FINITE-TIME SINGULARITY FORMATIONS FOR THE LANDAU-LIFSHITZ-GILBERT EQUATION IN DIMENSION TWO

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Abstract. We construct finite time blow-up solutions to the Landau-Lifshitz-Gilbert equation (LLG) from $\mathbb{R}^2$ into S^2

$$\begin{cases} u_t = a(\Delta u + |\nabla u|^2 u) - bu \wedge \Delta u & \text{in } \mathbb{R}^2 \times (0,T), \\ u(\cdot,0) = u_0 \in S^2 & \text{in } \mathbb{R}^2, \end{cases}$$

where $a^2 + b^2 = 1,\ a > 0,\ b \in \mathbb{R}$. Given any prescribed N points in $\mathbb{R}^2$ and small $T > 0$, we prove that there exists regular initial data such that the solution blows up precisely at these points at finite time $t = T$, taking around each point the profile of sharply scaled degree 1 harmonic map with the type II blow-up speed

$$\|\nabla u\|_{L^\infty} \sim \frac{|\ln(T-t)|^2}{T-t} \quad \text{as} \quad t \to T.$$

The proof is based on the *parabolic inner-outer gluing method*, developed in [12] for Harmonic Map Flow (HMF). However, substantial difficulties arise due to the coupling between HMF and Schrodinger Map Flow (SMF) in LLG, and such coupling produces both dissipative ($a > 0$) and dispersive ($b \neq 0$) features. A direct consequence of the presence of dispersion is the *lack of maximum principle* for suitable quantities, which makes the analysis more delicate even at the linearized level. The dispersion cannot be treated perturbatively even in the dissipation-dominating case $a/|b| >> 1$, and one has to include this as part of the leading order. To overcome these difficulties, we make use of two key technical ingredients: first, for the inner problem we employ the tool of *distorted Fourier transform*, as developed by Krieger, Miao, Schlag and Tataru [34, 35]. Second, the linear theory for the outer problem is achieved by means of the sub-Gaussian estimate for the fundamental solution of parabolic system in non-divergence form with coefficients of Dini mean oscillation in space ($\mathsf{DMO_x}$), which was proved by Dong, Kim and Lee [20] recently.

Contents

1. Introduction

Let $\mathcal{M}$ be a m-dimensional Riemannian manifold of the metric g and S^2 be the 2-sphere embedded in $\mathbb{R}^3$. The Landau-Lifshitz-Gilbert equation (LLG) on $\mathcal{M}$ is given by

$$\begin{cases} u_t = -au\wedge(u\wedge\Delta_{\mathcal{M}}u) - bu\wedge\Delta_{\mathcal{M}}u \ \text{ in } \ \mathcal{M}\times(0,T) \\ u(\cdot,0)=u_0\in S^2 \ \text{ in } \ \mathcal{M}, \end{cases} \tag{1.1}$$

where $a^2+b^2=1$, $a\geq 0$, $b\in\mathbb{R}$, $\Delta_{\mathcal{M}}u = \frac{1}{\sqrt{g}}\partial_{x_\beta}(g^{\alpha\beta}\sqrt{g}\partial_{x_\alpha}u)$ is the Laplace-Beltrami operator and $u=(u_1,u_2,u_3)$ is a 3-vector with normalized length which is a mapping $u(x,t):\mathcal{M}\times(0,T)\to S^2$. First formulated by Landau and Lifshitz [36] in 1935, LLG (1.1) is an important system modeling the effects of a magnetic field on ferromagnetic materials in micromagnetics, and it describes the evolution of spin fields in continuum ferromagnetism. See also Gilbert [22]. LLG (1.1) can be viewed as a bridge between the harmonic map flow (HMF) when $a=1, b=0$ and the Schrödinger map flow (SMF) when $a=0, b=-1$.

In the context of HMF, Struwe [51] proved the existence and uniqueness of weak solution with at most finitely many singular points when $\mathcal{M}$ is a Riemann surface. See [21] for further generalizations and [9, 52] for higher dimensional cases. Chang, Ding and Ye [7] first proved the existence of finite time blow-up solutions for HMF from disk into S^2. See also [8, 10, 16, 37, 45, 47, 54, 57] and the references therein. In [55], van den Berg, Hulshof and King used formal analysis to predict the existence of blow-up solutions with quantized rates

$$\lambda_k(t)\sim\frac{|T-t|^k}{|\ln(T-t)|^{\frac{2k}{2k-1}}}, \quad k\in\mathbb{N}^+ \tag{1.2}$$

for the two-dimensional HMF into S^2. For the case $\mathcal{M}=\mathbb{R}^2$ and the target manifold is a revolution surface, using degree 1 harmonic map Q_1 as the building block, Raphaël and Schweyer [48, 49] constructed finite time blow-up solutions with rates (1.2) for all $k\geq 1$ in the equivariant class, where the initial data can be taken arbitrarily close to Q_1 in the energy-critical topology. For the case that $\mathcal{M}\subset\mathbb{R}^2$ is a general bounded domain, Dávila, del Pino and Wei [12] constructed solutions which blow up at finite many points with the type II rate (1.2) for $k=1$, and they further investigated the stability and reverse bubbling phenomena. The construction in [12] can be generalized to the case $\mathcal{M}=\mathbb{R}^2$.

On the other hand, for SMF with $\mathcal{M}=\mathbb{R}^2$, Merle, Raphaël and Rodnianski [41] constructed the finite time blow-up solution with the rate (1.2) for $k=1$ in the 1-equivariant class. Analogous to the results in Krieger-Schlag-Tataru [32] for wave maps, Perelman [44] constructed finite time blow-up solutions with continuous rates. See also [3, 4, 5, 6, 29] and the references therein for the global well-posedness results and the dynamics of SMF near ground state.

For LLG, in the case $\mathcal{M} = \mathbb{R}^3, a > 0$, Alouges and Soyeur [1] proved the existence of weak solutions for (1.1) and constructed infinitely many weak solutions. The existence for the weak solution to LLG has been established by Guo and Hong [23, Theorem 4.2] when $\mathcal{M}$ is a closed Riemannian manifold with $m \geq 3$, while for the case that $\mathcal{M}$ is a closed Riemannian surface, the weak solution was shown to be unique and regular except for at most finitely many points [23, Theorem 3.13]. When $\mathcal{M} = \mathbb{R}^2$ and the target manifold is a smooth closed surface embedded in $\mathbb{R}^3$, approximation by discretization was used in [31] to construct a solution of LLG which is smooth away from a two-dimensional locally finite Hausdorff measure. In general, one cannot expect good partial regularity results for weak solutions in the higher dimensional case $m \geq 3$ without further regularity or energy minimizing assumptions. See the famous example by Rivière [50], where weakly harmonic maps from the ball $B^3 \subset \mathbb{R}^3$ into S^2 were constructed for which the singular set Sing u is the closed ball $\overline{B^3}$, and this result can be generalized to higher dimensions. In a similar spirit to the existence results for partially regular solution for HMF in higher dimensions of Chen and Struwe [9], Melcher [39] proved that for $\mathcal{M} = \mathbb{R}^m$ $(m = 3)$ there exists a global weak solution whose singular set has finite 3-dimensional parabolic Hausdorff measure. Later, this result was generalized to $m \leq 4$ by Wang [58]. With the additional stability assumption for the weak solution, for $m \leq 4$, Moser [42] proved better estimate for the singular set. The partial regularity of LLG (1.1) for $m \geq 5$ still remains open.

For $\mathcal{M} = \mathbb{R}^m$, the global existence, uniqueness and decay properties for the solution of (1.1) were established by Melcher [40] for $m \geq 3$ with initial data u_0 close to a fixed point in S^2 in the L^m norm. Lin, Lai and Wang [38] generalized the result to Morrey space and $m \geq 2$. For u_0 away from a fixed point in S^2 with BMO semi-norm sufficiently small, Gutiérrez and de Laire [27] proved the global existence, uniqueness and regularity results for LLG.

The study of the dynamics for LLG with initial data close to harmonic maps is of special significance and can provide hints on the mechanism of singularity formation. A series of works by Gustafson, Tsai and collaborators [24, 25, 26] are devoted to the behavior of the solutions to LLG with $\mathcal{M} = \mathbb{R}^2$ with initial data u_0 close to the harmonic map in the n-equivariant class. They found, among other things, that there is no finite time blow-up for LLG and HMF with u_0 close to n-equivariant harmonic maps for $n \geq 3$ and $n = 2$, respectively. See [26, Theorem 1.1], [26, Theorem 1.2].

The singularity formation for LLG is an important and challenging topic. For the case that $\mathcal{M}$ is a compact manifold with or without boundary in dimensions $m = 3, 4$, Ding and Wang [15] obtained the existence of a smooth finite time blow-up solution for LLG. For $\mathcal{M} \subset \mathbb{R}^2$, as an analogue of Qing [46] for HMF, Harpes [28] gave descriptions of solutions to LLG (1.1) near the singular points. For the energy critical case that $\mathcal{M}$ is a disk in $\mathbb{R}^2$, in an interesting paper [56], van den Berg and Williams predicted the existence of finite time blow-up by formal asymptotic analysis supported with numerical simulations. For $\mathcal{M} = \mathbb{R}^2$, Xu and Zhao [59] rigorously constructed a finite time blow-up solution to (1.1) in the 1-equivariant class.

In this paper, we consider the case with target manifold S^2, $\mathcal{M} = \mathbb{R}^2$, and positive damping parameter $a > 0$. LLG (1.1) can then be written as

$$\begin{cases} u_t = a(\Delta u + |\nabla u|^2 u) - bu \wedge \Delta u & \text{in } \mathbb{R}^2 \times (0, T), \\ u(\cdot, 0) = u_0 \in S^2 & \text{in } \mathbb{R}^2. \end{cases} \tag{1.3}$$

We are interested in the case of *multiple bubbles* to LLG (1.3) in the general *non-radially symmetric setting.* Our construction is based on the following degree 1 profile

$$W(y) = \frac{1}{|y|^2 + 1} \begin{bmatrix} 2y_1 \\ 2y_2 \\ |y|^2 - 1 \end{bmatrix}, \quad y = (y_1, y_2) \in \mathbb{R}^2,$$

and clearly

$$Q_\gamma W\left(\frac{x - \xi}{\lambda}\right)$$

solves the stationary equation of LLG (1.3) for any $\xi \in \mathbb{R}^2$, $\lambda > 0$, and any γ-rotation matrix around z-axis

$$Q_\gamma := \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us define $U_\infty = (0,0,1)^T$. Our main result is stated as follows.

Theorem 1. *For any prescribed N distinct points $q^{[j]} \in \mathbb{R}^2$, $j = 1, 2, \ldots, N$, $N \in \mathbb{Z}_+$ and T sufficiently small, there exists initial data u_0 such that the gradient of the solution u to LLG (1.3) with $a > 0$ blows up at these N points at finite time $t = T$. More precisely, the solution u takes the sharply scaled degree 1 profile around each point $q^{[j]}$*

$$u(x,t) = -(N-1)U_\infty + \sum_{j=1}^{N} Q_{\gamma_j} W\left(\frac{x-\xi^{[j]}}{\lambda_j}\right) + \Phi_{\rm per}$$

with

$$\lambda_j(t) = \kappa_j^* \frac{|\ln T|(T-t)}{|\ln(T-t)|^2}(1+o(1)),\ \xi^{[j]}(t) = q^{[j]}(1+o(1)),\ \gamma_j(t) = \gamma_j^*(1+o(1)) \quad as \quad t \to T$$

for some $\kappa_j^* = \kappa_j^*(a,b) \in \mathbb{R}_+$, $\gamma_j^* \in \mathbb{R}$, $o(1) \to 0$ as $t \to T$ and $\Phi_{\rm per}$ of smaller order.

Remark 1.1.

- *The positivity of the Gilbert damping parameter (a) plays a crucial role in our construction.*
- *The bubbling solution at multiple points constructed in Theorem 1 is type II, and for $j = 1, \ldots, N$,*

$$|\nabla u(q^{[j]}, t)| \sim \frac{|\ln(T-t)|^2}{|\ln T|(T-t)} \quad as \quad t \to T.$$

  Moreover, the dependence of the blow-up speed λ_j on the parameters a and b in (1.3) is in κ_j^*, which is of order 1.
- *The perturbative term $\Phi_{\rm per}$ in Theorem 1 is constructed in carefully designed weighted topologies. See Section 5.4.*

The proof of Theorem 1 is based on the *parabolic inner-outer gluing method*, which was recently developed in [11] and [12] to investigate the singularity formation for evolution PDEs. See also [13, 14] for elliptic analogues developed earlier. Our study of the singularity formation for LLG is motivated by the one for HMF [12]. However, substantial difficulties arise due to the coupling between HMF and SMF in LLG (1.3), and such coupling produces both dissipative ($a > 0$) and dispersive ($b \neq 0$) features. A direct consequence of the presence of dispersion is the lack of maximum principle for suitable quantities, which makes the analysis more delicate even at the linearized level. The dispersion cannot be treated perturbatively even in the dissipation-dominating case $a/|b| \gg 1$, and one has to include this as part of the leading order. In our inner-outer gluing construction, new linear theories for both inner and outer problems need to be developed, taking into account the dispersion.

The new linear theory for the inner problem is developed by analyzing each Fourier mode, which is the Fourier expansion of the complex form on each tangent plane of the bubble on S^2. Due to the absence of maximum principle, we first employ energy methods to get rough upper bounds for each mode. In order to refine the bounds and get better pointwise decay estimates, we perform another gluing procedure, called re-gluing process, at all the modes except mode -1. The re-gluing process was first used in the analysis of linearization of HMF at mode 0 in [12], and here we generalize this technique to all other modes except mode -1. For mode -1, using above method does give a solution, but this solution gets deteriorated in the innermost region and is not sufficient for the gluing to implement. Instead, we utilize the techniques of distorted Fourier transform for the dealing of mode -1. The use of distorted Fourier transform is motivated by a recent work of Krieger, Miao and Schlag [34] on the stability of blow-up for wave maps beyond the equivariant class. See also [32, 33, 35] by Krieger, Miao, Schlag, Tataru, and the references therein. Using the distorted Fourier transform, we develop linear theory at mode -1 with or without orthogonality conditions. The version with orthogonality removes the logarithmic loss compared to the one without orthogonality. See Section 9.6 for more details. In this paper, for mode -1, we only use the one without orthogonality since the introduction of two new modulation parameters will further complicate the interactions, and we control the logarithmic loss by Hölder properties inherited from the equations.

The outer problem turns out to be a quasi-linear parabolic system in non-divergence form. Different from the linearized outer problem in HMF, the one in LLG is a coupled system, and thus cannot be solved componentwisely. The coefficients of the coupled system for the linearized outer problem is in fact part of the blow-up profile. So one cannot expect good Hölder continuity in the coefficients and has to work in certain weaker class. The linear theory for the outer problem is achieved by means of the sub-Gaussian estimate for the fundamental solution of parabolic system in non-divergence form with coefficients of Dini mean oscillation in space ($\mathsf{DMO_x}$), which was proved by Dong, Kim and Lee [20] recently. We introduce Dini mean absolute oscillation in space ($|\mathsf{DMO}|_{\mathsf{x}}$), which is a subspace of $\mathsf{DMO}_{\mathsf{x}}$. Under some weak assumptions, the functions in $|\mathsf{DMO}|_{\mathsf{x}}$ are closed under arithmetic (see Lemma 7.1). This property makes it easier to achieve that the leading coefficients of the outer problem belong to $|\mathsf{DMO}|_{\mathsf{x}}$.

Another aspect in the construction is the dealing of slow decaying errors, usually present in lower dimensional problems. The improvement of these slow decaying errors involves finding good global/non-local corrections, which in turn make the dynamics for the parameters in the corresponding mode non-local. In the context of LLG, the mode with slow decaying error that we shall deal with is mode 0, which corresponds to the invariance of scaling and rotation around z-axis. To capture the precise blow-up dynamics, the non-local correction at mode 0 should be rather explicit. But due to the aforementioned structure of the outer problem, one cannot improve the error by solving the linearized system directly and has to extract part of the parabolic system instead. It turns out that the combination of the new error produced by the non-local correction and the remainder in the parabolic system together make the non-local equations for the scaling parameter λ and rotational parameter γ a well-structured complex system. See Section 6.1.

The construction of multiple bubbles also involves analyzing complicated interactions, and this reflects in the analysis on the tangent plane of each bubble. On the other hand, the ansatz for the solution u with multiple bubbles needs to be carefully chosen as the unit-length of the map $|u| = 1$ should be kept for all space-times. This further produces delicate interactions. Fortunately, we are able to control these in some well-designed topologies thanks to subtle cancellations as well as a trick that we call U_*-operation (see (5.6)), which simplifies analysis. This trick first appeared in [12] in the case of single bubble for HMF.

The rest of this paper is devoted to the proof of Theorem 1.

2. Sketch of the construction

Due to the complexities and technicalities in the construction, we sketch a roadmap of the major steps in this section.

• **Ansatz of multiple bubbles**

The construction begins with a careful choice of first approximation. Since the target is the 2-sphere, one has to choose some profile for multiple bubbles which is relatively reasonable to analyze. Here we choose

$$U_* = -(N-1)U_\infty + \sum_{j=1}^{N} Q_{\gamma_j} W\left(\frac{x-\xi^{[j]}}{\lambda_j}\right) := -(N-1)U_\infty + \sum_{j=1}^{N} U^{[j]}$$

as the first approximation. Notice that $|U_*| \approx 1$ at any space-times as those bubbles are essentially separated. Based on U_*, we then look for solution to LLG in the form

$$u(x,t) = (1+A)U_* + \Phi - (\Phi \cdot U_*)U_* \tag{2.1}$$

for some perturbation term Φ and scalar A. Here the purpose of the scalar A, depending on Φ, is to preserve the unit-length of the map $u(x,t)$ for any $(x,t) \in \mathbb{R}^2 \times (0,T)$. So here part of the interactions between bubbles get encoded in the scalar A. Let us denote the error of u as

$$S(u) := -u_t + a(\Delta_x u + |\nabla_x u|^2 u) - bu \wedge \Delta_x u.$$

An important observation here is that instead of solving $S(u) = 0$, we only need to solve

$$S(u) = \Xi(x,t)U_* \tag{2.2}$$

for some scalar function Ξ. Indeed, since $|u| = 1$ is kept for all $t \in (0,T)$ and $u = U_* + \tilde{w}$ where the perturbation $\tilde{w}$ is uniformly small, then

$$\Xi(U_* \cdot u) = S(u) \cdot u = -\frac{1}{2}\partial_t(|u|^2) + \frac{a}{2}\Delta|u|^2 = 0.$$

Thus $\Xi \equiv 0$ follows from $U_* \cdot u \geq \delta_0 > 0$. (5.6) provides us the flexibility to adjust the error terms in U_* direction, and we will call this U_*-*operation* throughout this paper. This operation can simplify analysis especially for the dealing of multiple bubbles.

• **Slow decaying errors and non-local corrections by approximate parabolic system**

The error $S(U_*)$ contains slowing decaying terms

$$\sum_{j=1}^{N} \mathcal{E}_0^{[j]} \notin L^2(\mathbb{R}^2)$$

which correspond to the re-scaling and rotation around z-axis. To improve the spatial decay of the error at remote region, we add well-designed global/non-local corrections around each bubble. Since the operator

$$-\partial_t + (a - bU^{[j]}\wedge)\Delta_x$$

depends on the blow-up profile U^j as well as the parameters λ_j, γ_j and $\xi^{[j]}$, one cannot expect explicit representation formula apriori without knowing the blow-up dynamics. We consider instead an approximate parabolic operator

$$-\partial_t + (a - bU_\infty\wedge)\Delta_x$$

and add correction $\Phi_0^{*[j]}$ around each bubble $U^{[j]}$ with

$$-\partial_t \Phi_0^{*[j]} + (a - bU_\infty\wedge)\Delta_x \Phi_0^{*[j]} + \mathcal{E}_0^{[j]} \approx 0.$$

Then the new error with corrections is given by those created by $\Phi_0^{*[j]}$ and the remainder $b(U_\infty - U^{[j]}) \wedge \Delta_x \Phi_0^{*[j]}$. This is rather important in the analysis of the non-local reduced problems.

• Formulation of the inner-outer gluing system

We next look for the perturbation Φ in (2.1) consisting of inner and outer parts with non-local corrections added

$$\Phi(x,t) = \sum_{j=1}^{N} \left(\eta_R^{[j]}(x,t) Q_{\gamma_j} \Phi_{\rm in}^{[j]}(y^{[j]},t) + \eta_{d_q}^{[j]}(x,t)\Phi_0^{*[j]} \right) + \Phi_{\rm out}(x,t)$$

where $\Phi_{\rm in}^{[j]}$ is on the tangent plane of $U^{[j]}$, $\eta_R^{[j]}$ and $\eta_{d_q}^{[j]}$ are suitable cut-off functions near $q^{[j]}$. Then u solving LLG implies a coupled inner-outer gluing system for $\Phi_{\rm in}^{[j]}$ and $\Phi_{\rm out}$, $j = 1, \dots, N$

$$\begin{cases} \lambda_j^2 \partial_t \Phi_{\rm in}^{[j]} = \left(a - bW^{[j]}\wedge\right) \left[\Delta_{y^{[j]}} \Phi_{\rm in}^{[j]} + |\nabla_{y^{[j]}} W^{[j]}|^2 \Phi_{\rm in}^{[j]} + 2\left(\nabla_{y^{[j]}} W^{[j]} \cdot \nabla_{y^{[j]}} \Phi_{\rm in}^{[j]} \right) W^{[j]} \right] \\ \qquad\qquad + \mathcal{H}^{[j]}[\Phi_{\rm in}^{[j]}, \Phi_{\rm out}, \lambda_j, \gamma_j, \xi^{[j]}] \ \text{ in } \ \mathsf{D}_{2R}, \\ \partial_t \Phi_{\rm out} = \mathbf{B}_{\Phi,U_*} \Delta_x \Phi_{\rm out} + \mathcal{G}[\Phi_{\rm in}^{[j]}, \Phi_{\rm out}, \lambda_j, \gamma_j, \xi^{[j]}] \ \text{ in } \ \mathbb{R}^2 \times (0,T), \end{cases}$$

where $W^{[j]}$ is the j-th bubble expressed in the rescaled variable $y^{[j]} = \frac{x - \xi^{[j]}}{\lambda_j}$, the right hand sides $\mathcal{H}^{[j]}$, $\mathcal{G}$ consists of the error terms, couplings and nonlinear terms depending on the parameters λ_j, γ_j, $\xi^{[j]}$, and $\mathbf{B}_{\Phi,U_*}$ is a matrix, defined in (5.26), that involves the perturbation Φ and the blow-up profile U_*.

For the full system above, finding blow-up of LLG at multiple points now gets reduced to finding well-behaved inner and outer profiles such that gluing procedure can be implemented. In other words, we need to devise appropriate weighted topologies in which the gluing system becomes weakly coupled and thus can be solved by fixed point arguments. For the outer problem, we make use of the sub-Guassian estimate recently proved in [20]. For the inner problem, good solutions with sufficient decay in space and time can only be captured with careful choices of the parameters λ_j, γ_j, $\xi^{[j]}$. We shall develop linear theory for the inner problems with orthogonality conditions, and these orthogonalities in turn determine the blow-up dynamics.

• Solving the inner problem

The linear theory for the inner problem is established by analyzing each Fourier mode. Decomposing the complex form in Fourier modes, one obtains the linearized operator at mode k of the form

$$\lambda_j^2 \partial_t - (a - ib) \left(\partial_{\rho_j \rho_j} + \frac{\partial_{\rho_j}}{\rho_j} - \frac{(k+1)^2\rho^4 + (2k^2 - 6)\rho^2 + (k-1)^2}{(\rho^2+1)^2} \frac{1}{\rho^2} \right), \tag{2.3}$$

where $\rho_j = |y^{[j]}|$. Then for all the modes $k \in \mathbb{Z} \backslash \{-1\}$, good inner solutions are found by the following

Step 1: we first use energy methods to get a rough pointwise upper bounds for the inner solutions ϕ_k;
Step 2: next we use Duhamel's formula to refine the pointwise bounds and further gain decay estimates;
Step 3: finally we perform re-gluing procedure to obtain better estimates in the innermost region.

As mentioned earlier, the treatment for mode $k = -1$ is different from the techniques that we employ for all the other modes. The reason is the following: as one can see from (2.3), mode -1 can be roughly viewed as a problem in 2D, which is worse than any other mode as one cannot gain spatial decay in Step 2 above. Fortunately, it turns out that the use of distorted Fourier transform can give us almost the optimal bound.

• **Non-local reduced problems**

The development of linear theory for the inner problem relies on orthogonalities which are achieved by adjusting modulation parameters λ_j, γ_j, $\xi^{[j]}$. The dynamics for $\xi^{[j]}$ turns out to be governed by an ODE, which is relatively straightforward to solve. However, the non-local feature in the corrections $\Phi_0^{*[j]}$ gets inherited by the mode 0 (λ_j and γ_j) of each bubble as the corrections are essentially for mode 0. Here one might expect the complex system involving both λ_j and γ_j is a rather sophisticated form due to the presence of dispersion. Fortunately, it turns out that the contribution of both $\Phi_0^{*[j]}$ and the remainder $b(U_\infty - U^{[j]}) \wedge \Delta_x \Phi_0^{*[j]}$ in the orthogonal equation at mode 0 results in the following well-ordered non-local problem

$$\int_0^{t-\lambda_j^2(t)} \frac{\dot{p}_j(s)}{t-s} ds \sim O(1),$$

where $p_j(t) = \lambda_j(t) e^{i\gamma_j(t)}$. This λ_j-γ_j system was first found and handled in the context of HMF [12]. Surprisingly, this comes with a similar form in LLG with the presence of dispersion.

• **Solving the outer problem**

The linear theory for the outer problem is done by using the sub-Gaussian estimate for the linearized parabolic system proved in [20], where $\mathsf{DMO_x}$-regularity is required for the entries in the matrix $\mathbf{B}_{\Phi,U_*}$. The dependence on Φ in the matrix shall be dealt with via Schauder fixed point theorem, and the key thing here is the dependence on the blow-up profile U_*. In fact, to ensure $\mathsf{DMO_x}$-regularity in U_*, type II blow-up rate plays a crucial rule. In other words, if U_* carries type I blow-up rate, then the matrix involving U_* is no longer $\mathsf{DMO_x}$. See Section 7.2.

3. Notations and preliminaries

We list in this section some notations and preliminaries that we shall use repeatedly throughout this paper.

