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Lecture 21: The one dimensional Wave Equation: D'Alembert's Solution

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In this lecture we discuss the one dimensional wave equation. We review some of the physical situations in which the wave equations describe the dynamics of the physical system, in particular, the vibrations of a guitar string and elastic waves in a bar. We describe the relationship between solutions to the the wave equation and transformation to a moving coordinate system known as the Galilean Transformation. The galilean transformation can be used to identify a general class of solutions to the wave equation requiring only that the solution be expressed in terms of functions that are sufficiently differentiable. We show how the second order wave equation can be decomposed into two first order wave operators, one representing a left-moving and the other a right moving wave. This decomposition is used to derive the classical D'Alembert Solution to the wave equation on the domain $(-\infty, \infty)$ with prescribed initial displacements and velocities. This solution fully describes the equations of motion of an infinite elastic string that has a prescribed shape and initial velocity.

Key Concepts: The one dimensional Wave Equation; Characteristics; Traveling Wave Solutions; Vibrations in a Bar; a Guitar String; Galilean Transformation; D'Alembert's Solution.

Reference Section: Boyce and Di Prima Section 10.7

21 The one dimensional Wave Equation

21.1 Types of boundary and initial conditions for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{21.1}$$

$$\begin{array}{lcll} \dfrac{\partial^2 u}{\partial t^2} & \rightarrow & \text{expect 2 initial conditions} & u(x,0) = f(x) \\ & & & \frac{\partial u}{\partial t}(x,0) = g(x) \\ \dfrac{\partial^2 u}{\partial x^2} & \rightarrow & \text{expect 2 boundary conditions} & u(0,t) = 0 \\ & & & u(L,t) = 0. \end{array} \tag{21.2}$$

21.2 Some examples of physical systems in which the wave equation governs the dynamics

21.2.1 *The Guitar String*

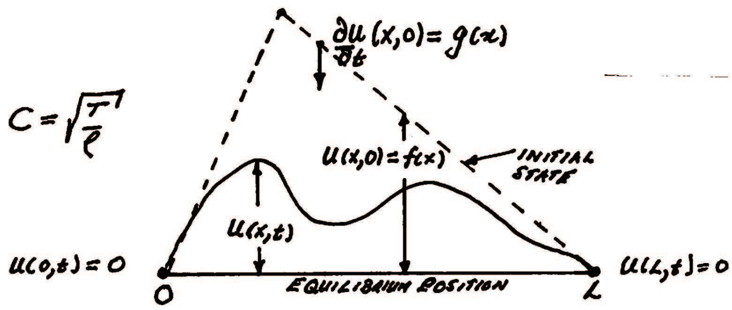


FIGURE 1. Initial condition and transient solution of the plucked guitar string, whose dynamics is governed by (21.1).

21.2.2 *Longitudinal Vibrations of an elastic bar*

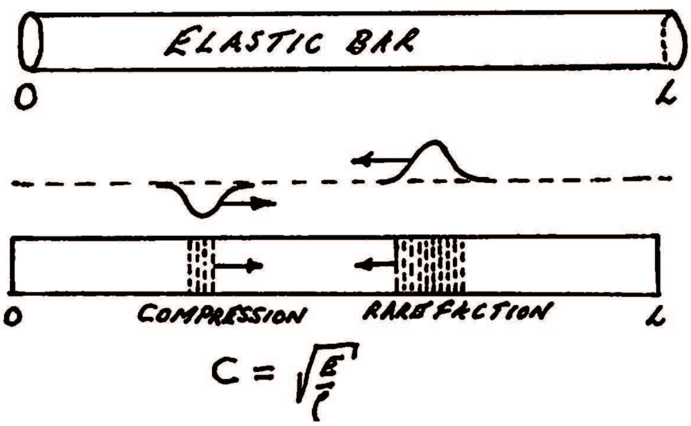


FIGURE 2. Compression and rarefaction waves in an elastic bar, whose dynamics is governed by (21.1).

21.3 A sneak preview - exponential solutions and the dispersion relation

To investigate the nature of the solutions to the wave equation that we might expect, let us look for exponential solutions of the form:

$$u = e^{ikx+\sigma t}$$

Substituting this trial solution into (21.1) yields

$$u_{tt} - c^2 u_{xx} = \left[\sigma^2 - c^2 (ik)^2\right] e^{ikx+\sigma t} = 0$$

Therefore in order that the exponential function (21.3 be a solution of (21.1), we require that σ satisfy the dispersion relation

$$\sigma^2 = -c^2 k^2$$

or

$$\sigma = \pm ikc$$

which implies that there are two solutions of the form

$$u = e^{ik(x \pm ct)} = e^{\pm ikct} e^{ikx}$$

We will now demonstrate physical significance of the argument $(x \pm ct)$ of the exponential and show that this leads to a much more general class of solutions. The products of time varying sinusoids with arguments $ikct$ with spatially varying sinusoids with arguments kx are precisely the same form as the solutions one would obtain by separation of variables for the wave equation defined on a finite domain. The selection of permissable wavenumbers k that apply in a particular problem will be determined by solving the appropriate eigenvalue problem.

