Method of separation of variables for inhomogeneous PDE

Let us apply the method of separation of variables to

$$\begin{cases} u_t = k u_{xx} + f(x,t), & 0<x<l,\ t>0 \\ u(0,t)=h(t), & u(l,t)=j(t) \\ u(x,0)=\phi(x) \end{cases}$$

Suppose $f=\phi=0$ first. Let

$$u = \sum_{n=1}^{+\infty} u_n(t)\sin\left(\frac{n\pi x}{l}\right)$$

Formally

$$\frac{du}{dt} = \sum_{n=1}^{\infty} \frac{du_n}{dt}\sin\left(\frac{n\pi x}{l}\right) \qquad \text{———} \ (*)$$

$$k u_{xx} = \sum_{n=1}^{\infty} u_n \left(-\left(\frac{n\pi}{l}\right)^2\right)\sin\left(\frac{n\pi x}{l}\right) \ \text{—} \ (**)$$

$$u_t - k u_{xx} = \sum \left(\frac{du_n}{dt} + k\left(\frac{n\pi}{l}\right)^2 u_n\right)\sin\frac{n\pi x}{l}$$

$$\Rightarrow \frac{du_n}{dt} + k\left(\frac{n\pi}{l}\right)^2 u_n = 0 \Rightarrow u_n = c_n e^{-k\left(\frac{n\pi}{l}\right)^2 t}$$

$$u = \sum c_n e^{-k\left(\frac{n\pi}{l}\right)^2 t}\sin\left(\frac{n\pi x}{l}\right)$$

BC $u(0,t)=h(t)$, $u(l,t)=j(t)$ $\Longrightarrow$ always not satisfied.

The problem is: $\dfrac{d^2}{dx^2}\sum_{n=1} u_n \sin\left(\frac{n\pi x}{l}\right) \neq \sum_{n=1}^{\infty} u_n\left(-\left(\frac{n\pi}{l}\right)^2\right)\sin\left(\frac{n\pi x}{l}\right)$

You can't differentiate Fourier expansion term by term.

<u>the right way</u>: Expand

$$u(x,t) = \sum_{n=1}^{+\infty} u_n(t)\sin\left(\frac{n\pi}{l}x\right), \quad u_n(t) = \frac{2}{l}\int_0^l u(x,t)\sin\left(\frac{n\pi}{l}x\right)$$

$$\frac{du}{dt} = \sum_{n=1}^{\infty} v_n(t)\sin\left(\frac{n\pi}{l}x\right), \quad v_n(t) = \frac{2}{l}\int_0^l \frac{du}{dt}\sin\left(\frac{n\pi}{l}x\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{+\infty} w_n(t) \sin\left(\frac{n\pi}{\ell}x\right), \quad w_n(t) = \frac{2}{\ell}\int_0^\ell \frac{\partial^2 u}{\partial x^2} \sin\left(\frac{n\pi x}{\ell}\right)$$

$$f(x,t) = \sum_{n=1}^{+\infty} f_n(t) \sin\left(\frac{n\pi}{\ell}x\right)$$

Then we have

$$v_n(t) = k w_n(t) + f_n(t)$$

$$v_n(t) = \frac{d}{dt}\left(\frac{2}{\ell}\int_0^\ell u \sin\left(\frac{n\pi x}{\ell}\right)dx\right) = \frac{d}{dt}u_n(t)$$

$$w_n(t) = \frac{2}{\ell}\int_0^\ell \left[\frac{\partial^2 u}{\partial x^2}\sin\left(\frac{n\pi x}{\ell}\right) + u\frac{\partial^2}{\partial x^2}\sin\left(\frac{n\pi x}{\ell}\right) + \left(\frac{n\pi}{\ell}\right)^2\left(\sin\frac{n\pi x}{\ell}\right)u(x,t)\right]$$

$$= \frac{2}{\ell}\left(u_x \sin\frac{n\pi x}{\ell} - u\left(\frac{n\pi}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)\right)\Big|_0^\ell - \left(\frac{n\pi}{\ell}\right)^2 \frac{2}{\ell}u_n(t)$$

$$= \frac{2}{\ell}\cdot\frac{n\pi}{\ell}\left(u(0,t) - u(\ell,t)\cos n\pi\right) - \left(\frac{n\pi}{\ell}\right)^2 \frac{2}{\ell}u_n(t)$$

$$= \frac{2}{\ell}\frac{n\pi}{\ell}\left(h(t) - (-1)^n j(t)\right) - \left(\frac{n\pi}{\ell}\right)^2 u_n(t)$$

