Lecture 22: Interpretating D'Alembert's Solution in Space-Time: characteristics, regions of influence and domains of dependence

(*Compiled 4 August 2017*)

In this lecture we discuss the physical interpretation of the D'Alembert solution in terms of space-time plots. In particular we identify the left and right-moving *characteristics* as well as the domain of dependence of a given point (x_0, t_0) in space-time and the region of influence of a given initial value specified at the point $x_1, 0)$. We discuss the evolution of a few simple pulses and track the regions in space-time that are carved out by the intersecting characteristics.

Key Concepts: The one dimensional Wave Equation; D'Alembert's Solution, Characteristics, Domain of Dependence, Region of Influence.

Reference Section: Boyce and Di Prima Section 10.7

22 Space-Time Interpretation of D'Alembert's Solution

In this lecture we discuss the interpretation of D'Alembert's solution

$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)ds \tag{22.1}$$

to the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{22.2}$$

22.1 Characteristics

In the $x-t$ plane the lines

$$x - ct = x_0 \text{ and } x + ct = x_0 \tag{22.3}$$

are called the *characteristics* that emanate from the point $(x_0, 0)$ in space-time (see figure 1). Characteristics are the lines (or curves of more general hyperbolic problems) along which information is propagated by the equation. To interpret the characteristic lines in the $x-t$ plane, it is useful to rewrite the characteristic equations in the form

$$\begin{aligned} x - ct = x_0 \quad &\Rightarrow \quad t = \frac{1}{c}x - \frac{1}{c}x_0 \\ x + ct = x_0 \quad &\Rightarrow \quad t = -\frac{1}{c}x + \frac{1}{c}x_0 \end{aligned} \tag{22.4}$$

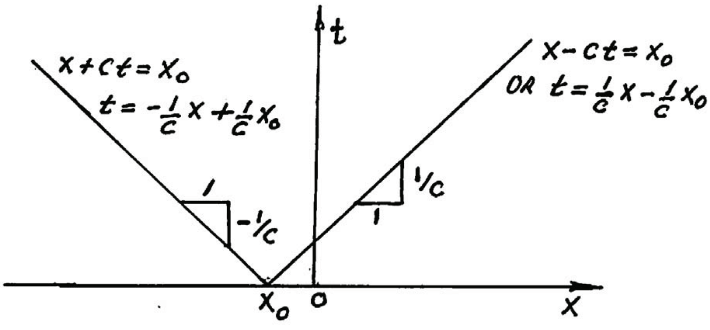

FIGURE 1. The characteristics that emanate from.

22.2 Region of Influence and Domain of Dependence

Region of influence: The lines $x + ct = x_1$ and $x - ct = x_1$ bound the region of influence of the function values at the initial point $(x_1, 0)$. Thus all the solution values $u(x, t)$ within this region can be influenced by the value at the point $(x_1, 0)$.

Domain of Dependence: The lines $x = x_0 - ct_0$ and $x = x_0 + ct_0$ that pass through the point (x_0, t_0) bound the domain of dependence. Thus the solution $u(x_0, t_0)$ depends on all the function values in the shaded region.

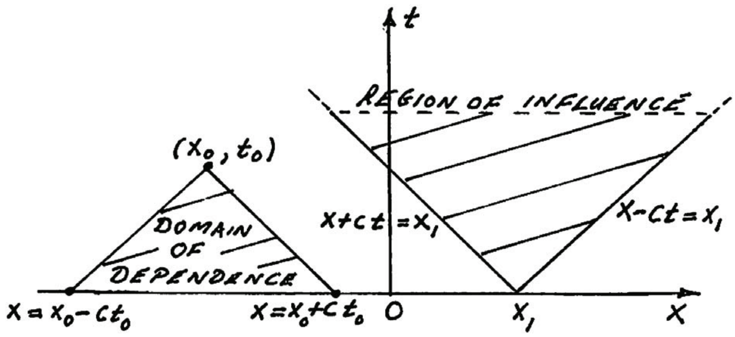

FIGURE 2. Space-time Region of Influence of the point $(x_1, 0)$ and Domain of Dependence of the point $(x0, t0)$, both of which can be determined from D'Alembert's Solution (22.1).

Example 22.1 *A Rectangular pulse Pulse:*

$$u(x,0) = \left\{ \begin{array}{ll} 1 & |x| < 1 \\ 0 & |x| > 1 \end{array} \right\} = u_0(x) \tag{22.5}$$

$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] \tag{22.6}$$

Let $c = 1$.

