1. (a) Let $\xi \in \partial \Omega$ and $w(x)$ be a barrier on $\Omega_1 \subset \subset \Omega$: (i) $w$ is superharmonic in $\Omega_1$; (ii) $w > 0$ in $\Omega_1 \setminus \{\xi\}$; $w(\xi) = 0$. Show that $w$ can be extended to a barrier in $\Omega$. (b) Let $\Omega = \{x^2 + y^2 < 1\} \setminus \{-1 \leq x \leq 0, y = 0\}$. Show that the function $w := -\text{Re} \left( \frac{1}{\log |z| + \eta} \right)$ is a local barrier at $\xi = 0$.

2. (a) Show that the problem of minimizing energy
\[ I[u] = \int_J x^2 |u'(x)|^2 \, dx, \]
for $u \in C(\bar{J})$ with piecewise continuous derivatives in $J := (-1, 1)$, satisfying the boundary conditions $u(-1) = 0, u(1) = 1$, is not attained. (b) Consider the problem of minimizing the energy
\[ I[u] = \int_0^1 (1 + |u'(x)|^2)^2 \, dx \]
for all $u \in C^1((0, 1)) \cap C([0, 1])$ satisfying $u(0) = 0, u(1) = 1$. Show that the minimum is 1 and is not attained.

3. Discuss Dirichlet Principle for
\[ \begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega \end{cases} \]

4. Let
\[ \Phi(x - y, t) = (4\pi t)^{-n/2} e^{-\frac{|x - y|^2}{4t}} \]
(a) Show that there exists a generic constant $C_n$ such that
\[ \Phi(x - y, t) \leq C_n |x - y|^{-n} \]
Hint: maximize the function in $t$.
(b) Let $n = 1$ and $f(x)$ be a bounded measurable function such that $f(x_0 -)$ and $f(x_0 +)$ exists. Show that
\[ \lim_{t \to 0} \int \Phi(x - x_0, t)f(y) \, dy = \frac{1}{2} (f(x_0 -) + f(x_0 +)) \]
(c) Let $u$ satisfy
\[ u_t = \Delta u, \quad u(x, 0) = f(x) \]
Suppose that $f$ is continuous and has compact support. Show that $\lim_{t \to +\infty} u(x, t) = 0$ for all $x$.

5. Derive a solution formula for
\[ u_t = \Delta u + cu + \phi(x, t), t > 0; u(x, 0) = g(x) \]

6. Consider the following general parabolic equation
\[ L[u] = a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u - u_t \]
where
\[ 0 < C_1 < a(x, t) < C_2, |b(x, t)| \leq C_3, c(x, t) \leq C_4 \]
(a) Show that $L[u] \geq 0$, then
\[ \max_{\Omega_T} u \leq e^{C_1T} \max_{\partial \Omega} u^+ \]
Here $\Omega_T = (0, L) \times (0, T), \partial \Omega_T = \partial \Omega \setminus ((0, L) \times \{T\})$ and $u^+ = \max(u, 0)$.
Hint: consider the function $v := ue^{-C_1t}$
(b) Prove the uniqueness of the initial value problem
\[ \begin{cases} L[u(x, t)] = f(x, t), & \text{in } \Omega_T; \\ u(x, 0) = \phi(x), x \in \Omega; \\ u(x, t) = g(x), & x \in \partial \Omega, t \in (0, T) \end{cases} \]

7. (a) Use d’Alembert’s formula to show that Maximum Principle does not hold for wave equation, i.e.,
\[ \begin{align*} u_{tt} &= u_{xx}, & -L < x < L, 0 < t < T \\ u(x, 0) &= f(x), & u_t(x, 0) = g(x) \end{align*} \]
\[ \max_{\overline{U}_t} u(x, t) > \max_{\overline{U}_t} u(x, t) \]

Hint: Let \( f = 0 \) and \( g \in C_0^\infty([\!-\!1, 1]) \), \( g \geq 0 \) and choose \( T \) small.

(b) Let \( u \) solve the initial value problem for the wave equation in one dimension
\[
\begin{cases}
  u_{tt} = u_{xx} & \text{in } \mathbb{R} \times (0, +\infty) \\
  u(x, 0) = f(x), \ u_t(x, 0) = g(x)
\end{cases}
\]
where \( f \) and \( g \) have compact support in \( \mathbb{R} \). Let \( k(t) = \frac{1}{2} \int u_t(x, t)^2 \, dx \) be the potential energy and \( p(t) = \frac{1}{2} \int u_x^2(x, t) \, dx \) be the potential energy. Show that
(a) \( k(t) + p(t) \) is constant in \( t \); (b) \( k(t) = p(t) \) for all large enough time \( t \).

8. Find the explicit formula for the following wave equation
\[
u_{tt} - \Delta u = 0, x \in \mathbb{R}^3, t > 0; u(x, 0) = 0, u_t(x, 0) = \begin{cases} 
  1, & \text{for } |x| < 1; \\
  0, & \text{for } |x| > 1
\end{cases}
\]

Hint: \( u \) is radially symmetric.