Thun

$$\frac{du_n}{dt} + k\left(\frac{n\pi}{\ell}\right)^2 u_n = \frac{2k\,n\pi}{\ell^2}\left(h(t) - (-1)^n j(t)\right) + f_n(t) \quad \text{— (1)}$$

Now initial condition

$$u(x,0) = \phi(x) = \sum_{n=1}^{+\infty} \phi_n \sin\left(\frac{n\pi}{\ell}x\right)$$

$$u_n(0) = \phi_n \qquad \text{— (2)}$$

Solving (1)-(2) together, we obtain u.

$$u_n(t) = \phi_n e^{-\lambda_n k t} + \frac{2n\pi}{\ell^2}k\int_0^t e^{-\lambda_n k(t-s)}\left[h(s) - (-1)^n j(s)\right]ds$$

As a second case, we can solve inhomogeneous wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(0,t) = h(t), \quad u(l,t) = k(t) \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \end{cases}$$

$u(x,t) = \sum u_n(t) \sin\left(\frac{n\pi x}{l}\right)$, $\quad u_n(t) = \frac{2}{l}\int_0^l u \sin\left(\frac{n\pi x}{l}\right) dx$

$u_{tt} = \sum v_n(t) \sin\left(\frac{n\pi x}{l}\right)$, $\quad v_n(t) = \frac{2}{l}\int_0^l u_{tt} \sin\left(\frac{n\pi}{l}x\right) dx$

$u_{xx} = \sum \omega_n(t) \sin\left(\frac{n\pi x}{l}\right)$, $\quad \omega_n(t) = \frac{2}{l}\int_0^l u_{xx} \sin\left(\frac{n\pi}{l}x\right) dx$

$f(x,t) = \sum f_n(t) \sin\left(\frac{n\pi x}{l}\right)$

$$\begin{cases} \dfrac{d^2 u_n}{dt^2} + c^2 \lambda_n u_n(t) = \dfrac{2n\pi}{l^2}\left[h(t) - (-1)^n k(t)\right] + f_n(t) \\ u_n(0) = \phi_n, \quad u_n'(0) = \psi_n \end{cases}$$

Method of shifting Data:

Suppose
$$\begin{cases} u_t = k u_{xx} + f(x), \quad 0 < x < l \\ u(0,t) = u_0, \quad u(l,t) = u_l \\ u(0,t) = \phi(x) \end{cases} \qquad (3)$$

1) Solve the steady-state problem first:

$$\begin{cases} k u^0_{xx} + f(x) = 0, \quad 0 < x < l \\ u^0(0) = u_0, \quad u^0(l) = u_l \end{cases} \qquad \text{— } (4)$$

2) $V(x,t) = u(x,t) - u^0(x)$ Then
$$\begin{cases} V_t = k V_{xx} \\ V(x,0) = \phi(x) - u^0(x) \\ V(0,t) = V(l,t) = 0 \end{cases}$$

Problem (3) is called the steady-state problem of (3). So the solution to (3) is given by

$$u(x,t) = U^{0}(x) + \sum_{n=1}^{+\infty} a_n e^{-k\lambda_n t} \sin\left(\frac{n\pi x}{l}\right)$$

with
$$a_n = \frac{2}{l}\int_0^l \left(\phi(x) - U^{0}(x)\right) \sin\left(\frac{n\pi x}{l}\right) dx$$

Conclusion: As $t \to +\infty$, $\quad u(x,t) \longrightarrow U^{0}(x)$.

