Lecture 3: Regular Singular points

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In this lecture we will define a Regular Singular Point about which a Taylor series will not work. We will also introduce the concept of the radius of convergence of the series and how it relates to the coefficient of the highest derivative of the ODE.

Key Concepts: Ordinary Points and Regular Singular Points, radius of convergence of power series.

3 Radius of Convergence and Nearest Singular Points

Example 1: $(1+x^2)y'' + 2xy' + 4x^2y = 0$.

(1) If we were given $y(0) = 0$ and $y'(0) = 1$ then we would want a power series expansion of the form

$$y = \sum_{n=0}^{\infty} c_n x^n \quad \text{about } x_0 = 0. \tag{3.1}$$

Roots of $1 + x^2 = 0$ are $x = \pm i$, so we expect the radius of convergence of the TS for $\frac{1}{1+x^2}$ to be 1 since

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots \lim \left|\frac{a_{n+2}}{a_n}\right| = 1 \quad \rho = 1 \tag{3.2}$$

(2) If we were given $y(1) = 1$, $y'(1) = 0$ then a power series expansion of the form $\sum c_n(x-1)^n$ is required. In this case $\rho = \sqrt{2}$.

Example 2: $(x-1)(2x-1)y'' + 2xy' - 2y = 0$.
$x = 0$ is an ordinary point. $x = 1$ and $x = \frac{1}{2}$ are singular points. One solution of this equation is

$$y(x) = \frac{1}{x-1} = -(1 + x + x^2 + \cdots) \quad \rho = 1. \tag{3.3}$$

This Taylor Series solution about the ordinary point $x = 0$ converges beyond the singular point $x = \frac{1}{2}$.
Example 3: $(x^2 - 2x)y'' + 5(x-1)y' + 3y = 0$ $y(1) = 7$ $y'(1) = 3$.
$x = 1$ is an ordinary point. $x = 0$ is a singular point $\left[(x-1)^2 - 1\right]y'' + 5(x-1)y' + 3y = 0$.

Let $t = x - 1$ so that $\frac{d}{dt} = \frac{d}{dx}$ and the equation is transformed to

$$
\begin{gathered}
(t^2-1)\ddot{y} + 5t\dot{y} + 3y = 0 \\
y = \sum_{n=0}^{\infty} c_n t^n, \quad \dot{y} = \sum_{n=1}^{\infty} c_n n t^{n-1}, \quad \ddot{y} = \sum_{n=2}^{\infty} c_n n(n-1) t^{n-2} \\
\sum_{n=2}^{\infty} n(n-1)c_n t^n - \sum_{n=2}^{\infty} n(n-1)c_n t^{n-2} + 5\sum_{n=1}^{\infty} n c_n t^n + 3\sum_{n=0}^{\infty} c_n t^n = 0 \\
m = n-2 \quad n = m+2 \quad n = 2 = m = 0 \\
\sum_{m=2}^{\infty} \left[-c_{m+2}(m+2)(m+1) + \{m(m-1)+5m+3\}\, c_m\right] t^m \\
-2c_2 + 3c_0 + \left[-c_3 3.2 + 5c_1 + 3c_1\right] t = 0
\end{gathered}
$$

$$
\begin{aligned}
t^0 > \qquad c_2 &= \tfrac{3}{2}c_0 \\
t^1 > \qquad c_3 &= \tfrac{8}{6}c_1 = \tfrac{4}{3}c_1 \\
t^m > \quad c_{m+2} &= \tfrac{c_m(m+1)(m+3)}{(m+1)(m+2)} \qquad m \geq 2.
\end{aligned}
\tag{3.4}
$$

c_0:

$$
\begin{aligned}
c_4 &= \tfrac{5c_2}{4} = \tfrac{5}{4}\left(\tfrac{3}{2}\right)c_0, \qquad\qquad c_6 = \tfrac{7}{6}c_4 = \tfrac{7}{6}\tfrac{5}{4}\tfrac{3}{2}c_0 \\
y_0(x) &= \sum_{n=0}^{\infty} \tfrac{357\ldots(2n+1)}{246\ldots(2n)}(x-1)^{2n}
\end{aligned}
\tag{3.5}
$$

c_1:

$$
\begin{aligned}
c_5 &= \tfrac{6}{5}c_3 = \tfrac{6}{5}\tfrac{4}{3}c_1 \qquad\qquad c_{2n+1} = \tfrac{46\ldots 2n+2}{35\ldots 2n+1}c_1 \\
y_1(x) &= \sum_{n=0}^{\infty} \tfrac{46\ldots 2n+2}{35\ldots 2n+1}(x-1)^{2n+1}
\end{aligned}
$$

$$
\lim_{n\to\infty} \frac{c_{m+2}}{c_m} = \frac{m+3}{m+1} = 1 \quad \rho = 1 \tag{3.6}
$$

$$
\begin{gathered}
y(x) = c_0 y_0(x) + c_1 y_1(x) \\
y(1) = c_0 = 7 \quad y'(1) = c_1 = 3.
\end{gathered}
\tag{3.7}
$$

3.1 Singular Points:

Consider

$$
P(x)y'' + Q(x)y' + R(x)y = 0. \tag{3.8}
$$

If P, Q and R are polynomials without common factors then singular points are points x_0 at which $P(x_0) = 0$.
Note: At singular points the solution is not necessarily analytic.
Examples:

(1) $x^2y'' + xy' = 0$
$y = x^r \to r(r-1) + r = 0 \to y = c_1 + c_2 \ln x$

The x^2y'' admits wild behaviour.

