# A Summary of 1st order PDEs

**Case 1.** $a(x,y)u_x + b(x,y)u_y = 0$

Characteristic curve Method: ① $\frac{dx}{a} = \frac{dy}{b} \Rightarrow y = y(x;\xi) \Rightarrow \xi = F(x,y)$ (characteristic curve)

② general sol'n, $u = f(\xi) = f(F(x,y))$

③ Plug in IC

**Case 2.** $a(x,y)u_x + b(x,y)u_y = d(x,y,u)$

Method 1 Parametrize IC and Initial Data $(x_0(\xi), y_0(\xi), u_0(\xi))$

$$\begin{cases} \frac{dx}{ds} = a(x,y),\ x(0) = x_0(\xi) \\ \frac{dy}{ds} = b(x,y),\ y(0) = y_0(\xi) \\ \frac{du}{ds} = d(x,y,u),\ u(0) = u_0(\xi) \end{cases} \Rightarrow \begin{matrix} x = x(s;\xi) \\ y = y(s;\xi) \end{matrix} \Rightarrow \begin{matrix} s = \cdots \\ \xi = \cdots \end{matrix}$$

$$u = u(s;\xi) \Rightarrow u = \cdots$$

Method 2. ① $\frac{dx}{a} = \frac{dy}{b} \Rightarrow \xi = F(x,y)$

② Change of variables $\begin{cases} x' = x \\ y' = F(x,y) \\ u = U \end{cases} \Rightarrow aU_{x'} = d \Rightarrow U_{x'} = \cdots \Rightarrow U = \cdots \Rightarrow u = \cdots$

③ Plug in IC

**Case 3.** $u_t + c(u)u_x = 0,\ u(x,0) = f(x)$

(1) $x - \xi = c(f(\xi))t,\ u = f(\xi)$

(2) breaking-up time, $t_0 = \dfrac{1}{\max\limits_{c'(f)f' < 0} |c'(f)f'(\xi)|}$

(3) Expansion Fan: $u = U(\Lambda),\ c(U) = \Lambda,\ \Lambda = \frac{x}{t}$

(4) shock Curve: $\frac{ds}{dt} = \frac{q(u^+) - q(u^-)}{u^+ - u^-},\ s(t_0) = x_0,\ q = \int c(u)du$

**Case 4.** $F(x,y,u,p,q) = 0,\ p = u_x,\ q = u_y$

① Parametrize ICurve and I Data, $(x_0(\xi), y_0(\xi), u_0(\xi))$

② $\begin{cases} F(x_0, y_0, u_0, p_0, q_0) = 0 \\ u_0' = x_0' p_0 + y_0' q_0 \end{cases} \Rightarrow p_0 = \cdots,\ q_0 = \cdots$

③ Charpit's Eqn

$$\begin{cases} \frac{dx}{ds} = F_p,\ x(0) = x_0 \\ \frac{dy}{ds} = F_q,\ y(0) = y_0 \\ \frac{dp}{ds} = -F_x - F_u p,\ p(0) = p_0 \end{cases} \quad \begin{matrix} \frac{dq}{ds} = -F_y - F_u q,\ q(0) = q_0 \\ \frac{du}{ds} = pF_p + qF_q,\ u(0) = u_0. \end{matrix}$$