

Lecture 2: Series solutions to ODE with variable coefficients

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In this lecture we will introduce series methods for the solution of variable coefficient ODE. We introduce the concepts of ordinary points about which Taylor series solutions are obtained and singular points about which more general solutions are required.

Key Concepts: Variable coefficient ODE, Series Solutions, Ordinary Points and Taylor Series, Singular Points, radius of convergence of power series.

2 Series Solution of ODEs

2.1 Power Series:

$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ polynomial approximation.

Idea: Extend the polynomial to include infinitely many terms.

$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \text{ Power Series} \\ &= \sum_{n=0}^{\infty} a_nx^n \end{aligned} \tag{2.1}$$

Example 1: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

More General Power Series:

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \tag{2.2}$$

2.2 Taylor series

If we know all the derivatives of a function $f(x)$ at a single point x_0 then we can solve for the coefficients of a power series that represents the function at neighboring points x as follows:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \\
f'(x) &= a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots + na_n(x-x_0)^n + \cdots \\
&\Rightarrow f'(x_0) = a_1 \\
f''(x) &= 2a_2 + 3\cdot 2a_3(x-x_0) + \cdots + n(n-1)a_n(x-x_0)^{n-2} + \cdots \\
&\Rightarrow f''(x_0) = 2a_2 \\
f^{(3)}(x) &= 3!a_3 + 4\cdot 3\cdot 2(x-x_0) + \cdots + n(n-1)(n-2)a_n(x-x_0)^{n-3} + \cdots \\
&\Rightarrow f^{(3)}(x_0) = 3!a_3 \\
f^{(n)}(x_0) &= n!a_n \Rightarrow a_n = \frac{f^{(n)}(x_0)}{n!} \\
\text{Therefore } f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \qquad (2.3)
\end{aligned}$$

Alternative Form of Taylor Series:

$$f(x_0+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} h^n \qquad (2.4)$$

Example 2: Taylor-Maclauren expansions of common functions

$$\begin{aligned}
\mathrm{e}^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad (2.5) \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}
\end{aligned}$$

$$\begin{aligned}
\mathrm{e}^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \cdots\right) \qquad (2.6) \\
&= \cos\theta + i\sin\theta
\end{aligned}$$

$$\begin{aligned}
\mathrm{e}^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
&= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(x + \frac{x^3}{3!} + \cdots\right) \\
&= \cosh x + \sinh x
\end{aligned}$$

$$\begin{aligned}
\mathrm{e}^{-x} &= \cosh x - \sinh x \\
\cosh x &= (\mathrm{e}^x + \mathrm{e}^{-x})/2 \\
\sinh x &= (\mathrm{e}^x - \mathrm{e}^{-x})/2
\end{aligned}$$

2.3 Series Solution to a constant coefficient ODE:

Example 3: In this example we use power series to solve the linear ODE

$$y' + 2y = 0 \tag{2.7}$$

which we solved by integrating factor in the previous lecture. Since the unknown solution $y(x)$ and all its derivatives are defined implicitly by the ODE $y' = -2y$, let us look for a series solution of the form: $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} \tag{2.8}$$

$$\text{Therefore} \quad y' + 2y = \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

In the first sum let

$$\begin{aligned} m &= n-1 \qquad n = 1 \Rightarrow m = 0 \\ n &= m+1 \end{aligned}$$

$$\begin{aligned} \text{Therefore } \sum_{m=0}^{\infty} a_{m+1}(m+1)x^m + \sum_{n=0}^{\infty} 2a_n x^n &= 0 \\ n \Leftrightarrow m: \quad \sum_{m=0}^{\infty} \{a_{m+1}(m+1) + 2a_m\} x^m &= 0 \end{aligned} \tag{2.9}$$

$$a_{m+1} = -\tfrac{2}{(m+1)} a_m$$

$$\begin{aligned} a_1 = -2a_0, a_2 = +\tfrac{2}{2}\tfrac{2}{1}a_0, a_3 = -\tfrac{2}{3} \cdot \tfrac{2}{2} \cdot \tfrac{2}{1}a_0 = (-1)^3 \tfrac{2^3}{3!}a_0, \\ \ldots, a_m = (-1)^m \tfrac{2^m}{m!}a_0 \end{aligned} \tag{2.10}$$

$$\text{Therefore } y(x) = a_0 \sum_{m=0}^{\infty} \tfrac{(-2x)^m}{m!} = a_0 \mathrm{e}^{-2x}$$

2.4 Variable coefficient ODE: ordinary and singular points

Example 4: We consider the following Cauchy-Euler equation:

$$(x-1)y'' + y' = 0 \tag{2.11}$$

whose solution is obtained as follows

$$y = (x-1)^r \Rightarrow r(r-1) + r = r^2 = 0 \quad r = 0, 0.$$

$$\text{So that } y(x) = A + B\ln|x-1| \text{ is the general solution} \tag{2.12}$$

Method I: The first method we consider for obtaining a series solution to (2.11) is to use the ODE to calculate all the derivatives of $y(x)$ by direct differentiation and then to substitute these derivatives into Taylor's formula. Although this method can, in principle, be applied to any suitable ODE, we will see that the computations can rapidly become tedious. However, this method does highlight when the power series method will fail.

