Remark: 1) Result holds if $f \in C_c^\alpha(\mathbb{R}^n)$

2) Result holds if f has sufficient decay

$$|f(y)| \leq C(1+|y|)^{-l}, \quad l > 2$$

Exercise.
$$\int \frac{1}{|x-y|^{n-2}} \frac{1}{(1+|y|)^l} dy \lesssim \begin{cases} \frac{1}{(1+|x|)^{l-2}}, & 2 < l < n \\ \frac{1}{(1+|x|)^{n-2}} \log|x|, & l = n \\ \frac{1}{(1+|x|)^{n-2}}, & l > n. \end{cases}$$

Our next aim: obtain a general solution formula for

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Suppose $f, g \in C^0$, Prove: $\exists$ and uniqueness

Properties of harmonic functions

Mean-Value-~~Property~~: If $u \in C^2(\Omega)$, $\Delta u = 0$, Then

$$u(x) = \frac{1}{|\partial B_\rho(x)|} \int_{\partial B_\rho(x)} u \, d\sigma = \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} u \, d\sigma, \quad B_\rho(x) \subset \Omega$$

Proof:
$$\int_{B_\rho} \Delta u = \int_{\partial B_\rho} \frac{\partial u}{\partial \nu} = \rho^{n-1} \int_{|\omega|=1} \frac{\partial u}{\partial \rho}(x+\rho\omega) \, d\sigma$$
$$= \rho^{n-1} \frac{\partial}{\partial \rho} \int_{|\omega|=1} u(x+\rho\omega) \, d\sigma$$

$$\frac{\partial}{\partial \rho} \int_{|\omega|=1} u(x+\rho\omega)\, d\sigma = 0.$$

Integrating from 0 to r we obtain

$$\int_{|\omega|=1} u(x+r\omega)\, d\sigma = \int_{|\omega|=1} u\, d\sigma = u(x)\,\omega_n$$

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y)\, d\sigma.$$

Remark: 1) If $\Delta u \leq 0$, then $\frac{\partial}{\partial \rho} \int_{|\omega|=1} u(x+\rho\omega)\, d\sigma \leq 0$

$$\Rightarrow \quad \cancel{\int_{|\omega|=1}} \quad \frac{1}{|\partial B_r(0)|} \int_{\partial B_r} \leq u(x)$$

2) M-V-P characterizes harmonic fcn

Theorem: If $u \in C^0(\Omega)$ has M-V-P, then u is smooth and harmonic in Ω.

Proof: Choose molifier $\varphi \in C_0^\infty(B_1(0))$, $\int_{B_1} \varphi = 1$

$$\varphi_\varepsilon(z) = \frac{1}{\varepsilon^n} \varphi\left(\frac{z}{\varepsilon}\right)$$

$$\int_\Omega u(y)\, \varphi_\varepsilon(y-x) = \int u(x+y)\, \varphi_\varepsilon(y)\, dy = \frac{1}{\varepsilon^n} \int_{|y|<\varepsilon} u(y+x)\, \varphi\left(\frac{y}{\varepsilon}\right) dy$$

$$= \int_{|y|<1} u(x+\varepsilon y)\, \varphi(y)\, dy$$

$$= \int_0^1 r^{n-1}\, dr \int_{\partial B_1} u(x+\varepsilon r\omega)\, \varphi(r\omega)\, d\sigma$$

$$= \int_0^1 \varphi(r)\, r^{n-1}\, dr \int_{|\omega|=1} u(x+\varepsilon r\omega)\, d\sigma$$

$$= u(x)\,\omega_n \int_0^1 \varphi(r) r^{n-1} dr = u(x)$$

$$u(x) = \varphi_\varepsilon * u \qquad \forall\, x \in \Omega_\varepsilon = \{ y \in \Omega \mid d(y, \partial\Omega) > \varepsilon \}$$

Hence $u \in C^\infty(\Omega)$

$$\int_{B_r(x)} \Delta u = r^{n-1} \frac{\partial}{\partial r} \int_{|w|=1} u(x + rw)\, d\sigma$$

$$= r^{n-1} \frac{\partial}{\partial r} (\omega_n u(x)) = 0, \quad \forall\, B_r(x) \subset \Omega.$$

Hence $\Delta u = 0$ in Ω

#

Consequences of M-V-P.

Thm (M.P.). If $u \in C^0(\bar\Omega)$ satisfies M-V-P, then u assumes its maximum and mininum only on $\partial\Omega$ unless u is constant

Corollary: Uniqueness of $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$

Proof: Maximum. Set

$$\Sigma = \{ x \in \Omega \mid u(x) = M = \max u \} \subset \Omega$$

Σ is closed. Next we show that Σ is open.

