1. (10pts) Find a solution formula for
\[ u_t = u_{xx} + u + f(x, t), \quad t > 0 \]
\[ u(x, 0) = \phi(x) \]
Hint: Consider \( u(x, t) = e^{t}v(x, t) \).

2. (10pts) Consider the following diffusion equation
\[ u_t = u_{xx} - e^{-x}, \quad -\infty < x < +\infty, \quad t > 0 \]
\[ u(x, 0) = 0 \]
(a) Use the solution formula to find a solution.
(b) Find a solution \( \psi(x) \) of \( \psi_{xx} - e^{-x} = 0 \). Let \( u(x, t) = \psi(x) + v(x, t) \) and then find a solution to \( v(x, t) \).

3. (10pts) (a) Solve the following diffusion equation by the method of extension:
\[ u_t = k u_{xx}, \quad x > 0, \quad t > 0 \]
\[ u(0, t) = -t \]
\[ u(x, 0) = 0 \]
Hint: let \( u(x, t) = -t + v(x, t) \). Write the solution in terms of integrals of \( Erf \) function.
(b) Use the energy method to show that the solution to (a) is unique, assuming that the solution has fast decay.

4. (10pts) Solve the following diffusion equation
\[ \begin{cases} 
  u_t = 2u_{xx}, & 0 < x < \pi \\
  u(x, 0) = \sin(x) \cos(x) + 2 \sin(20x) \\
  u(0, t) = 0, u(\pi, t) = 0 
\end{cases} \]

5. (20pts) (a) Find the eigenvalues and eigenfunctions of
\[ X'' + \lambda X = 0, \quad 0 < x < l, \quad X'(0) = 0, \quad X(l) = 0 \]
(b) Solve the following wave equation
\[ \begin{cases} 
  u_{tt} = c^2 u_{xx}, & 0 < x < 1 \\
  u(x, 0) = 2 \cos(\frac{3\pi}{2}x), u_x(0, 0) = \cos(\frac{\pi}{2}x) \\
  u_x(0, t) = 0, u(1, t) = 0 
\end{cases} \]

6. (20pts) (a) Solve
\[ \begin{cases} 
  u_t - u_{xx} = 0, & 0 < x < 1 \\
  u(x, 0) = \phi(x), & 0 < x < 1 \\
  u_x(0, t) + 2u(0, t) = 0, u_x(1, t) + u(1, t) = 0 
\end{cases} \]
by separation of variables. (b) Under what conditions on \( \phi(x) \), does the solution to (3) remain bounded as \( t \to +\infty \)?

7. (10pts) For the following eigenvalue problems, find out: (1) how many negative eigenvalues there are (2) the algebraic equations for all positive, zero and negative eigenvalues (c) the corresponding eigenfunctions

(a) \( X'' + \lambda X = 0, \quad 0 < x < \frac{1}{2}, \quad 2X'(0) + X(0) = 0, \quad X'(\frac{1}{2}) + X(\frac{1}{2}) = 0; \)
(b) \( X'' + \lambda X = 0, \quad 0 < x < 1, \quad X'(0) + 2X(0) = 0, \quad X'(1) - 2X(1) = 0 \)
(c) \( X'' + \lambda X = 0, \quad 0 < x < 2, \quad X'(0) + 3X(0) = 0, \quad X'(2) - 2X(2) = 0 \)

8. (10pts) For the following eigenvalue problems, transform it into standard Sturm-Liouville eigenvalue problem as
\[ (p(x)X')' - q(x)X(x) + \lambda w(x)X(x) = 0 \]
(a) \( X'' + x^2 X' + \lambda X = 0, \quad (b) \quad X'' + \frac{1}{x} X' - x X + \lambda X = 0 \)