Lecture 23: The wave equation on finite domains - solution by separation of variables

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In this lecture we discuss the solution of the one dimensional wave equation on a finite domain using the method of saparation of variables. The process proceeds in much the same was as with the heat equation. However, in this case the time equation is a second order ODE which has an indicial equation with complex roots, which lead to time functions that are sines and cosines rather than the exponential decay, which was the case with the heat equation. Depending on the boundary conditions for the spatial ODE we obtain the same eigenvalue problems as we did for the case of the heat equation. Each of these eigensolutions are associated with particular periodic extension, e.g. the Dirichlet BC give rise to eigenfunctions that are *sines* that are associated with the *odd* periodic extension of the solution defined on the domain $(0, L)$. We will demonstrate, using separation of variables, that the solution of the wave equation on a finite domain is none other than the D'Alembert solution in which the initial condition functions are the periodic extensions of the initial conditions that correspond to the boundary conditions that apply to the particular problem.

Key Concepts: The one dimensional Wave Equation; Finite Domains; Separation of Variables; Even and Odd Extensions and D'Alembert's solution for finite domains.

Reference Section: Boyce and Di Prima Section 10.7

23 Solution of the 1D wave equation on finite domains

23.1 Solution by separation of variables

Example 23.1

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < L, \quad t > 0 \tag{23.1}$$

$$\textit{BC: } u(0,t) = 0, \quad u(L,t) = 0 \tag{23.2}$$

$$\textit{IC: } u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{23.3}$$

For a guitar string $c = \sqrt{\dfrac{T_0}{\rho_0}}$ whereas for an elastic bar $c = \sqrt{\dfrac{E}{\rho}}$.

Separate Variables $u(x,t) = X(x)T(t)$

$$\frac{\ddot{T}(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 \tag{23.4}$$

$$\ddot{T}(t) + \lambda^2 c^2 T(t) = 0 \Rightarrow T(t) = c_1 \cos(\lambda ct) + c_2 \sin(\lambda ct) \tag{23.5}$$

$$\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(L) \end{array}\right\} \Rightarrow \left.\begin{array}{l} X(x) = A\cos(\lambda x) + B\sin\lambda x \\ X(0) = A = 0 \; X(L) = B\sin\lambda L = 0 \end{array}\right\}$$

$$\begin{array}{rcl} \lambda_n & = & \dfrac{n\pi}{L} \quad n = 1, 2, \ldots \\ X_n & = & \sin\left(\dfrac{n\pi x}{L}\right) \end{array}.$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \tag{23.6}$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \Rightarrow \boxed{A_n = \tfrac{2}{L}\int_0^L f(x)\sin\left(\tfrac{n\pi x}{L}\right)} \tag{23.7}$$

$$u_t(x,t) = \sum_{n=1}^{\infty} -A_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + B_n\left(\frac{n\pi c}{L}\right)$$

$$\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \tag{23.8}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = g(x) \Rightarrow \boxed{B_n\left(\tfrac{n\pi c}{L}\right) = \tfrac{2}{L}\int_0^L g(x)\sin\left(\tfrac{n\pi x}{L}\right)\,dx}. \tag{23.9}$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty}\left\{A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right\}\sin\left(\frac{n\pi x}{L}\right). \tag{23.10}$$

Observations

(1) *Period and Frequency of vibration:*

$$\cos\left(\frac{n\pi c}{L}(t+T)\right) = \cos\left(\frac{n\pi ct}{L}\right) \text{ provided } \frac{n\pi cT}{L} = 2\pi \tag{23.11}$$

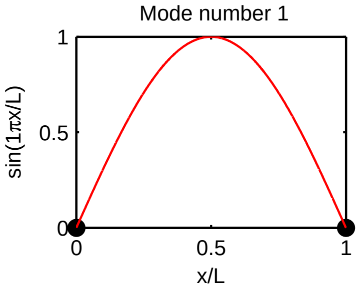
thus $T_n = \left(\dfrac{2L}{c}\right)\dfrac{1}{n}$ is the period (seconds per cycle) of mode n. $f_n = \dfrac{1}{T_n} = n\left(\dfrac{c}{2L}\right)$ are the natural frequencies of vibration.

(2) *Modes of Vibration: Standing waves of wavelength* $\boxed{\lambda_n = \dfrac{2L}{n}}$.

In the following four figures we plot the fist four modes of vibration. The first, known as the *fundamental mode of vibration*, is associated with the lowest frequency $f_1 = \dfrac{1}{T_1} = \left(\dfrac{c}{2L}\right)$. All higher frequencies, also known as *overtones*, are integer multiples of this fundamental frequency. The *nodes* in these modal plots are indicated by solid circles, which represent the points at which the displacement associated with a given mode is zero.

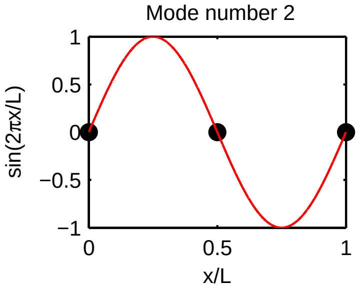
I: The fundamental mode of vibration with 2 nodes

$$X_1(x) = \sin\left(\frac{\pi x}{L}\right)$$



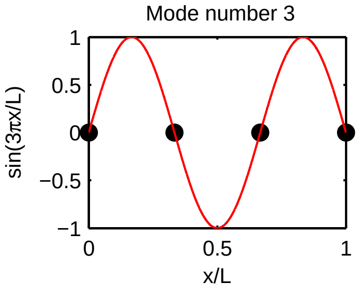
II: The second mode of vibration or first overtone with 3 nodes

