1. Consider the following ordinary differential equation
\[ ty' + 2y = t^{-1}e^t, \quad y(1) = 1 \]

(a) Write the equation in the following form
\[ y' + p(t)y = g(t). \]

(b) Compute \( \mu(t) = e^{\int p(t)dt} \) and \( \int \mu(t)g(t)dt. \)

(c) Find the solution and state the Interval of Existence.

(a) \[ y' + \frac{2}{t}y = t^{-2}e^t \]
\[ p(t) = \frac{2}{t}, \quad q(t) = \frac{1}{t^2}e^t \]

(b) \[ \mu(t) = e^{\int \frac{2}{t}dt} = t^2 \]
\[ \int \mu(t)g(t)dt = \int t^2 \frac{1}{t}e^t dt = e^t + c \]

(c) \[ y = \frac{1}{\mu(t)} \left( c + \int \mu(t)g(t)dt \right) \]
\[ = \frac{1}{t^2} \left( c + e^t \right) \]
\[ y(1) = 1 \Rightarrow c + 1 = 1 \Rightarrow c = 0 \]
\[ y(t) = \frac{1}{t^2}e^t + \frac{e^t}{t^2} \]

Interval of Existence: \((0, +\infty)\)
20 points) 2. Solve the following ordinary differential equation

\[ \frac{dy}{dt} = \frac{t}{2(y - y^3)}, \quad y(0) = -2 \]

and state the Interval of Existence.

\[ 2(y + y^2) \, dy = t \, dt \]

\[ \int 2(y + y^2) \, dy = \int t \, dt \]

\[ y^2 - \frac{y^4}{2} = \frac{t^2}{2} + C \]

\[ y^4 - 2y^2 = -t^2 + C \]

\[ t = 0, \quad y = -2, \quad C = 8 \]

\[ y^4 - 2y^2 = 8 - t^2 \]

\[ (y^2 - 1)^2 = 9 - t^2 \]

\[ y^2 = 1 \pm \sqrt{9 - t^2} \]

\[ y^2 = 1 + \sqrt{9 - t^2} \]

\[ y = -\sqrt{1 + \sqrt{9 - t^2}} \]

Interval of existence: \(-3 < t < 3\)
3. Consider the following ordinary differential equation

\[ y' = (y - 4) \log y, \quad y > 0. \]

(a) Find all critical points and classify the stability/instability of these critical points.

(b) Let \( y(0) = \frac{1}{2} \). What is the asymptotic behavior of \( y(t) \) as \( t \to +\infty \)?

(c) Let \( y(0) = 2 \). What is the asymptotic behavior of \( y(t) \) as \( t \to +\infty \)?

\[
\begin{align*}
(a) \quad f(y) &= (y - 4) \log y \\
\frac{f(y)}{y} &= 0 \implies y - 4 = 0 \text{ or } \log y = 0 \\
y_1 &= 4, \quad y_2 = 1 \quad \text{are critical points} \\
f'(y) &= + \log y + (y - 4) \frac{1}{y} \\
f'(y_1) &= + \log 4 \not= 0 \implies y_1 \text{ is unstable} \\
f'(y_2) &= - 3 < 0 \implies y_2 \text{ is stable} \\
\end{align*}
\]

(b) \( y(0) = \frac{1}{2}, \quad y_1 = 1 = y_2 \implies y(t) \to 1 \text{ as } t \to +\infty \)

(c) \( y(0) = 2, \quad f(y) < 0 \implies y(t) \to 1 \text{ as } t \to +\infty \)
4. Consider the following ordinary differential equation

\[ t^2 y'' - ty' + y = 0 \]

.5 points) (a) Write the equation in the following form:

\[ y'' + p(t)y' + q(t)y = 0 \]

2 points) (b) Find the Wronskian \( W \).

5 points) (c) Let \( y_1 = t \) be a solution. Use reduction of order to find \( y_2(t) = v(t)y_1(t) \). Hint: you may use the formula: \( v' = \frac{v}{y_1} \).

