# Lecture 25: More Rectangular Domains: Neumann Problems, mixed BC, and semi-infinite strip problems

*(Compiled 4 August 2017)*

In this lecture we Proceed with the solution of Laplace's equations on rectangular domains with Neumann, mixed boundary conditions, and on regions which comprise a semi-infinite strip.

**Key Concepts:** Laplace's equation; Rectangular domains; The Neumann Problem; Mixed BC and semi-infinite strip problems.

**Reference Section: Boyce and Di Prima Section 10.8**

## 25 More Rectangular Domains with mixed BC and semi-infinite strip problems

### 25.1 The Neumann Problem on a rectangle - only flux boundary conditions

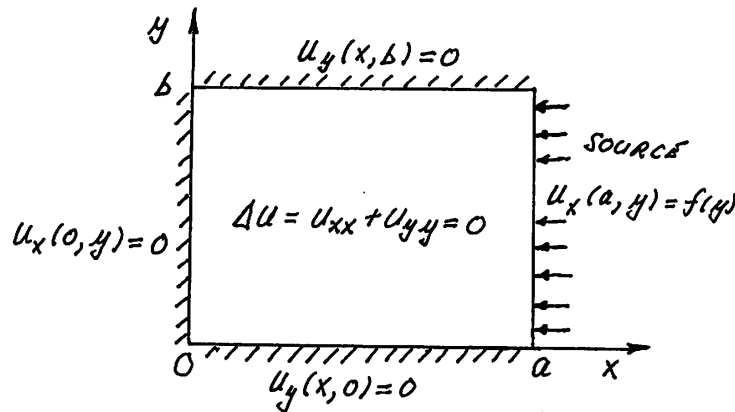**Example 25.1** *The Neumann Problem:*



FIGURE 1. Inhomogeneous Neumann Boundary conditions on a rectangular domain as prescribed in (**??**)

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a \quad 0 < y < b \tag{25.1}$$

$$u_x(0, y) = 0 \quad u_x(a, y) = f(y) \tag{25.2}$$

$$u_y(x, 0) = 0 = u_y(x, b). \tag{25.3}$$

Let $u(x, y) = X(x)Y(y)$.

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda^2 \tag{25.4}$$

$$\left. \begin{array}{l} Y''(y) + \lambda^2 Y(y) = 0 \\ Y'(0) = 0 = Y'(b) \end{array} \right\} \quad \begin{array}{rcl} Y & = & A\cos\lambda y + B\sin\lambda y \\ Y' & = & -A\lambda\sin\lambda y + B\lambda\cos\lambda y \end{array} \tag{25.5}$$

$$Y'(0) = \lambda B = 0 \quad \lambda = 0 \text{ or } B = 0. \tag{25.6}$$

$$Y'(b) = -A\lambda\sin\lambda b = 0 \quad \begin{array}{rcl} \lambda_n & = & (n\pi/b) \quad n = 0, 1, \dots \\ Y_n & = & \cos\left(\dfrac{n\pi y}{b}\right), \quad Y_0 = 1 \end{array} \tag{25.7}$$

$$X_n'' - \lambda^2 X_n = 0 \tag{25.8}$$

$$X_n'(0) = 0 \tag{25.9}$$

$n = 0$: $X_0'' = 0$, $X_0 = c_0 x + D_0 \Rightarrow X_0' = c_0 \Rightarrow X_0'(0) = c_0 = 0$.

Choose $D_0 = 1$: $X_0 = 1$

$$\begin{array}{rcl} n \geq 1 \quad X_n & = & c_n\cosh(\lambda_n x) + D_n\sinh(\lambda_n x) \\ X_n' & = & c_n\lambda\sinh(\lambda_n x) + D_n\lambda\cosh(\lambda_n x) \\ X_n'(0) & = & \lambda_n D_n = 0 \end{array} \tag{25.10}$$

Choose $c_n = 1$: $X_n = \cosh(\lambda_n x)$.

