Lecture 4: Frobenius Series about Regular Singular Points

(*Compiled 3 March 2014*)

In this lecture we will summarize the classification of expansion points x_0 for series as ordinary points for which Taylor Series approximations are appropriate, regular singular points for which Frobenius series expansions will work, and irregular singular points for which neither power series expansions work. We also discuss the radius of convergence of series expansions of ODE, which is at least as large as the minimum distance from x_0 to the nearest other singularity in the complex plane.

Key Concepts: Series Solutions; Ordinary Points and Taylor Series; Regular Singular Points and Frobenius Series; Irregular Singular Points; radii of convergence of power series solutions of ODE.

4 Fobenius series expansions about Regular Singular Points

4.1 Series Expansion Summary:

Consider

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{4.1}$$

Divide by $P(x)$:

$$Ly = y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0, \tag{4.2}$$

and define the functions $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$. We observe that (4.2) can be used to generate $y''(x_0)$ and all higher derivatives by differentiating the ODE repeatedly. This can be used to generate a Taylor series expansion, provided $P(x_0) \neq 0$, or more genreally if $p(x)$ and $q(x)$ are analytic at x_0, which leads to the following definition.

Ordinary Points:

x_0 is an ordinary point if $p(x)$ and $q(x)$ are analytic at x_0 i.e.,

$$\begin{aligned} p(x) &= p_0 + p_1(x - x_0) + \cdots \\ q(x) &= q_0 + q_1(x - x_0) + \cdots . \end{aligned} \tag{4.3}$$

About an ordinary point x_0 we can obtain 2 linearly independent solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{4.4}$$

whose radii of convergence are at least as large as those of p and q in (4.3) - i.e., the circle of convergence extends at least as far as the singularity closest to x_0 in the complex plane.

Singular Points: If $P(x_0) = 0$ then $p(x)$ and $q(x)$ may fail to be analytic in which case x_0 is a *singular point*.

Regular Singular Points - an example:
To motivate how to proceed near singular points let us consider the following example.
Example 1:

$$Ly = 2x^2y'' - xy' + (1+x)y = 0 \quad x = 0 \text{ is a RSP.} \tag{4.5}$$

We observe that this equation is "almost Equidimensional" but for the additional xy term. Let us rearrange this equation as follows

$$L_0y = 2x^2y'' - xy' + y = -xy \tag{4.6}$$

As $x \to 0$ we observe that $xy \ll y$, so that the term on the right side of (4.6) is much less than those on the left. Thus (throwing away the small term for the moment) we solve the homogeneous equidimensional equation $L_0y = 0$

$$L_0y = 2x^2y'' - xy' + y = 0, \text{ whose general solution is } y(x) = C_1x + C_2\sqrt{x} \tag{4.7}$$

Now take this solution and substitute it on the right side of (4.6)

$$L_0y = 2x^2y'' - xy' + y = -x\left(C_1x + C_2\sqrt{x}\right), \text{ whose general solution is } y(x) = C_1\left(x - \frac{1}{3}x^2\right) + C_2\sqrt{x}\,(1-x) \tag{4.8}$$

Substituting this term again on the right side of (4.6) we obtain

$$L_0y = 2x^2y'' - xy' + y = -\left(C_1\left(x - \frac{1}{3}x^2\right) + C_2\sqrt{x}\,(1-x)\right)$$
$$\text{whose general solution is } y(x) = C_1\left(x - \frac{1}{3}x^2 + \frac{1}{30}x^3\right) + C_2\sqrt{x}\left(1 - x + \frac{1}{6}x^2\right) \tag{4.9}$$

We observe that this procedure is generates a series expansion of the form:

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n \tag{4.10}$$

where r is one of the roots of the indicial equation associated with the equidimensional part of the ODE.
Observations:

(1) Singular points that can be treated by this procedure are known as *Regular Singular Points*
(2) The form of the series (4.10) that is suitable for determining the behavior of the solutions to ODE around such regular singular points are called *Frobenius Series.*
(3) Rather than proceed with this recursive approach, which can rapidly become complicated, we will adopt a procedure in which we substitute the series of the form (4.10) directly into the ODE and solve for the unknown coefficients. This procedure is illustrated in Example 4 below, which yields the same solution.

Regular Singular points for general second order linear ODE

Motivated by the previous example we consider the more general second order linear ODE. In this case we consider $(x-x_0)^2 Ly$

$$(x-x_0)^2 Ly = (x-x_0)^2 y'' + (x-x_0)\left\{(x-x_0)\frac{Q(x)}{P(x)}\right\} y' + \left\{(x-x_0)^2 \frac{R(x)}{P(x)}\right\} y = 0 \tag{4.11}$$

and define coefficient functions $p(x) = \left\{(x-x_0)\frac{Q(x)}{P(x)}\right\}$ and $q(x) = \left\{(x-x_0)^2\frac{R(x)}{P(x)}\right\}$. A point x_0 is a *regular singular point* if $p(x)$ and $q(x)$ are both analytic at x_0, i.e.,

$$\begin{aligned} p(x) &= (x-x_0)\frac{Q(x)}{P(x)} = p_0 + p_1(x-x_0) + \cdots \\ q(x) &= (x-x_0)^2\frac{R(x)}{P(x)} = q_0 + q_1(x-x_0) + \cdots \end{aligned} \tag{4.12}$$

