omega \setminus B_\rho(x_0)} G(x,y) f(y)\, dy$$

$$I_2 = \int_{\Omega \cap B_\rho(x_0)} G(x,y) f(y)\, dy$$

Let $M = \max_{\bar{\Omega}} |f(x)|$. Since

$$G(x,y) = \Phi(x-y) - \phi^x(y).$$

If $x \in B_\rho(x_0) \cap \Omega$, then $|I_2| \leq M \int_{\Omega \cap B_\rho(x_0)} \frac{dy}{|x-y|^{n-2}} + O(\rho^2)$

$$\leq c\rho^2$$

Consequently $\forall \varepsilon > 0$, $\exists \rho_0(\varepsilon) > 0$, $|I_2| < \frac{\varepsilon}{2}$, $0 < \rho \leq \rho_0$

For each fixed ρ, $0 < \rho \leq \rho_0$, we have

$$\lim_{x \to x_0,\, x \in \Omega} I_1(x) = 0$$

Since $G(x_0, y) = 0$ if $y \in \Omega \setminus B_\rho(x_0)$ and $G(x,y)$ is uniformly cts in $x \in B_{\rho/2}(x_0) \cap \Omega$ and $y \in \Omega \setminus B_\rho(x_0)$ #

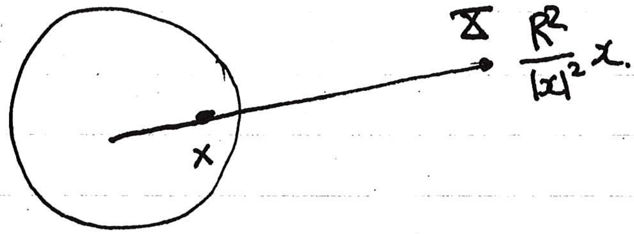
It remains to consider I_2. From now on, we consider.

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

We first prove the formula for $\Omega = B_R(0)$. Namely we compute the Green's Function in $B_R(x)$.

<u>Proposition</u>: $n \geq 3$, $G(x,y) = \Phi(x-y) - \Phi\left(\frac{R}{|x|}x - \frac{|x|}{R}y\right)$

$n = 2$, $G(x,y) = \Phi(x-y) - \Phi\left(\frac{R}{|x|}x - \frac{|x|}{R}y\right)$.



$$|y-x| = \frac{|x|}{R}|y-\bar{X}| = \left|\frac{|x|}{R}y - \frac{R}{|x|}x\right|, \quad \forall\, |y| = R$$

Corollary
$$\frac{\partial G}{\partial \nu} = \underbrace{\frac{R^2 - |x|^2}{\omega_n R |x-y|^n}}_{= K(x,y).}, \quad \forall x \in B_R, \ |y| = R.$$

Theorem: (Poisson Integral Formula). $\forall \varphi \in C(\partial B_R(0))$,

$$u(x) = \begin{cases} \int_{\partial B_R(0)} K(x,y)\varphi(y)\, dS_y, & |x| < R \\ \varphi(x) & |x| = R \end{cases}$$

Satisfies $u \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ and

$$\Delta u = 0 \quad \text{in } \Omega$$
$$u = \varphi \quad \text{on } \partial\Omega$$