(2) $x^2y'' - 2y = 0$
$y = x^r \to r(r-1) - 2 = 0 \quad r = 2, -1 \quad y = c_1x^2 + c_2x^{-1}$

Again the x^2y'' admits wild behaviour.

(3) $x^2y'' - 2xy' + 2y = 0$
$y = x^r \to r(r-1) - 2r + 2 = 0 \quad r = 1, 2 \quad y = c_1x + c_2x^2$

In this case both solutions are analytic.

3.2 Regular Singular Points - polynomial coefficients:

Notice that all these cases are equidimensional equations for which we can identify solutions of the form x^r or $x^r \log x$. There is a special class of singular points called regular singular points in which the singularities are no worse than those in the equidimensional equations.

$$x^2 y'' + \alpha x y' + \beta y = 0. \tag{3.9}$$

If P, Q and R are polynomials and suppose $P(x_0) = 0$ then x_0 is a regular singular point if

$$\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} \quad \text{and} \quad \lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} \quad \text{are finite.} \tag{3.10}$$

$$\begin{aligned} \text{I.E.} \quad (x - x_0) \frac{Q(x)}{P(x)} &= p_0 + p_1(x - x_0) + p_2(x - x_0)^2 + \cdots \\ &\to \text{ singularity no worse than } \tfrac{1}{x - x_0} \\ (x - x_0)^2 \frac{R(x)}{P(x)} &= q_0 + q_1(x - x_0) + q_2(x - x_0)^2 + \cdots \\ &\to \text{ singularity no worse than } \tfrac{1}{(x - x_0)^2} \end{aligned} \tag{3.11}$$

Examples:

(1)

$$\begin{gathered} (1 - x^2) y'' - 2xy' + 4y = 0 \\ P = 1 - x^2 \quad P(\pm 1) = 0 \quad Q = -2x \quad R = 4 \\ \lim_{x \to 1} (x - 1) \tfrac{(-2x)}{(1 - x)(1 + x)} = 1 \quad \lim_{x \to 1} (x - 1)^2 \tfrac{4}{(1 + x)(1 - x)} = 0 \end{gathered} \tag{3.12}$$

$x = 1$ is a R.S.P. (similarly for $x = -1$).

(2)

$$\begin{gathered} x^3 y'' - y = 0 \\ P(x) = x^3 \quad Q = 0 \quad R = -1 \\ \lim_{x \to 0} x^2 \left(\tfrac{-1}{x^3} \right) = \infty \end{gathered} \tag{3.13}$$

Thus $x = 0$ is an *irregular singular point*. Actually $y \sim x^{3/4} \mathrm{e}^{\pm 2/x^{1/2}}$ as $x \to 0+$ which is much wilder than the simple power law x^r or $x^r \log x$.

Note: Any singular point that is not a regular singular point is called an irregular singular point.

(3) $2(x - 2)^2 x y'' + 3xy' + (x - 2)y = 0$. Singular points at $x = 0, 2$. $x = 0$ is a regular singular point. $x = 2$ is an irregular singular point.

3.3 More General Definition of a Regular Singular Point:

If P, Q, and R are not limited to polynomials then consider

$$\begin{aligned} P(x)y'' + Q(x)y' + R(x)y &= 0 \\ \text{or} \quad & \\ x^2y'' + x\left(\tfrac{xQ}{P}\right)y' + \left(\tfrac{x^2R}{P}\right)y &= 0 \end{aligned} \tag{3.14}$$

$x = 0$ is a *regular singular point* if $\left(\dfrac{xQ}{P}\right)$ and $\left(\dfrac{x^2R}{P}\right)$ are analytic at $x = 0$, i.e.,

$$\frac{xQ}{P} = p(x) = p_0 + p_1x + \cdots \quad \text{and} \quad \frac{x^2R}{P} = q(x) = q_0 + q_1x + \cdots . \tag{3.15}$$

In this case

$$Ly = x^2y'' + xp_0y' + q_0y + \overbrace{x\left\{p_1xy' + q_1y + \cdots\right\}}^{\text{small as } x\to 0} = 0. \tag{3.16}$$

Then as $x \to 0$ $x^2y'' + xp_0y' + q_0y \approx 0$ which is an Euler Equation which has solutions of the form $y = x^r$.

Thus about a regular singular point we look for solutions of the form $y = x^r\displaystyle\sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^{n+r}$.

Our task is to determine:

(i) r

(ii) the coefficients a_n

(iii) the radius of convergence.

Example: $x^2y'' + 2(\mathrm{e}^x - 1)y' + \mathrm{e}^{-x}\cos xy = 0,\quad P = x^2,\ Q = 2(\mathrm{e}^x - 1),\ R = \mathrm{e}^{-x}\cos x.$

$x = 0$ is a singular point.

$$\begin{aligned} \lim_{x\to 0}\frac{xQ}{P} &= \lim_{k\to 0} x\frac{2(\mathrm{e}^x-1)}{x^2} = \lim_{x\to 0}\frac{2(\mathrm{e}^x-1)}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{2\mathrm{e}^x}{1} = 2 \text{ L'Hopital} \\ \lim_{x\to 0}\frac{x^2R}{P} &= \lim_{x\to 0} x^2\frac{\mathrm{e}^{-x}\cos x}{x^2} = 1 < \infty. \end{aligned} \tag{3.17}$$

Since the quotient functions $p = xQ/P$ and $q = x^2R/P$ have Taylor Expansions about $x = 0$, $x = 0$ is a regular singular point.