8 points)

(a) \[ y'' - \frac{1}{t} y' + \frac{1}{t^2} y = 0 \]

\[ p = -\frac{1}{t}, \quad q = \frac{1}{t^2} \]

(b) \[ w' + pw = 0 \implies W = e^{-\int p(t) dt} = c t \]

\[ \text{take } e. \]

(c) \[ v' = \frac{w}{y_1^2} = \frac{c t}{t^2} \]

\[ v = c \ln t \]

\[ y_2 = c(\ln t) \cdot t \]
5. Consider the following second order ordinary differential equation:

\[ y'' - y' - 2y = h(t) \]

(a) Find the solutions to the homogeneous problem

\[ y'' - y' - 2y = 0. \]

(b) Suppose \( h(t) = \cos(t) + 2e^t \). Use the method of undetermined coefficients to find the form of the special solution \( y_p \). Do not attempt to find the coefficients.

(c) Suppose \( h(t) = te^{2t} \). Use the method of undetermined coefficients to find the form of the special solution \( y_p \). Do not attempt to find the coefficients.

(d) Solve the following second order differential equation

\[ y'' - y' - 2y = t, y(0) = 0, y'(0) = 1 \]

(a) \( r^2 - r - 2 = 0 \) \( \Rightarrow \) \( r_1 = 2, \ r_2 = -1 \)

\[ y = c_1 e^{2t} + c_2 e^{-t} \]

(b) \( y_p = A \cos t + B \sin t + Ce^t \)

(c) \( y_p = t^s (A t + B) e^{2t}, s = 1 \)

(d) \[ y_p = At + B \]

\[ -A - 2At - 2B = t \] \( \Rightarrow \) \( A = -\frac{1}{3}, \ B = \frac{1}{3} \)

\[ y = c_1 e^{2t} + c_2 e^{-t} - \frac{1}{2} t + \frac{1}{4} \]

\[ y(0) = 0 \] \( \Rightarrow \) \( c_1 + c_2 = -\frac{1}{4} \)

\[ y'(0) = 1 \] \( \Rightarrow \) \( 2c_1 - c_2 = -\frac{1}{2} = 1 \)

\[ c_1 = \frac{5}{12}, \ c_2 = -\frac{8}{12} = -\frac{2}{3} \]

\[ y = \frac{5}{12} e^{2t} - \frac{2}{3} e^{-t} - \frac{1}{2} t + \frac{1}{4} \]
(15 points) 6. Use the method of variation of parameters to solve the inhomogeneous problem

\[ y'' + 9y = \frac{3}{\cos(3t)} - \frac{\pi}{6} < t < \frac{\pi}{6}. \]

Hint: You may use the formula \( \int \frac{\sin u}{\cos u} \, du = -\log \cos u + C. \)

\[ y'' + 9y = 0 \implies r^2 + 9 = 0 \implies r = \pm 3i \]
\[ \implies y_1 = \cos 3t, \quad y_2 = \sin 3t \]

\[ y_p = u_1 y_1 + u_2 y_2 \]

\[ \begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1 + u_2'y_2 = g(t) \end{cases} \]

\[ \implies \begin{cases} u_1' \cos 3t + u_2' \sin 3t = 0 \\ u_1'(-3 \sin 3t) + u_2'(3 \cos 3t) = \frac{3}{\cos t} \end{cases} \]

\[ \begin{align*}
    u_1' \cos 3t + u_2' \sin 3t &= 0 \\
    u_1'(-3 \sin 3t) + u_2'(3 \cos 3t) &= \frac{3}{\cos t} \\
    u_1' &= -\frac{\sin 3t}{\cos 3t} \implies u_1 = -\int \frac{\sin 3t}{\cos 3t} \\
    u_2' &= 1 \implies u_2 = t \\
\end{align*} \]

\[ u_1 = -\frac{1}{3} \log \cos 3t \]

\[ y_p = \frac{1}{3} \log \cos 3t \cos 3t + t \sin 3t \]

\[ u = c \cos 3t + c_2 \sin 3t + \frac{1}{3} \log \cos 3t \cos 3t + t \sin 3t \]