$$X_2(x) = \sin\left(\frac{2\pi x}{L}\right)$$



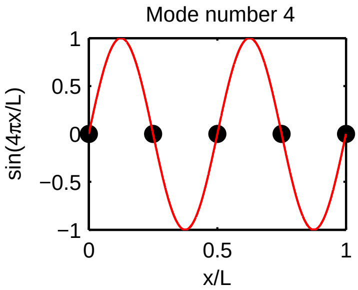
III: The third mode of vibration with 4 nodes

$$X_3(x) = \sin\left(\frac{3\pi x}{L}\right)$$



IV: The fourth mode of vibration with 5 nodes

$$X_4(x) = \sin\left(\frac{4\pi x}{L}\right)$$



23.2 Interpretation of the Fourier Series solution in terms of D'Alembert's Solution

Recall the double angle trigonometric identities

$$\begin{aligned}\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B;\\ \cos(A \pm B) &= \cos A \cos \mp \sin A \sin B,\end{aligned} \tag{23.12}$$

which we are going to use to interpret the solution (23.10) in terms of D'Alembert's Solution for an infinite domain. Using (23.12) we obtain

$$\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left\{\sin\frac{n\pi}{L}(x+ct) + \sin\left(\frac{n\pi}{L}\right)(x-ct)\right\} \tag{23.13}$$

$$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) = \frac{1}{2}\left\{\cos\frac{n\pi}{L}(x-ct) - \cos\frac{n\pi}{L}(x+ct)\right\} \tag{23.14}$$

Now

$$\begin{aligned}\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) &= \frac{1}{2}\sum_{n=1}^{\infty} A_n \left[\sin\left(\frac{n\pi}{L}\right)(x+ct)\right.\\ &\qquad \left. + \sin\left(\frac{n\pi}{L}\right)(x-ct)\right] &(23.15)\\ &= \frac{1}{2}\left[f_o(x+ct) + f_o(x-ct)\right] &(23.16)\end{aligned}$$

where f_0 is the odd periodic extension of f. Similarly,

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\sum_{n=1}^{\infty} B_n\left[\cos\frac{n\pi}{L}(x-ct) - \cos\frac{n\pi}{L}(x+ct)\right] = \frac{1}{2}\left[G(x-ct) - G(x+ct)\right] \tag{23.17}$$

where

$$G(x) := \frac{1}{2}\sum_{n=1}^{\infty} B_n \cos(\frac{n\pi}{L}x) \quad \text{and} \quad B_n = \frac{b_n^g}{\lambda_n cL} = \int_0^L g(x)\sin\lambda_n x dx \tag{23.18}$$

$$\begin{aligned}G(x) \ :&= \sum_{n=1}^{\infty} B_n \cos(\frac{n\pi}{L}x) \quad \text{and} \quad B_n = \frac{b_n^g}{\lambda_n c} = \frac{2}{\lambda_n cL}\int_0^L g(x)\sin\lambda_n x dx\\ &= \sum_{n=1}^{\infty} \frac{b_n^g}{\lambda_n c}\cos(\frac{n\pi}{L}x) \quad \text{and} \quad b_n^g = \frac{2}{L}\int_0^L g(x)\sin\lambda_n x dx\end{aligned}$$

therefore

$$G'(x) = -\frac{1}{c}\sum_{n=1}^{\infty} b_n^g \sin(\frac{n\pi}{L}x) = -\frac{1}{c}g_o(x)$$

Thus

$$G(x) = -\frac{1}{c}\int_0^x g_o(s)ds + D$$

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left[G(x-ct) - G(x+ct)\right] \tag{23.19}$$

$$= \frac{1}{2c}\left\{\left(-\int_0^{x-ct} g_o(s)\,ds + D\right) - \left(-\int_0^{x+ct} g_o(s)\,ds + D\right)\right\} \tag{23.20}$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} g_o(s)\,ds \tag{23.21}$$

Therefore, combining (23.16) and (23.21) we obtain the following expression for the solution of the wave equation for a finite domain in the form of D'ALembert's solution

$$\boxed{u(x,t) = \frac{1}{2}\left[f_o(x+ct) + f_o(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g_o(s)\,ds} \tag{23.22}$$

where f_o and g_o are the odd periodic extensions of f and g on $[0, L]$ i.e.

$$f_o(x) = \begin{cases} f(x) & 0 < x < L \\ -f(-x) & -L < x < 0 \end{cases} \quad \text{and} \quad f_0(x+2L) = f_0(x_0) \tag{23.23}$$

$$g_o(x) = \begin{cases} g(x) & 0 < x < L \\ -g(-x) & -L < x < 0 \end{cases} \quad \text{and} \quad g_o(x+2L) = g_o(x) \quad . \tag{23.24}$$

Observations

(1) Equation (23.22) above shows that the Wave Equation Solution for a string tied down at its ends is given by D'Alembert's Solution (see (23.25) in Lecture 23) in which the initial displacement function is given by the odd periodic extension f_0 of the initial displacement of the string, and the initial velocity function is given by the odd periodic extension of g_0.

(2) Information is carried along the characteristic curves $x + ct = \text{const}$ $\quad x - ct = \text{const}$.

(3) Observe that the time dependence of the solution involves $\sin\left(\frac{n\pi ct}{L}\right)$ and $\cos\left(\frac{n\pi ct}{L}\right)$ which do not decay with time. Thus the solutions to the Wave Equation persist with time, whereas the solutions to the Heat Equation typically decay exponentially with time.