21.4 The Galilean Transformation and solutions to the wave equation

Claim 1 *The Galilean transformation $x' = x + ct$ associated with a coordinate system $O'x'$ moving to the left at a speed c relative to the coordinates Ox, yields a solution to the wave equation: i.e., $u(x,t) = G(x+ct)$ is a solution to (21.1)*

$$u_t = cG' \quad u_{tt} = c^2 G'' \tag{21.3}$$

$$u_x = \ G' \quad u_{xx} = G''. \tag{21.4}$$

Therefore

$$u_{tt} - c^2 u_{xx} = c^2 G'' - c^2 G'' = 0. \tag{21.5}$$

Similarly $u(x,t) = F(x - ct)$ is also a solution to (21.1) associated with a right moving coordinate system $O'x'$ such that $x' = x - ct$. Is the sum of two solutions also a solution?

Claim 2 *Because the wave equation is linear, superposition applies: i.e., If u_1 and u_2 are solutions to (21.1) then $u(x,t) = \alpha_1 u_1(x,t) + \alpha_2 u_2(x,t)$ is also a solution.*

$$\begin{aligned} \frac{\partial^2}{\partial t^2}(\alpha_1 u_1 + \alpha_2 u_2) &= \alpha_1 \frac{\partial^2 u_1}{\partial t^2} + \alpha_2 \frac{\partial^2 u_2}{\partial t^2} \\ &= \alpha_1 c^2 \frac{\partial^2 u_1}{\partial x^2} + \alpha_2 c^2 \frac{\partial^2 u_2}{\partial x^2} \quad \text{since } u_1 \text{ and } u_2 \text{ solve (21.1)} \end{aligned}$$

Thus

$$\frac{\partial^2}{\partial t^2}(\alpha_1 u_1 + \alpha_2 u_2) = c^2 \frac{\partial^2}{\partial x^2}(\alpha_1 u_1 + \alpha_2 u_2).$$

Therefore, the general solution to the one dimensional wave equation (21.1) can be written in the form

$$u(x,t) = F(x - ct) + G(x + ct) \tag{21.6}$$

provided F and G are sufficiently differentiable functions.

Observations:

(1) This property is due to the linearity of $u_{tt} = c^2 u_{xx}$ (21.1).

(2) Every solution for (21.1) on $(-\infty, \infty)$ is of this form.

21.4.1 *Decomposition of the wave operator into left and right moving waves*

We observe that the wave operator can be decomposed as follows:

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u(x,t) = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u(x,t) = 0. \tag{21.7}$$

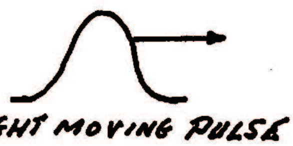
Let $w = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u$ then solving the wave equation can be reduced to solving the following system of first order wave equations:

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = w \text{ and } \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0. \tag{21.8}$$

In Lecture 2 we used the Galilean Transformation to interpret and identify solutions to these two first order wave operators.

In particular,

$\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} \rightarrow \{\text{right moving pulse}\} \implies$



and

$\frac{\partial}{\partial t} - c\frac{\partial}{\partial x} \rightarrow \{\text{left moving pulse}\} \implies$

21.5 D'Alembert's Solution

Motivated by the left and right moving coordinate systems we consider the following change of variables.

$$\begin{array}{rclcrcl} r &=& x + ct & \quad & s &=& x - ct \\ x &=& \frac{1}{2}(r+s) & & t &=& \frac{1}{2c}(r-s) \end{array}. \tag{21.9}$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial t}\frac{\partial t}{\partial r} = \frac{1}{2c}\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) \tag{21.10}$$

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial}{\partial t}\frac{\partial t}{\partial s} = -\frac{1}{2c}\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) \tag{21.11}$$

Therefore

$$-4c^2 \frac{\partial^2 u}{\partial r \partial s} = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0. \tag{21.12}$$

Therefore

$$\frac{\partial^2 u}{\partial r \partial s}(r,s) = 0 \tag{21.13}$$

$$\Rightarrow \frac{\partial u}{\partial s}(r,s) = \bar{\phi}_1(s) \tag{21.14}$$

$$\Rightarrow u(r,s) = \int \bar{\phi}_1(s)\,ds + \phi_2(r) = \phi_1(s) + \phi_2(r). \tag{21.15}$$

Say we have the IC:

$$u(x,0) = u_0(x) \quad \text{displacement} \tag{21.16}$$

$$\frac{\partial u}{\partial t}(x,0) = v_0(x) \quad \text{velocity} \tag{21.17}$$

$$u(x,t) = F(x-ct) + G(x+ct) \tag{21.18}$$

$$u(x,0) = F(x) + G(x) = u_0(x) \tag{21.19}$$

$$\frac{\partial u}{\partial t}(x,0) = -cF'(x) + cG'(x) = v_0(x) \tag{21.20}$$

$$-cF(x) + cG(x) = \int\limits_0^x v_0(\xi)\,d\xi + A \tag{21.21}$$

$$\begin{bmatrix} 1 & 1 \\ -c & c \end{bmatrix} \quad \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} u_0 \\ \int\limits_0^x v_0(\xi)\,d\xi + A \end{bmatrix} \tag{21.22}$$

$$F = \frac{1}{2c}\left\{ cu_0 - \left(\int\limits_0^x v_0(\xi)\,d\xi + A \right) \right\} \tag{21.23}$$

$$G = \frac{1}{2c}\left\{ \int\limits_0^x v_0(\xi)\,d\xi + A + cu_0 \right\} \tag{21.24}$$

Therefore

$$\boxed{u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\,d\xi} \tag{21.25}$$

D'Alembert's Solution to the wave equation on $(-\infty, \infty)$.