Assume that $y(0)$ and $y'(0)$ are given, then

$$y'' = -\frac{y'}{x-1} \Rightarrow y''(0) = y'(0)$$
$$y^{(3)} = -\frac{y''}{x-1} + \frac{y'}{(x-1)^2} \Longrightarrow y^{(3)}(0) = +y''(0) + y'(0) = 2y'(0)$$

Substituting this into Taylor's formula $y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{3!}y^{(3)}(0) + \cdots$

$$y(x) = y(0) + y'(0)\left\{x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right\} \tag{2.13}$$

We observe that this process works for equation (2.11) using the expansion point $x_0 = 0$, but will not work for $x_0 = 1$, which is called a *singular point.* In fact, a power series expansion is possible for all points $x_0 \neq 1$, which are called *ordinary points.*

Method II: We now consider an alternative, and more convenient, method for determining the coefficients of a power series solution to the ODE, by substituting the power series into the ODE and determining recursion formulae for these coefficients. Expand around the ordinary point $x_0 = 0$:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} \tag{2.14}$$

$$(x-1)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n-1} = 0$$
$$-\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} c_n \{n(n-1) + n\} x^{n-1} + c_1 = 0 \tag{2.15}$$
$$m - 1 = n - 2 \Rightarrow m = n - 1 \quad n = 2 \Rightarrow m = 1 \quad n = m + 1$$
$$-c_2 \cdot 2 \cdot 1 + c_1 + \sum_{m=2}^{\infty} \left[-c_{m+1}(m+1)m + c_m m^2\right] x^{m-1} = 0$$

where c_0 and c_1 are arbitrary:

$$c_{m+1} = \frac{m}{m+1}c_m \quad m \geq 2 \quad c_2 = \frac{c_1}{2}$$
$$c_3 = \frac{2}{3}c_2 = \frac{c_1}{3} \quad c_4 = \frac{3}{4}c_3 = \frac{c_1}{4} \ldots c_n = \frac{c_1}{n}$$

Therefore

$$y(x) = c_0 + c_1 \sum_{n=1}^{\infty} \frac{x^n}{n}. \tag{2.16}$$

We observe that the solutions (2.13) and (2.16) obtained by the two different methods are identical.

Recall

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \quad \int \frac{1}{1-x}\,dx = -\ln|1-x| = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$
$$y(x) = A + B\ln|x-1| \tag{2.17}$$

Thus the series solution is identical to the solution (2.12) provided $|x| \leq 1$. We note that the radius of convergence of convergence for the power series (2.13) is 1, which corresponds to the distance between the expansion point $x_0 = 0$ and the nearest singular point $x = 1$.

Power series solution of general variable coefficient linear ODE:
Consider solving variable coefficient linear ODEs of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0 \text{ Homogeneous Eq.} \tag{2.18}$$

Divide through by $P(x)$:

$$Ly = y'' + p(x)y' + q(x)y = 0 \quad p(x) = Q/P, \quad q(x) = R/P \tag{2.19}$$

In order to calculate the higher derivatives of $y(x)$ to substitute into Taylor's formula, we rewrite (2.19) as follows

$$y'' = -p(x)y' - q(x)y$$

If $y(x_0)$ and $y'(x_0)$ are given, then $y''(x_0)$ can be obtained directly from the ODE. Higher derivatives of y can, in turn, be obtained by differentiating the ODE repeatedly. This process will be successful provided $p(x)$ and $q(x)$ are infinitely differentiable at $x = x_0$. In this case $p(x)$ and $q(x)$ are said to be analytic at x_0 and have Taylor expansions of the form

$$\begin{aligned} \text{i.e.} \quad p(x) = p_0 + p_1(x-x_0) + \cdots &= \sum_{k=0}^{\infty} p_k (x-x_0)^k \\ q(x) = q_0 + q_1(x-x_0) + \cdots &= \sum_{k=0}^{\infty} q_k (x-x_0)^k \end{aligned}$$