Thus

$$\left. \begin{array}{rcl} u_n(x,y) & = & X_n Y_n = \cosh(\lambda_n x)\cos(\lambda_n y) \\ u_0(x,y) & = & X_0 Y_0 = 1 \end{array} \right\} \text{ satisfy homog. BC.} \tag{25.11}$$

Therefore

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n\cosh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right). \tag{25.12}$$

Now $f(y) = u_x(a, y)$.

$$u_x(x,y) = \sum_{n=1}^{\infty} A_n\left(\frac{n\pi}{b}\right)\sinh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right) \tag{25.13}$$

$$u_x(a,y) = \sum_{n=1}^{\infty}\left\{A_n\left(\frac{n\pi}{b}\right)\sinh\left(\frac{n\pi a}{b}\right)\right\}\cos\left(\frac{n\pi y}{b}\right) = f(y)\dots \tag{25.14}$$

This is like a Fourier Cosine Series for $f(y)$ but without the constant term $a_0$.

Recall

$$f(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi y}{b}\right), \quad a_n = \frac{2}{b}\int_0^b f(y)\cos\left(\frac{n\pi y}{b}\right)dy. \tag{25.15}$$

Thus the expansion (25.14) is consistent only if $a_0 = 0$. For this to be true we require that

$$\int_0^b f(y)\,dy = 0 \tag{25.16}$$

if $\displaystyle\int_0^b f(y)\,dy \neq 0$ then there is no solution to the boundary value problem 1.

**Note**

(1) If $\displaystyle\int_0^b f(y)\,dy \neq 0$ there is a **net flux** into the domain through the right hand boundary and, since the other boundaries are insulated, there can be no steady solution – the temperature will continually change with time.

(2) If $\displaystyle\int_0^b f(y)\,dy = 0$ there is no net flux through the boundary and a steady state can exist. i.e. It is possible that $u_{xx} + u_{yy} = u_t = 0$. If $\displaystyle\int_0^b f(y)\,dy = 0$ then

$$A_n \left(\frac{n\pi}{b}\right) \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b} \int_0^b f(y) \cos\left(\frac{n\pi y}{b}\right) dy. \tag{25.17}$$

Therefore

$$A_n = \frac{2}{n\pi \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y) \cos\left(\frac{n\pi y}{b}\right) dy \quad n \geq 1 \tag{25.18}$$

and

$$u_\infty(x,y) = A_0 + \sum_{n=1}^\infty A_n \cosh\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \tag{25.19}$$

where $A_0$ is undetermined. $u(x,y)$ is said to be known up to an arbitrary constant.

(3) If $u_\infty(x,y)$ is the steady state of a 2D Heat Equation $u_t = u_{xx} + u_{yy}$ with $u(x,y,0) = u_0(x,y)$ then

$$\int_D u_t\,dx\,dy = \int_D \nabla \cdot \nabla u\,dx\,dy = \int_{\partial D} \frac{\partial u}{\partial n}\,ds = 0. \tag{25.20}$$

Therefore

$$\frac{\partial}{\partial t}\left(\int_D u\,dx\,dy\right) = 0 \Rightarrow \int_D u\,dx\,dy = \text{ const for all time } = \int_D u_0(x,y)\,dx\,dy. \tag{25.21}$$

Now

$$\int_D u_\infty(x,y)dxdy = A_0 \times \text{ area}(D) = \int_D u_0(x,y)\,dx \tag{25.22}$$

Which is the condition that determines $A_0$.

## 25.2 Rectangular domains with mixed BC

**Example 25.2** *Insulating BC along two sides and specified temperatures on the others:*

$$\Delta u = u_{xx} + u_{yy} = 0 \tag{25.23}$$

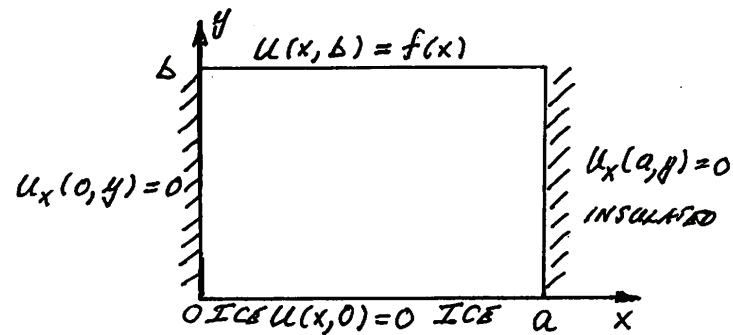$$0 = u_x(0,y) = u_x(a,y) = u(x,0) \tag{25.24}$$

$$u(x,b) = f(x). \tag{25.25}$$

FIGURE 2. Mixed Boundary conditions on a rectangular domain as prescribed in (25.24)