- We assume $a \lesssim b$ if there exists a positive constant C such that $a \le Cb$. Denote $a \sim b$ if $a \lesssim b \lesssim a$. All constants stated in the paper are independent of T.
- For any $c \in \mathbb{R}$, we use the notation $c-$ to denote a constant less than c and can be chosen close to c arbitrarily.
- For $f \in C^m(\mathbb{R}^d)$, the symbol $D_x^k f$ with $k \in \{0, 1, \dots m\}$ is used to denote $\partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \cdots \partial_{x_d}^{k_d} f$ for some $\sum_{j=1}^d k_j = k$. We will omit the subscript "$_x$" and adopt $D^k f$ if no ambiguity.
- Write the indicator function 1_Ω of a set Ω as
$$1_\Omega(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}.$$
- Set $\eta(x)$ as a smooth cut-off function satisfying $0 \le \eta(x) \le 1$ and
$$\eta(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| \ge 2 \end{cases}.$$
- Denote $\Gamma_d^\natural$ as the fundamental solution of
$$\partial_t u = (a - ib)\Delta u \quad \text{in } \mathbb{R}^d$$
and $\Gamma_d^\natural$ is given by
$$\Gamma_d^\natural(x,t) = (a-ib)^{-\frac{d}{2}} (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4(a-ib)t}} \tag{3.1}$$
where $(a-ib)^{-\frac{d}{2}} = e^{-i\theta\frac{d}{2}}$ if $a - ib = e^{i\theta}$.
It is easy to have
$$|\Gamma_d^\natural(x,t)| \lesssim t^{-\frac{d}{2}} e^{-\frac{a|x|^2}{4t}}.$$

Given a fundamental solution $\Gamma(x,y,t,s)$ for a parabolic system and some admissible functions $f(x)$, $h(x,t)$, denote

$$(\Gamma * f)(x,t,s) := \int_{\mathbb{R}^d} \Gamma(x,y,t,s) f(y)dy, \quad (\Gamma ** h)(x,t,t_0) := \int_{t_0}^{t}\int_{\mathbb{R}^d} \Gamma(x,y,t,s)h(y,s)dyds.$$

- For any vector $a = (a_1, a_2, a_3)^T \in \mathbb{R}^3$, where "$T$" means the transpose of matrix. It is equivalent to regarding $a = (a_1 + ia_2, a_3)^T$.
- For any matrix $A = (a_{ij})_{n\times m}$, denote $|A| = (\sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}|^2)^{\frac{1}{2}}$. Specially, for $a \in \mathbb{C}$, $|a|$ is the usual absolute value.
- For any smooth function $f(x,t)$ and $x = x(t)$ depending on t smoothly, denote $\partial_t f(x(t),t) = (\partial_t f)(x(t),t)$ and $\partial_t(f(x(t),t)) = (\partial_t f)(x(t),t) + x'(t)\cdot(\nabla_x f)(x(t),t)$.
- For $f \in C^1(\mathbb{R}^2,\mathbb{R})$, $\vec{v} \in C^1(\mathbb{R}^2,\mathbb{R}^3)$, denote

$$\nabla f \nabla \vec{v} = \left[\nabla f\cdot\nabla v_1,\ \nabla f\cdot\nabla v_2,\ \nabla f\cdot\nabla v_3\right]^T.$$

Denote

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \left[a_1 b_{11} + a_2 b_{21} + a_3 b_{31}, a_1 b_{12} + a_2 b_{22} + a_3 b_{32}\right].$$

We consider the Landau-Lifshitz-Gilbert equation given in (1.3). Let W be the least energy harmonic map

$$W(y) = \frac{1}{|y|^2+1}\begin{bmatrix} 2y_1 \\ 2y_2 \\ |y|^2 - 1 \end{bmatrix}, \quad y \in \mathbb{R}^2,$$

which solves the stationary equation of LLG (1.3). Since we shall consider the case of multiple bubbles, subscripts or superscripts "$_j$", "$_{[j]}$" will be used to distinguish different bubbles and their associated tangent planes. In the (rescaled) polar coordinates around $\xi^{[j]} \in \mathbb{R}^2$

$$y^{[j]} = \frac{x-\xi^{[j]}}{\lambda_j} = \rho_j e^{i\theta_j}, \quad x = \xi^{[j]} + \lambda_j \rho_j e^{i\theta_j}, \quad \theta_j = \arctan\left(\frac{x_2 - \xi_2^{[j]}}{x_1 - \xi_1^{[j]}}\right)$$

we can write for $j = 1, 2, \ldots, N$,

$$W^{[j]} := W(y^{[j]}) = \begin{bmatrix} \cos\theta_j \sin w(\rho_j) \\ \sin\theta_j \sin w(\rho_j) \\ \cos w(\rho_j) \end{bmatrix} := \begin{bmatrix} e^{i\theta_j} \sin w(\rho_j) \\ \cos w(\rho_j) \end{bmatrix} \tag{3.2}$$

with

$$w(\rho_j) = \pi - 2\arctan(\rho_j),$$

and we have

$$w_{\rho_j} = -\frac{2}{\rho_j^2+1},\ \sin w(\rho_j) = -\rho_j w_{\rho_j} = \frac{2\rho_j}{\rho_j^2+1},\ \cos w(\rho_j) = \frac{\rho_j^2-1}{\rho_j^2+1},\ |\nabla_{y^{[j]}} W(y^{[j]})|^2 = 2w_{\rho_j}^2 = \frac{8}{(\rho_j^2+1)^2}.$$

We denote the Frenet basis associated to $W^{[j]}$ as

$$E_1^{[j]} = \begin{bmatrix} \cos\theta_j \cos w(\rho_j) \\ \sin\theta_j \cos w(\rho_j) \\ -\sin w(\rho_j) \end{bmatrix} := \begin{bmatrix} e^{i\theta_j}\cos w(\rho_j) \\ -\sin w(\rho_j) \end{bmatrix},\ E_2^{[j]} = \begin{bmatrix} -\sin\theta_j \\ \cos\theta_j \\ 0 \end{bmatrix} := \begin{bmatrix} ie^{i\theta_j} \\ 0 \end{bmatrix}, \tag{3.3}$$

so

$$W^{[j]} \wedge E_1^{[j]} = E_2^{[j]}, \quad W^{[j]} \wedge E_2^{[j]} = -E_1^{[j]}, \quad E_1^{[j]} \wedge E_2^{[j]} = W^{[j]}. \tag{3.4}$$

It is direct to check that in the polar coordinates $x = \xi^{[j]} + \lambda_j \rho_j e^{i\theta_j}$, $\lambda_j \rho_j = r_j = |x - \xi^{[j]}|$,

$$\begin{aligned}
&\partial_{\rho_j} W^{[j]} = w_{\rho_j} E_1^{[j]},\ \partial_{\rho_j\rho_j} W^{[j]} = w_{\rho_j\rho_j} E_1^{[j]} - w_{\rho_j}^2 W^{[j]},\ \partial_{\theta_j} W^{[j]} = \sin w(\rho_j) E_2^{[j]},\\
&\partial_{\theta_j\theta_j} W^{[j]} = -\sin w(\rho_j)\left(\sin w(\rho_j) W^{[j]} + \cos w(\rho_j) E_1^{[j]}\right),\\
&\partial_{\rho_j} E_1^{[j]} = -w_{\rho_j} W^{[j]},\ \partial_{\rho_j\rho_j} E_1^{[j]} = -w_{\rho_j\rho_j} W^{[j]} - w_{\rho_j}^2 E_1^{[j]},\ \partial_{\theta_j} E_1^{[j]} = \cos w(\rho_j) E_2^{[j]},\\
&\partial_{\theta_j\theta_j} E_1^{[j]} = -\cos w(\rho_j)\left(\sin w(\rho_j) W^{[j]} + \cos w(\rho_j) E_1^{[j]}\right),\\
&\partial_{\rho_j} E_2^{[j]} = \partial_{\rho_j\rho_j} E_2^{[j]} = 0,\ \partial_{\theta_j} E_2^{[j]} = -\sin w(\rho_j) W^{[j]} - \cos w(\rho_j) E_1^{[j]},\ \partial_{\theta_j\theta_j} E_2^{[j]} = -E_2^{[j]}.
\end{aligned} \tag{3.5}$$

The linearization of the harmonic map operator around $W^{[j]}$ is the elliptic operator

$$L_W^{[j]}[\phi] := \Delta_{y^{[j]}} \phi + |\nabla_{y^{[j]}} W^{[j]}|^2 \phi + 2\left(\nabla_{y^{[j]}} W^{[j]} \cdot \nabla_{y^{[j]}} \phi\right) W^{[j]}, \tag{3.6}$$

whose kernel functions are given by

$$\begin{cases}
Z_{0,1}^{[j]}(y^{[j]}) = \rho_j w_{\rho_j}(\rho_j) E_1^{[j]}(y^{[j]}),\\
Z_{0,2}^{[j]}(y^{[j]}) = \rho_j w_{\rho_j}(\rho_j) E_2^{[j]}(y^{[j]}),\\
Z_{1,1}^{[j]}(y^{[j]}) = w_{\rho_j}(\rho_j)[\cos\theta_j E_1^{[j]}(y^{[j]}) + \sin\theta_j E_2^{[j]}(y^{[j]})],\\
Z_{1,2}^{[j]}(y^{[j]}) = w_{\rho_j}(\rho_j)[\sin\theta_j E_1^{[j]}(y^{[j]}) - \cos\theta_j E_2^{[j]}(y^{[j]})],\\
Z_{-1,1}^{[j]}(y^{[j]}) = \rho_j^2 w_{\rho_j}(\rho_j)[\cos\theta_j E_1^{[j]}(y^{[j]}) - \sin\theta_j E_2^{[j]}(y^{[j]})],\\
Z_{-1,2}^{[j]}(y^{[j]}) = \rho_j^2 w_{\rho_j}(\rho_j)[\sin\theta_j E_1^{[j]}(y^{[j]}) + \cos\theta_j E_2^{[j]}(y^{[j]})].
\end{cases} \tag{3.7}$$

We see that

$$L_W^{[j]}[Z_{p,q}^{[j]}] = 0 \ \text{ for } p = \pm 1, 0,\ q = 1, 2.$$

Clearly,

$$U^{[j]}(x,t) := Q_{\gamma_j} W\left(\frac{x - \xi^{[j]}}{\lambda_j}\right) \tag{3.8}$$

solves the harmonic map equation, where Q_{γ_j} is the γ_j-rotation matrix around z-axis

$$Q_{\gamma_j} := \begin{bmatrix} \cos\gamma_j & -\sin\gamma_j & 0\\ \sin\gamma_j & \cos\gamma_j & 0\\ 0 & 0 & 1 \end{bmatrix}. \tag{3.9}$$

For $\mathbf{f} = [f_1, f_2, f_3]^T$, we have

$$Q_{\gamma_j}\mathbf{f} = \Big[\mathrm{Re}\left(e^{i\gamma_j}(f_1 + i f_2)\right), \mathrm{Im}\left(e^{i\gamma_j}(f_1 + i f_2)\right), f_3\Big]. \tag{3.10}$$

By basic linear algebra, $(M\mathbf{f}) \wedge (M\mathbf{g}) = (\det M)(M^{-1})^T(\mathbf{f} \wedge \mathbf{g})$ where M is a 3×3 matrix and $(M^{-1})^T$ is the transpose of the inverse, $\mathbf{f}, \mathbf{g} \in \mathbb{R}^3$. Specially, $(Q_{\gamma_j}\mathbf{f}) \wedge (Q_{\gamma_j}\mathbf{g}) = Q_{\gamma_j}(\mathbf{f} \wedge \mathbf{g})$. Combining this with (3.4), we have

$$U^{[j]} \wedge (Q_{\gamma_j} E_1^{[j]}) = Q_{\gamma_j} E_2^{[j]}, \quad U^{[j]} \wedge (Q_{\gamma_j} E_2^{[j]}) = -Q_{\gamma_j} E_1^{[j]}, \quad (Q_{\gamma_j} E_1^{[j]}) \wedge (Q_{\gamma_j} E_2^{[j]}) = U^{[j]}. \tag{3.11}$$

For the purpose of dealing linearization near concentration zones, it will be convenient to use complex notations as all the analysis will be done on the associated tangent plane. For any $\mathbf{f} \in \mathbb{R}^3$ satisfying $\mathbf{f} \cdot U^{[j]} = 0$, we define the equivalent complex form of $\mathbf{f}$ as

$$\mathbf{f}_{\mathcal{C}_j} := \mathbf{f} \cdot (Q_{\gamma_j} E_1^{[j]}) + i\ \mathbf{f} \cdot (Q_{\gamma_j} E_2^{[j]}).$$

For any complex-valued function f, we define

$$f_{\mathcal{C}_j^{-1}} := (\mathrm{Re} f)\, Q_{\gamma_j} E_1^{[j]} + (\mathrm{Im} f)\, Q_{\gamma_j} E_2^{[j]}. \tag{3.12}$$

By (3.11),

$$U^{[j]} \wedge f_{\mathcal{C}_j^{-1}} = (\mathrm{Re} f)\, Q_{\gamma_j} E_2^{[j]} - (\mathrm{Im} f)\, Q_{\gamma_j} E_1^{[j]} = (if)_{\mathcal{C}_j^{-1}}. \tag{3.13}$$

Similarly, for any $\mathbf{g} \in \mathbb{R}^3$ satisfying $\mathbf{g} \cdot W^{[j]} = 0$, the equivalent complex form of $\mathbf{f}$ is defined as

$$\mathbf{g}_{\mathbb{C}_j} := \mathbf{g} \cdot E_1^{[j]} + i\mathbf{g} \cdot E_2^{[j]}.$$

Notice for any $\mathbf{f}=[f_1,f_2,f_3]^T\in\mathbb{R}^3$,

$$\begin{aligned}\left(\Pi_{U^{[j]\perp}}\mathbf{f}\right)_{\mathcal{C}_j}&=\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[(f_1+if_2)\,e^{-i(\theta_j+\gamma_j)}\right]-\frac{2\rho_j}{\rho_j^2+1}f_3,\\ \mathbf{f}\cdot U^{[j]}&=\frac{2\rho_j}{\rho_j^2+1}\mathrm{Re}\left[(f_1+if_2)\,e^{-i(\theta_j+\gamma_j)}\right]+\frac{\rho_j^2-1}{\rho_j^2+1}f_3.\end{aligned}\tag{3.14}$$

The linearization around $U^{[j]}$ is given by

$$L_{U^{[j]}}[\phi]:=\Delta_x\phi+|\nabla_xU^{[j]}|^2\phi+2(\nabla_xU^{[j]}\cdot\nabla_x\phi)U^{[j]}.\tag{3.15}$$

It is easy to have

$$L_{U^{[j]}}[Q_{\gamma_j}\mathbf{f}(y^{[j]})]=\lambda_j^{-2}Q_{\gamma_j}L_{W^{[j]}}[\mathbf{f}(y^{[j]})],\quad\text{where}\quad y^{[j]}=\frac{x-\xi^{[j]}}{\lambda_j}.$$

In the sequel, it is of significance to compute the action of $L_U^{[j]}$ on functions whose value is orthogonal to $U^{[j]}$ pointwisely. For any $\mathbf{f},\mathbf{g}\in\mathbb{R}^3$, we define

$$\Pi_{\mathbf{g}^\perp}\mathbf{f}:=\mathbf{f}-(\mathbf{f}\cdot\mathbf{g})\mathbf{g}.\tag{3.16}$$

Specially, when $|\mathbf{g}|=1$, $\Pi_{\mathbf{g}^\perp}$ is the usual orthogonal projection on $\mathbf{g}^\perp$.

We now give several useful formulas whose proof is similar to that of [12, Section 3]. For any vector-valued function $\mathbf{f}:\mathbb{R}^2\to\mathbb{R}^3$, we set

$$\tilde{L}_{U^{[j]}}[\mathbf{f}]:=|\nabla_xU^{[j]}|^2\Pi_{U^{[j]\perp}}\mathbf{f}-2\nabla_x(\mathbf{f}\cdot U^{[j]})\nabla_xU^{[j]}\tag{3.17}$$

with

$$\nabla_x(\mathbf{f}\cdot U^{[j]})\nabla_xU^{[j]}=\sum_{k=1}^{2}\partial_{x_k}(\mathbf{f}\cdot U^{[j]})\partial_{x_k}U^{[j]}.$$

Similarly, we set

$$\tilde{L}_{W^{[j]}}[\mathbf{f}]:=|\nabla_{y^{[j]}}W^{[j]}|^2\Pi_{W^{[j]\perp}}\mathbf{f}-2\nabla_{y^{[j]}}(\mathbf{f}\cdot W^{[j]})\nabla_{y^{[j]}}W^{[j]}.$$

Then it is straightforward to get

$$\tilde{L}_{U^{[j]}}[Q_{\gamma_j}\mathbf{f}(y^{[j]})]=\lambda_j^{-2}Q_{\gamma_j}\tilde{L}_{W^{[j]}}[\mathbf{f}(y^{[j]})].$$

$$L_{U^{[j]}}[\Pi_{U^{[j]\perp}}\mathbf{f}]=\Pi_{U^{[j]\perp}}\Delta_x\mathbf{f}+\tilde{L}_{U^{[j]}}[\mathbf{f}].$$

For $\mathbf{f}=[f_1,f_2,f_3]^T$, we have

$$\begin{aligned}&\tilde{L}_{U^{[j]}}[Q_{\gamma_j}\mathbf{f}]\\ &=\lambda_j^{-1}\Big\{\rho_jw_{\rho_j}^2(\rho_j)(\partial_{x_1}f_1+\partial_{x_2}f_2)-2w_{\rho_j}(\rho_j)\cos w(\rho_j)(\cos\theta_j\partial_{x_1}f_3+\sin\theta_j\partial_{x_2}f_3)\\ &\quad+\rho_jw_{\rho_j}^2(\rho_j)\left[\cos(2\theta_j)(\partial_{x_1}f_1-\partial_{x_2}f_2)+\sin(2\theta_j)(\partial_{x_2}f_1+\partial_{x_1}f_2)\right]\Big\}Q_{\gamma_j}E_1^{[j]}\\ &\quad+\lambda_j^{-1}\Big\{-\rho_jw_{\rho_j}^2(\rho_j)(\partial_{x_2}f_1-\partial_{x_1}f_2)-2w_{\rho_j}(\rho_j)\cos w(\rho_j)(\sin\theta_j\partial_{x_1}f_3-\cos\theta_j\partial_{x_2}f_3)\\ &\quad+\rho_jw_{\rho_j}^2(\rho_j)\left[\sin(2\theta_j)(\partial_{x_1}f_1-\partial_{x_2}f_2)-\cos(2\theta_j)(\partial_{x_2}f_1+\partial_{x_1}f_2)\right]\Big\}Q_{\gamma_j}E_2^{[j]}\\ &=\lambda_j^{-1}\Big\{\rho_jw_{\rho_j}^2(\rho_j)(\partial_{x_1}f_1+\partial_{x_2}f_2)-e^{i\theta_j}w_{\rho_j}(\rho_j)\cos w(\rho_j)(\partial_{x_1}f_3-i\partial_{x_2}f_3)\\ &\quad-e^{-i\theta_j}w_{\rho_j}(\rho_j)\cos w(\rho_j)(\partial_{x_1}f_3+i\partial_{x_2}f_3)+e^{2i\theta_j}\frac{1}{2}\rho_jw_{\rho_j}^2(\rho_j)\left[(\partial_{x_1}f_1-\partial_{x_2}f_2)-i(\partial_{x_2}f_1+\partial_{x_1}f_2)\right]\\ &\quad+e^{-2i\theta_j}\frac{1}{2}\rho_jw_{\rho_j}^2(\rho_j)\left[(\partial_{x_1}f_1-\partial_{x_2}f_2)+i(\partial_{x_2}f_1+\partial_{x_1}f_2)\right]\Big\}Q_{\gamma_j}E_1^{[j]}\\ &\quad+\lambda_j^{-1}\Big\{-\rho_jw_{\rho_j}^2(\rho_j)(\partial_{x_2}f_1-\partial_{x_1}f_2)+e^{i\theta_j}w_{\rho_j}(\rho_j)\cos w(\rho_j)(\partial_{x_2}f_3+i\partial_{x_1}f_3)\\ &\quad+e^{-i\theta_j}w_{\rho_j}(\rho_j)\cos w(\rho_j)(\partial_{x_2}f_3-i\partial_{x_1}f_3)-e^{2i\theta_j}\frac{1}{2}\rho_jw_{\rho_j}^2(\rho_j)\left[(\partial_{x_2}f_1+\partial_{x_1}f_2)+i(\partial_{x_1}f_1-\partial_{x_2}f_2)\right]\\ &\quad+e^{-2i\theta_j}\frac{1}{2}\rho_jw_{\rho_j}^2(\rho_j)\left[-(\partial_{x_2}f_1+\partial_{x_1}f_2)+i(\partial_{x_1}f_1-\partial_{x_2}f_2)\right]\Big\}Q_{\gamma_j}E_2^{[j]}.\end{aligned}$$

Then the corresponding complex form is given by

$$\begin{aligned}(\tilde{L}_{U^{[j]}}[Q_{\gamma_j}\mathbf{f}])_{\mathcal{C}_j} &= \lambda_j^{-1}\big\{\rho_j w_{\rho_j}^2(\rho_j)\left[(\partial_{x_1}f_1+\partial_{x_2}f_2)-i(\partial_{x_2}f_1-\partial_{x_1}f_2)\right]\\ &+e^{i\theta_j}2w_{\rho_j}(\rho_j)\cos w(\rho_j)(-\partial_{x_1}f_3+i\partial_{x_2}f_3)+e^{2i\theta_j}\rho_j w_{\rho_j}^2(\rho_j)\left[(\partial_{x_1}f_1-\partial_{x_2}f_2)-i(\partial_{x_2}f_1+\partial_{x_1}f_2)\right]\big\}.\end{aligned} \tag{3.18}$$

Specially,

$$(\tilde{L}_{U^{[j]}}[\mathbf{f}])_{\mathcal{C}_j} = (\tilde{L}_{U^{[j]}}[\mathbf{f}])_{\mathcal{C}_j,0}+e^{i\theta_j}(\tilde{L}_{U^{[j]}}[\mathbf{f}])_{\mathcal{C}_j,1}+e^{2i\theta_j}(\tilde{L}_{U^{[j]}}[\mathbf{f}])_{\mathcal{C}_j,2}, \tag{3.19}$$

where

$$\begin{aligned}(\tilde{L}_{U^{[j]}}[\mathbf{f}])_{\mathcal{C}_j,0} &:= \lambda_j^{-1}\rho_j w_{\rho_j}^2(\rho_j)\left[\partial_{x_1}(Q_{-\gamma_j}\mathbf{f})_1+\partial_{x_2}(Q_{-\gamma_j}\mathbf{f})_2-i\left(\partial_{x_2}(Q_{-\gamma_j}\mathbf{f})_1-\partial_{x_1}(Q_{-\gamma_j}\mathbf{f})_2\right)\right]\\ &= \lambda_j^{-1}\rho_j w_{\rho_j}^2(\rho_j)e^{-i\gamma_j}\left[\partial_{x_1}f_1+\partial_{x_2}f_2+i\left(\partial_{x_1}f_2-\partial_{x_2}f_1\right)\right],\\ (\tilde{L}_{U^{[j]}}[\mathbf{f}])_{\mathcal{C}_j,1} &:= 2\lambda_j^{-1}w_{\rho_j}(\rho_j)\cos w(\rho_j)\left(-\partial_{x_1}(Q_{-\gamma_j}\mathbf{f})_3+i\partial_{x_2}(Q_{-\gamma_j}\mathbf{f})_3\right)\\ &= 2\lambda_j^{-1}w_{\rho_j}(\rho_j)\cos w(\rho_j)\left(-\partial_{x_1}f_3+i\partial_{x_2}f_3\right),\\ (\tilde{L}_{U^{[j]}}[\mathbf{f}])_{\mathcal{C}_j,2} &:= \lambda_j^{-1}\rho_j w_{\rho_j}^2(\rho_j)\left[\partial_{x_1}(Q_{-\gamma_j}\mathbf{f})_1-\partial_{x_2}(Q_{-\gamma_j}\mathbf{f})_2-i\left(\partial_{x_2}(Q_{-\gamma_j}\mathbf{f})_1+\partial_{x_1}(Q_{-\gamma_j}\mathbf{f})_2\right)\right]\\ &= \lambda_j^{-1}\rho_j w_{\rho_j}^2(\rho_j)e^{i\gamma_j}\left[\partial_{x_1}f_1-\partial_{x_2}f_2-i\left(\partial_{x_1}f_2+\partial_{x_2}f_1\right)\right],\end{aligned} \tag{3.20}$$

where we have used

$$\begin{aligned}&\partial_{x_1}(Q_{-\gamma_j}\mathbf{f})_1+\partial_{x_2}(Q_{-\gamma_j}\mathbf{f})_2-i\left(\partial_{x_2}(Q_{-\gamma_j}\mathbf{f})_1-\partial_{x_1}(Q_{-\gamma_j}\mathbf{f})_2\right)\\ =\ &\partial_{x_1}\mathrm{Re}\left[e^{-i\gamma_j}\left(f_1+if_2\right)\right]+\partial_{x_2}\mathrm{Im}\left[e^{-i\gamma_j}\left(f_1+if_2\right)\right]\\ &-i\partial_{x_2}\mathrm{Re}\left[e^{-i\gamma_j}\left(f_1+if_2\right)\right]+i\partial_{x_1}\mathrm{Im}\left[e^{-i\gamma_j}\left(f_1+if_2\right)\right]\\ =\ &e^{-i\gamma_j}\left[\partial_{x_1}\left(f_1+if_2\right)-i\partial_{x_2}\left(f_1+if_2\right)\right]=e^{-i\gamma_j}\left[\partial_{x_1}f_1+\partial_{x_2}f_2+i\left(\partial_{x_1}f_2-\partial_{x_2}f_1\right)\right],\end{aligned}$$

$$\begin{aligned}&\partial_{x_1}(Q_{-\gamma_j}\mathbf{f})_1-\partial_{x_2}(Q_{-\gamma_j}\mathbf{f})_2-i\left(\partial_{x_2}(Q_{-\gamma_j}\mathbf{f})_1+\partial_{x_1}(Q_{-\gamma_j}\mathbf{f})_2\right)\\ =\ &\partial_{x_1}\mathrm{Re}\left[e^{-i\gamma_j}\left(f_1+if_2\right)\right]-\partial_{x_2}\mathrm{Im}\left[e^{-i\gamma_j}\left(f_1+if_2\right)\right]\\ &-i\partial_{x_2}\mathrm{Re}\left[e^{-i\gamma_j}\left(f_1+if_2\right)\right]-i\partial_{x_1}\mathrm{Im}\left[e^{-i\gamma_j}\left(f_1+if_2\right)\right]\\ =\ &e^{i\gamma_j}\left[\partial_{x_1}f_1-\partial_{x_2}f_2-i\left(\partial_{x_1}f_2+\partial_{x_2}f_1\right)\right].\end{aligned}$$

By (3.11),

$$\begin{aligned}&Q_{-\gamma_j}\left\{\left(a-bU^{[j]}\wedge\right)\left[|\nabla_x U^{[j]}|^2\Pi_{U^{[j]\perp}}\Phi_{\mathrm{out}}-2\nabla_x\left(U^{[j]}\cdot\Phi_{\mathrm{out}}\right)\nabla_x U^{[j]}\right]\right\}\\ =\ &Q_{-\gamma_j}\left[\left(a-bU^{[j]}\wedge\right)\tilde{L}_{U^{[j]}}[\Phi_{\mathrm{out}}]\right]\\ =\ &Q_{-\gamma_j}\left[\left(a-bU^{[j]}\wedge\right)\left\{\mathrm{Re}\left[\left(\tilde{L}_{U^{[j]}}[\Phi_{\mathrm{out}}]\right)_{\mathcal{C}_j}\right]Q_{\gamma_j}E_1^{[j]}+\mathrm{Im}\left[\left(\tilde{L}_{U^{[j]}}[\Phi_{\mathrm{out}}]\right)_{\mathcal{C}_j}\right]Q_{\gamma_j}E_2^{[j]}\right\}\right]\\ =\ &\mathrm{Re}\left[\left(\tilde{L}_{U^{[j]}}[\Phi_{\mathrm{out}}]\right)_{\mathcal{C}_j}\right]\left(aE_1^{[j]}-bE_2^{[j]}\right)+\mathrm{Im}\left[\left(\tilde{L}_{U^{[j]}}[\Phi_{\mathrm{out}}]\right)_{\mathcal{C}_j}\right]\left(aE_2^{[j]}+bE_1^{[j]}\right).\end{aligned}$$

Thus,

$$\begin{aligned}&\left\{Q_{-\gamma_j}\left[\left(a-bU^{[j]}\wedge\right)\tilde{L}_{U^{[j]}}[\Phi_{\mathrm{out}}]\right]\right\}_{\mathbb{C}_j}=(a-ib)\left(\tilde{L}_{U^{[j]}}[\Phi_{\mathrm{out}}]\right)_{\mathcal{C}_j}\\ =\ &(a-ib)\left[\left(\tilde{L}_{U^{[j]}}[\Phi_{\mathrm{out}}]\right)_{\mathcal{C}_j,0}+e^{i\theta_j}\left(\tilde{L}_{U^{[j]}}[\Phi_{\mathrm{out}}]\right)_{\mathcal{C}_j,1}+e^{2i\theta_j}\left(\tilde{L}_{U^{[j]}}[\Phi_{\mathrm{out}}]\right)_{\mathcal{C}_j,2}\right]\end{aligned} \tag{3.21}$$

where we used (3.19) in the last equality.

