Two examples on Method of Reflection

Ex. 1

$$\begin{cases} u_{tt} = c^2 u_{xx} + t, & x>0 \\ u(x,0) = u_t(x,0) = 0, & x>0 \end{cases}$$

Solution: $f(x,t) = t,\ x>0$, $\phi(x) = \psi(x) = 0,\ x>0$

$$f_{ext}(x,t) = \begin{cases} t, & x>0 \\ -f(-x,t) = -t, & x<0 \end{cases}, \quad \phi_{ext} = \psi_{ext} = 0$$

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{ext}(y,s)\, dy\, ds$$

Case 1 $x > ct$

In this case, $x+c(t-s) > x-c(t-s) > 0$, $f_{ext}(y,s) = s$

$$u(x,t) = \frac{1}{2c} \int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} s\, dy \right) ds$$

$$= \frac{1}{2c} \int_0^t 2c(t-s)s\, ds$$

$$= \int_0^t (t-s)s\, ds = \frac{t^3}{6}$$

Case 2. $x < ct$

In this case, $x-c(t-s) = x-ct+cs \le 0$ if $s \le t - \frac{x}{c}$

So

$$u(x,t) = \underbrace{\frac{1}{2c}\int_0^{t-\frac{x}{c}} \int_{x-c(t-s)}^{x+c(t-s)} f_{ext}(y,s)\,dy\,ds}_{I_1} + \underbrace{\frac{1}{2c}\int_{t-\frac{x}{c}}^{t} \int_{x-c(t-s)}^{x+c(t-s)} f_{ext}(y,s)\,dy\,ds}_{I_2}$$

For I_2, $x-c(t-s)>0$, $f_{ext}(y,s)=s$

$$I_2 = \frac{1}{2c}\int_{t-\frac{x}{c}}^{t} \int_{x-c(t-s)}^{x+c(t-s)} s\,dy\,ds$$

$$= \frac{1}{2c}\int_{t-\frac{x}{c}}^{t} 2c(t-s)s\,ds = \int_{t-\frac{x}{c}}^{t} (t-s)s\,ds$$

$$= t\frac{s^2}{2} - \frac{s^3}{3}\Big|_{t-\frac{x}{c}}^{t} = \frac{t^3}{6} - \left[t\frac{(t-\frac{x}{c})^2}{2} - \frac{(t-\frac{x}{c})^3}{3}\right]$$

For I_1, $x-c(t-s)<0$, so

$$u(x,t) = \frac{1}{2c}\int_0^{t-\frac{x}{c}} \int_{c(t-s)-x}^{x+c(t-s)} f_{ext}(y,s)\,dy\,ds$$

$$= \frac{1}{2c}\int_0^{t-\frac{x}{c}} \int_{c(t-s)-x}^{x+c(t-s)} s\,dy\,ds$$

$$= \frac{1}{2c}\int_0^{t-\frac{x}{c}} (2x)\,s\,ds = \frac{x}{c}\frac{1}{2}(t-\frac{x}{c})^2$$

Thus

$$u(x,t) = \frac{t^3}{6} - \left[(t-\frac{x}{c})^3\frac{1}{2} - \frac{1}{3}(t-\frac{x}{c})^3\right]$$

$$= \frac{t^3}{6} - \frac{1}{6}(t-\frac{x}{c})^3$$

So $u(x,t) = \begin{cases} \frac{t^3}{6}, & x>ct \\ \frac{t^3}{6} - \frac{1}{6}(t-\frac{x}{c})^3, & x<ct \end{cases}$

Ex. 2

$$\begin{cases} u_{tt} = u_{xx}, \quad 0<x<1 \\ u(x,0)=0, \; u_t(x,0)=1 \\ u(0,t)=u(1,t)=0 \end{cases}, \quad \text{Find } u(\tfrac{1}{2}, \tfrac{7}{4}), \quad u(\tfrac{1}{2}, 3)$$

Sol'n: $\phi = 0, \; \psi = 0$

$$\phi_{ext} = 0, \quad \psi_{ext} = \begin{cases} 1, & 0<x<1 \\ -1, & -1<x<0 \\ \psi_{ext}(x \pm 2) & \end{cases}$$

So $u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{ext}(s)\, ds$

$$u(\tfrac{1}{2}, \tfrac{7}{4}) = \frac{1}{2} \int_{\frac{1}{2}-\frac{7}{4}}^{\frac{1}{2}+\frac{7}{4}} \psi_{ext}(s)\,ds = \frac{1}{2} \int_{-\frac{5}{4}}^{\frac{9}{4}} \psi_{ext}(s)\,ds$$

$$= \frac{1}{2} \left[\int_{-\frac{5}{4}}^{-\frac{5}{4}+2} + \int_{-\frac{5}{4}+2}^{\frac{9}{4}-2} + \int_{\frac{9}{4}-2}^{\frac{9}{4}} \right] (\psi_{ext}(s))\, ds$$

$$= \frac{1}{2} \int_{-\frac{5}{4}+2}^{\frac{9}{4}-2} \psi_{ext}(s)\,ds = \frac{1}{2} \int_{\frac{3}{4}}^{\frac{1}{4}} \psi_{ext}(s)\,ds$$

$$= -\frac{1}{2} \int_{\frac{1}{4}}^{\frac{3}{4}} \psi_{ext}(s)\,ds = -\frac{1}{2} \int_{\frac{1}{4}}^{\frac{3}{4}} 1\, ds = -\frac{1}{4}$$

$$u(\tfrac{1}{2}, 3) = \frac{1}{2} \int_{\frac{1}{2}-3}^{\frac{1}{2}+3} \psi_{ext}(s)\,ds = \frac{1}{2} \int_{\frac{1}{2}-3+2}^{\frac{1}{2}+3-2}$$

$$= \frac{1}{2} \int_{+\frac{3}{2}}^{\frac{3}{2}} \psi_{ext}(s)\,ds = 0$$

[Another way, $\frac{1}{2}-3 = \frac{1}{2}+3-(6)$ so this is zero]

For odd functions with period $2L$, we have

$$\int_a^{a+2L} \psi = 0, \quad \int_{a-2L}^{a} \psi = 0, \quad \int_{-a}^{a} \psi = 0$$