Lecture 29: The heat equation with Robin BC

(*Compiled 4 August 2017*)

In this lecture we demonstrate the use of the Sturm-Liouville eigenfunctions in the solution of the heat equation. We first discuss the expansion of an arbitrary function $f(x)$ in terms of the eigenfunctions $\{\phi_n(x)\}$ associated with the Robins boundary conditions. This is a generalization of the Fourier Series approach and entails establishing the appropriate normalizing factors for these eigenfunctions. We then uses the new generalized Fourier Series to determine a solution to the heat equation when subject to Robins boundary conditions.

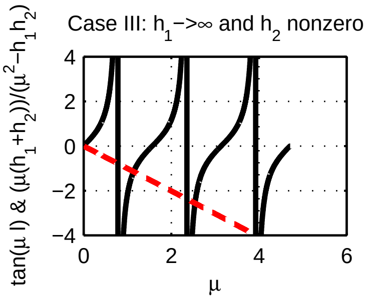
Key Concepts: Eigenvalue Problems, Sturm-Liouville Boundary Value Problems; Robin Boundary conditions.

Reference Section: Boyce and Di Prima Section 11.1 and 11.2

29 Solving the heat equation with Robin BC

29.1 Expansion in Robin Eigenfunctions

In this subsection we consider a Robin problem in which $\ell = 1$, $\mathbf{h_1} \to \infty$, and $\mathbf{h_2} = \mathbf{1}$, which is a Case III problem as considered in lecture 30. In particular:



$$\left.\begin{array}{c}\phi'' + \mu^2\phi = 0\\ \phi(0) = 0, \phi'(1) = -\phi(1)\end{array}\right\} \Longrightarrow \left\{\begin{array}{c}\phi_n = \sin(\mu_n x),\\ \tan(\mu_n) = -\mu_n\\ \mu_n \sim \left[\left(\frac{2n+1}{2}\right)\pi\right] \text{ as } n \to \infty\end{array}\right.$$

Assume that we can expand $f(x)$ in terms of $\phi_n(x)$:

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \tag{29.1}$$

$$\int_0^1 f(x)\sin(\mu_n x)\,dx = c_n \int_0^1 \left[\phi_n(x)\right]^2 dx \tag{29.2}$$

$$= c_n \frac{1}{2}\left[1 + \cos^2 \mu_n\right] \tag{29.3}$$

Therefore

$$c_n = \frac{2}{[1 + \cos^2 \mu_n]} \int_0^1 f(x)\sin(\mu_n x)\,dx. \tag{29.4}$$

If $f(x) = x$ then

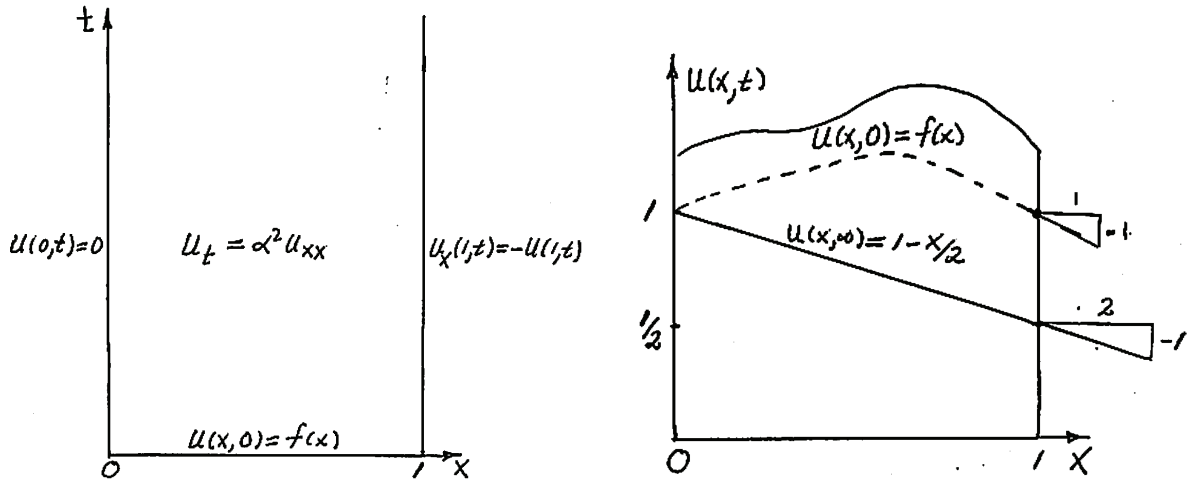
$$\begin{aligned}\int_0^1 x\sin(\mu_n x)\,dx &= -\frac{\cos(\mu_n x)}{\mu_n} - x\Big|_0^1 + \frac{1}{\mu_n}\int_0^1 \cos\mu_n x\,dx \\ &= -\frac{\cos(\mu_n)}{\mu_n} + \frac{\sin\mu_n x}{\mu_n^2}\Big|_0^1 \\ \text{but } -\mu_n\cos\mu_n = \sin\mu_n \qquad &= \frac{\sin\mu_n - \mu_n\cos\mu_n}{\mu_n^2} = 2\frac{\sin\mu_n}{\mu_n^2}.\end{aligned} \tag{29.5}$$