4. Approximations and improvement

4.1. **First approximation.** We consider the case of multiple N bubbles for any $N\in\mathbb{Z}_+$. We take the approximation

$$U_*(x,t) := -(N-1)U_\infty+\sum_{j=1}^{N}U^{[j]}(x,t),\quad x\in\mathbb{R}^2, \tag{4.1}$$

where $U^{[j]}$ is defined in (3.8) and $U_\infty = [0,0,1]^T$. Notice that

$$|U_*| = 1 + O(\sum_{j=1}^N \lambda_j) \quad \text{when} \quad \min_{k\neq m} |\xi^{[k]} - \xi^{[m]}| > 0. \tag{4.2}$$

Let us denote the error

$$S(u) := -u_t + a(\Delta_x u + |\nabla_x u|^2 u) - bu \wedge \Delta_x u.$$

The error of the approximate solution is

$$\mathcal{S}(U_*) = -\sum_{j=1}^N \partial_t U^{[j]} + a(\Delta_x U_* + |\nabla_x U_*|^2 U_*) - bU_* \wedge \Delta_x U_*.$$

Notice

$$-\partial_t U^{[j]} = \mathcal{E}_0^{[j]} + \mathcal{E}_1^{[j]}, \quad \mathcal{E}_0^{[j]} := -\dot\lambda_j \partial_{\lambda_j} U^{[j]} - \dot\gamma_j \partial_{\gamma_j} U^{[j]}, \quad \mathcal{E}_1^{[j]} := -\dot\xi_1^{[j]} \partial_{\xi_1^{[j]}} U^{[j]} - \dot\xi_2^{[j]} \partial_{\xi_2^{[j]}} U^{[j]} \tag{4.3}$$

where

$$\begin{cases} \partial_{\lambda_j} U^{[j]}(x) = -\lambda_j^{-1} Q_{\gamma_j} Z_{0,1}^{[j]}(y^{[j]}), \\ \partial_{\gamma_j} U^{[j]}(x) = -Q_{\gamma_j} Z_{0,2}^{[j]}(y^{[j]}), \\ \partial_{\xi_1^{[j]}} U^{[j]}(x) = -\lambda_j^{-1} Q_{\gamma_j} Z_{1,1}^{[j]}(y^{[j]}), \\ \partial_{\xi_2^{[j]}} U^{[j]}(x) = -\lambda_j^{-1} Q_{\gamma_j} Z_{1,2}^{[j]}(y^{[j]}) \end{cases}$$

with $Z_{p,q}^{[j]}$, $E_1^{[j]}$, $E_2^{[j]}$ given in (3.7), (3.3), respectively. It is straightforward to see

$$\begin{aligned} \mathcal{E}_0^{[j]} &= Q_{\gamma_j} \left(\lambda_j^{-1} \dot\lambda_j Z_{0,1}^{[j]}(y^{[j]}) + \dot\gamma_j Z_{0,2}^{[j]}(y^{[j]}) \right) = \rho_j w_{\rho_j}(\rho_j) Q_{\gamma_j} \left(\lambda_j^{-1} \dot\lambda_j E_1^{[j]} + \dot\gamma_j E_2^{[j]} \right) \\ &= \frac{-2\rho_j}{\rho_j^2+1} \begin{bmatrix} \left(\lambda_j^{-1} \dot\lambda_j \cos w(\rho_j) + i\dot\gamma_j \right) e^{i(\theta_j+\gamma_j)} \\ -\lambda_j^{-1} \dot\lambda_j \sin w(\rho_j) \end{bmatrix}, \end{aligned} \tag{4.4}$$

$$(\mathcal{E}_0^{[j]})_{\mathcal{C}_j} = \frac{-2\rho_j}{\rho_j^2+1} \left(\lambda_j^{-1} \dot\lambda_j + i\dot\gamma_j \right), \tag{4.5}$$

$$\begin{aligned} \mathcal{E}_1^{[j]} &= \dot\xi_1^{[j]} \lambda_j^{-1} Q_{\gamma_j} Z_{1,1}(y^{[j]}) + \dot\xi_2^{[j]} \lambda_j^{-1} Q_{\gamma_j} Z_{1,2}(y^{[j]}) \\ &= \dot\xi_1^{[j]} \lambda_j^{-1} Q_{\gamma_j} w_{\rho_j} \left(\cos\theta_j E_1^{[j]} + \sin\theta_j E_2^{[j]} \right) + \dot\xi_2^{[j]} \lambda_j^{-1} Q_{\gamma_j} w_{\rho_j} \left(\sin\theta_j E_1^{[j]} - \cos\theta_j E_2^{[j]} \right) \\ &= \lambda_j^{-1} w_{\rho_j} \left(\dot\xi_1^{[j]} \cos\theta_j + \dot\xi_2^{[j]} \sin\theta_j \right) Q_{\gamma_j} E_1^{[j]} + \lambda_j^{-1} w_{\rho_j} \left(\dot\xi_1^{[j]} \sin\theta_j - \dot\xi_2^{[j]} \cos\theta_j \right) Q_{\gamma_j} E_2^{[j]} \\ &= \frac{-2\lambda_j^{-1}}{\rho_j^2+1} \mathrm{Re} \left[\left(\dot\xi_1^{[j]} - i\dot\xi_2^{[j]} \right) e^{i\theta_j} \right] Q_{\gamma_j} E_1^{[j]} - \frac{2\lambda_j^{-1}}{\rho_j^2+1} \mathrm{Im} \left[\left(\dot\xi_1^{[j]} - i\dot\xi_2^{[j]} \right) e^{i\theta_j} \right] Q_{\gamma_j} E_2^{[j]}, \end{aligned} \tag{4.6}$$

$$(\mathcal{E}_1^{[j]})_{\mathcal{C}_j} = \frac{-2\lambda_j^{-1} (\dot\xi_1^{[j]} - i\dot\xi_2^{[j]})}{\rho_j^2+1} e^{i\theta_j}. \tag{4.7}$$

Combining (4.4) and (4.6), we have

$$|\partial_t U^{[j]}| \lesssim \left(\lambda_j^{-1} |\dot\lambda_j| + |\dot\gamma_j| \right) \langle \rho_j \rangle^{-1} + \lambda_j^{-1} |\dot\xi^{[j]}| \langle \rho_j \rangle^{-2}. \tag{4.8}$$

Notice that $S(U_*)$ contains errors $\mathcal{E}_0^{[j]}$ with slow decay in space, which is not in $L^2(\mathbb{R}^2)$. We shall introduce global corrections to improve the error.

4.2. **Global corrections by parabolic systems.** In this part, we will transfer slow decay terms by parabolic systems.

Around each bubble, the slow decaying error of (4.4) is given by

$$\mathcal{E}_0^{[j]} \approx -\frac{2}{z_j}\begin{bmatrix}\dot{p}_j(t)e^{i\theta_j}\\ 0\end{bmatrix},$$

where

$$z_j=(\lambda_j^2(t)+r_j^2)^{1/2},\quad r_j=|x^{[j]}|,\quad x^{[j]}=x-\xi^{[j]}(t),\quad p_j(t)=\lambda_j(t)e^{i\gamma_j(t)}.$$

We aim to find global corrections $\Phi_0^{*[j]}(r_j,t)$ such that

$$-\partial_t(\Phi_0^{*[j]})+(a-bU_\infty\wedge)\,\Delta_x\Phi_0^{*[j]}-\frac{2}{z_j}\begin{bmatrix}\dot{p}_j(t)e^{i\theta_j}\\ 0\end{bmatrix}\approx 0$$

with the form

$$\Phi_0^{*[j]}(r_j,t):=\frac{r_j^2}{r_j^2+\lambda_j^2}\begin{bmatrix}\Phi_0^{[j]}(\sqrt{r_j^2+\lambda_j^2},t)e^{i\theta_j}\\ 0\end{bmatrix}=\begin{bmatrix}\frac{\rho_j^2}{\rho_j^2+1}\Phi_0^{[j]}(z_j,t)e^{i\theta_j}\\ 0\end{bmatrix}. \tag{4.9}$$

Formally, the approximate calculation is the following

$$\Delta_x\begin{bmatrix}\Phi_0^{[j]}e^{i\theta_j}\\ 0\end{bmatrix}\approx\begin{bmatrix}\left(\partial_{z_jz_j}\Phi_0^{[j]}+z_j^{-1}\partial_{z_j}\Phi_0^{[j]}-z_j^{-2}\Phi_0^{[j]}\right)e^{i\theta_j}\\ 0\end{bmatrix}.$$

Since for any $v_1,v_2\in\mathbb{R}$,

$$(a-bU_\infty\wedge)\begin{bmatrix}v_1\\ v_2\\ 0\end{bmatrix}=\begin{bmatrix}\mathrm{Re}\,[(a-ib)(v_1+iv_2)]\\ \mathrm{Im}\,[(a-ib)(v_1+iv_2)]\\ 0\end{bmatrix}, \tag{4.10}$$

we then have

$$\begin{aligned}&-\partial_t(\Phi_0^{*[j]})+(a-bU_\infty\wedge)\,\Delta_x\Phi_0^{*[j]}-\frac{2}{z_j}\begin{bmatrix}\dot{p}_j(t)e^{i\theta_j}\\ 0\end{bmatrix}\\ \approx&\begin{bmatrix}-\partial_t\Phi_0^{[j]}e^{i\theta_j}+(a-ib)\left(\partial_{z_jz_j}\Phi_0^{[j]}+z_j^{-1}\partial_{z_j}\Phi_0^{[j]}-z_j^{-2}\Phi_0^{[j]}\right)e^{i\theta_j}\\ 0\end{bmatrix}-\frac{2}{z_j}\begin{bmatrix}\dot{p}_j(t)e^{i\theta_j}\\ 0\end{bmatrix}.\end{aligned}$$

For this reason, we choose $\Phi_0^{[j]}(z_j,t)$ to solve

$$(a+ib)\partial_t\Phi_0^{[j]}=\partial_{z_jz_j}\Phi_0^{[j]}+\frac{1}{z_j}\partial_{z_j}\Phi_0^{[j]}-\frac{1}{z_j^2}\Phi_0^{[j]}-\frac{2(a+ib)\dot{p}_j(t)}{z_j}. \tag{4.11}$$

The special choice of $\frac{r_j^2}{r_j^2+\lambda_j^2}\Phi_0^{[j]}(\sqrt{r_j^2+\lambda_j^2},t)$ aims to avoid singular (in space) terms when calculating new errors.

To analyze (4.11), first we look for self-similar profile to

$$(a+ib)\partial_t\varphi_0^{[j]}=\partial_{z_jz_j}\varphi_0^{[j]}+\frac{1}{z_j}\partial_{z_j}\varphi_0^{[j]}-\frac{1}{z_j^2}\varphi_0^{[j]}+\frac{1}{z_j}$$

with

$$\varphi_0^{[j]}(z_j,t)=t^{1/2}q_0\Big(\frac{z_j}{t^{1/2}}\Big).$$

Then q_0 satisfies

$$q_0''(\xi_j)+\left(\frac{1}{\xi_j}+\frac{a+ib}{2}\xi_j\right)q_0'(\xi_j)-\left(\frac{1}{\xi_j^2}+\frac{a+ib}{2}\right)q_0(\xi_j)+\frac{1}{\xi_j}=0,\quad \xi_j=\frac{z_j}{t^{1/2}}.$$

Observe that ξ_j is a homogeneous solution, so we have a solution

$$q_0(\xi_j)=\xi_j\int_{\xi_j}^{\infty}\frac{e^{-\frac{a+ib}{4}\eta^2}}{\eta^3}d\eta\int_0^{\eta}se^{\frac{a+ib}{4}s^2}ds=\frac{2\xi_j}{a+ib}\int_{\xi_j}^{\infty}\frac{1-e^{-\frac{a+ib}{4}\eta^2}}{\eta^3}d\eta,$$

and

$$|q_0(\xi_j)|\lesssim\begin{cases}-\xi_j\ln\xi_j, & \xi_j\to 0,\\ \xi_j^{-1}, & \xi_j\to\infty.\end{cases}$$

Then by Duhamel's formula, one has a solution to (4.11)

$$\begin{aligned}\Phi_0^{[j]}(z_j,t) = & \int_0^t \dot g_j(s)\varphi_0^{[j]}(z_j,t-s)ds + g_j(0)\varphi_0^{[j]}(z_j,t) = \int_0^t g_j(s)\partial_t\varphi_0^{[j]}(z_j,t-s)ds \\ = & \int_0^t g_j(s)\frac{1}{a+ib}\frac{1-e^{-\frac{a+ib}{4}\frac{z_j^2}{t-s}}}{z_j}ds,\end{aligned}$$

where $g_j(t) = -2(a+ib)\dot p_j(t)$. Rearranging terms, one has

$$\Phi_0^{[j]}(z_j,t) = -z_j\int_0^t \frac{\dot p_j(s)}{t-s}K_0(\frac{z_j^2}{t-s})ds, \tag{4.12}$$

where

$$K_0(\zeta_j) = 2\frac{1-e^{-\frac{(a+ib)\zeta_j}{4}}}{\zeta_j}, \quad \zeta_j = \frac{z_j^2}{t-s}.$$

It is straightforward to compute

$$\begin{aligned}K_0(\zeta_j) = & \left(\frac{a+ib}{2} + O(\zeta_j)\right)\mathbf{1}_{\{\zeta_j\le 1\}} + \left(2\zeta_j^{-1} + O\left(\zeta_j^{-1}e^{-\frac{a}{4}\zeta_j}\right)\right)\mathbf{1}_{\{\zeta_j>1\}} \\ = & \left(\frac{a+ib}{2} + O(\zeta_j)\right)\mathbf{1}_{\{\zeta_j\le 1\}} + O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}, \\ K_{0\zeta_j}(\zeta_j) = & \left[-\left(\frac{a+ib}{4}\right)^2 + O(\zeta_j)\right]\mathbf{1}_{\{\zeta_j\le 1\}} + \left(-2\zeta_j^{-2} + O\left(\zeta_j^{-1}e^{-\frac{a}{4}\zeta_j}\right)\right)\mathbf{1}_{\{\zeta_j>1\}} \\ = & \left[-\left(\frac{a+ib}{4}\right)^2 + O(\zeta_j)\right]\mathbf{1}_{\{\zeta_j\le 1\}} + O\left(\zeta_j^{-2}\right)\mathbf{1}_{\{\zeta_j>1\}}, \\ \zeta_jK_{0\zeta_j}(\zeta_j) = & O(\zeta_j)\mathbf{1}_{\{\zeta_j\le 1\}} + O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}, \\ K_{0\zeta_j\zeta_j}(\zeta_j) = & \left[\frac{2}{3}\left(\frac{a+ib}{4}\right)^3 + O(\zeta_j)\right]\mathbf{1}_{\{\zeta_j\le 1\}} + \left(4\zeta_j^{-3} + O\left(\zeta_j^{-1}e^{-\frac{a}{4}\zeta_j}\right)\right)\mathbf{1}_{\{\zeta_j>1\}} \\ = & \left[\frac{2}{3}\left(\frac{a+ib}{4}\right)^3 + O(\zeta_j)\right]\mathbf{1}_{\{\zeta_j\le 1\}} + O\left(\zeta_j^{-3}\right)\mathbf{1}_{\{\zeta_j>1\}} \\ \zeta_j^2K_{0\zeta_j\zeta_j}(\zeta_j) = & O\left(\zeta_j^2\right)\mathbf{1}_{\{\zeta_j\le 1\}} + O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}.\end{aligned} \tag{4.13}$$

For

$$\zeta_j = \frac{z_j^2}{t-s} = \frac{\lambda_j^2(t)(\rho_j^2+1)}{t-s} = \iota_j(\rho_j^2+1), \quad \iota_j := \frac{\lambda_j^2(t)}{t-s}, \tag{4.14}$$

$$\Phi_0^{[j]}(z_j,t) = -z_j\int_0^t \frac{\dot p_j(s)}{t-s}K_0(\zeta_j)ds = -\lambda_j(\rho_j^2+1)^{\frac12}\int_0^t \frac{\dot p_j(s)}{t-s}K_0(\zeta_j)ds, \tag{4.15}$$

$$\begin{aligned}\partial_{z_j}\Phi_0^{[j]}(z_j,t) = & -\int_0^t \frac{\dot p_j(s)}{t-s}(K_0(\frac{z_j^2}{t-s}) + \frac{2z_j^2}{t-s}K_{0\zeta_j}(\frac{z_j^2}{t-s}))ds \\ = & -\int_0^t \frac{\dot p_j(s)}{t-s}\left(K_0(\zeta_j) + 2\zeta_jK_{0\zeta_j}(\zeta_j)\right)ds,\end{aligned} \tag{4.16}$$

$$\begin{aligned}\partial_{z_jz_j}\Phi_0^{[j]}(z_j,t) = & -\int_0^t \frac{\dot p_j(s)}{t-s}(\frac{2z_j}{t-s}K_{0\zeta_j}(\frac{z_j^2}{t-s}) + \frac{4z_j}{t-s}K_{0\zeta_j}(\frac{z_j^2}{t-s}) + \frac{4z_j^3}{(t-s)^2}K_{0\zeta_j\zeta_j}(\frac{z_j^2}{t-s}))ds \\ = & -z_j^{-1}\int_0^t \frac{\dot p_j(s)}{t-s}(6\zeta_jK_{0\zeta_j}(\zeta_j) + 4\zeta_j^2K_{0\zeta_j\zeta_j}(\zeta_j))ds \\ = & -\lambda_j^{-1}(\rho_j^2+1)^{-\frac12}\int_0^t \frac{\dot p_j(s)}{t-s}\left(6\zeta_jK_{0\zeta_j}(\zeta_j) + 4\zeta_j^2K_{0\zeta_j\zeta_j}(\zeta_j)\right)ds.\end{aligned} \tag{4.17}$$

4.2.1. *The upper bound of the nonlocal terms.* Since $|\dot p| \lesssim |\dot\lambda_*|$ and (4.13),

$$
\begin{aligned}
|\Phi_0^{[j]}| &\lesssim z_j \int_0^t \frac{|\dot\lambda_*(s)|}{t-s} \left(\mathbf{1}_{\{\zeta_j\le 1\}} + O(\zeta_j^{-1})\mathbf{1}_{\{\zeta_j>1\}}\right) ds,\\
|\partial_{z_j}\Phi_0^{[j]}| &\lesssim \int_0^t \frac{|\dot\lambda_*(s)|}{t-s} \left(\mathbf{1}_{\{\zeta_j\le 1\}} + O(\zeta_j^{-1})\mathbf{1}_{\{\zeta_j>1\}}\right) ds, \qquad (4.18)\\
|\partial_{z_jz_j}\Phi_0^{[j]}| &\lesssim z_j^{-1} \int_0^t \frac{|\dot\lambda_*(s)|}{t-s} \left(O(\zeta_j)\mathbf{1}_{\{\zeta_j\le 1\}} + O(\zeta_j^{-1})\mathbf{1}_{\{\zeta_j>1\}}\right) ds.
\end{aligned}
$$

Claim:

$$
\begin{aligned}
&\int_0^t \frac{|\dot\lambda_*(s)|}{t-s} \left(\mathbf{1}_{\{\zeta_j\le 1\}} + \zeta_j^{-1}\mathbf{1}_{\{\zeta_j>1\}}\right)\big|_{\zeta_j=z_j^2(t-s)^{-1}} ds\\
&\lesssim \begin{cases} t|\ln T|^{-1} z_j^{-2}, & z_j^2\ge t\\ \begin{cases} |\ln T|^{-1}\langle \ln(\frac{t}{z_j^2})\rangle, & t\le \frac{T}{2},\\ 1-\frac{|\ln T|}{|\ln(2(T-t))|} + |\dot\lambda_*(t)|\langle\ln(\frac{T-t}{z_j^2})\rangle, & t>\frac{T}{2}, z_j^2<T-t,\\ 1-\frac{|\ln T|}{|\ln(T-t+z_j^2)|} + |\ln T|(\ln z_j)^{-2}, & t>\frac{T}{2}, z_j^2\ge T-t, \end{cases} & z_j^2<t \end{cases} \qquad (4.19)\\
&\lesssim \mathbf{1}_{\{z_j^2<t\}} + t|\ln T|^{-1} z_j^{-2}\mathbf{1}_{\{z_j^2\ge t\}}.
\end{aligned}
$$

By (4.18) and (4.19), we have

$$
|\Phi_0^{[j]}| + z_j|\partial_{z_j}\Phi_0^{[j]}| + z_j^2|\partial_{z_jz_j}\Phi_0^{[j]}| \lesssim z_j\mathbf{1}_{\{z_j^2<t\}} + t|\ln T|^{-1}z_j^{-1}\mathbf{1}_{\{z_j^2\ge t\}}. \qquad (4.20)
$$

Recalling (4.9), (4.23), (4.25) and using (4.20), we have

$$
|\Phi_0^{*[j]}| + z_j|\nabla_x\Phi_0^{*[j]}| + z_j^2|\Delta_x\Phi_0^{*[j]}| \lesssim z_j\mathbf{1}_{\{z_j^2<t\}} + t|\ln T|^{-1}z_j^{-1}\mathbf{1}_{\{z_j^2\ge t\}}. \qquad (4.21)
$$

Proof of Claim (4.19). Denote

$$
g(z_j,t) := \int_0^t \frac{|\dot\lambda_*(s)|}{t-s} \left(\mathbf{1}_{\{\zeta_j\le 1\}} + \zeta_j^{-1}\mathbf{1}_{\{\zeta_j>1\}}\right)\big|_{\zeta_j=z_j^2(t-s)^{-1}} ds.
$$

For $z_j^2\ge t$,

$$
g(z_j,t) = z_j^{-2}\int_0^t |\dot\lambda_*(s)| ds \sim z_j^{-2}|\ln T|\int_0^t |\ln(T-s)|^{-2} ds \sim t|\ln T|^{-1}z_j^{-2}
$$

since if $t\le\frac{T}{2}$,

$$
\int_0^t |\ln(T-s)|^{-2} ds \sim t|\ln T|^{-2};
$$

if $\frac{T}{2}<t\le T$,

$$
\int_0^t |\ln(T-s)|^{-2} ds = \left(\int_0^{\frac{T}{2}} + \int_{\frac{T}{2}}^t\right)|\ln(T-s)|^{-2}ds \sim T|\ln T|^{-2} + \int_{T-t}^{\frac{T}{2}} (\ln z)^{-2} dz \sim T|\ln T|^{-2} \sim t|\ln T|^{-2}.
$$

For $z_j^2<t$,

$$
g(z_j,t) = \int_0^{t-z_j^2} \frac{|\dot\lambda_*(s)|}{t-s} ds + z_j^{-2}\int_{t-z_j^2}^t |\dot\lambda_*(s)| ds.
$$

If $t\le\frac{T}{2}$,

$$
g(z_j,t) \sim |\ln T|^{-1}\langle\ln(\frac{t}{z_j^2})\rangle.
$$

If $t>\frac{T}{2}$ and $z_j^2<T-t$,

$$
g(z_j,t) \sim 1 - \frac{|\ln T|}{|\ln(2(T-t))|} + |\dot\lambda_*(t)|(\ln(\frac{T-t}{z_j^2})+1)
$$

since

$$\int_0^{t-z_j^2} \frac{|\dot\lambda_*(s)|}{t-s} ds = \left(\int_0^{t-(T-t)} + \int_{t-(T-t)}^{t-z_j^2}\right) \frac{|\dot\lambda_*(s)|}{t-s} ds \sim \int_0^{t-(T-t)} \frac{|\dot\lambda_*(s)|}{T-s} ds + |\dot\lambda_*(t)| \int_{t-(T-t)}^{t-z_j^2} \frac{1}{t-s} ds$$

$$\sim |\ln T| \int_0^{t-(T-t)} \frac{1}{(T-s)|\ln(T-s)|^2} ds + |\dot\lambda_*(t)| \ln(\frac{T-t}{z_j^2})$$

$$= 1 - \frac{|\ln T|}{|\ln(2(T-t))|} + |\dot\lambda_*(t)| \ln(\frac{T-t}{z_j^2}),$$

$$z_j^{-2} \int_{t-z_j^2}^t |\dot\lambda_*(s)| ds \sim |\dot\lambda_*(t)|.$$

If $t > \frac{T}{2}$ and $T - t \le z_j^2 < t$,

$$g(z_j, t) \lesssim 1 - \frac{|\ln T|}{|\ln(T-t+z_j^2)|} + |\ln T| (\ln z_j)^{-2}$$

since

$$\int_0^{t-z_j^2} \frac{|\dot\lambda_*(s)|}{t-s} ds \sim \int_0^{t-z_j^2} \frac{|\dot\lambda_*(s)|}{T-s} ds \sim |\ln T| \int_0^{t-z_j^2} \frac{1}{(T-s)|\ln(T-s)|^2} ds = 1 - \frac{|\ln T|}{|\ln(T-t+z_j^2)|},$$

$$z_j^{-2} \int_{t-z_j^2}^t |\dot\lambda_*(s)| ds \sim z_j^{-2} |\ln T| \int_{t-z_j^2}^t (\ln(T-s))^{-2} ds = z_j^{-2} |\ln T| \int_{T-t}^{T-t+z_j^2} (\ln v)^{-2} dv$$

$$\lesssim z_j^{-2} |\ln T| (T-t+z_j^2)(\ln(T-t+z_j^2))^{-2} \sim |\ln T| (\ln z_j)^{-2}.$$

Collecting above estimates, we get the first part of (4.19).

