1. (30pts) Put the following three problems in standard Sturm-Liouville form, identify the weight function \( w(x) \), and calculate the eigenvalues and eigenfunctions. (You can use Bessel function of order zero \( J_0(r) \): \( J_0'' + \frac{1}{r} J_0' + J_0 = 0, J_0(0) = 1 \) ) Write down the general formula for the expansion of a general function \( f(x) \) in terms of the eigenfunctions.

\[(a) \quad x^2 X'' + x X' + \lambda X = 0, 1 < x < 2, \quad X(1) = 0, \quad X(2) = 0\]

\[(b) \quad X'' - 2X' + \lambda X = 0, 0 < x < 1; \quad X(0) = 0, \quad X(1) = 0; \quad (c) \quad X'' + \frac{1}{x} X' + \lambda X = 0, 0 < x < 1, X'(1) + X(1) = 0\]

2. (10pts) Solve the following diffusion equation

\[
\begin{cases}
  u_t = x^2 u_{xx} + xu_x, & 1 < x < 2 \\
  u(x, 0) = 1, \\
  u(1, t) = u(2, t) = 0
\end{cases}
\]

You are allowed to use results in Problem 1.

3. (10pts) Solve the following wave equation

\[
\begin{cases}
  u_{tt} = u_{xx} - 2u_x, & 0 < x < 1 \\
  u(x, 0) = x, u_t(x, 0) = 0 \\
  u(0, t) = u(1, t) = 0
\end{cases}
\]

You are allowed to use results in Problem 1.

4. (10pts) Solve the following wave equation

\[
\begin{cases}
  u_t = u_{rr} + \frac{1}{r} u_r, & 0 < r < 1 \\
  u(r, 0) = r^2 \\
  u_r(0, t) = 0, u_r(1, t) + u(1, t) = 0
\end{cases}
\]

You are allowed to use results in Problem 1. Write your solution in terms of Bessel function of order zero \( J_0 \).

5. (10pts) Use the method of separation of variables to solve

\[
\begin{cases}
  u_t = u_{xx} + e^t \sin(3x), & 0 < x < \pi \\
  u(x, 0) = \sin(2x) \\
  u(0, t) = t, \quad u(\pi, t) = 0
\end{cases}
\]

6. (10pts) Use the method of separation of variables to solve

\[
\begin{cases}
  u_{tt} = u_{xx} + e^{-t} \sin(x), & 0 < x < \pi \\
  u(x, 0) = \sin(3x), u_t(x, 0) = 0 \\
  u(0, t) = 1, \quad u(\pi, t) = 0
\end{cases}
\]

7. (20pts) (a) Use the method of separation of variables to solve

\[
\begin{cases}
  u_{xx} + u_{yy} = 0, 0 < x < \pi, \quad 0 < y < \pi, \\
  u(0, y) = u_x(\pi, y) = u(x, 0) = 0, \\
  u(x, \pi) = \sin \frac{x}{2} - 2 \sin \frac{3x}{2}
\end{cases}
\]

(b) Use divergence theorem to show the solutions to (a) are unique.