Therefore

$$c_n = \frac{4\sin\mu_n}{\mu_n^2[1+\cos^2\mu_n]} \tag{29.6}$$

$$f(x) = 4\sum_{n=1}^{\infty}\frac{\sin\mu_n\sin(\mu_n x)}{\mu_n^2[1+\cos^2\mu_n]} \tag{29.7}$$

29.2 Solving the Heat Equation with Robin BC



(b) Solution profiles $u(x,t)$ at various times

FIGURE 1. Left: Initial and boundary conditions; Right:Solution profiles $u(x,t)$

$$u_t = \alpha^2 u_{xx} \quad 0 < x < 1 \tag{29.8}$$

$$u(0,t) = 1 \quad u_x(1,t) + u(1,t) = 0 \tag{29.9}$$

$$u(x,0) = f(x). \tag{29.10}$$

Look for a steady state solution $v(x)$

$$\left.\begin{aligned} &v''(x) = 0 \\ &v(0) = 1 \quad v'(1) + v(1) = 0 \end{aligned}\right\} \tag{29.11}$$

$$\begin{gathered} v = Ax + B \quad v(0) = B = 1 \quad v'(x) = A \quad v'(1) + v(1) = A + (A+1) = 0 \\ A = -1/2 \end{gathered} \tag{29.12}$$

Therefore

$$v(x) = 1 - x/2. \tag{29.13}$$

Now let $u(x,t) = v(x) + w(x,t)$

$$u_t = w_t = \alpha^2(\cancel{v''} + w_{xx}) \Rightarrow w_t = \alpha^2 w_{xx}$$

$$1 = u(0,t) = v(0) + w(0,t) = 1 + w(0,t) \Rightarrow w(0,t) = 0$$

$$\begin{aligned} 0 = u_x(1,t) + u(1,t) &= \{\cancel{v'(1) + v(1)}\} \quad + w_x(1,t) + w(1,t) &&\Rightarrow \quad w_x(1,t) + w(1,t) = 0 \\ f(x) = u(x,0) &= v(x) + w(x,0) &&\Rightarrow \quad w(x,0) = f(x) - v(x). \end{aligned}$$

Let

$$w(x,t) = X(x)T(t) \tag{29.14}$$

$$\frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''}{X} = -\mu^2 \tag{29.15}$$

$$T(t) = ce^{-\alpha^2\mu^2 t} \tag{29.16}$$

$$\left.\begin{array}{l} X'' + \mu^2 X = 0 \\ X(0) = 0 \quad X'(1) + X(1) = 0 \end{array}\right\} \quad \begin{array}{l} \text{The } \mu_n \text{ are solutions of the transcendental} \\ \text{equation: } \tan\mu_n = -\mu_n. \end{array} \tag{29.17}$$

$$X_n(x) = \sin(\mu_n x) \tag{29.18}$$

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2\mu_n^2 t}\sin(\mu_n x) \tag{29.19}$$

where

$$f(x) - v(x) = w(x,0) = \sum_{n=1}^{\infty} c_n \sin(\mu_n x) \tag{29.20}$$

$$\Rightarrow c_n = \frac{2}{[1 + \cos^2\mu_n]}\int_0^1 [f(x) - v(x)]\sin(\mu_n x)\,dx \tag{29.21}$$

$$u(x,t) = 1 - \frac{x}{2} + \sum_{n=1}^{\infty} c_n e^{-\alpha^2\mu_n^2 t}\sin(\mu_n x). \tag{29.22}$$