Specially, for $z_j^2 < t, t \le \frac{T}{2}$,

$$t > z_j^2 \gtrsim (\frac{|\ln T|(T-t)}{|\ln(T-t)|^2})^2 \sim T^2 |\ln T|^{-2}.$$

Thus

$$|\ln T|^{-1} \langle \ln(\frac{t}{z_j^2}) \rangle \lesssim 1.$$

For $z_j^2 < t, t > \frac{T}{2}, z_j^2 < T - t$,

$$|\dot\lambda_*(t)| \langle \ln(\frac{T-t}{z_j^2}) \rangle \lesssim \frac{|\ln T|}{|\ln(T-t)|^2} \langle \ln(\frac{T-t}{\lambda_*^2(t)}) \rangle \lesssim 1.$$

Thus we have the second part of (4.19).

□

4.2.2. *New errors produced by the global corrections.* Next we calculate the new errors produced by $\Phi_0^{*[j]}$ defined in (4.9), that is,

$$\begin{aligned} \mathcal{S}^{[j]} := & -\partial_t(\Phi_0^{*[j]}) + \left(a - bU^{[j]}\wedge\right) \left[\Delta_x \Phi_0^{*[j]} + |\nabla_x U^{[j]}|^2 \Phi_0^{*[j]} - 2\nabla_x \left(U^{[j]} \cdot \Phi_0^{*[j]}\right) \nabla_x U^{[j]}\right] - \partial_t U^{[j]} \\ = & -\partial_t(\Phi_0^{*[j]}) + (a - bU_\infty\wedge) \Delta_x \Phi_0^{*[j]} - \partial_t U^{[j]} \\ & - b\left(U^{[j]} - U_\infty\right) \wedge \Delta_x \Phi_0^{*[j]} + a|\nabla_x U^{[j]}|^2 \Phi_0^{*[j]} + b|\nabla_x U^{[j]}|^2 \Phi_0^{*[j]} \wedge U^{[j]} \\ & + \left(a - bU^{[j]}\wedge\right) \left[-2\nabla_x \left(U^{[j]} \cdot \Phi_0^{*[j]}\right) \nabla_x U^{[j]}\right]. \end{aligned}$$

We present some formulas we will use frequently.

$$z_j = \sqrt{r_j^2 + \lambda_j^2} = \sqrt{|x - \xi^{[j]}(t)|^2 + \lambda_j^2(t)},\ \partial_{r_j}(\sqrt{r_j^2+\lambda_j^2}) = \frac{r_j}{\sqrt{r_j^2+\lambda_j^2}},\ \partial_{r_j r_j}(\sqrt{r_j^2+\lambda_j^2}) = \frac{\lambda_j^2}{(r_j^2+\lambda_j^2)^{\frac{3}{2}}},$$

$$\partial_t(\sqrt{r_j^2+\lambda_j^2}) = \frac{\dot\lambda_j \lambda_j - \dot\xi^{[j]} \cdot (x - \xi^{[j]})}{\sqrt{r_j^2+\lambda_j^2}},$$

$$\partial_{r_j}(\frac{r_j^2}{r_j^2+\lambda_j^2}) = \frac{2\lambda_j^2 r_j}{(r_j^2+\lambda_j^2)^2},\ \partial_{r_jr_j}(\frac{r_j^2}{r_j^2+\lambda_j^2}) = 2\lambda_j^2\frac{\lambda_j^2-3r_j^2}{(r_j^2+\lambda_j^2)^3},$$
$$\partial_t(\frac{r_j^2}{r_j^2+\lambda_j^2}) = -\frac{2\lambda_j^2\dot{\xi}^{[j]}\cdot(x-\xi^{[j]})+2\dot{\lambda}_j\lambda_j r_j^2}{(r_j^2+\lambda_j^2)^2},$$
$$\theta_j = \arctan(\frac{x_2-\xi_2^{[j]}}{x_1-\xi_1^{[j]}}), \quad \partial_t\theta_j = \frac{-\dot{\xi}_2^{[j]}(x_1-\xi_1^{[j]})+\dot{\xi}_1^{[j]}(x_2-\xi_2^{[j]})}{r_j^2}.$$
$$\partial_t(\frac{r_j^2}{r_j^2+\lambda_j^2}\Phi_0^{[j]}(z_j,t)) = \frac{r_j^2}{r_j^2+\lambda_j^2}(\partial_t\Phi_0^{[j]}+\frac{\dot{\lambda}_j\lambda_j-\dot{\xi}^{[j]}\cdot(x-\xi^{[j]})}{\sqrt{r_j^2+\lambda_j^2}}\partial_{z_j}\Phi_0^{[j]})$$
$$-\frac{2\lambda_j^2\dot{\xi}^{[j]}\cdot(x-\xi^{[j]})+2\dot{\lambda}_j\lambda_j r_j^2}{(r_j^2+\lambda_j^2)^2}\Phi_0^{[j]},$$
$$\partial_{r_j}(\frac{r_j^2}{r_j^2+\lambda_j^2}\Phi_0^{[j]}(z_j,t)) = \frac{r_j^2}{r_j^2+\lambda_j^2}\partial_{z_j}\Phi_0^{[j]}\frac{r_j}{\sqrt{r_j^2+\lambda_j^2}}+\frac{2\lambda_j^2 r_j}{(r_j^2+\lambda_j^2)^2}\Phi_0^{[j]},$$
$$\partial_{r_jr_j}(\frac{r_j^2}{r_j^2+\lambda_j^2}\Phi_0^{[j]}(z_j,t))$$
$$= \frac{r_j^4}{(r_j^2+\lambda_j^2)^2}\partial_{z_jz_j}\Phi_0^{[j]}+\frac{r_j^2}{r_j^2+\lambda_j^2}\partial_{z_j}\Phi_0^{[j]}\frac{\lambda_j^2}{(r_j^2+\lambda_j^2)^{\frac{3}{2}}}+2\lambda_j^2\frac{\lambda_j^2-3r_j^2}{(r_j^2+\lambda_j^2)^3}\Phi_0^{[j]}+\frac{4\lambda_j^2 r_j}{(r_j^2+\lambda_j^2)^2}\partial_{z_j}\Phi_0^{[j]}\frac{r_j}{\sqrt{r_j^2+\lambda_j^2}}.$$
$$\begin{aligned}\partial_{r_j}\Phi_0^{*[j]} &= \begin{bmatrix}\big[\frac{2\lambda_j^2 r_j}{(r_j^2+\lambda_j^2)^2}\Phi_0^{[j]}(z_j,t)+\frac{r_j^3}{(r_j^2+\lambda_j^2)^{\frac{3}{2}}}\partial_{z_j}\Phi_0^{[j]}(z_j,t)\big]e^{i\theta_j}\\ 0\end{bmatrix}\\ &= \begin{bmatrix}\big[\frac{2\lambda_j^{-1}\rho_j}{(\rho_j^2+1)^2}\Phi_0^{[j]}(z_j,t)+\frac{\rho_j^3}{(\rho_j^2+1)^{\frac{3}{2}}}\partial_{z_j}\Phi_0^{[j]}(z_j,t)\big]e^{i\theta_j}\\ 0\end{bmatrix}, \\ \partial_{\theta_j}\Phi_0^{*[j]} &= \begin{bmatrix}\frac{r_j^2}{r_j^2+\lambda_j^2}\Phi_0^{[j]}(z_j,t)ie^{i\theta_j}\\ 0\end{bmatrix} = \begin{bmatrix}\frac{\rho_j^2}{\rho_j^2+1}\Phi_0^{[j]}(z_j,t)ie^{i\theta_j}\\ 0\end{bmatrix}.\end{aligned} \tag{4.22}$$

It follows that

$$\begin{aligned}&|\nabla_x\Phi_0^{*[j]}|^2 = |\nabla_{x^{[j]}}\Phi_0^{*[j]}|^2 = |\partial_{r_j}\Phi_0^{*[j]}|^2+r_j^{-2}|\partial_{\theta_j}\Phi_0^{*[j]}|^2\\ =& \left|\frac{2\lambda_j^{-1}\rho_j}{(\rho_j^2+1)^2}\Phi_0^{[j]}(z_j,t)+\frac{\rho_j^3}{(\rho_j^2+1)^{\frac{3}{2}}}\partial_{z_j}\Phi_0^{[j]}(z_j,t)\right|^2+\lambda_j^{-2}\rho_j^{-2}\left|\frac{\rho_j^2}{\rho_j^2+1}\Phi_0^{[j]}(z_j,t)\right|^2.\end{aligned} \tag{4.23}$$

Then recalling (4.9), we have

$$
\begin{aligned}
&-\partial_t(\Phi_0^{*[j]}) = \Big[-\partial_t\big(\frac{r_j^2}{r_j^2+\lambda_j^2}\Phi_0^{[j]}(z_j,t)e^{i\theta_j}\big),0\Big]^T\\
&=\Big[\frac{-r_j^2}{r_j^2+\lambda_j^2}\big[\partial_t\Phi_0^{[j]}+\frac{\dot\lambda_j\lambda_j-\dot\xi^{[j]}\cdot(x-\xi^{[j]})}{\sqrt{r_j^2+\lambda_j^2}}\partial_{z_j}\Phi_0^{[j]}\big]e^{i\theta_j}+\frac{2\lambda_j^2\dot\xi^{[j]}\cdot(x-\xi^{[j]})+2\dot\lambda_j\lambda_j r_j^2}{(r_j^2+\lambda_j^2)^2}\Phi_0^{[j]}e^{i\theta_j}\\
&\quad-\frac{r_j^2}{r_j^2+\lambda_j^2}\Phi_0^{[j]}i\frac{-\dot\xi_2^{[j]}(x_1-\xi_1^{[j]})+\dot\xi_1^{[j]}(x_2-\xi_2^{[j]})}{r_j^2}e^{i\theta_j},0\Big]^T\\
&=\Big[\Big\{\frac{-r_j^2}{r_j^2+\lambda_j^2}\partial_t\Phi_0^{[j]}-\frac{r_j^2[\dot\lambda_j\lambda_j-\dot\xi^{[j]}\cdot(x-\xi^{[j]})]}{(r_j^2+\lambda_j^2)^{\frac32}}\partial_{z_j}\Phi_0^{[j]}+\big[\frac{2\lambda_j^2\dot\xi^{[j]}\cdot(x-\xi^{[j]})+2\dot\lambda_j\lambda_j r_j^2}{(r_j^2+\lambda_j^2)^2}\\
&\quad-i\frac{-\dot\xi_2^{[j]}(x_1-\xi_1^{[j]})+\dot\xi_1^{[j]}(x_2-\xi_2^{[j]})}{r_j^2+\lambda_j^2}\big]\Phi_0^{[j]}\Big\}e^{i\theta_j},0\Big]^T\\
&=\Big[\Big\{\frac{-\rho_j^2}{\rho_j^2+1}\partial_t\Phi_0^{[j]}+\frac{\rho_j^2(\dot\xi^{[j]}\cdot y^{[j]}-\dot\lambda_j)}{(\rho_j^2+1)^{\frac32}}\partial_{z_j}\Phi_0^{[j]}+\big[\frac{2\lambda_j^{-1}(\dot\xi^{[j]}\cdot y^{[j]}+\dot\lambda_j\rho_j^2)}{(\rho_j^2+1)^2}\\
&\quad+\frac{i\lambda_j^{-1}(\dot\xi_2^{[j]}y_1^{[j]}-\dot\xi_1^{[j]}y_2^{[j]})}{\rho_j^2+1}\big]\Phi_0^{[j]}\Big\}e^{i\theta_j},0\Big]^T.
\end{aligned}
\tag{4.24}
$$

-

$$
\begin{aligned}
&\Delta_x\Phi_0^{*[j]}=\Delta_{x^{[j]}}\Phi_0^{*[j]}=\Big[\big(\partial_{r_jr_j}+\frac{1}{r_j}\partial_{r_j}+\frac{1}{r_j^2}\partial_{\theta_j\theta_j}\big)\big(\frac{r_j^2}{r_j^2+\lambda_j^2}\Phi_0^{[j]}(z_j,t)e^{i\theta_j}\big),0\Big]^T\\
&=\Big[\big[\frac{r_j^4}{(r_j^2+\lambda_j^2)^2}\partial_{z_jz_j}\Phi_0^{[j]}+\frac{r_j^2}{r_j^2+\lambda_j^2}\partial_{z_j}\Phi_0^{[j]}\frac{\lambda_j^2}{(r_j^2+\lambda_j^2)^{\frac32}}+2\lambda_j^2\frac{\lambda_j^2-3r_j^2}{(r_j^2+\lambda_j^2)^3}\Phi_0^{[j]}+\frac{4\lambda_j^2r_j}{(r_j^2+\lambda_j^2)^2}\partial_{z_j}\Phi_0^{[j]}\frac{r_j}{\sqrt{r_j^2+\lambda_j^2}}\big]e^{i\theta_j}\\
&\quad+\big[\frac{r_j^2}{r_j^2+\lambda_j^2}\partial_{z_j}\Phi_0^{[j]}\frac{1}{\sqrt{r_j^2+\lambda_j^2}}+\frac{2\lambda_j^2}{(r_j^2+\lambda_j^2)^2}\Phi_0^{[j]}\big]e^{i\theta_j}-\frac{1}{r_j^2+\lambda_j^2}\Phi_0^{[j]}e^{i\theta_j},0\Big]^T\\
&=\Big[\Big\{\frac{r_j^4}{(r_j^2+\lambda_j^2)^2}\partial_{z_jz_j}\Phi_0^{[j]}+\big[\frac{5\lambda_j^2r_j^2}{(r_j^2+\lambda_j^2)^{\frac52}}+\frac{r_j^2}{(r_j^2+\lambda_j^2)^{\frac32}}\big]\partial_{z_j}\Phi_0^{[j]}+\big[\frac{4\lambda_j^4-4\lambda_j^2r_j^2}{(r_j^2+\lambda_j^2)^3}-\frac{1}{r_j^2+\lambda_j^2}\big]\Phi_0^{[j]}\Big\}e^{i\theta_j},0\Big]^T\\
&=\Big[\Big\{\frac{\rho_j^4}{(\rho_j^2+1)^2}\partial_{z_jz_j}\Phi_0^{[j]}+\big[\frac{5\lambda_j^{-1}\rho_j^2}{(\rho_j^2+1)^{\frac52}}+\frac{\lambda_j^{-1}\rho_j^2}{(\rho_j^2+1)^{\frac32}}\big]\partial_{z_j}\Phi_0^{[j]}+\big[\frac{4\lambda_j^{-2}(1-\rho_j^2)}{(\rho_j^2+1)^3}-\frac{\lambda_j^{-2}}{\rho_j^2+1}\big]\Phi_0^{[j]}\Big\}e^{i\theta_j},0\Big]^T.
\end{aligned}
\tag{4.25}
$$

• By (4.24), (4.25) and (4.10), we have

$$
\begin{aligned}
&-\partial_t(\Phi_0^{*[j]}) + (a - bU_\infty\wedge)\,\Delta_x\Phi_0^{*[j]} - \partial_t U^{[j]}\\
=&\Big[\Big\{\frac{-\rho_j^2}{\rho_j^2+1}\partial_t\Phi_0^{[j]} + \frac{\rho_j^2(\dot\xi^{[j]}\cdot y^{[j]} - \dot\lambda_j)}{(\rho_j^2+1)^{\frac32}}\partial_{z_j}\Phi_0^{[j]} + \Big[\frac{2\lambda_j^{-1}(\dot\xi^{[j]}\cdot y^{[j]} + \dot\lambda_j\rho_j^2)}{(\rho_j^2+1)^2}\\
&+\frac{i\lambda_j^{-1}(\dot\xi_2^{[j]}y_1^{[j]} - \dot\xi_1^{[j]}y_2^{[j]})}{\rho_j^2+1}\Big]\Phi_0^{[j]}\Big\}e^{i\theta_j},0\Big]^T\\
&+\Big[(a-ib)\Big\{\Big[\frac{\rho_j^2}{\rho_j^2+1} - \frac{\rho_j^2}{(\rho_j^2+1)^2}\Big]\partial_{z_jz_j}\Phi_0^{[j]} + \Big[\frac{5\lambda_j^{-1}\rho_j^2}{(\rho_j^2+1)^{\frac52}} + \frac{\lambda_j^{-1}\rho_j^2}{(\rho_j^2+1)^{\frac32}}\Big]\partial_{z_j}\Phi_0^{[j]}\\
&+\Big[\frac{4\lambda_j^{-2}(1-\rho_j^2)}{(\rho_j^2+1)^3} - \frac{\lambda_j^{-2}}{(\rho_j^2+1)^2} - \frac{\lambda_j^{-2}\rho_j^2}{(\rho_j^2+1)^2}\Big]\Phi_0^{[j]}\Big\}e^{i\theta_j},0\Big]^T + \mathcal{E}_0^{[j]} + \mathcal{E}_1^{[j]}\\
=&\Big[\Big\{\frac{\dot\xi^{[j]}\cdot y^{[j]}\rho_j^2}{(\rho_j^2+1)^{\frac32}}\partial_{z_j}\Phi_0^{[j]} + \Big[\frac{2\lambda_j^{-1}\dot\xi^{[j]}\cdot y^{[j]}}{(\rho_j^2+1)^2} + \frac{i\lambda_j^{-1}(\dot\xi_2^{[j]}y_1^{[j]} - \dot\xi_1^{[j]}y_2^{[j]})}{\rho_j^2+1}\Big]\Phi_0^{[j]}\Big\}e^{i\theta_j},0\Big]^T\\
&+\Big[\Big[\frac{-\dot\lambda_j\rho_j^2}{(\rho_j^2+1)^{\frac32}}\partial_{z_j}\Phi_0^{[j]} + \frac{2\lambda_j^{-1}\dot\lambda_j\rho_j^2}{(\rho_j^2+1)^2}\Phi_0^{[j]}\Big]e^{i\theta_j},0\Big]^T\\
&+\Big[(a-ib)\Big[\frac{-\rho_j^2}{(\rho_j^2+1)^2}\partial_{z_jz_j}\Phi_0^{[j]} + \frac{5\lambda_j^{-1}\rho_j^2}{(\rho_j^2+1)^{\frac52}}\partial_{z_j}\Phi_0^{[j]} + \frac{\lambda_j^{-2}(3-5\rho_j^2)}{(\rho_j^2+1)^3}\Phi_0^{[j]}\Big]e^{i\theta_j},0\Big]^T\\
&+\mathcal{E}_0^{[j]} + \Big[\frac{2\lambda_j^{-1}\dot p_j(t)\rho_j^2}{(\rho_j^2+1)^{\frac32}}e^{i\theta_j},0\Big]^T + \mathcal{E}_1^{[j]},
\end{aligned}
\tag{4.26}
$$

where we have used (4.11) in the last equality.

Also we have

$$
\begin{aligned}
\dot\xi^{[j]}\cdot y^{[j]} &= 2^{-1}\rho_j\left[(\dot\xi_1^{[j]} - i\dot\xi_2^{[j]})e^{i\theta_j} + (\dot\xi_1^{[j]} + i\dot\xi_2^{[j]})e^{-i\theta_j}\right],\\
\dot\xi_2^{[j]}y_1^{[j]} - \dot\xi_1^{[j]}y_2^{[j]} &= 2^{-1}\rho_j\left[(\dot\xi_2^{[j]} + i\dot\xi_1^{[j]})e^{i\theta_j} + (\dot\xi_2^{[j]} - i\dot\xi_1^{[j]})e^{-i\theta_j}\right],
\end{aligned}
\tag{4.27}
$$

$$
\begin{aligned}
&\mathcal{E}_0^{[j]} + \Big[\frac{2\lambda_j^{-1}\dot p_j(t)\rho_j^2}{(\rho_j^2+1)^{\frac32}}e^{i\theta_j},0\Big]^T = \mathcal{E}_0^{[j]} + \Big[\frac{2(\lambda_j^{-1}\dot\lambda_j + i\dot\gamma_j)\rho_j^2e^{i(\theta_j+\gamma_j)}}{(\rho_j^2+1)^{\frac32}},0\Big]^T\\
=&-\frac{2\rho_j}{\rho_j^2+1}\Big[\Big[\lambda_j^{-1}\dot\lambda_j\Big(1-\frac{2}{\rho_j^2+1}\Big) + i\dot\gamma_j\Big]e^{i(\theta_j+\gamma_j)}, -\frac{2\lambda_j^{-1}\dot\lambda_j\rho_j}{\rho_j^2+1}\Big]^T + \Big[\frac{2\rho_j^2}{(\rho_j^2+1)^{\frac32}}(\lambda_j^{-1}\dot\lambda_j + i\dot\gamma_j)e^{i(\theta_j+\gamma_j)},0\Big]^T\\
=&\Big[-\frac{2\rho_j(\lambda_j^{-1}\dot\lambda_j + i\dot\gamma_j)e^{i(\theta_j+\gamma_j)}}{[\rho_j + (\rho_j^2+1)^{\frac12}](\rho_j^2+1)^{\frac32}} + \frac{4\rho_j\lambda_j^{-1}\dot\lambda_je^{i(\theta_j+\gamma_j)}}{(\rho_j^2+1)^2}, \frac{4\rho_j^2\lambda_j^{-1}\dot\lambda_j}{(\rho_j^2+1)^2}\Big]^T\\
=&\Big[\Big\{\frac{-2i\dot\gamma_j\rho_j}{[\rho_j + (\rho_j^2+1)^{\frac12}](\rho_j^2+1)^{\frac32}} + \frac{2\lambda_j^{-1}\dot\lambda_j\rho_j[2\rho_j + (\rho_j^2+1)^{\frac12}]}{[\rho_j + (\rho_j^2+1)^{\frac12}](\rho_j^2+1)^2}\Big\}e^{i(\theta_j+\gamma_j)}, \frac{4\lambda_j^{-1}\dot\lambda_j\rho_j^2}{(\rho_j^2+1)^2}\Big]^T.
\end{aligned}
$$

Then by (3.14),

$$
\begin{aligned}
&\left(\Pi_{U^{[j]\perp}}\left(\mathcal{E}_0^{[j]} + \Big[\frac{2\lambda_j^{-1}\dot p_j(t)\rho_j^2}{(\rho_j^2+1)^{\frac32}}e^{i\theta_j},0\Big]^T\right)\right)_{\mathcal{C}_j}\\
=&\left(1 - \frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\frac{-2i\dot\gamma_j\rho_j}{[\rho_j + (\rho_j^2+1)^{\frac12}](\rho_j^2+1)^{\frac32}} + \frac{2\lambda_j^{-1}\dot\lambda_j\rho_j[2\rho_j + (\rho_j^2+1)^{\frac12}]}{[\rho_j + (\rho_j^2+1)^{\frac12}](\rho_j^2+1)^2}\right\} - \frac{8\lambda_j^{-1}\dot\lambda_j\rho_j^3}{(\rho_j^2+1)^3}\\
=&\frac{-2i\dot\gamma_j\rho_j}{[\rho_j + (\rho_j^2+1)^{\frac12}](\rho_j^2+1)^{\frac32}} - \lambda_j^{-1}\dot\lambda_j\frac{4\rho_j^4 + 6\rho_j^3(\rho_j^2+1)^{\frac12} + 4\rho_j^2 + 2\rho_j(\rho_j^2+1)^{\frac12}}{[\rho_j + (\rho_j^2+1)^{\frac12}](\rho_j^2+1)^3}\\
=&\frac{-2(\lambda_j^{-1}\dot\lambda_j + i\dot\gamma_j)\rho_j}{[\rho_j + (\rho_j^2+1)^{\frac12}](\rho_j^2+1)^{\frac32}} - \lambda_j^{-1}\dot\lambda_j\frac{4\rho_j^2[\rho_j^2 + \rho_j(\rho_j^2+1)^{\frac12} + 1]}{[\rho_j + (\rho_j^2+1)^{\frac12}](\rho_j^2+1)^3},
\end{aligned}
\tag{4.28}
$$

$$\left(\mathcal{E}_0^{[j]} + \Big[\frac{2\lambda_j^{-1}\dot{p}_j(t)\rho_j^2}{(\rho_j^2+1)^{\frac{3}{2}}}e^{i\theta_j}, 0\Big]^T\right)\cdot U^{[j]} = 4\lambda_j^{-1}\dot{\lambda}_j\rho_j^2\left\{\frac{2\rho_j + (\rho_j^2+1)^{\frac{1}{2}}}{[\rho_j+(\rho_j^2+1)^{\frac{1}{2}}](\rho_j^2+1)^3} + \frac{\rho_j^2-1}{(\rho_j^2+1)^3}\right\}. \tag{4.29}$$