<u>Example 1</u>: Solve

$$\begin{cases} u_t = k u_{xx}, & 0 < x < l \\ u(0,t) = e^t, \ u(l,t) = 0 \\ u(x,0) = 1 \end{cases}$$

Solution: $u(x,t) = \sum_{n=1}^{+\infty} u_n(t) \sin\left(\frac{n\pi}{l}x\right)$, $\quad u_n(0) = \frac{2}{l}\int_0^l \sin\left(\frac{n\pi}{l}x\right)dx = \frac{2}{n\pi}\left(1 - (-1)^n\right)$.

So we have

$$\begin{cases} \dfrac{du_n}{dt} + \lambda_n k u_n = \dfrac{2kn\pi}{l} e^t \\ u_n(0) = \dfrac{2}{n\pi}\left(1 - (-1)^n\right) \end{cases}$$

$$u_n = u_n(0) e^{-\lambda_n k t} + \frac{2n\pi}{l^2} k \int_0^t e^{-\lambda_n k (t-s)} e^{s} ds$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right) e^{-\lambda_n k t} + \frac{2n\pi}{l^2} k \cdot e^{-\lambda_n k t} \cdot \frac{1}{1+\lambda_n k}\left[e^{(\lambda_n k + 1)t} - 1\right]$$

#

7. Let $\int_{-\pi}^{\pi} [|f(x)|^2 + |g(x)|^2]\, dx$ be finite, where $g(x) = f(x)/(e^{ix} - 1)$. Let c_n be the coefficients of the full complex Fourier series of $f(x)$. Show that $\sum_{n=-N}^{N} c_n \to 0$ as $N \to \infty$.
8. Prove that both integrals in (12) tend to zero.
9. Fill in the missing steps in the proof of uniform convergence.
10. Prove the theorem on uniform convergence for the case of the Fourier sine series and for the Fourier cosine series.
11. Prove that the full Fourier series of the function $|x|$ in the interval $(-\pi, \pi)$ converges uniformly to $|x|$ in $[-\pi, \pi]$.
12. Show that if $f(x)$ is a C^1 function in $[-\pi, \pi]$ and if $\int_{-\pi}^{\pi} f(x)\, dx = 0$, then $\int_{-\pi}^{\pi} |f|^2\, dx \le \int_{-\pi}^{\pi} |f'|^2\, dx$. (*Hint:* Use Parseval's equality.)
13. A very slick proof of the pointwise convergence of Fourier series, due to P. Chernoff (*American Mathematical Monthly,* May 1980), goes as follows.
    (a) Let $f(x)$ be a C^1 function of period 2π. First show that we may as well assume that $f(0) = 0$ and we need only show that the Fourier series converges to zero at $x = 0$.
    (b) Let $g(x) = f(x)/(e^{ix} - 1)$. Show that $g(x)$ is a continuous function.
    (c) Let C_n be the (complex) Fourier coefficients of $f(x)$ and D_n the coefficients of $g(x)$. Show that $D_n \to 0$.
    (d) Show that $C_n = D_{n-1} - D_n$ so that the series $\sum C_n$ is telescoping.
    (e) Deduce that the Fourier series of $f(x)$ at $x = 0$ converges to zero.

5.6 INHOMOGENEOUS BOUNDARY CONDITIONS

In this section we consider problems with sources given at the boundary. We shall see that naive use of the separation of variables technique will not work.

Let's begin with the *diffusion equation with sources at both endpoints.*

$$\begin{gathered} u_t = ku_{xx} \qquad 0 < x < l, \quad t > 0 \\ u(0, t) = h(t) \qquad u(l, t) = j(t) \\ u(x, 0) \equiv 0. \end{gathered} \tag{1}$$

A separated solution $u = X(x)T(t)$ just will not fit the boundary conditions. So we try a slightly different approach.