Ordinary points: The expansion point x_0 is said to be an ordinary point of (2.19) if $p(x) = Q/P$ and $q(x) = R/P$ are analytic at x_0. If x_0 is an ordinary point it is possible to obtain power series expansions of the solution $y(x)$ of the form:

$$y(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n. \tag{2.20}$$

The idea is to substitute the expansion (2.20) into (2.19) and solve for the unknown coefficients c_n in order to determine a solution.
Observations:

- If P, Q and R are polynomials then a point x_0 such that $P(x_0) \neq 0$ is an ordinary point.
- If $x_0 = 0$ is an ordinary point then we assume

$$y = \sum_{n=0}^{\infty} c_n x^n, y_n' = \sum_{n=1}^{\infty} c_n n x^{n-1}, y_n'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

$$\begin{aligned} 0 = Ly = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} &+ \left(\sum_{n=0}^{\infty} p_n x^n\right) \sum_{n=1}^{\infty} n c_n x^{n-1} \\ &+ \left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} c_n x^n\right) \end{aligned} \tag{2.21}$$

$$\begin{aligned} &\sum_{m=0}^{\infty} \left\{(m+2)(m+1)c_{m+2} + \left(p_0(m+1)c_{m+1} + \cdots + p_m c_1\right)\right. \\ &\quad \left. + \; \left(q_0 c_m + \cdots + q_m c_0\right)\right\} x^m = 0 \end{aligned} \tag{2.22}$$

yields a non-degenerate recursion for the c_m. At an ordinary point x_0 we can obtain two linearly independent solutions of the form (2.20).

Singular Points: If $p(x)$ or $q(x)$ are not analytic at x_0, then x_0 is said to be a singular point of (2.19). For example if P, Q and R are polynomials and $P(x_0) = 0$ and $Q(x_0) \neq 0$ or $R(x_0) \neq 0$ then x_0 is a singular point. Or if $p(x) = \sqrt{(}x)$ and $q(x) = 2$, then $x_0 = 0$ is a singular point because $p(x)$ is not differentiable at $x = 0$.
The radius of convergence of (2.20) is at least as large as the radius of convergence of each of the series expansions for $p(x) = Q/P$ and $q(x) = R/P$, i.e., up to the closest singularity to x_0.

Example 5 *The Airy equation:* Consider the Airy equation, which arises in Quantum Mechanics:

$$Ly = y'' - xy = 0 \tag{2.23}$$

We observe that $x = 0$ is an ordinary point.

$$\begin{gathered} y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} c_n n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} \\ \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} c_n x^{n+1} \\ m+1 = n-2 \quad n = m+3 \quad n = 2 \Rightarrow m = -1 \\ c_2 2x^0 + \sum_{m=0}^{\infty} \left[c_{m+3}(m+3)(m+2) - c_m\right] x^{m+1} = 0 \\ c_2 = 0 \quad c_{m+3} = \tfrac{c_m}{(m+3)(m+2)} \quad m = 0, 1, \dots \end{gathered} \tag{2.24}$$

(1) $c_0 \to c_3 \to c_6$.

$$\begin{aligned} c_3 &= \frac{c_0}{3.2}, c_6 = \frac{c_3}{6.5} = \frac{c_0}{6.5.3.2}, c_9 = \frac{c_0}{9.8.6.5.3.2} \\ c_{3n} &= \frac{c_0}{(3n)(3n-1)(3n-3)(3n-4)\dots 9.8.6.5.3.2} \\ y_0(x) &= 1 + \frac{x^3}{3.2} + \frac{x^6}{6.5.3.2} + \dots + \frac{x^{3n}}{(3n)(3n-1)\dots 3.2} + \dots \end{aligned} \tag{2.25}$$

(2) $c_1 \to c_4 \to c_7 \to$.

$$c_4 = \frac{c_1}{4.3} \quad c_7 = \frac{c_1}{7.6.4.3} \quad c_{10} = \frac{c_1}{(10.9)(7.6)(4.3)} \tag{2.26}$$

$$\begin{aligned} c_{3n+1} &= \frac{c_1}{(3n+1)(3n)(3n-2)(3n-3)\dots(7.6)(4.3)} \\ y_1(x) &= x + \frac{x^4}{4.3} + \frac{x^7}{7.6.4.3} + \dots + \frac{x^{3n+1}}{(3n+1)(3n)\dots 4.3} \\ y(x) &= c_0 y_0(x) + c_1 y_1(x) \end{aligned} \tag{2.27}$$