-

$$\begin{aligned}
&-b(U^{[j]}-U_\infty)\wedge\Delta_x\Phi_0^{*[j]}\\
=&\frac{-2b}{|y^{[j]}|^2+1}\Big[y_1^{[j]}\cos\gamma_j - y_2^{[j]}\sin\gamma_j, y_1^{[j]}\sin\gamma_j + y_2^{[j]}\cos\gamma_j, -1\Big]^T\wedge\Delta_x\Phi_0^{*[j]}\\
=&\frac{-2b}{\rho_j^2+1}\Big[(\Delta_x\Phi_0^{*[j]})_2, -(\Delta_x\Phi_0^{*[j]})_1, (y_1^{[j]}\cos\gamma_j - y_2^{[j]}\sin\gamma_j)(\Delta_x\Phi_0^{*[j]})_2 - (y_1^{[j]}\sin\gamma_j + y_2^{[j]}\cos\gamma_j)(\Delta_x\Phi_0^{*[j]})_1\Big]^T\\
=&\frac{-2b}{\rho_j^2+1}\Big[(\Delta_x\Phi_0^{*[j]})_2, -(\Delta_x\Phi_0^{*[j]})_1, \rho_j\mathrm{Re}\Big[\Big((\Delta_x\Phi_0^{*[j]})_2 - i(\Delta_x\Phi_0^{*[j]})_1\Big)e^{-i(\theta_j+\gamma_j)}\Big]\Big]^T
\end{aligned} \tag{4.30}$$

where in the last equality, we have used the following formula. For any $a_1, a_2 \in \mathbb{R}$,

$$\begin{aligned}
&\Big(y_1^{[j]}\cos\gamma_j - y_2^{[j]}\sin\gamma_j\Big)a_1 - \Big(y_1^{[j]}\sin\gamma_j + y_2^{[j]}\cos\gamma_j\Big)a_2 = \rho_j\left(a_1\cos(\theta_j+\gamma_j) - a_2\sin(\theta_j+\gamma_j)\right)\\
=&\rho_j\mathrm{Re}\Big[(a_1-ia_2)e^{-i(\theta_j+\gamma_j)}\Big] = \rho_j\mathrm{Im}\Big[(a_2+ia_1)e^{-i(\theta_j+\gamma_j)}\Big].
\end{aligned} \tag{4.31}$$

Then by (3.14), we have

$$\begin{aligned}
&\Big(\Pi_{U^{[j]\perp}}\Big[-b(U^{[j]}-U_\infty)\wedge\Delta_x\Phi_0^{*[j]}\Big]\Big)_{\mathcal{C}_j}\\
=&\frac{-2b}{\rho_j^2+1}\Big\{\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\Big[\Big((\Delta_x\Phi_0^{*[j]})_2 - i(\Delta_x\Phi_0^{*[j]})_1\Big)e^{-i(\theta_j+\gamma_j)}\Big]\\
&-\frac{2\rho_j}{\rho_j^2+1}\rho_j\mathrm{Re}\Big[\Big((\Delta_x\Phi_0^{*[j]})_2 - i(\Delta_x\Phi_0^{*[j]})_1\Big)e^{-i(\theta_j+\gamma_j)}\Big]\Big\}\\
=&\frac{2ib}{\rho_j^2+1}\overline{\Big((\Delta_x\Phi_0^{*[j]})_1 + i(\Delta_x\Phi_0^{*[j]})_2\Big)e^{-i(\theta_j+\gamma_j)}}\\
=&\frac{2ib}{\rho_j^2+1}\left\{\frac{\rho_j^4}{(\rho_j^2+1)^2}\partial_{z_jz_j}\overline{\Phi_0^{[j]}} + \left[\frac{5\lambda_j^{-1}\rho_j^2}{(\rho_j^2+1)^{\frac{5}{2}}} + \frac{\lambda_j^{-1}\rho_j^2}{(\rho_j^2+1)^{\frac{3}{2}}}\right]\partial_{z_j}\overline{\Phi_0^{[j]}} + \left[\frac{4\lambda_j^{-2}(1-\rho_j^2)}{(\rho_j^2+1)^3} - \frac{\lambda_j^{-2}}{\rho_j^2+1}\right]\overline{\Phi_0^{[j]}}\right\}e^{i\gamma_j}\\
=&\frac{2ib}{\rho_j^2+1}\left[\frac{\rho_j^4}{(\rho_j^2+1)^2}\partial_{z_jz_j}\overline{\Phi_0^{[j]}} + \frac{\lambda_j^{-1}(\rho_j^4+6\rho_j^2)}{(\rho_j^2+1)^{\frac{5}{2}}}\partial_{z_j}\overline{\Phi_0^{[j]}} - \frac{\lambda_j^{-2}(\rho_j^4+6\rho_j^2-3)}{(\rho_j^2+1)^3}\overline{\Phi_0^{[j]}}\right]e^{i\gamma_j}\\
=&\left[\frac{2ib\rho_j^4}{(\rho_j^2+1)^3}\partial_{z_jz_j}\overline{\Phi_0^{[j]}} + \frac{2ib\lambda_j^{-1}(\rho_j^4+6\rho_j^2)}{(\rho_j^2+1)^{\frac{7}{2}}}\partial_{z_j}\overline{\Phi_0^{[j]}} - \frac{2ib\lambda_j^{-2}(\rho_j^4+6\rho_j^2-3)}{(\rho_j^2+1)^4}\overline{\Phi_0^{[j]}}\right]e^{i\gamma_j}
\end{aligned} \tag{4.32}$$

where we have used (4.25).

$$\begin{aligned}
&\Big[-b(U^{[j]}-U_\infty)\wedge\Delta_x\Phi_0^{*[j]}\Big]\cdot U^{[j]}\\
=&\frac{-2b}{\rho_j^2+1}\Big\{\frac{2\rho_j}{\rho_j^2+1}\mathrm{Re}\Big[\Big((\Delta_x\Phi_0^{*[j]})_2 - i(\Delta_x\Phi_0^{*[j]})_1\Big)e^{-i(\theta_j+\gamma_j)}\Big]\\
&+\frac{\rho_j^2-1}{\rho_j^2+1}\rho_j\mathrm{Re}\Big[\Big((\Delta_x\Phi_0^{*[j]})_2 - i(\Delta_x\Phi_0^{*[j]})_1\Big)e^{-i(\theta_j+\gamma_j)}\Big]\Big\}\\
=&\frac{-2b\rho_j}{\rho_j^2+1}\mathrm{Im}\Big[\Big((\Delta_x\Phi_0^{*[j]})_1 + i(\Delta_x\Phi_0^{*[j]})_2\Big)e^{-i(\theta_j+\gamma_j)}\Big].
\end{aligned} \tag{4.33}$$

-

$$a|\nabla_x U^{[j]}|^2\Phi_0^{*[j]} = a\lambda_j^{-2}|\nabla_{y^{[j]}}U^{[j]}|^2\Phi_0^{*[j]} = \Big[\frac{8a\lambda_j^{-2}\rho_j^2}{(\rho_j^2+1)^3}\Phi_0^{[j]}e^{i\theta_j}, 0\Big]^T.$$

Then

$$\begin{aligned}&\left(\Pi_{U^{[j]\perp}}\left(a|\nabla_x U^{[j]}|^2\Phi_0^{*[j]}\right)\right)_{\mathcal{C}}=\frac{8a\lambda_j^{-2}\rho_j^2}{(\rho_j^2+1)^3}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)(\Phi_0^{[j]}e^{-i\gamma_j}),\\&\left(a|\nabla_x U^{[j]}|^2\Phi_0^{*[j]}\right)\cdot U^{[j]}=\frac{16a\lambda_j^{-2}\rho_j^3}{(\rho_j^2+1)^4}\mathrm{Re}(\Phi_0^{[j]}e^{-i\gamma_j}).\end{aligned} \tag{4.34}$$

•

$$\begin{aligned}&b|\nabla_x U^{[j]}|^2\Phi_0^{*[j]}\wedge U^{[j]}=\frac{8b\lambda_j^2}{(r_j^2+\lambda_j^2)^2}\Phi_0^{*[j]}\wedge U^{[j]}\\&=\frac{8b\lambda_j^2}{(r_j^2+\lambda_j^2)^2}\Phi_0^{*[j]}\wedge\frac{1}{1+|y^{[j]}|^2}\left[2y_1^{[j]}\cos\gamma_j-2y_2^{[j]}\sin\gamma_j,2y_1^{[j]}\sin\gamma_j+2y_2^{[j]}\cos\gamma_j,|y^{[j]}|^2-1\right]^T\\&=\frac{8b\lambda_j^{-2}}{(\rho_j^2+1)^3}\Big[(\rho_j^2-1)(\Phi_0^{*[j]})_2,-(\rho_j^2-1)(\Phi_0^{*[j]})_1,\\&\quad\left(2y_1^{[j]}\sin\gamma_j+2y_2^{[j]}\cos\gamma_j\right)(\Phi_0^{*[j]})_1-\left(2y_1^{[j]}\cos\gamma_j-2y_2^{[j]}\sin\gamma_j\right)(\Phi_0^{*[j]})_2\Big]^T\\&=\frac{8b\lambda_j^{-2}}{(\rho_j^2+1)^3}\left[(\rho_j^2-1)(\Phi_0^{*[j]})_2,-(\rho_j^2-1)(\Phi_0^{*[j]})_1,-2\rho_j\mathrm{Re}\left[\left((\Phi_0^{*[j]})_2-i(\Phi_0^{*[j]})_1\right)e^{-i(\theta_j+\gamma_j)}\right]\right]^T\end{aligned} \tag{4.35}$$

where we have used (4.31) in the last equality.

It is easy to see

$$\left(b|\nabla_x U^{[j]}|^2\Phi_0^{*[j]}\wedge U^{[j]}\right)\cdot U^{[j]}=0.$$

By (3.14), we get

$$\begin{aligned}&\left(b|\nabla_x U^{[j]}|^2\Phi_0^{*[j]}\wedge U^{[j]}\right)_{\mathcal{C}_j}\\&=\frac{8b\lambda_j^{-2}}{(\rho_j^2+1)^3}\Big\{\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[(\rho_j^2-1)\left((\Phi_0^{*[j]})_2-i(\Phi_0^{*[j]})_1\right)e^{-i(\theta_j+\gamma_j)}\right]\\&\quad+\frac{2\rho_j}{\rho_j^2+1}2\rho_j\mathrm{Re}\left[\left((\Phi_0^{*[j]})_2-i(\Phi_0^{*[j]})_1\right)e^{-i(\theta_j+\gamma_j)}\right]\Big\}\\&=\frac{8b\lambda_j^{-2}}{(\rho_j^2+1)^3}\left(\rho_j^2-1+2\mathrm{Re}\right)\left[\left((\Phi_0^{*[j]})_2-i(\Phi_0^{*[j]})_1\right)e^{-i(\theta_j+\gamma_j)}\right]\\&=\frac{8b\lambda_j^{-2}}{(\rho_j^2+1)^3}\left(\rho_j^2-1+2\mathrm{Re}\right)\left[-i\frac{\rho_j^2}{\rho_j^2+1}\Phi_0^{[j]}e^{i\theta_j}e^{-i(\theta_j+\gamma_j)}\right]\\&=\frac{-8ib\lambda_j^{-2}\rho_j^2(\rho_j^2-1)}{(\rho_j^2+1)^4}\Phi_0^{[j]}e^{-i\gamma_j}+\frac{16b\lambda_j^{-2}\rho_j^2}{(\rho_j^2+1)^4}\mathrm{Im}\left(\Phi_0^{[j]}e^{-i\gamma_j}\right)\\&=\frac{-8ib\lambda_j^{-2}\rho_j^2}{(\rho_j^2+1)^3}\Phi_0^{[j]}e^{-i\gamma_j}+\frac{16ib\lambda_j^{-2}\rho_j^2}{(\rho_j^2+1)^4}\mathrm{Re}\left(\Phi_0^{[j]}e^{-i\gamma_j}\right)\\&=\frac{-8ib\lambda_j^{-2}\rho_j^2}{(\rho_j^2+1)^3}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left(\Phi_0^{[j]}e^{-i\gamma_j}\right),\end{aligned} \tag{4.36}$$

where we have used (4.9) in the third equality.

• By (4.22), we calculate

$$
\begin{aligned}
& -2\nabla_x(\Phi_0^{*[j]}\cdot U^{[j]})\nabla_x U^{[j]} = -2\nabla_{x^{[j]}}(\Phi_0^{*[j]}\cdot U^{[j]})\nabla_{x^{[j]}}U^{[j]}\\
= & -2\partial_{r_j}(\Phi_0^{*[j]}\cdot U^{[j]})\partial_{r_j}U^{[j]} - 2r_j^{-2}\partial_{\theta_j}(\Phi_0^{*[j]}\cdot U^{[j]})\partial_{\theta_j}U^{[j]}\\
= & -2\left(\partial_{r_j}\Phi_0^{*[j]}\cdot U^{[j]} + \Phi_0^{*[j]}\cdot\partial_{r_j}U^{[j]}\right)\partial_{r_j}U^{[j]} - 2r_j^{-2}\left(\partial_{\theta_j}\Phi_0^{*[j]}\cdot U^{[j]} + \Phi_0^{*[j]}\cdot\partial_{\theta_j}U^{[j]}\right)\partial_{\theta_j}U^{[j]}\\
= & -2\left(\partial_{r_j}\Phi_0^{*[j]}\cdot U^{[j]} + \Phi_0^{*[j]}\cdot\lambda_j^{-1}w_{\rho_j}Q_{\gamma_j}E_1^{[j]}\right)\lambda_j^{-1}w_{\rho_j}Q_{\gamma_j}E_1^{[j]}\\
& -2r_j^{-2}\left(\partial_{\theta_j}\Phi_0^{*[j]}\cdot U^{[j]} + \Phi_0^{*[j]}\cdot\sin w(\rho_j)Q_{\gamma_j}E_2^{[j]}\right)\sin w(\rho_j)Q_{\gamma_j}E_2^{[j]}\\
= & -2\Big\{\Big[\big[\frac{2\lambda_j^{-1}\rho_j}{(\rho_j^2+1)^2}\Phi_0^{[j]} + \frac{\rho_j^3}{(\rho_j^2+1)^{\frac{3}{2}}}\partial_{z_j}\Phi_0^{[j]}\big]e^{i\theta_j},0\Big]^T\cdot U^{[j]}\\
& + \Big[\frac{\rho_j^2}{\rho_j^2+1}\Phi_0^{[j]}e^{i\theta_j},0\Big]^T\cdot\lambda_j^{-1}w_{\rho_j}Q_{\gamma_j}E_1^{[j]}\Big\}\lambda_j^{-1}w_{\rho_j}Q_{\gamma_j}E_1^{[j]}\\
& -2r_j^{-2}\left(\Big[\frac{\rho_j^2}{\rho_j^2+1}\Phi_0^{[j]}ie^{i\theta_j},0\Big]^T\cdot U^{[j]} + \Big[\frac{\rho_j^2}{\rho_j^2+1}\Phi_0^{[j]}e^{i\theta_j},0\Big]^T\cdot\sin w(\rho_j)Q_{\gamma_j}E_2^{[j]}\right)\sin w(\rho_j)Q_{\gamma_j}E_2^{[j]}\\
= & -2\Bigg\{\mathrm{Re}\left\{\left[\frac{2\lambda_j^{-1}\rho_j}{(\rho_j^2+1)^2}\Phi_0^{[j]} + \frac{\rho_j^3}{(\rho_j^2+1)^{\frac{3}{2}}}\partial_{z_j}\Phi_0^{[j]}\right]\sin w(\rho_j)e^{-i\gamma_j}\right\}\\
& + \frac{\lambda_j^{-1}\rho_j^2 w_{\rho_j}}{\rho_j^2+1}\mathrm{Re}\left(\Phi_0^{[j]}\cos w(\rho_j)e^{-i\gamma_j}\right)\Bigg\}\lambda_j^{-1}w_{\rho_j}Q_{\gamma_j}E_1^{[j]}\\
& -2r_j^{-2}\left[\frac{\rho_j^2}{\rho_j^2+1}\mathrm{Re}(\Phi_0^{[j]}\sin w(\rho_j)ie^{-i\gamma_j}) + \frac{\rho_j^2\sin w(\rho_j)}{\rho_j^2+1}\mathrm{Im}(\Phi_0^{[j]}e^{-i\gamma_j})\right]\sin w(\rho_j)Q_{\gamma_j}E_2^{[j]}\\
= & \left[\frac{8\lambda_j^{-2}(3\rho_j^2-\rho_j^4)}{(\rho_j^2+1)^4}\mathrm{Re}(\Phi_0^{[j]}e^{-i\gamma_j}) + \frac{8\lambda_j^{-1}\rho_j^4}{(\rho_j^2+1)^{\frac{7}{2}}}\mathrm{Re}(\partial_{z_j}\Phi_0^{[j]}e^{-i\gamma_j})\right]Q_{\gamma_j}E_1^{[j]}
\end{aligned}
\tag{4.37}
$$

where we have used

$$
\partial_{r_j}U^{[j]} = \lambda_j^{-1}\partial_{\rho_j}U^{[j]} = \lambda_j^{-1}w_{\rho_j}Q_{\gamma_j}E_1^{[j]},\quad \partial_{\theta_j}U^{[j]} = \sin w(\rho_j)Q_{\gamma_j}E_2^{[j]}.
$$

It is easy to see

$$
\left\{\left(a - bU^{[j]}\wedge\right)\left[-2\nabla_x\left(U^{[j]}\cdot\Phi_0^{*[j]}\right)\nabla_x U^{[j]}\right]\right\}\cdot U^{[j]} = 0.
$$

By (3.11), one has

$$
\begin{aligned}
& \left(a - bU^{[j]}\wedge\right)\left[-2\nabla_x\left(U^{[j]}\cdot\Phi_0^{*[j]}\right)\nabla_x U^{[j]}\right]\\
= & \left[\frac{8\lambda_j^{-2}(3\rho_j^2-\rho_j^4)}{(\rho_j^2+1)^4}\mathrm{Re}(\Phi_0^{[j]}e^{-i\gamma_j}) + \frac{8\lambda_j^{-1}\rho_j^4}{(\rho_j^2+1)^{\frac{7}{2}}}\mathrm{Re}(\partial_{z_j}\Phi_0^{[j]}e^{-i\gamma_j})\right]\left(aQ_{\gamma_j}E_1^{[j]} - bQ_{\gamma_j}E_2^{[j]}\right).
\end{aligned}
\tag{4.38}
$$

In summary, by (3.14), (4.27), and (4.26), (4.28), (4.29), (4.6), (4.32), (4.33), (4.34), (4.36), (4.38), we conclude that

$$
\begin{aligned}
&\mathcal{S}^{[j]}\cdot U^{[j]}\\
&=\frac{2\rho_j}{\rho_j^2+1}\mathrm{Re}\Bigg\{\Big\{\frac{\dot\xi^{[j]}\cdot y^{[j]}\rho_j^2}{(\rho_j^2+1)^{\frac32}}\partial_{z_j}\Phi_0^{[j]}+\left[\frac{2\lambda_j^{-1}\dot\xi^{[j]}\cdot y^{[j]}}{(\rho_j^2+1)^2}+\frac{i\lambda_j^{-1}(\dot\xi_2^{[j]}y_1^{[j]}-\dot\xi_1^{[j]}y_2^{[j]})}{\rho_j^2+1}\right]\Phi_0^{[j]}\\
&\quad+\frac{-\dot\lambda_j\rho_j^2}{(\rho_j^2+1)^{\frac32}}\partial_{z_j}\Phi_0^{[j]}+\frac{2\lambda_j^{-1}\dot\lambda_j\rho_j^2}{(\rho_j^2+1)^2}\Phi_0^{[j]}\\
&\quad+(a-ib)\left[\frac{-\rho_j^2}{(\rho_j^2+1)^2}\partial_{z_jz_j}\Phi_0^{[j]}+\frac{5\lambda_j^{-1}\rho_j^2}{(\rho_j^2+1)^{\frac52}}\partial_{z_j}\Phi_0^{[j]}+\frac{\lambda_j^{-2}(3-5\rho_j^2)}{(\rho_j^2+1)^3}\Phi_0^{[j]}\right]\Big\}e^{-i\gamma_j}\Bigg\}\\
&\quad+4\lambda_j^{-1}\dot\lambda_j\rho_j^2\left\{\frac{2\rho_j+(\rho_j^2+1)^{\frac12}}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^3}+\frac{\rho_j^2-1}{(\rho_j^2+1)^3}\right\}\\
&\quad-\frac{2b\rho_j}{\rho_j^2+1}\mathrm{Im}\left[\Big((\Delta_x\Phi_0^{*[j]})_1+i(\Delta_x\Phi_0^{*[j]})_2\Big)e^{-i(\theta_j+\gamma_j)}\right]+\frac{16a\lambda_j^{-2}\rho_j^3}{(\rho_j^2+1)^4}\mathrm{Re}(\Phi_0^{[j]}e^{-i\gamma_j}).
\end{aligned}
\tag{4.39}
$$

By (4.20), (4.21), we have

$$
|\mathcal{S}^{[j]}\cdot U^{[j]}|\lesssim|\dot\xi^{[j]}|\langle\rho_j\rangle^{-1}+|\lambda_j|^{-1}\langle\rho_j\rangle^{-2},\tag{4.40}
$$

and

$$
\begin{aligned}
&\left(\Pi_{U^{[j]\perp}}\mathcal{S}^{[j]}\right)_{\mathcal{C}_j}\\
&=\left[(\dot\xi_1^{[j]}-i\dot\xi_2^{[j]})e^{i\theta_j}+(\dot\xi_1^{[j]}+i\dot\xi_2^{[j]})e^{-i\theta_j}\right]\\
&\quad\times\left[\frac{\rho_j^3}{2(\rho_j^2+1)^{\frac32}}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)(\partial_{z_j}\Phi_0^{[j]}e^{-i\gamma_j})+\frac{\lambda_j^{-1}\rho_j}{(\rho_j^2+1)^2}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)(\Phi_0^{[j]}e^{-i\gamma_j})\right]\\
&\quad+\left[(\dot\xi_2^{[j]}+i\dot\xi_1^{[j]})e^{i\theta_j}+(\dot\xi_2^{[j]}-i\dot\xi_1^{[j]})e^{-i\theta_j}\right]\frac{\lambda_j^{-1}\rho_j}{2(\rho_j^2+1)}\left(i+\frac{2}{\rho_j^2+1}\mathrm{Im}\right)(\Phi_0^{[j]}e^{-i\gamma_j})\\
&\quad+\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\Bigg\{\Big\{\frac{-\dot\lambda_j\rho_j^2}{(\rho_j^2+1)^{\frac32}}\partial_{z_j}\Phi_0^{[j]}+\frac{2\lambda_j^{-1}\dot\lambda_j\rho_j^2}{(\rho_j^2+1)^2}\Phi_0^{[j]}\\
&\quad+(a-ib)\left[\frac{-\rho_j^2}{(\rho_j^2+1)^2}\partial_{z_jz_j}\Phi_0^{[j]}+\frac{5\lambda_j^{-1}\rho_j^2}{(\rho_j^2+1)^{\frac52}}\partial_{z_j}\Phi_0^{[j]}+\frac{\lambda_j^{-2}(3-5\rho_j^2)}{(\rho_j^2+1)^3}\Phi_0^{[j]}\right]\Big\}e^{-i\gamma_j}\Bigg\}\\
&\quad-\frac{2(\lambda_j^{-1}\dot\lambda_j+i\dot\gamma_j)\rho_j}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^{\frac32}}-\lambda_j^{-1}\dot\lambda_j\frac{4\rho_j^2[\rho_j^2+\rho_j(\rho_j^2+1)^{\frac12}+1]}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^3}-\frac{2\lambda_j^{-1}(\dot\xi_1^{[j]}-i\dot\xi_2^{[j]})}{\rho_j^2+1}e^{i\theta_j}\\
&\quad+\left[\frac{2ib\rho_j^4}{(\rho_j^2+1)^3}\partial_{z_jz_j}\overline{\Phi_0^{[j]}}+\frac{2ib\lambda_j^{-1}(\rho_j^4+6\rho_j^2)}{(\rho_j^2+1)^{\frac72}}\partial_{z_j}\overline{\Phi_0^{[j]}}-\frac{2ib\lambda_j^{-2}(\rho_j^4+6\rho_j^2-3)}{(\rho_j^2+1)^4}\overline{\Phi_0^{[j]}}\right]e^{i\gamma_j}\\
&\quad+\frac{8a\lambda_j^{-2}\rho_j^2}{(\rho_j^2+1)^3}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)(\Phi_0^{[j]}e^{-i\gamma_j})-\frac{8ib\lambda_j^{-2}\rho_j^2}{(\rho_j^2+1)^3}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left(\Phi_0^{[j]}e^{-i\gamma_j}\right)\\
&\quad+(a-ib)\left[\frac{8\lambda_j^{-2}(3\rho_j^2-\rho_j^4)}{(\rho_j^2+1)^4}\mathrm{Re}(\Phi_0^{[j]}e^{-i\gamma_j})+\frac{8\lambda_j^{-1}\rho_j^4}{(\rho_j^2+1)^{\frac72}}\mathrm{Re}(\partial_{z_j}\Phi_0^{[j]}e^{-i\gamma_j})\right]
\end{aligned}
$$

$$
\begin{aligned}
= & \left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{(a-ib)e^{-i\gamma_j}\left[\frac{-\rho_j^2}{(\rho_j^2+1)^2}\partial_{z_jz_j}\Phi_0^{[j]}+\frac{5\lambda_j^{-1}\rho_j^2}{(\rho_j^2+1)^{\frac{5}{2}}}\partial_{z_j}\Phi_0^{[j]}+\frac{\lambda_j^{-2}(3-5\rho_j^2)}{(\rho_j^2+1)^3}\Phi_0^{[j]}\right]\right\} \\
& +\left[\frac{2ib\rho_j^4}{(\rho_j^2+1)^3}\partial_{z_jz_j}\overline{\Phi_0^{[j]}}+\frac{2ib\lambda_j^{-1}(\rho_j^4+6\rho_j^2)}{(\rho_j^2+1)^{\frac{7}{2}}}\partial_{z_j}\overline{\Phi_0^{[j]}}-\frac{2ib\lambda_j^{-2}(\rho_j^4+6\rho_j^2-3)}{(\rho_j^2+1)^4}\overline{\Phi_0^{[j]}}\right]e^{i\gamma_j} \\
& +(a-ib)\frac{8\lambda_j^{-2}\rho_j^2}{(\rho_j^2+1)^3}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)(\Phi_0^{[j]}e^{-i\gamma_j}) \\
& +(a-ib)\left[\frac{8\lambda_j^{-1}\rho_j^4}{(\rho_j^2+1)^{\frac{7}{2}}}\mathrm{Re}(\partial_{z_j}\Phi_0^{[j]}e^{-i\gamma_j})+\frac{8\lambda_j^{-2}(3\rho_j^2-\rho_j^4)}{(\rho_j^2+1)^4}\mathrm{Re}(\Phi_0^{[j]}e^{-i\gamma_j})\right] \\
& +\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[\frac{-\dot\lambda_j\rho_j^2}{(\rho_j^2+1)^{\frac{3}{2}}}\partial_{z_j}\Phi_0^{[j]}e^{-i\gamma_j}+\frac{2\lambda_j^{-1}\dot\lambda_j\rho_j^2}{(\rho_j^2+1)^2}\Phi_0^{[j]}e^{-i\gamma_j}\right] \\
& -\frac{2(\lambda_j^{-1}\dot\lambda_j+i\dot\gamma_j)\rho_j}{[\rho_j+(\rho_j^2+1)^{\frac{1}{2}}](\rho_j^2+1)^{\frac{3}{2}}}-\lambda_j^{-1}\dot\lambda_j\frac{4\rho_j^2[\rho_j^2+\rho_j(\rho_j^2+1)^{\frac{1}{2}}+1]}{[\rho_j+(\rho_j^2+1)^{\frac{1}{2}}](\rho_j^2+1)^3} \\
& +\Bigg\{-\frac{2\lambda_j^{-1}(\dot\xi_1^{[j]}-i\dot\xi_2^{[j]})}{\rho_j^2+1} \\
& +(\dot\xi_1^{[j]}-i\dot\xi_2^{[j]})\left[\frac{\rho_j^3}{2(\rho_j^2+1)^{\frac{3}{2}}}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)(\partial_{z_j}\Phi_0^{[j]}e^{-i\gamma_j})+\frac{\lambda_j^{-1}\rho_j}{(\rho_j^2+1)^2}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)(\Phi_0^{[j]}e^{-i\gamma_j})\right] \\
& +(\dot\xi_2^{[j]}+i\dot\xi_1^{[j]})\frac{\lambda_j^{-1}\rho_j}{2(\rho_j^2+1)}\left(i+\frac{2}{\rho_j^2+1}\mathrm{Im}\right)(\Phi_0^{[j]}e^{-i\gamma_j})\Bigg\}e^{i\theta_j} \\
& +\Bigg\{(\dot\xi_1^{[j]}+i\dot\xi_2^{[j]})\left[\frac{\rho_j^3}{2(\rho_j^2+1)^{\frac{3}{2}}}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)(\partial_{z_j}\Phi_0^{[j]}e^{-i\gamma_j})+\frac{\lambda_j^{-1}\rho_j}{(\rho_j^2+1)^2}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)(\Phi_0^{[j]}e^{-i\gamma_j})\right] \\
& +(\dot\xi_2^{[j]}-i\dot\xi_1^{[j]})\frac{\lambda_j^{-1}\rho_j}{2(\rho_j^2+1)}\left(i+\frac{2}{\rho_j^2+1}\mathrm{Im}\right)(\Phi_0^{[j]}e^{-i\gamma_j})\Bigg\}e^{-i\theta_j}.
\end{aligned}
$$