EXPANSION METHOD

We already know that for the corresponding homogeneous problem the correct expansion is the Fourier sine series. For each t, we certainly can expand

$$\boxed{u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{l}} \tag{2}$$

for *some* coefficients $u_n(t)$, because the completeness theorems guarantee that any function in $(0, l)$ can be so expanded. The coefficients are necessarily given by

$$u_n(t) = \frac{2}{l} \int_0^l u(x, t) \sin \frac{n\pi x}{l}\, dx. \tag{3}$$

You may object that each term in the series vanishes at both endpoints and thereby violates the boundary conditions. The answer is that we simply do not insist that the series converge at the endpoints but only inside the interval. In fact, we are exactly in the situation of Theorems 3 and 4 but not of Theorem 2 of Section 5.4.

Now differentiating the series (2) term by term, we get

$$0 = u_t - ku_{xx} = \sum \left[\frac{du_n}{dt} + ku_n(t) \left(\frac{n\pi}{l} \right)^2 \right] \sin \frac{n\pi x}{l}.$$

So the PDE seems to require that $du_n/dt + k\lambda_n u_n = 0$, so that $u_n(t) = A_n e^{k\lambda_n t}$. There is no way for this to fit the boundary conditions. Our method fails! What's the moral? It is that you can't differentiate term by term. See Example 3

Let's start over again but avoid direct differentiation of the Fourier series. The expansion (2) with the coefficients (3) must be valid, by the completeness theorem 5.4.3, say, provided that $u(x, t)$ is a continuous function. Clearly, the initial condition requires that $u_n(0) = 0$. If the derivatives of $u(x, t)$ are also continuous, let's expand them, too. Thus

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} v_n(t) \sin \frac{n\pi x}{l} \tag{4}$$

with

$$v_n(t) = \frac{2}{l} \int_0^l \frac{\partial u}{\partial t} \sin \frac{n\pi x}{l}\, dx = \frac{du_n}{dt}. \tag{5}$$

The last equality is valid since we can differentiate under an integral sign if the new integrand is continuous (see Section A.3). We also expand

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} w_n(t) \sin \frac{n\pi x}{l} \tag{6}$$

with the coefficients

$$w_n(t) = \frac{2}{l} \int_0^l \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi x}{l}\, dx. \tag{7}$$

By Green's second identity (5.3.3) the last expression equals

$$\frac{-2}{l} \int_0^l \left(\frac{n\pi}{l} \right)^2 u(x, t) \sin \frac{n\pi x}{l}\, dx + \frac{2}{l} \left(u_x \sin \frac{n\pi x}{l} - \frac{n\pi}{l} u \cos \frac{n\pi x}{l} \right) \Bigg|_0^l.$$

Here come the boundary conditions. The sine factor vanishes at both ends. The last term will involve the boundary conditions. Thus

$$w_n(t) = -\lambda_n u_n(t) - 2n\pi l^{-2}(-1)^n j(t) + 2n\pi l^{-2} h(t), \tag{8}$$

where $\lambda_n = (n\pi/l)^2$. Now by (5) and (7) the PDE requires

$$v_n(t) - kw_n(t) = \frac{2}{l}\int_0^l (u_t - ku_{xx}) \sin\frac{n\pi x}{l}\,dx = \int_0^l 0 = 0.$$

So from (5) and (8) we deduce that $u_n(t)$ satisfies

$$\frac{du_n}{dt} = k\{-\lambda_n u_n(t) - 2n\pi l^{-2}[(-1)^n j(t) - h(t)]\}. \tag{9}$$

This is just an ordinary differential equation, to be solved together with the initial condition $u_n(0) = 0$ from (1). The solution of (9) is

$$\boxed{u_n(t) = Ce^{-\lambda_n kt} - 2n\pi l^{-2}k \int_0^t e^{-\lambda_n k(t-s)}\,[(-1)^n j(s) - h(s)]\,ds.} \tag{10}$$

As a second case, let's solve the *inhomogeneous wave problem*

$$\begin{gathered} u_{tt} - c^2 u_{xx} = f(x, t) \\ \boldsymbol{u(0, t) = h(t) \qquad u(l, t) = k(t)} \\ u(x, 0) = \phi(x) \qquad u_t(x, 0) = \psi(x). \end{gathered} \tag{11}$$