$\mathbf{t = \frac{1}{2}}$**:**

$$\begin{array}{llllll} x_r - \dfrac{1}{2} = 1 & \Rightarrow & x_r = \dfrac{3}{2} & x_R + \dfrac{1}{2} = 1 & & x_R = \dfrac{1}{2} \\ x_\ell - \dfrac{1}{2} = -1 & \Rightarrow & x_\ell = -\dfrac{1}{2} & x_L + \dfrac{1}{2} = -1 & & x_L = -\dfrac{3}{2} \end{array} \tag{22.7}$$

$\mathbf{t = 1}$**:**

$$\begin{array}{llllll} x_r - 1 = 1 & \Rightarrow & x_r = 2 & x_R + 1 = 1 & \Rightarrow & x_R = 0 \\ x_\ell - 1 = -1 & \Rightarrow & x_\ell = 0 & x_\ell + 1 = -1 & \Rightarrow & x_L = -2 \end{array} \tag{22.8}$$

$\mathbf{t = 2}$**:**

$$\begin{array}{llllll} x_r - 2 = 1 & \Rightarrow & x_r = 3 & x_R + 2 = 1 & \Rightarrow & x_R = -1 \\ x_\ell - 2 = -1 & \Rightarrow & x_\ell = 1 & x_L + 2 = -1 & \Rightarrow & x_L = -3 \end{array} \tag{22.9}$$

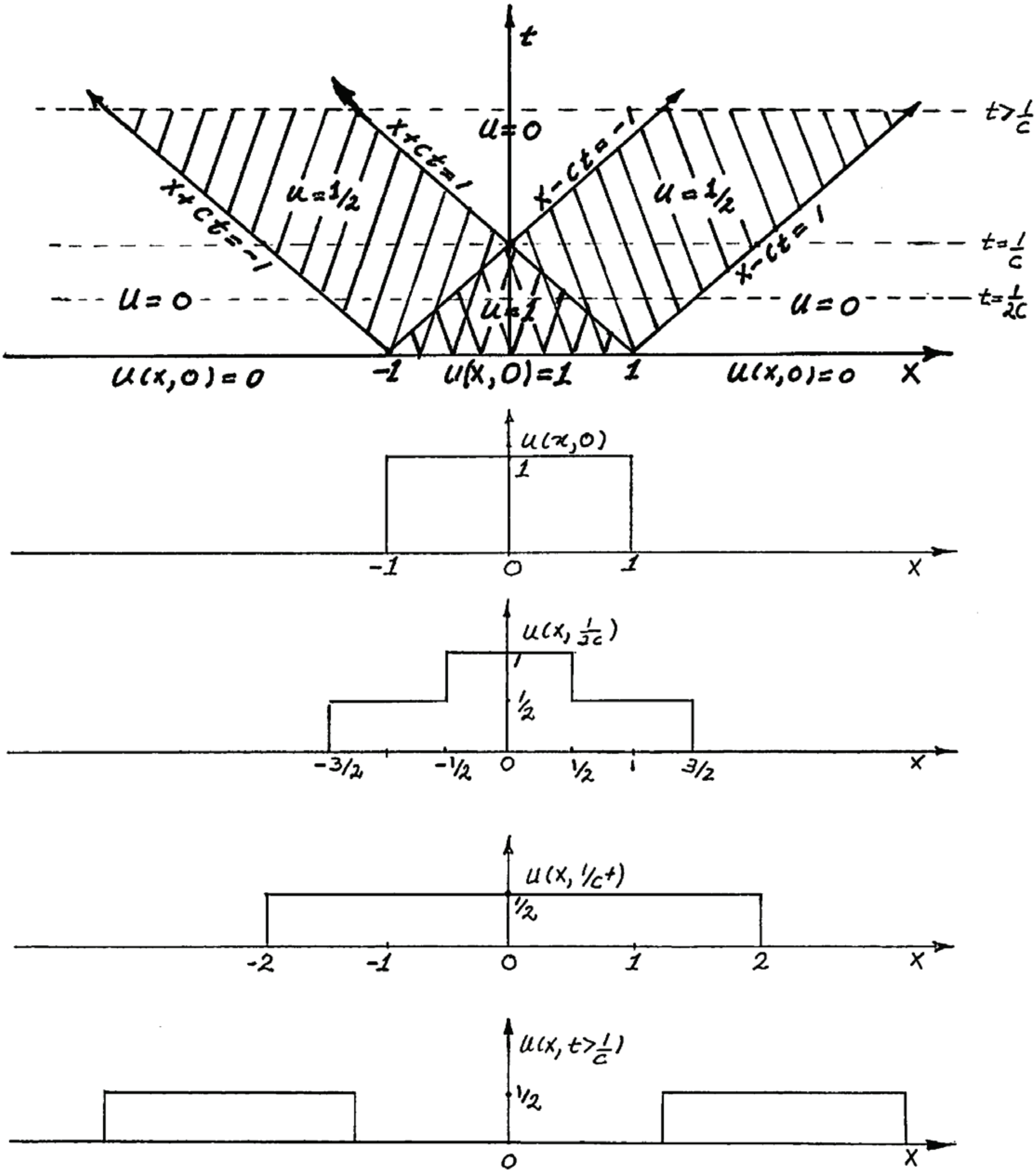

FIGURE 3. Top: Space-time representation of the regions in which the solution takes on different values for the rectangular pulse (22.5). Bottom: Cross sections of the solution $u(x,t)$ at times $t = 0,\ 1/2c,\ 1/c,$, and $t > 1/c$