Then by (4.15), (4.16), (4.17), we get

$$
\begin{aligned}
= &\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\Bigg\{(a-ib)\Bigg[\frac{\lambda_j^{-1}\rho_j^2}{(\rho_j^2+1)^{\frac52}}\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left(6\zeta_jK_{0\zeta_j}(\zeta_j)+4\zeta_j^2K_{0\zeta_j\zeta_j}(\zeta_j)\right)ds\\
&-\frac{5\lambda_j^{-1}\rho_j^2}{(\rho_j^2+1)^{\frac52}}\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left(K_0(\zeta_j)+2\zeta_jK_{0\zeta_j}(\zeta_j)\right)ds-\frac{\lambda_j^{-1}(3-5\rho_j^2)}{(\rho_j^2+1)^{\frac52}}\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}K_0(\zeta_j)ds\Bigg]\Bigg\}\\
&-\frac{2ib\lambda_j^{-1}\rho_j^4}{(\rho_j^2+1)^{\frac72}}\int_0^t\overline{\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}}\left(\overline{6\zeta_jK_{0\zeta_j}(\zeta_j)+4\zeta_j^2K_{0\zeta_j\zeta_j}(\zeta_j)}\right)ds\\
&-\frac{2ib\lambda_j^{-1}(\rho_j^4+6\rho_j^2)}{(\rho_j^2+1)^{\frac72}}\int_0^t\overline{\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}}\left(\overline{K_0(\zeta_j)+2\zeta_jK_{0\zeta_j}(\zeta_j)}\right)ds\\
&+\frac{2ib\lambda_j^{-1}(\rho_j^4+6\rho_j^2-3)}{(\rho_j^2+1)^{\frac72}}\int_0^t\overline{\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}}\overline{K_0(\zeta_j)}ds\\
&-(a-ib)\frac{8\lambda_j^{-1}\rho_j^2}{(\rho_j^2+1)^{\frac52}}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left(\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}K_0(\zeta_j)ds\right)\\
&+(a-ib)\Bigg[\frac{8\lambda_j^{-1}\rho_j^4}{(\rho_j^2+1)^{\frac72}}\mathrm{Re}\left(-\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left(K_0(\zeta_j)+2\zeta_jK_{0\zeta_j}(\zeta_j)\right)ds\right)\\
&-\frac{8\lambda_j^{-1}(3\rho_j^2-\rho_j^4)}{(\rho_j^2+1)^{\frac72}}\mathrm{Re}\left(\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}K_0(\zeta_j)ds\right)\Bigg]\\
&+\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\Bigg[\frac{\dot\lambda_j\rho_j^2}{(\rho_j^2+1)^{\frac32}}\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left(K_0(\zeta_j)+2\zeta_jK_{0\zeta_j}(\zeta_j)\right)ds\\
&-\frac{2\dot\lambda_j\rho_j^2}{(\rho_j^2+1)^{\frac32}}\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}K_0(\zeta_j)ds\Bigg]\\
&-\frac{2(\lambda_j^{-1}\dot\lambda_j+i\dot\gamma_j)\rho_j}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^{\frac32}}-\lambda_j^{-1}\dot\lambda_j\frac{4\rho_j^2[\rho_j^2+\rho_j(\rho_j^2+1)^{\frac12}+1]}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^3}\\
&\quad+\Bigg\{-\frac{2\lambda_j^{-1}(\dot\xi_1^{[j]}-i\dot\xi_2^{[j]})}{\rho_j^2+1}\\
&\quad+(\dot\xi_1^{[j]}-i\dot\xi_2^{[j]})\Bigg[\frac{\rho_j^3}{2(\rho_j^2+1)^{\frac32}}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left(-\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left(K_0(\zeta_j)+2\zeta_jK_{0\zeta_j}(\zeta_j)\right)ds\right)\\
&\quad+\frac{\rho_j}{(\rho_j^2+1)^{\frac32}}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left(-\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}K_0(\zeta_j)ds\right)\Bigg]\\
&\quad+(\dot\xi_2^{[j]}+i\dot\xi_1^{[j]})\frac{\rho_j}{2(\rho_j^2+1)^{\frac12}}\left(i+\frac{2}{\rho_j^2+1}\mathrm{Im}\right)\left(-\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}K_0(\zeta_j)ds\right)\Bigg\}e^{i\theta_j}\\
&\quad+\Bigg\{(\dot\xi_1^{[j]}+i\dot\xi_2^{[j]})\Bigg[\frac{\rho_j^3}{2(\rho_j^2+1)^{\frac32}}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left(-\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left(K_0(\zeta_j)+2\zeta_jK_{0\zeta_j}(\zeta_j)\right)ds\right)\\
&\quad+\frac{\rho_j}{(\rho_j^2+1)^{\frac32}}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left(-\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}K_0(\zeta_j)ds\right)\Bigg]\\
&\quad+(\dot\xi_2^{[j]}-i\dot\xi_1^{[j]})\frac{\rho_j}{2(\rho_j^2+1)^{\frac12}}\left(i+\frac{2}{\rho_j^2+1}\mathrm{Im}\right)\left(-\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}K_0(\zeta_j)ds\right)\Bigg\}e^{-i\theta_j}\\
&=M_0^{[j]}(\rho_j,t)+e^{i\theta_j}M_1^{[j]}(\rho_j,t)+e^{-i\theta_j}M_{-1}^{[j]}(\rho_j,t)
\end{aligned}
$$

where

$$
\begin{aligned}
&M_0^{[j]}(\rho_j,t)\\
:=\ &\lambda_j^{-1}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{(a-ib)\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left[\frac{-3K_0(\zeta_j)}{(\rho_j^2+1)^{\frac52}}-\frac{4\rho_j^2\zeta_jK_{0\zeta_j}(\zeta_j)}{(\rho_j^2+1)^{\frac52}}+\frac{4\rho_j^2\zeta_j^2K_{0\zeta_j\zeta_j}(\zeta_j)}{(\rho_j^2+1)^{\frac52}}\right]ds\right\}\\
&-ib\lambda_j^{-1}\left\{\int_0^t \frac{\overline{\dot p_j(s)e^{-i\gamma_j(t)}}}{t-s}\left[\frac{6\overline{K_0(\zeta_j)}}{(\rho_j^2+1)^{\frac72}}+\frac{8(2\rho_j^4+3\rho_j^2)\zeta_j\overline{K_{0\zeta_j}(\zeta_j)}}{(\rho_j^2+1)^{\frac72}}+\frac{8\rho_j^4\zeta_j^2\overline{K_{0\zeta_j\zeta_j}(\zeta_j)}}{(\rho_j^2+1)^{\frac72}}\right]ds\right\}\\
&-(a-ib)\lambda_j^{-1}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\frac{8\rho_j^2K_0(\zeta_j)}{(\rho_j^2+1)^{\frac52}}ds\right]\\
&-(a-ib)\lambda_j^{-1}\mathrm{Re}\left\{\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left[\frac{24\rho_j^2K_0(\zeta_j)}{(\rho_j^2+1)^{\frac72}}+\frac{16\rho_j^4\zeta_jK_{0\zeta_j}(\zeta_j)}{(\rho_j^2+1)^{\frac72}}\right]ds\right\}\\
&+\dot\lambda_j\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left[\frac{-\rho_j^2K_0(\zeta_j)}{(\rho_j^2+1)^{\frac32}}+\frac{2\rho_j^2\zeta_jK_{0\zeta_j}(\zeta_j)}{(\rho_j^2+1)^{\frac32}}\right]ds\right\}\\
&-\frac{2(\lambda_j^{-1}\dot\lambda_j+i\dot\gamma_j)\rho_j}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^{\frac32}}-\lambda_j^{-1}\dot\lambda_j\frac{4\rho_j^2[\rho_j^2+\rho_j(\rho_j^2+1)^{\frac12}+1]}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^3}\\
=\ &\lambda_j^{-1}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[(a-ib)\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\right.\\
&\times\left\{\frac{-3}{(\rho_j^2+1)^{\frac52}}\left[\left(\frac{a+ib}{2}+O(\zeta_j)\right)\mathbf{1}_{\{\zeta_j\le1\}}+O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}\right]\right.\\
&\left.\left.+\langle\rho_j\rangle^{-3}\left(O(\zeta_j)\mathbf{1}_{\{\zeta_j\le1\}}+O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}\right)\right\}ds\right]\\
&-ib\lambda_j^{-1}\int_0^t \frac{\overline{\dot p_j(s)e^{-i\gamma_j(t)}}}{t-s}\left\{\frac{6}{(\rho_j^2+1)^{\frac72}}\left[\left(\frac{a-ib}{2}+O(\zeta_j)\right)\mathbf{1}_{\{\zeta_j\le1\}}+O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}\right]\right.\\
&\left.+\langle\rho_j\rangle^{-3}\left(O(\zeta_j)\mathbf{1}_{\{\zeta_j\le1\}}+O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}\right)\right\}ds\\
&-(a-ib)\lambda_j^{-1}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\right.\\
&\left.\times\frac{8\rho_j^2}{(\rho_j^2+1)^{\frac52}}\left[\left(\frac{a+ib}{2}+O(\zeta_j)\right)\mathbf{1}_{\{\zeta_j\le1\}}+O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}\right]ds\right\}\\
&-(a-ib)\lambda_j^{-1}\mathrm{Re}\left[\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left\{\frac{24\rho_j^2}{(\rho_j^2+1)^{\frac72}}\left[\left(\frac{a+ib}{2}+O(\zeta_j)\right)\mathbf{1}_{\{\zeta_j\le1\}}+O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}\right]\right.\right.\\
&\left.\left.+\langle\rho_j\rangle^{-3}\left(O(\zeta_j)\mathbf{1}_{\{\zeta_j\le1\}}+O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}\right)\right\}ds\right]\\
&+\dot\lambda_j\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left\{\frac{-\rho_j^2}{(\rho_j^2+1)^{\frac32}}\left[\left(\frac{a+ib}{2}+O(\zeta_j)\right)\mathbf{1}_{\{\zeta_j\le1\}}+O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}\right]\right.\right.\\
&\left.\left.+\langle\rho_j\rangle^{-1}\left(O(\zeta_j)\mathbf{1}_{\{\zeta_j\le1\}}+O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}\right)\right\}ds\right]\\
&-\frac{2\lambda_j^{-1}\dot p_je^{-i\gamma_j}\rho_j}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^{\frac32}}-\lambda_j^{-1}\dot\lambda_j\frac{4\rho_j^2[\rho_j^2+\rho_j(\rho_j^2+1)^{\frac12}+1]}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^3}
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda_j^{-1}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left\{\left[\frac{-3}{2(\rho_j^2+1)^{\frac52}}+\frac{O(\zeta_j)}{\langle\rho_j\rangle^3}\right]\mathbf{1}_{\{\zeta_j\le 1\}}+\frac{O(\zeta_j^{-1})}{\langle\rho_j\rangle^3}\mathbf{1}_{\{\zeta_j>1\}}\right\}ds\right]\\
&\quad - ib\lambda_j^{-1}\int_0^t \frac{\overline{\dot p_j(s)e^{-i\gamma_j(t)}}}{t-s}\left\{\left[\frac{3(a-ib)}{(\rho_j^2+1)^{\frac72}}+\frac{O(\zeta_j)}{\langle\rho_j\rangle^3}\right]\mathbf{1}_{\{\zeta_j\le 1\}}+\frac{O(\zeta_j^{-1})}{\langle\rho_j\rangle^3}\mathbf{1}_{\{\zeta_j>1\}}\right\}ds\\
&\quad -(a-ib)\lambda_j^{-1}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\right.\\
&\quad \times\left.\left\{\left[\frac{4(a+ib)\rho_j^2}{(\rho_j^2+1)^{\frac52}}+\frac{O(\zeta_j)}{\langle\rho_j\rangle^3}\right]\mathbf{1}_{\{\zeta_j\le 1\}}+\frac{O(\zeta_j^{-1})}{\langle\rho_j\rangle^3}\mathbf{1}_{\{\zeta_j>1\}}\right\}ds\right]\\
&\quad -(a-ib)\lambda_j^{-1}\mathrm{Re}\left[\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left\{\left[\frac{12(a+ib)\rho_j^2}{(\rho_j^2+1)^{\frac72}}+\frac{O(\zeta_j)}{\langle\rho_j\rangle^3}\right]\mathbf{1}_{\{\zeta_j\le 1\}}+\frac{O(\zeta_j^{-1})}{\langle\rho_j\rangle^3}\mathbf{1}_{\{\zeta_j>1\}}\right\}ds\right]\\
&\quad +\dot\lambda_j\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left\{\left[\frac{-(a+ib)\rho_j^2}{2(\rho_j^2+1)^{\frac32}}+\frac{O(\zeta_j)}{\langle\rho_j\rangle}\right]\mathbf{1}_{\{\zeta_j\le 1\}}+\frac{O(\zeta_j^{-1})}{\langle\rho_j\rangle}\mathbf{1}_{\{\zeta_j>1\}}\right\}ds\right]\\
&\quad -\frac{2\lambda_j^{-1}\dot p_j e^{-i\gamma_j}\rho_j}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^{\frac32}}-\lambda_j^{-1}\dot\lambda_j\frac{4\rho_j^2[\rho_j^2+\rho_j(\rho_j^2+1)^{\frac12}+1]}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^3}
\end{aligned}
$$

where we have used (4.13).

Then by (4.19), it follows that

$$
\begin{aligned}
|M_0^{[j]}(\rho_j,t)| &\lesssim \left(\lambda_*^{-1}\langle\rho_j\rangle^{-3}+|\dot\lambda_*|\langle\rho_j\rangle^{-1}\right)\int_0^t \frac{|\dot\lambda_*(s)|}{t-s}\left(\mathbf{1}_{\{\zeta_j\le 1\}}+O(\zeta_j^{-1})\mathbf{1}_{\{\zeta_j>1\}}\right)ds+\lambda_*^{-1}|\dot\lambda_*|\langle\rho_j\rangle^{-3}\\
&\lesssim \lambda_*^{-1}\langle\rho_j\rangle^{-3}+|\dot\lambda_*|\langle\rho_j\rangle^{-1}.
\end{aligned}
\tag{4.41}
$$

$$
\begin{aligned}
&M_1^{[j]}(\rho_j,t)\\
:=\ & -(\dot\xi_1^{[j]}-i\dot\xi_2^{[j]})\left[\frac{2\lambda_j^{-1}}{\rho_j^2+1}+\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left[\frac{(\rho_j^3+2\rho_j)K_0(\zeta_j)}{2(\rho_j^2+1)^{\frac32}}+\frac{\rho_j^3\zeta_j K_{0\zeta_j}(\zeta_j)}{(\rho_j^2+1)^{\frac32}}\right]ds\right\}\right]\\
& -(i\dot\xi_1^{[j]}+\dot\xi_2^{[j]})\frac{\rho_j}{2(\rho_j^2+1)^{\frac12}}\left(i+\frac{2}{\rho_j^2+1}\mathrm{Im}\right)\left(\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}K_0(\zeta_j)ds\right)\\
=\ & -(\dot\xi_1^{[j]}-i\dot\xi_2^{[j]})\left\{\frac{2\lambda_j^{-1}}{\rho_j^2+1}\right.\\
& +\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left\{\frac{\rho_j^3+2\rho_j}{2(\rho_j^2+1)^{\frac32}}\left[\left(\frac{a+ib}{2}+O(\zeta_j)\right)\mathbf{1}_{\{\zeta_j\le 1\}}+O(\zeta_j^{-1})\mathbf{1}_{\{\zeta_j>1\}}\right]\right.\right.\\
& \left.\left.\left.+\frac{\rho_j^3}{(\rho_j^2+1)^{\frac32}}\left(O(\zeta_j)\mathbf{1}_{\{\zeta_j\le 1\}}+O(\zeta_j^{-1})\mathbf{1}_{\{\zeta_j>1\}}\right)\right\}ds\right]\right\}\\
& -(i\dot\xi_1^{[j]}+\dot\xi_2^{[j]})\frac{\rho_j}{2(\rho_j^2+1)^{\frac12}}\\
& \times\left(i+\frac{2}{\rho_j^2+1}\mathrm{Im}\right)\left(\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left[\left(\frac{a+ib}{2}+O(\zeta_j)\right)\mathbf{1}_{\{\zeta_j\le 1\}}+O(\zeta_j^{-1})\mathbf{1}_{\{\zeta_j>1\}}\right]ds\right)\\
=\ & -(\dot\xi_1^{[j]}-i\dot\xi_2^{[j]})\frac{2\lambda_j^{-1}}{\rho_j^2+1}+\tilde M_1^{[j]}(\rho_j,t)
\end{aligned}
$$

where we have used (4.13) and

$$\begin{aligned}
&\tilde{M}_1^{[j]}(\rho_j,t) := -(\dot{\xi}_1^{[j]} - i\dot{\xi}_2^{[j]})\Bigg\{ \left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\Bigg[\int_0^t \frac{\dot{p}_j(s)e^{-i\gamma_j(t)}}{t-s} \\
&\times \left\{ \left[\frac{(a+ib)(\rho_j^3+2\rho_j)}{4(\rho_j^2+1)^{\frac{3}{2}}} + O(\zeta_j)\right]\mathbf{1}_{\{\zeta_j\le 1\}} + O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}\right\} ds\Bigg]\Bigg\} \\
&-(i\dot{\xi}_1^{[j]} + \dot{\xi}_2^{[j]}) \\
&\times \left(i+\frac{2}{\rho_j^2+1}\mathrm{Im}\right)\left[\int_0^t \frac{\dot{p}_j(s)e^{-i\gamma_j(t)}}{t-s}\left\{\left[\frac{(a+ib)\rho_j}{4(\rho_j^2+1)^{\frac{1}{2}}}+O(\zeta_j)\right]\mathbf{1}_{\{\zeta_j\le 1\}} + O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}\right\} ds\right].
\end{aligned} \tag{4.42}$$

Then by (4.19), one has

$$|\tilde{M}_1^{[j]}| \lesssim |\dot{\xi}^{[j]}| \int_0^t \frac{|\dot{\lambda}_j(s)|}{t-s}\left(\mathbf{1}_{\{\zeta_j\le 1\}} + O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}\right) ds \lesssim |\dot{\xi}^{[j]}|, \quad |M_1^{[j]}| \lesssim |\dot{\xi}^{[j]}|\lambda_*^{-1}\langle\rho_j\rangle^{-2} + |\dot{\xi}^{[j]}|. \tag{4.43}$$

$$\begin{aligned}
&M_{-1}^{[j]}(\rho_j,t) \\
:= &-(\dot{\xi}_1^{[j]} + i\dot{\xi}_2^{[j]})\left[\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\int_0^t \frac{\dot{p}_j(s)e^{-i\gamma_j(t)}}{t-s}\left[\frac{(\rho_j^3+2\rho_j)K_0(\zeta_j)}{2(\rho_j^2+1)^{\frac{3}{2}}} + \frac{\rho_j^3\zeta_j K_{0\zeta_j}(\zeta_j)}{(\rho_j^2+1)^{\frac{3}{2}}}\right] ds\right\}\right] \\
&-(\dot{\xi}_2^{[j]} - i\dot{\xi}_1^{[j]})\frac{\rho_j}{2(\rho_j^2+1)^{\frac{1}{2}}}\left(i+\frac{2}{\rho_j^2+1}\mathrm{Im}\right)\left(\int_0^t \frac{\dot{p}_j(s)e^{-i\gamma_j(t)}}{t-s}K_0(\zeta_j)ds\right).
\end{aligned} \tag{4.44}$$

By (4.13) and (4.19), we get

$$|M_{-1}^{[j]}| \lesssim (|\dot{\xi}_1^{[j]}| + |\dot{\xi}_2^{[j]}|)\int_0^t \frac{|\dot{\lambda}_*(s)|}{t-s}\left(\mathbf{1}_{\{\zeta_j\le 1\}} + O\left(\zeta_j^{-1}\right)\mathbf{1}_{\{\zeta_j>1\}}\right) ds \lesssim |\dot{\xi}^{[j]}|. \tag{4.45}$$

As a result of (4.41), (4.43) and (4.45), we have

$$\left|\left(\Pi_{U^{[j]\perp}}\mathcal{S}^{[j]}\right)_{\mathcal{C}}\right| \lesssim \lambda_*^{-1}\langle\rho_j\rangle^{-3} + |\dot{\lambda}_*|\langle\rho_j\rangle^{-1} + |\dot{\xi}^{[j]}|\left(\lambda_*^{-1}\langle\rho_j\rangle^{-2}+1\right). \tag{4.46}$$

Integrating (4.40) and (4.46), we have

$$\left|\mathcal{S}^{[j]}\right| \lesssim \lambda_*^{-1}\langle\rho_j\rangle^{-2} + |\dot{\lambda}_*|\langle\rho_j\rangle^{-1} + |\dot{\xi}^{[j]}|. \tag{4.47}$$

5. Gluing system

In this section, we formulate the inner–outer gluing system such that solution with desired asymptotics can be found.

5.1. Ansatz for the multi-bubble solution u. We look for solution u of the form

$$\begin{aligned}
&u = (1+A)U_* + \Phi - (\Phi\cdot U_*)U_*, \\
&\Phi(x,t) = \sum_{j=1}^N \left(\eta_R^{[j]}(x,t)Q_{\gamma_j}\Phi_{\text{in}}^{[j]}(y^{[j]},t) + \eta_{d_q}^{[j]}(x,t)\Phi_0^{*[j]}(r_j,t)\right) + \Phi_{\text{out}}(x,t)
\end{aligned} \tag{5.1}$$

where $\Phi_{\text{in}}^{[j]}(y^{[j]},t)\cdot W^{[j]} = 0$ for all $t\in(0,T)$, $j=1,2,\dots,N$; η is smooth cut-off function,

$$\eta(s) = \begin{cases} 1, & \text{for } s<1, \\ 0, & \text{for } s>2, \end{cases} \quad \eta_R^{[j]}(x,t) = \eta\left(\frac{x-\xi^{[j]}(t)}{\lambda_j(t)R(t)}\right),\ \eta_{d_q}^{[j]}(x,t) = \eta\left(\frac{x-\xi^{[j]}(t)}{d_q}\right), \tag{5.2}$$

$$d_q := \frac{1}{9}\min_{k\neq m}|q^{[k]}-q^{[m]}|.$$

where A is a scalar function, $\Phi_{\text{in}}^{[j]}$ and Φ_{out} will be solved in the inner-outer system, $\Phi_0^{*[j]}(r_j,t)$ is defined in (4.12).

Set $p_j(t)=\lambda_j(t)e^{i\gamma_j(t)}$. Throughout this paper, we choose the following ansatzes for all $j=1,2,\dots,N$,

$$
\begin{aligned}
&C_\lambda^{-1}\lambda_*(t)\le |p_j(t)|=\lambda_j(t)\le C_\lambda\lambda_*(t)\ \text{ with }\ \lambda_*(t)=\frac{|\ln T|(T-t)}{\ln^2(T-t)},\\
&C_\lambda^{-1}\frac{|\ln T|}{\ln^2(T-t)}\le|\dot p_j(t)|\le C_\lambda\frac{|\ln T|}{\ln^2(T-t)},\quad |\dot\gamma_j(t)|\lesssim C_\gamma(T-t)^{-1},\quad |\dot\xi_j(t)|\le C_\xi\lambda_*^{\epsilon_\xi}(t),\\
&R(t)=\lambda_*^{-\beta}(t),\quad |\Phi|\ll 1
\end{aligned}
\tag{5.3}
$$

where $C_\lambda\ge 1$, $C_\xi>0$, $C_\gamma>0$; $\epsilon_\xi>0$ is small; $0<\beta<1$ will be chosen later; $\Phi_{\rm in}^{[j]}$ solves the inner problem near each bubble $U^{[j]}$, while $\Phi_{\rm out}$ handles the region away from the concentration zones. Notice that

$$\eta_{d_q}^{[j]}\equiv 1\ \text{ in }\ |x-\xi^{[j]}(t)|\le 2\lambda_j(t)R(t).$$

Suitable A will be chosen in (5.1) to make $|u|=1$. Indeed,

$$
\begin{aligned}
&|u|^2=1\Longleftrightarrow(1+A)^2|U_*|^2+2(1+A)(\Phi\cdot U_*)(1-|U_*|^2)+|\Phi-(\Phi\cdot U_*)U_*|^2=1\\
&\Longleftrightarrow(1+A)^2+2(1+A)\frac{(\Phi\cdot U_*)(1-|U_*|^2)}{|U_*|^2}=\frac{1-|\Phi-(\Phi\cdot U_*)U_*|^2}{|U_*|^2}\\
&\Longleftrightarrow\left[1+A+\frac{(\Phi\cdot U_*)(1-|U_*|^2)}{|U_*|^2}\right]^2=\frac{1-|\Phi-(\Phi\cdot U_*)U_*|^2}{|U_*|^2}+\left[\frac{(\Phi\cdot U_*)(1-|U_*|^2)}{|U_*|^2}\right]^2.
\end{aligned}
$$

We take

$$A=\left\{1+\frac{1-|U_*|^2-|\Phi-(\Phi\cdot U_*)U_*|^2}{|U_*|^2}+\left[\frac{(\Phi\cdot U_*)(1-|U_*|^2)}{|U_*|^2}\right]^2\right\}^{\frac12}-1-\frac{(\Phi\cdot U_*)(1-|U_*|^2)}{|U_*|^2}. \tag{5.4}$$

By (4.2) and (5.3), we have

$$A=\left(1+O(\lambda_*+|\Phi|^2)+O(\lambda_*^2|\Phi|^2)\right)^{\frac12}-1+O(\lambda_*|\Phi|)=O(\lambda_*+\lambda_*|\Phi|+|\Phi|^2)=O(\lambda_*+|\Phi|^2) \tag{5.5}$$

under the assumption $|\Phi|\ll 1$ in (5.3).