Let $u(x, y) = X(x)Y(y)$.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \pm \lambda^2. \tag{25.26}$$

*Since we have homogeneous BC on $X'(0) = 0 = X'(a)$ choose $-\lambda^2$.*

(1) $X'' + \lambda^2 X = 0 \quad X'(0) = 0 = X'(a)$.

$$\begin{array}{llll}
X(x) & = & A \cos \lambda x + B \sin \lambda x \quad X'(x) & = & -A\lambda \sin(\lambda x) + B\lambda \cos(\lambda x) \\
X'(0) & = & B\lambda = 0 \Rightarrow B = 0 \quad X'(a) & = & -A\lambda \sin(\lambda a) = 0
\end{array} \tag{25.27}$$

*Therefore*

$$\lambda_n = (n\pi/a) \quad n = 0, 1, 2, \ldots \quad X_n(x) = \cos\left(\frac{n\pi y}{a}\right) \tag{25.28}$$

*are eigenfunctions and eigenvalues.*

(2) $\lambda_n \neq 0$: $Y'' - \lambda^2 Y = 0$ *and* $Y(0) = 0 \Rightarrow Y_n(y) = A \sinh\left(\frac{n\pi y}{a}\right)$ $n \neq 0$. *Thus*

$$u_n(x, y) = \cos\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) \tag{25.29}$$

*satisfy homogeneous BC.*

$\lambda_0 = 0$: *In this case the ODE for $Y_0$ is:*

$$Y_0'' = 0 \Rightarrow Y(y) = c_1 y + c_2 \tag{25.30}$$

$$Y_0(0) = c_2 = 0 \Rightarrow Y_0(y) = y \tag{25.31}$$

*and $u_0(x, y) = y \cdot 1$ satisfies the homogeneous BC.*

*Therefore*

$$u(x,y) = c_0 y + \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi y}{a}\right) \cos\left(\frac{n\pi x}{a}\right) \qquad (25.32)$$

$$u(x,b) = \frac{(2c_0 b)}{2} + \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi b}{a}\right) \cos\left(\frac{n\pi x}{a}\right) = f(x) \qquad (25.33)$$

$$(2c_0 b) = \frac{2}{a} \int_0^a f(x)\,dx; \quad c_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a f(x) \cos\left(\frac{n\pi x}{a}\right)\,dx \qquad (25.34)$$

$$c_0 = \frac{1}{ab} \int_0^a f(x)\,dx; \quad c_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)\,dx \qquad (25.35)$$

$$u(x,y) = c_0 y + \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi y}{a}\right) \cos\left(\frac{n\pi x}{a}\right). \qquad (25.36)$$

### 25.3 Semi-infinite strip problems

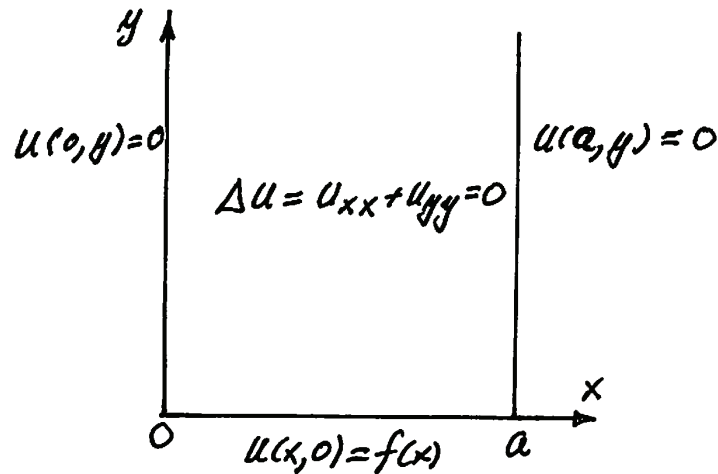**Example 25.3** *A Semi-infinite strip with specified temperatures:*