#

Thm. (Gradient Estimate) $u \in C(\overline{B}_R)$ is harmonic in $B_R(x_0)$. Then

$$|Du(x_0)| \leq \frac{n}{R} \max |u|.$$

Proof: $\Delta u = 0 \Rightarrow u \in C^\infty(B_R)$. $\quad \Delta(D_{x_i} u) = 0$.

$$D_{x_i} u(x_0) = \frac{n}{\omega_n R^n} \int_{B_R(x_0)} D_{x_i} u = \frac{n}{\omega_n R^n} \int_{\partial B_R} u\, \nu_i\, d\sigma$$

which implies

$$|D_{x_i} u(x_0)| \le \frac{n}{\omega_n R^n} \max|u| \; \omega_n R^{n-1} \le \frac{n}{R} \max|u|$$

Corollary: $\Delta u = 0$ in R, $|u| \le (1+|y|)^l$, $l<1$ $\Rightarrow u \equiv C$

Lemma I If $u \ge 0$, $\Delta u = 0$. then

$$|Du(x_0)| \le \frac{n}{R} u(x_0)$$

Corollary: $\Delta u = 0$ in R^n, $u \le C$ $\Rightarrow$ $u \equiv C$

Thm $u \in C(\overline{B_R})$, $\Delta u = 0$ in B_R

Then $$|D^\alpha u(x_0)| \le \frac{n^m e^{m-1} m!}{R^m} \max|u|$$

Proof: By induction. $m=1$ ✓

Assume it holds for m. Consider $m+1$.

For $0<\theta<1$, $r = (1-\theta)R$

$$|D^{m+1} u(x_0)| \le \frac{n}{r} \max_{\overline{B}_r} |D^m u|$$

$$\max_{B_r} |D^m u| \le \frac{n^m e^{m-1} m!}{(R-r)^m} \max|u|$$

$$|D^{m+1} u(x_0)| \le \frac{n}{r} \frac{n^m e^{m-1} m!}{(R-r)^m} \max|u| = \frac{n^{m+1} e^{m-1} m!}{R^{m+1} \theta^m (1-\theta)} \max_{B_R} |u|$$

Take $\theta = \frac{m}{m+1}$, $\frac{1}{\theta^m(1-\theta)} = \left(1+\frac{1}{m}\right)^m (m+1) < e(m+1)$ #

Thm. Harmonic function is analytic

Proof: $u(x+h) = u(x) + \sum_{i=1}^{m-1} \frac{1}{i!} \left[(h_1 \partial_1 + \cdots + h_n \partial_n)^i u\right](x) + R_m(h)$

$$R_m(h) = \frac{1}{m!} \left[(h_1 \partial_1 + \cdots + h_n \partial_n)^m u\right](x_1 + \theta h_1, \cdots, x_n + \theta h_n)$$

$$|R_m(h)| \le \frac{1}{m!} |h|^m n^m \frac{n^m e^{m-1} m!}{R^m} \max_{B_R} |u|$$

$$\le \left(\frac{|h| n^2 e}{R}\right)^m \max |u|$$

$\forall h$, $|h| n^2 e < R/2$, then $R_m(h) \to 0$ as $m \to +\infty$

Thm. (Harnack Inequality).

$\Delta u = 0$ in Ω.

$$\frac{1}{C} u(y) \le u(x) \le C u(y), \quad \forall x, y \in K$$

By M-V-P. $B_{4R}(x_0) \subset \Omega$,

$$\frac{1}{c} u(y) \le u(x) \le c u(y), \quad x, y \in B_R(x_0)$$

$c = c(n)$.

Now for any given compact subset K, take $x_1, \cdots, x_N \in K$, such that $\{B_R(x_i)\}$ covers K with $4R < \mathrm{dist}(K, \partial\Omega)$, $C = c^N$

Thm (Harnack Convergence Thm) Let u_n be a monotone increasing sequence of harmonic fcns in Ω. Suppose at some point $y \in \Omega$ $\{u_n(y)\}$ is bdd. Then the sequence converges uniformly $\forall \Omega' \subset$

to a harmonic fcn

Proof : $\{u_n(y)\}$ converge, $\forall \varepsilon>0, \exists N$ s.t $\forall n \geq N$

$$0 \leq u_m(y)-u_n(y) < \varepsilon, \ N < n \leq m$$

By Harnack

$$\sup_{\Omega'} |u_m(x)-u_n(x)| < C\varepsilon,$$

So $u_m(x)$ ~~is a~~ Cauchy sequence, $u_m \to u_0(x)$, M-V-P, harmonic #