One important insight is that we only need to solve

$$S(u)=\Xi(x,t)U_* \tag{5.6}$$

for some scalar function Ξ. Indeed, since $|u|=1$ is kept for all $t\in(0,T)$ and $u=U_*+\tilde w$ where the perturbation $\tilde w$ is uniformly small, then

$$\Xi(U_*\cdot u)=S(u)\cdot u=-\frac12\partial_t(|u|^2)+\frac a2\Delta|u|^2=0.$$

Thus $\Xi\equiv 0$ follows from $U_*\cdot u\ge\delta_0>0$. (5.6) provides us the flexibility to adjust the error terms in U_* direction and we will call this **U_*-operation** throughout this paper.

We compute

$$
\begin{aligned}
-\partial_t\Phi=&-\partial_t\Phi_{\rm out}-\sum_{j=1}^2\partial_t(\Phi_0^{*[j]})+\sum_{j=1}^2\eta_R^{[j]}Q_{\gamma_j}\left[-\partial_t\Phi_{\rm in}^{[j]}+\left(\lambda_j^{-1}\dot\lambda_j y^{[j]}+\lambda_j^{-1}\dot\xi^{[j]}\right)\cdot\nabla_{y^{[j]}}\Phi_{\rm in}^{[j]}-\dot\gamma_jJ\Phi_{\rm in}^{[j]}\right]\\
&-\sum_{j=1}^2\partial_t\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]},
\end{aligned}
$$

$$\Delta_x\Phi=\Delta\Phi_{\rm out}+\sum_{j=1}^2\Delta_x\Phi_0^{*[j]}+\sum_{j=1}^2\eta_R^{[j]}Q_{\gamma_j}\Delta_x\Phi_{\rm in}^{[j]}+\sum_{j=1}^2Q_{\gamma_j}\left(\Phi_{\rm in}^{[j]}\Delta_x\eta_R^{[j]}+2\nabla_x\eta_R^{[j]}\cdot\nabla_x\Phi_{\rm in}^{[j]}\right)$$

where we have used $\partial_t(Q_{\gamma_j})=\dot\gamma_jJQ_{\gamma_j}=\dot\gamma_jQ_{\gamma_j}J$,

$$J:=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}. \tag{5.7}$$

Notice

- $$\begin{aligned}
&U_*\Delta_x A + (1+A)\Delta_x U_* + 2\nabla_x A\nabla_x U_* + \Delta_x\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + \left|\nabla_x\left[(1+A)U_* + \Phi - (\Phi\cdot U_*)U_*\right]\right|^2\left[(1+A)U_* + \Phi - (\Phi\cdot U_*)U_*\right]\\
&= \Delta_x\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + \left|\nabla_x\left[(1+A)U_*\right]\right|^2\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + \Big\{2\nabla_x\left[(1+A)U_*\right]\cdot\nabla_x\left[\Phi - (\Phi\cdot U_*)U_*\right] + \left|\nabla_x\left[\Phi - (\Phi\cdot U_*)U_*\right]\right|^2\Big\}\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + 2\nabla_x A\nabla_x U_* + (1+A)\Delta_x U_*\\
&\quad + U_*\Big[\Delta_x A + \left|\nabla_x\left[(1+A)U_* + \Phi - (\Phi\cdot U_*)U_*\right]\right|^2(1+A)\Big]\\
&= \Delta_x\left[\Phi - (\Phi\cdot U_*)U_*\right] + \left|\nabla_x U_*\right|^2\left[\Phi - (\Phi\cdot U_*)U_*\right] + 2\left\{\nabla_x U_*\cdot\nabla_x\left[\Phi - (\Phi\cdot U_*)U_*\right]\right\}U_*\\
&\quad + \left|\nabla_x\left[(1+A)U_*\right]\right|^2\left[\Phi - (\Phi\cdot U_*)U_*\right] - \left|\nabla_x U_*\right|^2\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + \Big\{2\nabla_x\left[(1+A)U_*\right]\cdot\nabla_x\left[\Phi - (\Phi\cdot U_*)U_*\right] + \left|\nabla_x\left[\Phi - (\Phi\cdot U_*)U_*\right]\right|^2\Big\}\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + 2\nabla_x A\nabla_x U_* + (1+A)\Delta_x U_*\\
&\quad + U_*\Big\{\Delta_x A + \left|\nabla_x\left[(1+A)U_* + \Phi - (\Phi\cdot U_*)U_*\right]\right|^2(1+A) - 2\nabla_x U_*\cdot\nabla_x\left[\Phi - (\Phi\cdot U_*)U_*\right]\Big\}\\
&= \Delta_x\Phi - 2\nabla_x(\Phi\cdot U_*)\nabla_x U_* + \left|\nabla_x U_*\right|^2\Phi\\
&\quad + \left|\nabla_x\left[(1+A)U_*\right]\right|^2\left[\Phi - (\Phi\cdot U_*)U_*\right] - \left|\nabla_x U_*\right|^2\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + \Big\{2\nabla_x\left[(1+A)U_*\right]\cdot\nabla_x\left[\Phi - (\Phi\cdot U_*)U_*\right] + \left|\nabla_x\left[\Phi - (\Phi\cdot U_*)U_*\right]\right|^2\Big\}\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + 2\nabla_x A\nabla_x U_* + \left[1 + A - (\Phi\cdot U_*)\right]\Delta_x U_*\\
&\quad + U_*\Big\{\Delta_x A + \left|\nabla_x\left[(1+A)U_* + \Phi - (\Phi\cdot U_*)U_*\right]\right|^2(1+A) - \left|\nabla_x U_*\right|^2(\Phi\cdot U_*) - \Delta_x(\Phi\cdot U_*)\Big\}.
\end{aligned}$$

- $$\begin{aligned}
&\left[(1+A)U_* + \Phi - (\Phi\cdot U_*)U_*\right]\wedge\Delta_x\left[(1+A)U_* + \Phi - (\Phi\cdot U_*)U_*\right]\\
&= \left[\Phi - (\Phi\cdot U_*)U_*\right]\wedge\Delta_x\left[(1+A)U_*\right]\\
&\quad + \left[(1+A)U_*\right]\wedge\Delta_x\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + \left[\Phi - (\Phi\cdot U_*)U_*\right]\wedge\Delta_x\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + (1+A)U_*\wedge\Delta_x\left[(1+A)U_*\right]\\
&= \left[\Phi - (\Phi\cdot U_*)U_*\right]\wedge\Delta_x U_* + U_*\wedge\Delta_x\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + \left[\Phi - (\Phi\cdot U_*)U_*\right]\wedge\Delta_x(AU_*) + AU_*\wedge\Delta_x\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + \left[\Phi - (\Phi\cdot U_*)U_*\right]\wedge\Delta_x\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + (1+A)U_*\wedge\Delta_x\left[(1+A)U_*\right]\\
&= \Phi\wedge\Delta_x U_* + U_*\wedge\left[\Delta_x\Phi - 2\nabla_x(\Phi\cdot U_*)\nabla_x U_*\right]\\
&\quad + \left[\Phi - (\Phi\cdot U_*)U_*\right]\wedge\Delta_x(AU_*) + AU_*\wedge\Delta_x\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + \left[\Phi - (\Phi\cdot U_*)U_*\right]\wedge\Delta_x\left[\Phi - (\Phi\cdot U_*)U_*\right]\\
&\quad + (1+A)U_*\wedge\Delta_x\left[(1+A)U_*\right] - 2(\Phi\cdot U_*)U_*\wedge\Delta_x U_*.
\end{aligned}$$

Next we calculate

$$
\begin{aligned}
& S(u) \\
= & -U_*\partial_t A-(1+A)\partial_t U_*-\partial_t\Phi+(\Phi\cdot U_*)\partial_t U_*+U_*\partial_t(\Phi\cdot U_*) \\
& +a\Big\{U_*\Delta_x A+(1+A)\Delta_x U_*+2\nabla_x A\cdot\nabla_x U_*+\Delta_x[\Phi-(\Phi\cdot U_*)U_*] \\
& +\left|\nabla_x\left[(1+A)U_*+\Phi-(\Phi\cdot U_*)U_*\right]\right|^2\left[(1+A)U_*+\Phi-(\Phi\cdot U_*)U_*\right]\Big\} \\
& -b\left[(1+A)U_*+\Phi-(\Phi\cdot U_*)U_*\right]\wedge\Delta_x\left[(1+A)U_*+\Phi-(\Phi\cdot U_*)U_*\right] \\
= & -(1+A)\partial_t U_*-\partial_t\Phi+(\Phi\cdot U_*)\partial_t U_*+U_*\left[\partial_t(\Phi\cdot U_*)-\partial_t A\right] \\
& +a\Big\{\Delta_x\Phi-2\nabla_x(\Phi\cdot U_*)\nabla_x U_*+\left|\nabla_x U_*\right|^2\Phi \\
& +\left|\nabla_x\left[(1+A)U_*\right]\right|^2\left[\Phi-(\Phi\cdot U_*)U_*\right]-\left|\nabla_x U_*\right|^2\left[\Phi-(\Phi\cdot U_*)U_*\right] \\
& +\Big\{2\nabla_x\left[(1+A)U_*\right]\cdot\nabla_x\left[\Phi-(\Phi\cdot U_*)U_*\right]+\left|\nabla_x\left[\Phi-(\Phi\cdot U_*)U_*\right]\right|^2\Big\}\left[\Phi-(\Phi\cdot U_*)U_*\right] \\
& +2\nabla_x A\nabla_x U_*+\left[1+A-(\Phi\cdot U_*)\right]\Delta_x U_* \\
& +U_*\left\{\Delta_x A+\left|\nabla_x\left[(1+A)U_*+\Phi-(\Phi\cdot U_*)U_*\right]\right|^2(1+A)-\left|\nabla_x U_*\right|^2(\Phi\cdot U_*)-\Delta_x(\Phi\cdot U_*)\right\}\Big\} \\
& -b\Big\{\Phi\wedge\Delta_x U_*+U_*\wedge\left[\Delta_x\Phi-2\nabla_x(\Phi\cdot U_*)\nabla_x U_*\right] \\
& +\left[\Phi-(\Phi\cdot U_*)U_*\right]\wedge\Delta_x\left(AU_*\right)+AU_*\wedge\Delta_x\left[\Phi-(\Phi\cdot U_*)U_*\right] \\
& +\left[\Phi-(\Phi\cdot U_*)U_*\right]\wedge\Delta_x\left[\Phi-(\Phi\cdot U_*)U_*\right] \\
& +(1+A)U_*\wedge\Delta_x\left[(1+A)U_*\right]-2(\Phi\cdot U_*)U_*\wedge\Delta_x U_*\Big\} \\
= & -\partial_t\Phi+a\left\{\Delta_x\Phi-2\nabla_x(\Phi\cdot U_*)\nabla_x U_*+\left|\nabla_x U_*\right|^2\Phi\right\}-b\left\{\Phi\wedge\Delta_x U_*+U_*\wedge\left[\Delta_x\Phi-2\nabla_x(\Phi\cdot U_*)\nabla_x U_*\right]\right\} \\
& -\partial_t U_*+(\Phi\cdot U_*)\partial_t U_*+\mathcal{N}[\Phi]+\mathcal{B}[\Phi]U_* \\
= & -\partial_t\Phi+(a-bU_*\wedge)\left[\Delta_x\Phi-2\nabla_x(\Phi\cdot U_*)\nabla_x U_*\right]+a\left|\nabla_x U_*\right|^2\Phi-b\Phi\wedge\Delta_x U_* \\
& -\partial_t U_*+(\Phi\cdot U_*)\partial_t U_*+\mathcal{N}[\Phi]+\mathcal{B}[\Phi]U_* \\
= & -\partial_t\Phi+(a-bU_*\wedge)\left[\Delta_x\Phi-2\nabla_x(\Phi\cdot U_*)\nabla_x U_*\right]+a\Phi\sum_{j=1}^N|\nabla_x U^{[j]}|^2+b\Phi\wedge\sum_{j=1}^N|\nabla_x U^{[j]}|^2U^{[j]} \\
& -\partial_t U_*+a\Phi\sum_{j\neq k}\nabla_x U^{[j]}\cdot\nabla_x U^{[k]}+(\Phi\cdot U_*)\partial_t U_*+\mathcal{N}[\Phi]+\mathcal{B}[\Phi]U_* \\
= & -\partial_t\Phi+(a-bU_*\wedge)\left[\Delta_x\Phi-2\nabla_x(\Phi\cdot U_*)\nabla_x U_*\right]+\sum_{j=1}^N|\nabla_x U^{[j]}|^2\left(a-bU^{[j]}\wedge\right)\Phi \\
& -\partial_t U_*+a\Phi\sum_{j\neq k}\nabla_x U^{[j]}\cdot\nabla_x U^{[k]}+(\Phi\cdot U_*)\partial_t U_*+\mathcal{N}[\Phi]+\mathcal{B}[\Phi]U_* \\
= & -\partial_t\Phi+(a-bU_*\wedge)\left\{\Delta_x\Phi-2\sum_{j=1}^N\nabla_x\left[\Phi\cdot\left(U^{[j]}+U_*-U^{[j]}\right)\right]\nabla_x U^{[j]}\right\}+\sum_{j=1}^N|\nabla_x U^{[j]}|^2\left(a-bU^{[j]}\wedge\right)\Phi \\
& -\partial_t U_*+a\Phi\sum_{j\neq k}\nabla_x U^{[j]}\cdot\nabla_x U^{[k]}+(\Phi\cdot U_*)\partial_t U_*+\mathcal{N}[\Phi]+\Xi[\Phi]U_*
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
= & -\partial_t\Phi + (a-bU_*\wedge)\left[\Delta_x\Phi - 2\sum_{j=1}^N \nabla_x\left(\Phi\cdot U^{[j]}\right)\nabla_x U^{[j]}\right] + \sum_{j=1}^N |\nabla_x U^{[j]}|^2\left(a-bU^{[j]}\wedge\right)\Phi \\
& -\partial_t U_* + (a-bU_*\wedge)\left\{-2\sum_{j=1}^N \nabla_x\left[\Phi\cdot\left(U_*-U^{[j]}\right)\right]\nabla_x U^{[j]}\right\} \\
& + a\Phi\sum_{j\neq k}\nabla_x U^{[j]}\cdot\nabla_x U^{[k]} + (\Phi\cdot U_*)\partial_t U_* + \mathcal{N}[\Phi] + \mathcal{B}[\Phi]U_* \\
= & -\partial_t\Phi_{\rm out} - \sum_{j=1}^N \partial_t(\eta_{d_q}^{[j]}\Phi_0^{*[j]}) + \sum_{j=1}^N \eta_R^{[j]}Q_{\gamma_j}\left[-\partial_t\Phi_{\rm in}^{[j]} + \left(\lambda_j^{-1}\dot\lambda_j y^{[j]} + \lambda_j^{-1}\dot\xi^{[j]}\right)\cdot\nabla_{y^{[j]}}\Phi_{\rm in}^{[j]} - \dot\gamma_j J\Phi_{\rm in}^{[j]}\right] \\
& - \sum_{j=1}^N \partial_t\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]} \\
& + (a-bU_*\wedge)\Bigg\{\Delta_x\Phi_{\rm out} + \sum_{j=1}^N\Delta_x(\eta_{d_q}^{[j]}\Phi_0^{*[j]}) + \sum_{j=1}^N \eta_R^{[j]}Q_{\gamma_j}\Delta_x\Phi_{\rm in}^{[j]} + \sum_{j=1}^N Q_{\gamma_j}\left(\Phi_{\rm in}^{[j]}\Delta_x\eta_R^{[j]} + 2\nabla_x\eta_R^{[j]}\cdot\nabla_x\Phi_{\rm in}^{[j]}\right) \\
& - 2\sum_{j=1}^N\nabla_x\left\{U^{[j]}\cdot\left[\sum_{k=1}^N\left(\eta_R^{[k]}Q_{\gamma_k}\Phi_{\rm in}^{[k]} + \eta_{d_q}^{[k]}\Phi_0^{*[k]}\right) + \Phi_{\rm out}\right]\right\}\nabla_x U^{[j]}\Bigg\} \\
& + \sum_{j=1}^N|\nabla_x U^{[j]}|^2\left(a-bU^{[j]}\wedge\right)\left[\sum_{k=1}^N\left(\eta_R^{[k]}Q_{\gamma_k}\Phi_{\rm in}^{[k]} + \eta_{d_q}^{[k]}\Phi_0^{*[k]}\right) + \Phi_{\rm out}\right] \\
& -\partial_t U_* + (a-bU_*\wedge)\left\{-2\sum_{j=1}^N \nabla_x\left[\Phi\cdot\left(U_*-U^{[j]}\right)\right]\nabla_x U^{[j]}\right\} \\
& + a\Phi\sum_{j\neq k}\nabla_x U^{[j]}\cdot\nabla_x U^{[k]} + (\Phi\cdot U_*)\partial_t U_* + \mathcal{N}[\Phi] + \Xi[\Phi]U_*
\end{aligned}
$$

$$
\begin{aligned}
= & -\partial_t \Phi_{\text{out}} + (a - bU_* \wedge)\, \Delta_x \Phi_{\text{out}} + \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \left(a - bU^{[j]} \wedge\right) \Phi_{\text{out}} \\
& - \sum_{j=1}^{N} \partial_t (\eta_{d_q}^{[j]} \Phi_0^{*[j]}) + \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \left(a - bU^{[j]} \wedge\right) \eta_{d_q}^{[j]} \Phi_0^{*[j]} \\
& - \sum_{j=1}^{N} \eta_R^{[j]} Q_{\gamma_j} \partial_t \Phi_{\text{in}}^{[j]} + \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \left(a - bU^{[j]} \wedge\right) \eta_R^{[j]} Q_{\gamma_j} \Phi_{\text{in}}^{[j]} \\
& + \sum_{j=1}^{N} \eta_R^{[j]} Q_{\gamma_j} \left[\left(\lambda_j^{-1} \dot{\lambda}_j y^{[j]} + \lambda_j^{-1} \dot{\xi}^{[j]} \right) \cdot \nabla_{y^{[j]}} \Phi_{\text{in}}^{[j]} - \dot{\gamma}_j J \Phi_{\text{in}}^{[j]} \right] \\
& - \sum_{j=1}^{N} \partial_t \eta_R^{[j]} Q_{\gamma_j} \Phi_{\text{in}}^{[j]} \\
& + \sum_{j=1}^{N} \left[a - b \left(U^{[j]} + U_* - U^{[j]} \right) \wedge \right] \Bigg\{ \Delta_x (\eta_{d_q}^{[j]} \Phi_0^{*[j]}) + \eta_R^{[j]} Q_{\gamma_j} \Delta_x \Phi_{\text{in}}^{[j]} + Q_{\gamma_j} \left(\Phi_{\text{in}}^{[j]} \Delta_x \eta_R^{[j]} + 2 \nabla_x \eta_R^{[j]} \cdot \nabla_x \Phi_{\text{in}}^{[j]} \right) \\
& - 2 \nabla_x \left(U^{[j]} \cdot \Phi_{\text{out}} \right) \nabla_x U^{[j]} - 2 \nabla_x \left[U^{[j]} \cdot \left(\eta_R^{[j]} Q_{\gamma_j} \Phi_{\text{in}}^{[j]} + \eta_{d_q}^{[j]} \Phi_0^{*[j]} \right) \right] \nabla_x U^{[j]} \Bigg\} \\
& - \partial_t U_* + (a - bU_* \wedge) \left\{ -2 \sum_{j=1}^{N} \nabla_x \left[\Phi \cdot \left(U_* - U^{[j]} \right) \right] \nabla_x U^{[j]} \right\} \\
& + (a - bU_* \wedge) \left\{ -2 \sum_{j=1}^{N} \nabla_x \left[U^{[j]} \cdot \sum_{k \neq j} \left(\eta_R^{[k]} Q_{\gamma_k} \Phi_{\text{in}}^{[k]} + \eta_{d_q}^{[k]} \Phi_0^{*[k]} \right) \right] \nabla_x U^{[j]} \right\} \\
& + \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \left(a - bU^{[j]} \wedge\right) \sum_{k \neq j} \left(\eta_R^{[k]} Q_{\gamma_k} \Phi_{\text{in}}^{[k]} + \eta_{d_q}^{[k]} \Phi_0^{*[k]} \right) \\
& + a \Phi \sum_{j \neq k} \nabla_x U^{[j]} \cdot \nabla_x U^{[k]} + (\Phi \cdot U_*) \partial_t U_* + \mathcal{N}[\Phi] + \Xi[\Phi] U_*
\end{aligned}
$$

$$
\begin{aligned}
= & -\partial_t\Phi_{\rm out} + (a-bU_*\wedge)\,\Delta_x\Phi_{\rm out} + \sum_{j=1}^N |\nabla_x U^{[j]}|^2\left(a-bU^{[j]}\wedge\right)\Phi_{\rm out}\\
& + \sum_{j=1}^N \left(a-bU^{[j]}\wedge\right)\left[-2\nabla_x\left(U^{[j]}\cdot\Phi_{\rm out}\right)\nabla_x U^{[j]}\right]\\
& - \sum_{j=1}^N \partial_t(\eta_{d_q}^{[j]}\Phi_0^{*[j]}) + \sum_{j=1}^N \left(a-bU^{[j]}\wedge\right)\Delta_x(\eta_{d_q}^{[j]}\Phi_0^{*[j]}) + \sum_{j=1}^N |\nabla_x U^{[j]}|^2\left(a-bU^{[j]}\wedge\right)\eta_{d_q}^{[j]}\Phi_0^{*[j]}\\
& + \sum_{j=1}^N \left(a-bU^{[j]}\wedge\right)\left[-2\nabla_x\left(U^{[j]}\cdot\eta_{d_q}^{[j]}\Phi_0^{*[j]}\right)\nabla_x U^{[j]}\right]\\
& - \sum_{j=1}^N \eta_R^{[j]}Q_{\gamma_j}\partial_t\Phi_{\rm in}^{[j]} + \sum_{j=1}^N \left(a-bU^{[j]}\wedge\right)\eta_R^{[j]}Q_{\gamma_j}\Delta_x\Phi_{\rm in}^{[j]} + \sum_{j=1}^N |\nabla_x U^{[j]}|^2\left(a-bU^{[j]}\wedge\right)\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\\
& + \sum_{j=1}^N \left(a-bU^{[j]}\wedge\right)\left\{-2\nabla_x\left[U^{[j]}\cdot\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\nabla_x U^{[j]}\right\}\\
& + \sum_{j=1}^N \eta_R^{[j]}Q_{\gamma_j}\left[\left(\lambda_j^{-1}\dot\lambda_j y^{[j]} + \lambda_j^{-1}\dot\xi^{[j]}\right)\cdot\nabla_{y^{[j]}}\Phi_{\rm in}^{[j]} - \dot\gamma_j J\Phi_{\rm in}^{[j]}\right]\\
& - \sum_{j=1}^N \partial_t\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]} + \sum_{j=1}^N \left(a-bU^{[j]}\wedge\right)\left[Q_{\gamma_j}\left(\Phi_{\rm in}^{[j]}\Delta_x\eta_R^{[j]} + 2\nabla_x\eta_R^{[j]}\cdot\nabla_x\Phi_{\rm in}^{[j]}\right)\right]\\
& - \partial_t U_* - \sum_{j=1}^N b\left(U_*-U^{[j]}\right)\wedge\left\{\Delta_x(\eta_{d_q}^{[j]}\Phi_0^{*[j]}) + \eta_R^{[j]}Q_{\gamma_j}\Delta_x\Phi_{\rm in}^{[j]} + Q_{\gamma_j}\left(\Phi_{\rm in}^{[j]}\Delta_x\eta_R^{[j]} + 2\nabla_x\eta_R^{[j]}\cdot\nabla_x\Phi_{\rm in}^{[j]}\right)\right.\\
& \left. - 2\nabla_x\left(U^{[j]}\cdot\Phi_{\rm out}\right)\nabla_x U^{[j]} - 2\nabla_x\left[U^{[j]}\cdot\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]} + \eta_{d_q}^{[j]}\Phi_0^{*[j]}\right)\right]\nabla_x U^{[j]}\right\}\\
& + (a-bU_*\wedge)\left\{-2\sum_{j=1}^N\nabla_x\left[\Phi\cdot\left(U_*-U^{[j]}\right)\right]\nabla_x U^{[j]}\right\}\\
& + (a-bU_*\wedge)\left\{-2\sum_{j=1}^N\nabla_x\left[U^{[j]}\cdot\sum_{k\neq j}\left(\eta_R^{[k]}Q_{\gamma_k}\Phi_{\rm in}^{[k]} + \eta_{d_q}^{[k]}\Phi_0^{*[k]}\right)\right]\nabla_x U^{[j]}\right\}\\
& + \sum_{j=1}^N |\nabla_x U^{[j]}|^2\left(a-bU^{[j]}\wedge\right)\sum_{k\neq j}\left(\eta_R^{[k]}Q_{\gamma_k}\Phi_{\rm in}^{[k]} + \eta_{d_q}^{[k]}\Phi_0^{*[k]}\right)\\
& + a\Phi\sum_{j\neq k}\nabla_x U^{[j]}\cdot\nabla_x U^{[k]} + (\Phi\cdot U_*)\partial_t U_* + \mathcal{N}[\Phi] + \Xi[\Phi]U_*
\end{aligned}
$$