Now substituting the expansions defined in (4.12) into (4.11) and collecting powers of $(x-x_0)$, we obtain:

$$(x-x_0)^2 Ly = (x-x_0)^2 y'' + (x-x_0)p_0 y' + q_0 y + \overbrace{(x-x_0)\left\{p_1(x-x_0)y' + q_1 y + p_2(x-x_0)^2 y' + q_2(x-x_0)y + \cdots\right\}}^{\text{small as } x\to x_0} = 0 \tag{4.13}$$

Thus, in the limit as $x \to x_0$ the operator $(x-x_0)^2 Ly \approx L_0 y$, where

$$L_0 y = (x-x_0)^2 y'' + (x-x_0)p_0 y' + q_0 y = 0 \tag{4.14}$$

This implies that close to the expansion point x_0, the operator Ly has singularities no worse than the Euler Equation (4.14). Since the higher order terms (designated as small in (4.13)) cannot introduce more singular terms but rather corrections that are higher powers in $(x-x_0)$, we are motivated to look for solutions of the form

$$y(x) = (x-x_0)^r \sum_{n=0}^{\infty} c_n (x-x_0)^n = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}, \tag{4.15}$$

which is known as a *Frobenius Series.*

Radius of convergence: The radius of convergence of this series is greater than or equal to the distance between x_0 and the nearest singular point in the complex plane.

Irregular Singular Points:

If x_0 is a singular point and $p(x) = (x-x_0)\frac{Q(x)}{P(x)}$ and $(x-x_0)^2\frac{R(x)}{P(x)}$ are not analytic, then x_0 is called an irregular singular point.

Example 2: $x = 0$ is an irregular point of the second order equation

$$Ly = x^2 y'' + (1+3x)y' + y = 0$$

In this case the leading behavior of $y(x)$ as $x \to 0$ is $y(x) \approx cxe^{1/x}$, which could not be captured by a Frobenius expansion.

Example 3: $x = 0$ is an irregular point of the first order equation

$$Ly = x^2 y' + y = 0$$

The solution of this first order linear equation can be obtained by means of an integrating factor $F = e^{-1/x}$, which yields the solution $y(x) = ce^{1/x}$, which could not be captured by a Frobenius expansion about $x_0 = 0$.

4.2 Frobenius Series Expansion

Example 4:

We revisit Example 1 by using a Frobenius series to solve the equations directly.

$$\begin{aligned} Ly &= 2x^2y'' - xy' + (1+x)y = 0 \quad x = 0 \text{ is a RSP.} \\ y &= \sum_{n=0}^{\infty} a_n x^{n+r} \end{aligned} \tag{4.16}$$

$$\begin{aligned} Ly = 2x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} &- x \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \\ &+ (1+x) \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \\ \sum_{n=0}^{\infty} a_n \left\{2(n+r)(n+r-1) - (n+r) + 1\right\} x^{n+r} & \\ &+ \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0 \\ &m = n+1 \quad n = 0 \rightarrow m = 1 \\ &n = m-1 \\ \text{Therefore} \qquad a_0 \left\{2r(r-1) - r + 1\right\} x^r &+ \sum_{n=1}^{\infty} \left[a_n \left\{2(n+r)(n+r-1)\right.\right. \\ &- \left.\left.(n+r)+1\right\} + a_{n-1}\right] x^{n+r} = 0. \end{aligned} \tag{4.17}$$

$x^r >$ Indicial Equation: $2r^2 - 3r + 1 = (2r-1)(r-1) = 0 \quad r = \frac{1}{2}, \quad r = 1.$

a_0 arbitrary

Recursion

$$a_n = \frac{-a_{n-1}}{(2n+2r-3)(n+r)+1} \tag{4.18}$$

Let $r = 1/2$:

$$\begin{aligned} a_n &= \frac{-a_{n-1}}{(2n-2)(n+1/2)+1} = \frac{-a_{n-1}}{(n-1)(2n+1)+1} = \frac{-a_{n-1}}{n(2n-1)} \\ n = 1: \quad a_1 &= \frac{-a_0}{1}; \quad n = 2: a_2 = \frac{-a_1}{2.3} = \frac{+a_0}{2.3} \\ a_3 &= \frac{-a_2}{3.5} = \frac{-a_0}{1.(2.3)(3.5)}; \quad a_4 = \frac{-a_3}{4.7} = \frac{+a_0}{1(2.3)(3.5)(4.7)} \\ a_n &= \frac{(-1)^n a_0}{n!1.3.5.(2n-1)} = \frac{(-1)^n 2^{(n-1)} a_0}{n(2n-1)!} \\ y_1(x) &= x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{(n-1)}}{n(2n-1)!} x^n \end{aligned} \tag{4.19}$$

$r = 1$:

$$
\begin{aligned}
a_n &= \frac{-a_{n-1}}{(2n-1)(n+1)+1} = \frac{-a_{n-1}}{(2n+1)n} \\
a_1 &= \frac{-a_0}{3.1}, \quad a_2 = \frac{-a_1}{5.2} = \frac{+a_0}{(1.3)(2.5)}; \quad a_3 = \frac{-a_2}{3.7} = \frac{-a_0}{(1.3)(2.5)(3.7)} \\
a_n &= \frac{(-1)^n a_0}{n!3.5.7.(2n+1)} = \frac{(-1)^n 2^n a_0}{(2n+1)!} \qquad (4.20) \\
y_2(x) &= x \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n
\end{aligned}
$$

General Solution: $y(x) = c_1 y_1(x) + c_2 y_2(x)$

Radius of Convergence ∞.