Again we expand everything in the eigenfunctions of the corresponding homogeneous problem:

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin\frac{n\pi x}{l},$$

$u_{tt}(x, t)$ with coefficients $v_n(t)$, $u_{xx}(x, t)$ with coefficients $w_n(t)$, $f(x, t)$ with coefficients $f_n(t)$, $\phi(x)$ with coefficients ϕ_n, and $\psi(x)$ with coefficients ψ_n. Then

$$v_n(t) = \frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial t^2} \sin\frac{n\pi x}{l}\,dx = \frac{d^2 u_n}{dt^2}$$

and, just as before,

$$\begin{aligned} w_n(t) &= \frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{l}\,dx \\ &= -\lambda_n u_n(t) + 2n\pi l^{-2}[h(t) - (-1)^n k(t)]. \end{aligned}$$

From the PDE we also have

$$v_n(t) - c^2 w_n(t) = \frac{2}{l}\int_0^l (u_{tt} - c^2 u_{xx}) \sin\frac{n\pi x}{l}\,dx = f_n(t).$$

Therefore,

$$\frac{d^2u_n}{dt^2} + c^2\lambda_n u_n(t) = -2n\pi l^{-2}[(-1)^n k(t) - h(t)] + f_n(t) \tag{12}$$

with the initial conditions

$$u_n(0) = \phi_n \qquad u_n'(0) = \psi_n.$$

The solution can be written explicitly (see Exercise 11).

METHOD OF SHIFTING THE DATA

By subtraction, the data can be shifted from the boundary to another spot in the problem. *The boundary conditions can be made homogeneous by subtracting any known function that satisfies them.* Thus for the problem (11) treated above, the function

$$\mathcal{U}(x, t) = \left(1 - \frac{x}{l}\right)h(t) + \frac{x}{l}\,k(t)$$

obviously satisfies the BCs. If we let

$$v(x, t) = u(x, t) - \mathcal{U}(x, t),$$

then $v(x, t)$ satisfies the same problem but with zero boundary data, with initial data $\phi(x) - \mathcal{U}(x, 0)$ and $\psi(x) - \mathcal{U}_t(x, 0)$, and with right-hand side f replaced by $f - \mathcal{U}_{tt}$.

The boundary condition and the differential equation can simultaneously be made homogeneous by subtracting any known function that satisfies them. One case when this can surely be accomplished is the case of "stationary data" when h, k, and $f(x)$ all are independent of time. Then it is easy to find a solution of

$$-c^2\mathcal{U}_{xx} = f(x) \qquad \mathcal{U}(0) = h \qquad \mathcal{U}(l) = k.$$

Then $v(x, t) = u(x, t) - \mathcal{U}(x)$ solves the problem with zero boundary data, zero right-hand side, and initial data $\phi(x) - \mathcal{U}(x)$ and $\psi(x)$.

For another example, take problem (11) for a simple periodic case:

$$f(x, t) = F(x)\cos\omega t \qquad h(t) = H\cos\omega t \qquad k(t) = K\cos\omega t,$$

that is, with the same time behavior in all the data. We wish to subtract a solution of

$$\mathcal{U}_{tt} - c^2\mathcal{U}_{xx} = F(x)\cos\omega t$$
$$\mathcal{U}(0, t) = H\cos\omega t \qquad \mathcal{U}(l, t) = K\cos\omega t.$$

A good guess is that $\mathcal{U}$ should have the form $\mathcal{U}(x, t) = \mathcal{U}_0(x)\cos\omega t$. This will happen if $\mathcal{U}_0(x)$ satisfies

$$-c^2\mathcal{U}_0 - \omega^2\mathcal{U}_0'' = F(x) \qquad \mathcal{U}_0(0) = H \qquad \mathcal{U}_0(l) = K. \qquad \square$$

There is also the method of Laplace transforms, which can be found in Section 12.5.