FIGURE 3. Diriclet Boundary conditions on a semi-infinite strip as prescribed in (25.39)

$$u_{xx} + u_{yy} = 0 \qquad 0 < x < a, \quad 0 < y < \infty \qquad (25.37)$$

$$u(0,y) = 0 = u(a,y) \qquad (25.38)$$

$$u(x,0) = f(x) \qquad u(x,y) \to 0 \ as \ y \to \infty \qquad (25.39)$$

*Let $u(x,t) = X(x)T(t)$ and plug into (1a?):*

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda^2 \ since \ we \ have \ homogeneous \ BC \ on \ X. \qquad (25.40)$$

(1)

$$\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(a) \end{array}\right\} \quad \begin{array}{rcl} \lambda_n &=& n\pi/a \quad n = 1, 2, \ldots \\ X_n &=& \sin\left(\dfrac{n\pi x}{a}\right) \end{array} \tag{25.41}$$

(2) $Y'' - \lambda^2 Y = 0$ $\quad Y(y) = Ae^{-\lambda y} + Be^{\lambda y}$. *Since $u(x,y) \to 0$ as $y \to \infty$ we require $B = 0$. Therefore*

$$u_n(x, y) = e^{-\lambda_n y} \sin\left(\frac{n\pi x}{a}\right) \tag{25.42}$$

*satisfy the homogeneous BC and the BC at $\infty$. Thus*

$$u(x, y) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{a}\right)y} \sin\left(\frac{n\pi x}{a}\right). \tag{25.43}$$

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) \Rightarrow c_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\,dx. \tag{25.44}$$
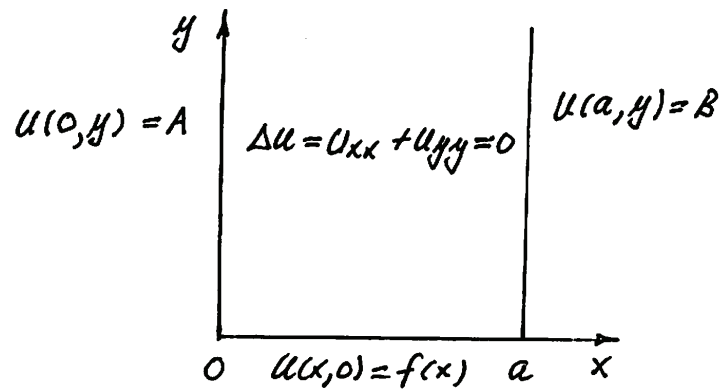
**Example 25.4** *Semi-infinite strip with inhomogeneous BC:*



FIGURE 4. Diriclet Boundary conditions on a semi-infinite strip as prescribed in (25.47)

$$u_{xx} + u_{yy} = 0 \qquad 0 < x < a, \quad 0 < y < \infty \tag{25.45}$$

$$u(0, y) = A, \quad B = u(a, y) \tag{25.46}$$

$$u(x, 0) = f(x) \qquad u(x, y) \to 0 \text{ as } y \to \infty \tag{25.47}$$

*Look for a function $v(x)$ for which $v'' = 0$ and which satisfies the inhomogeneous BC.*

$v = \alpha x + \beta \quad v(0) = A = \beta \quad v(a) = \alpha a + A = B$

*Therefore $v(x) = \left(\dfrac{B - A}{a}\right)x + A$.*

*Now let* $u(x, y) = v(x) + w(x, y)$.

$$0 = u_{xx} + u_{yy} = v_{\cancel{xx}} + w_{xx} + v_{\cancel{yy}} + w_{yy} \Rightarrow \Delta w = 0 \tag{25.48}$$

$$A = u(0, y) = v(0) + w(0, y) \Rightarrow w(0, y) = 0 \tag{25.49}$$

$$B = u(a, y) = v(a) + w(a, y) \Rightarrow w(a, y) = 0 \tag{25.50}$$

$$f(x) = u(x, 0) = v(x) + w(x, 0) \Rightarrow w(x, 0) = f(x) - v(x). \tag{25.51}$$

*Thus $w$ satisfies the same BVP as does $u$ in Eg. 3 above.*

*Therefore*

$$u(x, y) = (B - A)(x/a) + A + \sum_{n=1}^{\infty} d_n e^{-\left(\frac{n\pi}{a}\right)y} \sin\left(\frac{n\pi x}{a}\right) \tag{25.52}$$

*where*

$$d_n = \frac{2}{a} \int_0^a \{f(x) - v(x)\} \sin\left(\frac{n\pi x}{a}\right) dx. \tag{25.53}$$