Radius of Convergence:

$$\lim_{m\to\infty} \frac{c_{m+3}}{c_m}|x|^3 = \lim_{m\to\infty} \frac{|x|^3}{(m+3)(m+2)} = 0 < 1 \quad \rho = \infty. \tag{2.28}$$

See B&D for expansion of Airy Solution about $x_0 = 1$:, i.e. $y(x) = \sum a_n (x-1)^n$. It is useful to write $x = (x-1)+1$.

$$y'' = (x-1)y + y \tag{2.29}$$

Example 6 *The Hermite Equation*

Consider the Hermite equation, which has application in Quantum mechanics and numerical analysis:

$$Ly = y'' - 2xy' + \lambda y = 0 \tag{2.30}$$

Since $x = 0$ is an ordinary point let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ then

$$Ly = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2\sum_{n=1}^{\infty} a_n n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0. \tag{2.31}$$

$$\begin{array}{lll} m = n-2 \rightarrow n = m+2 & m \leftarrow n & m \leftarrow n \\ n = 2 \Rightarrow m = 0 & & \end{array}$$

Therefore

$$\sum_{m=1}^{\infty} \left[a_{m+2}(m+2)(m+1) - 2a_m m + \lambda a_m\right] x^m + \left[a_2 2 + \lambda a_0\right] x^0 = 0. \tag{2.32}$$

x^0 :

$$a_2 = -\lambda a_0/2 \tag{2.33}$$

x^m :

$$a_{m+2} = \frac{(2m-\lambda)a_m}{(m+1)(m+2)} \quad m \geq 1 \tag{2.34}$$

a_0:

$$\begin{aligned} a_2 &= -\frac{\lambda}{2}a_0, a_4 = \frac{(4-\lambda)}{4.3}a_2 = \frac{(4-\lambda)(-\lambda)}{4.3.2}a_0, a_6 = \frac{(8-\lambda)(4-\lambda)(-\lambda)}{6.5.4.3.2}a_0 \\ a_{2k} &= \frac{[4(k-1)-\lambda][4(k-2)-\lambda]\ldots[-\lambda]}{(2k)!}a_0 \\ y_0 &= a_0\left[1 - \frac{\lambda}{2}x^2 + \frac{(\lambda-4)\lambda}{4!}x^4 + \frac{(8-\lambda)(4-\lambda)(-\lambda)}{6!}x^6 + \cdots\right] \end{aligned} \tag{2.35}$$

a_1:

$$\begin{aligned} a_3 &= \frac{(2-\lambda)}{3.2}a_1; a_5 = \frac{(6-\lambda)}{5!}(2-\lambda)a_1; a_7 = \frac{(10-\lambda)(6-\lambda)(2-\lambda)}{7!}a_1, \ldots \\ y_1 &= a_1\left[x + \frac{(2-\lambda)}{3!}x^3 + \frac{(6-\lambda)(2-\lambda)}{5!}x^5 + \frac{(10-\lambda)(6-\lambda)(2-\lambda)x^7}{7!} + \cdots\right] \end{aligned} \tag{2.36}$$

The general solution is of the form

$$y(x) = Ay_0(x) + By_1(x) \tag{2.37}$$

Note:

(a) If $\lambda = 2n$ then the recursion yields $a_{m+2} = 0 = a_{m+4} = \cdots$ for $m = n$. Thus if n is an even integer then the series solution y_0 will terminate and become a polynomial of degree n.

In this case:

$$\begin{aligned} y_0(x) = a_0 \Bigg[&1 - nx^2 + n(n-2)2^2\frac{x^4}{4!} - n(n-2)(n-4)\frac{2^3x^6}{6!} + \cdots \\ &+ (-1)^{n/2}n(n-2)\ldots 2.\left(2^{n/2}\right)\frac{x^n}{n!}\Bigg]. \end{aligned} \tag{2.38}$$

On the other hand if n is an odd integer then the series solution $y_1(x)$ will terminate and become a polynomial of degree n. In this case

$$\begin{aligned} y_1(x) = a_1 \Big[& x - 2(n-1)\frac{x^3}{3!} + 2^2(n-1)(n-3)\frac{x^5}{5!} \\ & - (n-1)(n-3)(n-5)2^3\frac{x^7}{7!} + \cdots \\ & + (n-1)(n-3)\ldots 3.1(-2)^{\frac{(n-1)}{2}}\frac{x^n}{n!}\Big] \end{aligned} \tag{2.39}$$

(b) For example in the special case $\lambda = 4 = 2n$ then $n = 2$.

$$y_0(x) = a_0[1 - 2x^2]. \tag{2.40}$$