$$
\begin{aligned}
= & -\partial_t\Phi_{\rm out}+(a-bU_*\wedge)\,\Delta_x\Phi_{\rm out}\\
&+\sum_{j=1}^N\left(1-\eta_R^{[j]}\right)\left(a-bU^{[j]}\wedge\right)\left[|\nabla_xU^{[j]}|^2\Phi_{\rm out}-2\nabla_x\left(U^{[j]}\cdot\Phi_{\rm out}\right)\nabla_xU^{[j]}\right]\\
&+\sum_{j=1}^N\left(1-\eta_R^{[j]}\right)\Big\{-\partial_t(\eta_{d_q}^{[j]}\Phi_0^{*[j]})+\left(a-bU^{[j]}\wedge\right)\Big[\Delta_x(\eta_{d_q}^{[j]}\Phi_0^{*[j]})+|\nabla_xU^{[j]}|^2\eta_{d_q}^{[j]}\Phi_0^{*[j]}\\
&-2\nabla_x\left(U^{[j]}\cdot\eta_{d_q}^{[j]}\Phi_0^{*[j]}\right)\nabla_xU^{[j]}\Big]-\partial_tU^{[j]}\Big\}\\
&+\sum_{j=1}^N\eta_R^{[j]}Q_{\gamma_j}\Bigg\{-\partial_t\Phi_{\rm in}^{[j]}+\lambda_j^{-2}\left(a-bW^{[j]}\wedge\right)\Big[\Delta_{y^{[j]}}\Phi_{\rm in}^{[j]}+|\nabla_{y^{[j]}}W^{[j]}|^2\Phi_{\rm in}^{[j]}-2\nabla_{y^{[j]}}\left(W^{[j]}\cdot\Phi_{\rm in}^{[j]}\right)\nabla_{y^{[j]}}W^{[j]}\\
&+2\left(\nabla_{y^{[j]}}W^{[j]}\cdot\nabla_{y^{[j]}}\Phi_{\rm in}^{[j]}\right)W^{[j]}\Big]\\
&+Q_{-\gamma_j}\left\{\left(a-bU^{[j]}\wedge\right)\left[|\nabla_xU^{[j]}|^2\Pi_{U^{[j]\perp}}\Phi_{\rm out}-2\nabla_x\left(U^{[j]}\cdot\Phi_{\rm out}\right)\nabla_xU^{[j]}\right]\right\}\\
&+Q_{-\gamma_j}\Pi_{U^{[j]\perp}}\Big\{-\partial_t(\Phi_0^{*[j]})+\left(a-bU^{[j]}\wedge\right)\Big[\Delta_x\Phi_0^{*[j]}+|\nabla_xU^{[j]}|^2\Phi_0^{*[j]}-2\nabla_x\left(U^{[j]}\cdot\Phi_0^{*[j]}\right)\nabla_xU^{[j]}\Big]\\
&-\partial_tU^{[j]}\Big\}\Bigg\}\\
&+\sum_{j=1}^N\eta_R^{[j]}Q_{\gamma_j}\left[\left(\lambda_j^{-1}\dot\lambda_jy^{[j]}+\lambda_j^{-1}\dot\xi^{[j]}\right)\cdot\nabla_{y^{[j]}}\Phi_{\rm in}^{[j]}-\dot\gamma_jJ\Phi_{\rm in}^{[j]}\right]\\
&+\sum_{j=1}^NQ_{\gamma_j}\left\{-\Phi_{\rm in}^{[j]}\partial_t\eta_R^{[j]}+\left(a-bW^{[j]}\wedge\right)\left[\Phi_{\rm in}^{[j]}\Delta_x\eta_R^{[j]}+2\nabla_x\eta_R^{[j]}\nabla_x\Phi_{\rm in}^{[j]}-\left(W^{[j]}\cdot\Phi_{\rm in}^{[j]}\right)\left(2\nabla_x\eta_R^{[j]}\nabla_xW^{[j]}\right)\right]\right\}\\
&-\sum_{j=1}^Nb\left(U_*-U^{[j]}\right)\wedge\Bigg\{\Delta_x(\eta_{d_q}^{[j]}\Phi_0^{*[j]})+\eta_R^{[j]}Q_{\gamma_j}\Delta_x\Phi_{\rm in}^{[j]}+Q_{\gamma_j}\left(\Phi_{\rm in}^{[j]}\Delta_x\eta_R^{[j]}+2\nabla_x\eta_R^{[j]}\nabla_x\Phi_{\rm in}^{[j]}\right)\\
&-2\nabla_x\left(U^{[j]}\cdot\Phi_{\rm out}\right)\nabla_xU^{[j]}-2\nabla_x\left[U^{[j]}\cdot\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}+\eta_{d_q}^{[j]}\Phi_0^{*[j]}\right)\right]\nabla_xU^{[j]}\Bigg\}\\
&+(a-bU_*\wedge)\left\{-2\sum_{j=1}^N\nabla_x\left[\Phi\cdot\left(U_*-U^{[j]}\right)\right]\nabla_xU^{[j]}\right\}\\
&+(a-bU_*\wedge)\left\{-2\sum_{j=1}^N\nabla_x\left[U^{[j]}\cdot\sum_{k\neq j}\left(\eta_R^{[k]}Q_{\gamma_k}\Phi_{\rm in}^{[k]}+\eta_{d_q}^{[k]}\Phi_0^{*[k]}\right)\right]\nabla_xU^{[j]}\right\}\\
&+\sum_{j=1}^N|\nabla_xU^{[j]}|^2\left(a-bU^{[j]}\wedge\right)\sum_{k\neq j}\left(\eta_R^{[k]}Q_{\gamma_k}\Phi_{\rm in}^{[k]}+\eta_{d_q}^{[k]}\Phi_0^{*[k]}\right)\\
&+a\Phi\sum_{j\neq k}\nabla_xU^{[j]}\cdot\nabla_xU^{[k]}+\left[(\Phi\cdot U_*)-A\right]\partial_tU_*+\mathcal N[\Phi]+\Xi[\Phi]U_*\\
&\qquad+\sum_{j=1}^N\eta_R^{[j]}\left(U^{[j]}-U_*+U_*\right)\Bigg\{-2a\left(\nabla_xW^{[j]}\cdot\nabla_x\Phi_{\rm in}^{[j]}\right)+a|\nabla_xU^{[j]}|^2\left(U^{[j]}\cdot\Phi_{\rm out}\right)\\
&\qquad+\Big\{-\partial_t(\Phi_0^{*[j]})+\left(a-bU^{[j]}\wedge\right)\Big[\Delta_x\Phi_0^{*[j]}+|\nabla_xU^{[j]}|^2\Phi_0^{*[j]}-2\nabla_x\left(U^{[j]}\cdot\Phi_0^{*[j]}\right)\nabla_xU^{[j]}\Big]\\
&\qquad-\partial_tU^{[j]}\Big\}\cdot U^{[j]}\Bigg\}
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathcal{N}[\Phi] \\
&:= a\Big\{ \Big\{ |\nabla_x [(1+A)U_*]|^2 - |\nabla_x U_*|^2 + 2\nabla_x [(1+A)U_*]\cdot \nabla_x \Pi_{U_*^\perp}\Phi + \left|\nabla_x \Pi_{U_*^\perp}\Phi\right|^2 \Big\} \Pi_{U_*^\perp}\Phi \\
&\quad + 2\nabla_x A \nabla_x U_* + [1 + A - (\Phi\cdot U_*)]\,\Delta_x U_* \Big\} \\
&\quad - b\Big\{ \Pi_{U_*^\perp}\Phi \wedge \Delta_x (AU_*) + AU_* \wedge \Delta_x \Pi_{U_*^\perp}\Phi + \Pi_{U_*^\perp}\Phi \wedge \Delta_x \Pi_{U_*^\perp}\Phi \\
&\quad + (1+A)U_* \wedge \Delta_x [(1+A)U_*] - 2(\Phi\cdot U_*)U_*\wedge \Delta_x U_* \Big\}, \\
&\Xi[\Phi] := \partial_t(\Phi\cdot U_*) - \partial_t A \\
&+ a\left\{ \Delta_x A + |\nabla_x [(1+A)U_* + \Phi - (\Phi\cdot U_*)U_*]|^2 (1+A) - |\nabla_x U_*|^2 (\Phi\cdot U_*) - \Delta_x(\Phi\cdot U_*) \right\}.
\end{aligned}
\tag{5.9}
$$

5.2. **Simplification of $\mathcal{N}[\Phi]$.** In this part, we will simplify the nonlinear terms and extract $\Delta_x\Phi$ in $\mathcal{N}[\Phi]$ for later purpose.

$$
\begin{aligned}
&\Pi_{U_*^\perp}\Phi \wedge \Delta_x (AU_*) + AU_*\wedge \Delta_x \Pi_{U_*^\perp}\Phi + \Pi_{U_*^\perp}\Phi\wedge \Delta_x\Pi_{U_*^\perp}\Phi \\
=\ & [\Phi - (\Phi\cdot U_*)U_*] \wedge [(\Delta_x A)U_* + A\Delta_x U_* + 2\nabla_x A\nabla_x U_*] \\
&+ AU_*\wedge \Delta_x[\Phi - (\Phi\cdot U_*)U_*] + [\Phi - (\Phi\cdot U_*)U_*]\wedge \Delta_x[\Phi - (\Phi\cdot U_*)U_*] \\
=\ & (\Phi\wedge U_*)\Delta_x A + \Pi_{U_*^\perp}\Phi \wedge [A\Delta_x U_* + 2\nabla_x A\nabla_x U_*] \\
&+ AU_*\wedge\Delta_x\Phi - AU_*\wedge[(\Phi\cdot U_*)\Delta_x U_* + 2\nabla_x(\Phi\cdot U_*)\nabla_x U_*] \\
&+ \Pi_{U_*^\perp}\Phi\wedge\Delta_x\Phi - (\Phi\wedge U_*)\Delta_x(\Phi\cdot U_*) - \Pi_{U_*^\perp}\Phi\wedge[(\Phi\cdot U_*)\Delta_x U_* + 2\nabla_x(\Phi\cdot U_*)\nabla_x U_*] \\
=\ & (\Phi\wedge U_*)\Delta_x A + AU_*\wedge\Delta_x\Phi + \Pi_{U_*^\perp}\Phi\wedge\Delta_x\Phi - (\Phi\wedge U_*)\Delta_x(\Phi\cdot U_*) \\
&+ \Pi_{U_*^\perp}\Phi\wedge(A\Delta_x U_* + 2\nabla_x A\nabla_x U_*) - AU_*\wedge[(\Phi\cdot U_*)\Delta_x U_* + 2\nabla_x(\Phi\cdot U_*)\nabla_x U_*] \\
&- \Pi_{U_*^\perp}\Phi\wedge[(\Phi\cdot U_*)\Delta_x U_* + 2\nabla_x(\Phi\cdot U_*)\nabla_x U_*] \\
=\ & (\Phi\wedge U_*)\Delta_x A + AU_*\wedge\Delta_x\Phi + \Pi_{U_*^\perp}\Phi\wedge\Delta_x\Phi - (\Phi\wedge U_*)\Delta_x(\Phi\cdot U_*) \\
&+ \Pi_{U_*^\perp}\Phi\wedge[2\nabla_x A\nabla_x U_* - 2\nabla_x(\Phi\cdot U_*)\nabla_x U_*] - AU_*\wedge[(\Phi\cdot U_*)\Delta_x U_* + 2\nabla_x(\Phi\cdot U_*)\nabla_x U_*] \\
&+ [A - (\Phi\cdot U_*)]\Phi\wedge\Delta_x U_* + [(\Phi\cdot U_*)^2 - A(\Phi\cdot U_*)]U_*\wedge\Delta_x U_* \\
=\ & (\Phi\wedge U_*)\Delta_x A + AU_*\wedge\Delta_x\Phi + \Pi_{U_*^\perp}\Phi\wedge\Delta_x\Phi - (\Phi\wedge U_*)\Delta_x(\Phi\cdot U_*) \\
&- (\Pi_{U_*^\perp}\Phi + AU_*)\wedge[2\nabla_x(\Phi\cdot U_*)\nabla_x U_*] + [A - (\Phi\cdot U_*)]\Phi\wedge\Delta_x U_* \\
&+ \Pi_{U_*^\perp}\Phi\wedge(2\nabla_x A\nabla_x U_*) + [(\Phi\cdot U_*)^2 - 2A(\Phi\cdot U_*)]U_*\wedge\Delta_x U_*,
\end{aligned}
\tag{5.10}
$$

where

$$
\Delta_x(\Phi\cdot U_*) = U_*\cdot\Delta_x\Phi + 2\nabla_x\Phi\cdot\nabla_x U_* + \Phi\cdot\Delta_x U_*.
$$

Next, we give explicit formula for $\nabla_x A$ and $\Delta_x A$, with interactions of bubbles encoded. Due to the choice of (5.1), $|u| = 1$ is equivalent to

$$
(1+A)^2|U_*|^2 + 2(1+A)\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right) + |\Pi_{U_*^\perp}\Phi|^2 = 1. \tag{5.11}
$$

Taking ∇_x for (5.11), we get

$$
2(1+A)|U_*|^2\nabla_x A + (1+A)^2\nabla_x(|U_*|^2) + \nabla_x(|\Pi_{U_*^\perp}\Phi|^2) + 2(1+A)\nabla_x(U_*\cdot\Pi_{U_*^\perp}\Phi) + 2(U_*\cdot\Pi_{U_*^\perp}\Phi)\nabla_x A = 0.
$$

So

$$
\nabla_x A = -\frac{(1+A)^2\nabla_x(|U_*|^2) + \nabla_x(|\Pi_{U_*^\perp}\Phi|^2) + 2(1+A)\nabla_x(U_*\cdot\Pi_{U_*^\perp}\Phi)}{2(1+A)|U_*|^2 + 2(U_*\cdot\Pi_{U_*^\perp}\Phi)}. \tag{5.12}
$$

Taking Δ_x for (5.11), we have

$$
\begin{aligned}
&(1+A)^2\Delta_x(|U_*|^2) + |U_*|^2\Delta_x[(1+A)^2] + 2\nabla_x(|U_*|^2)\cdot\nabla_x[(1+A)^2] + 2(1+A)\Delta_x\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right) \\
&+ 2\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\Delta_x A + 4\nabla_x\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\cdot\nabla_x A + \Delta_x(|\Pi_{U_*^\perp}\Phi|^2) = 0,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&(1+A)^2\Delta_x(|U_*|^2) + |U_*|^2\left[2(1+A)\Delta_x A + 2|\nabla_x A|^2\right] + 4(1+A)\nabla_x(|U_*|^2)\cdot\nabla_x A \\
&+ 2(1+A)\Delta_x\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right) + 2\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\Delta_x A + 4\nabla_x\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\cdot\nabla_x A + \Delta_x\left(|\Pi_{U_*^\perp}\Phi|^2\right) = 0.
\end{aligned}
$$

Thus, we have

$$\begin{aligned}\Delta_x A = & -2^{-1}\left[(1+A)|U_*|^2+\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\right]^{-1}\Big[\Delta_x\left(|\Pi_{U_*^\perp}\Phi|^2\right)+2(1+A)\Delta_x\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\\ &+4\nabla_x\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\cdot\nabla_x A+2|U_*|^2|\nabla_x A|^2+4(1+A)\nabla_x(|U_*|^2)\cdot\nabla_x A+(1+A)^2\Delta_x(|U_*|^2)\Big].\end{aligned}\tag{5.13}$$

We further expand

$$\begin{aligned}\Delta_x(|\Pi_{U_*^\perp}\Phi|^2) &= \Delta_x\left[|\Phi|^2+(|U_*|^2-2)(\Phi\cdot U_*)^2\right]\\ &= 2\Phi\cdot\Delta_x\Phi+(|U_*|^2-2)\Delta_x[(\Phi\cdot U_*)^2]\\ &\quad+2\nabla_x(|U_*|^2)\cdot\nabla_x[(\Phi\cdot U_*)^2]+2|\nabla_x\Phi|^2+(\Phi\cdot U_*)^2\Delta_x(|U_*|^2)\\ &= 2\Phi\cdot\Delta_x\Phi+2(|U_*|^2-2)\left[(\Phi\cdot U_*)\Delta_x(\Phi\cdot U_*)+|\nabla_x(\Phi\cdot U_*)|^2\right]\\ &\quad+2\nabla_x(|U_*|^2)\cdot\nabla_x[(\Phi\cdot U_*)^2]+2|\nabla_x\Phi|^2+(\Phi\cdot U_*)^2\Delta_x(|U_*|^2)\end{aligned}$$

and

$$\begin{aligned}\Delta_x(U_*\cdot\Pi_{U_*^\perp}\Phi) &= \Delta_x[(1-|U_*|^2)(\Phi\cdot U_*)]\\ &= (1-|U_*|^2)\Delta_x(\Phi\cdot U_*)-(\Phi\cdot U_*)\Delta_x(|U_*|^2)-2\nabla_x(|U_*|^2)\cdot\nabla_x(\Phi\cdot U_*).\end{aligned}$$

We arrange terms in (5.13) as follows

$$\begin{aligned}&\Delta_x(|\Pi_{U_*^\perp}\Phi|^2)+2(1+A)\Delta_x(U_*\cdot\Pi_{U_*^\perp}\Phi)+4\nabla_x(U_*\cdot\Pi_{U_*^\perp}\Phi)\cdot\nabla_x A\\ &\quad+2|U_*|^2|\nabla_x A|^2+4(1+A)\nabla_x(|U_*|^2)\cdot\nabla_x A+(1+A)^2\Delta_x(|U_*|^2)\\ &=2\Phi\cdot\Delta_x\Phi+\left[2(|U_*|^2-2)(\Phi\cdot U_*)+2(1+A)(1-|U_*|^2)\right]\Delta_x(\Phi\cdot U_*)\\ &\quad+2(|U_*|^2-2)|\nabla_x(\Phi\cdot U_*)|^2+2\nabla_x(|U_*|^2)\cdot\nabla_x[(\Phi\cdot U_*)^2]+2|\nabla_x\Phi|^2+(\Phi\cdot U_*)^2\Delta_x(|U_*|^2)\\ &\quad-2(1+A)\left[(\Phi\cdot U_*)\Delta_x(|U_*|^2)+2\nabla_x(|U_*|^2)\cdot\nabla_x(\Phi\cdot U_*)\right]\\ &\quad+4\nabla_x(U_*\cdot\Pi_{U_*^\perp}\Phi)\cdot\nabla_x A+2|U_*|^2|\nabla_x A|^2+4(1+A)\nabla_x(|U_*|^2)\cdot\nabla_x A+(1+A)^2\Delta_x(|U_*|^2)\\ &=2\Phi\cdot\Delta_x\Phi+\left[2(|U_*|^2-2)(\Phi\cdot U_*)+2(1+A)(1-|U_*|^2)\right](U_*\cdot\Delta_x\Phi+2\nabla_x\Phi\cdot\nabla_x U_*+\Phi\cdot\Delta_x U_*)\\ &\quad+2(|U_*|^2-2)|\nabla_x(\Phi\cdot U_*)|^2+2|\nabla_x\Phi|^2+4[(\Phi\cdot U_*)-(1+A)]\nabla_x(|U_*|^2)\cdot\nabla_x(\Phi\cdot U_*)\\ &\quad+2|U_*|^2|\nabla_x A|^2+4\nabla_x(U_*\cdot\Pi_{U_*^\perp}\Phi)\cdot\nabla_x A+4(1+A)\nabla_x(|U_*|^2)\cdot\nabla_x A\\ &\quad+[(\Phi\cdot U_*)-(1+A)]^2\Delta_x(|U_*|^2).\end{aligned}\tag{5.14}$$

Combining (5.13) and (5.14), we have

$$
\begin{aligned}
&(\Phi\wedge U_*)\Delta_x A + AU_*\wedge\Delta_x\Phi + \Pi_{U_*^\perp}\Phi\wedge\Delta_x\Phi - (\Phi\wedge U_*)\Delta_x(\Phi\cdot U_*)\\
=\;& -2^{-1}(\Phi\wedge U_*)\left[(1+A)|U_*|^2 + \left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\right]^{-1}\Big\{2\Phi\cdot\Delta_x\Phi\\
&+\left[2(|U_*|^2-2)(\Phi\cdot U_*) + 2(1+A)(1-|U_*|^2)\right](U_*\cdot\Delta_x\Phi + 2\nabla_x\Phi\cdot\nabla_x U_* + \Phi\cdot\Delta_x U_*)\\
&+2(|U_*|^2-2)|\nabla_x(\Phi\cdot U_*)|^2 + 2|\nabla_x\Phi|^2 + 4[(\Phi\cdot U_*)-(1+A)]\nabla_x(|U_*|^2)\cdot\nabla_x(\Phi\cdot U_*)\\
&+2|U_*|^2|\nabla_x A|^2 + 4\nabla_x(U_*\cdot\Pi_{U_*^\perp}\Phi)\cdot\nabla_x A + 4(1+A)\nabla_x(|U_*|^2)\cdot\nabla_x A\\
&+[(\Phi\cdot U_*)-(1+A)]^2\Delta_x(|U_*|^2)\Big\}\\
&+AU_*\wedge\Delta_x\Phi + \Pi_{U_*^\perp}\Phi\wedge\Delta_x\Phi - (\Phi\wedge U_*)(U_*\cdot\Delta_x\Phi + 2\nabla_x\Phi\cdot\nabla_x U_* + \Phi\cdot\Delta_x U_*)\\
=\;& -2^{-1}(\Phi\wedge U_*)\left[(1+A)|U_*|^2 + \left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\right]^{-1}\\
&\times\left\{2\Phi\cdot\Delta_x\Phi + \left[2(|U_*|^2-2)(\Phi\cdot U_*) + 2(1+A)(1-|U_*|^2)\right](U_*\cdot\Delta_x\Phi)\right\}\\
&+AU_*\wedge\Delta_x\Phi + \Pi_{U_*^\perp}\Phi\wedge\Delta_x\Phi - (\Phi\wedge U_*)(U_*\cdot\Delta_x\Phi)\\
&-2^{-1}(\Phi\wedge U_*)\left[(1+A)|U_*|^2 + \left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\right]^{-1}\\
&\times\Big\{\left[2(|U_*|^2-2)(\Phi\cdot U_*) + 2(1+A)(1-|U_*|^2)\right](2\nabla_x\Phi\cdot\nabla_x U_* + \Phi\cdot\Delta_x U_*)\\
&+2(|U_*|^2-2)|\nabla_x(\Phi\cdot U_*)|^2 + 2|\nabla_x\Phi|^2 + 4[(\Phi\cdot U_*)-(1+A)]\nabla_x(|U_*|^2)\cdot\nabla_x(\Phi\cdot U_*)\\
&+2|U_*|^2|\nabla_x A|^2 + 4\nabla_x(U_*\cdot\Pi_{U_*^\perp}\Phi)\cdot\nabla_x A + 4(1+A)\nabla_x(|U_*|^2)\cdot\nabla_x A\\
&+[(\Phi\cdot U_*)-(1+A)]^2\Delta_x(|U_*|^2)\Big\}\\
&-(\Phi\wedge U_*)(2\nabla_x\Phi\cdot\nabla_x U_* + \Phi\cdot\Delta_x U_*)\\
=\;& -2^{-1}(\Phi\wedge U_*)\left[(1+A)|U_*|^2 + \left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\right]^{-1}\left[2\Phi\cdot\Delta_x\Phi + 2\left(1+A-\Phi\cdot U_*\right)(U_*\cdot\Delta_x\Phi)\right]\\
&+AU_*\wedge\Delta_x\Phi + \Pi_{U_*^\perp}\Phi\wedge\Delta_x\Phi\\
&-2^{-1}(\Phi\wedge U_*)\left[(1+A)|U_*|^2 + \left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\right]^{-1}\Big\{2\left(1+A-\Phi\cdot U_*\right)(2\nabla_x\Phi\cdot\nabla_x U_* + \Phi\cdot\Delta_x U_*)\\
&+2(|U_*|^2-2)|\nabla_x(\Phi\cdot U_*)|^2 + 2|\nabla_x\Phi|^2 + 4[(\Phi\cdot U_*)-(1+A)]\nabla_x(|U_*|^2)\cdot\nabla_x(\Phi\cdot U_*)\\
&+2|U_*|^2|\nabla_x A|^2 + 4\nabla_x(U_*\cdot\Pi_{U_*^\perp}\Phi)\cdot\nabla_x A + 4(1+A)\nabla_x(|U_*|^2)\cdot\nabla_x A\\
&+[(\Phi\cdot U_*)-(1+A)]^2\Delta_x(|U_*|^2)\Big\}.
\end{aligned}
\tag{5.15}
$$

Combining (5.9), (5.10) and (5.15), we get

$$
\begin{aligned}
&\mathcal{N}[\Phi]\\
=\,& a\Big\{ \Big\{ |\nabla_x [(1+A)U_*]|^2 - |\nabla_x U_*|^2 + 2\nabla_x [(1+A)U_*]\cdot \nabla_x \Pi_{U_*^\perp}\Phi + \big|\nabla_x \Pi_{U_*^\perp}\Phi\big|^2 \Big\} \Pi_{U_*^\perp}\Phi \\
&+ 2\nabla_x A \nabla_x U_* + (1+A-\Phi\cdot U_*)\,\Delta_x U_* \Big\}\\
&- b\Big\{ -2^{-1}(\Phi\wedge U_*)\left[(1+A)|U_*|^2 + \left(U_*\cdot \Pi_{U_*^\perp}\Phi\right)\right]^{-1} [2\Phi\cdot\Delta_x\Phi + 2\,(1+A-\Phi\cdot U_*)\,(U_*\cdot\Delta_x\Phi)]\\
&+ AU_*\wedge\Delta_x\Phi + \Pi_{U_*^\perp}\Phi\wedge\Delta_x\Phi\\
&- 2^{-1}(\Phi\wedge U_*)\left[(1+A)|U_*|^2 + \left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\right]^{-1} \Big\{ 2\,(1+A-\Phi\cdot U_*)\,(2\nabla_x\Phi\cdot\nabla_x U_* + \Phi\cdot\Delta_x U_*)\\
&+ 2(|U_*|^2-2)|\nabla_x(\Phi\cdot U_*)|^2 + 2|\nabla_x\Phi|^2 + 4[(\Phi\cdot U_*) - (1+A)]\nabla_x(|U_*|^2)\cdot\nabla_x(\Phi\cdot U_*)\\
&+ 2|U_*|^2|\nabla_x A|^2 + 4\nabla_x(U_*\cdot\Pi_{U_*^\perp}\Phi)\cdot\nabla_x A + 4(1+A)\nabla_x(|U_*|^2)\cdot\nabla_x A\\
&+ [(\Phi\cdot U_*) - (1+A)]^2\Delta_x(|U_*|^2)\Big\}\\
&- (\Pi_{U_*^\perp}\Phi + AU_*)\wedge[2\nabla_x(\Phi\cdot U_*)\nabla_x U_*] + [A - (\Phi\cdot U_*)]\Phi\wedge\Delta_x U_*\\
&+ \Pi_{U_*^\perp}\Phi\wedge(2\nabla_x A\nabla_x U_*) + [(\Phi\cdot U_*)^2 - 2A(\Phi\cdot U_*)]U_*\wedge\Delta_x U_*\\
&+ (1+A)U_*\wedge[(1+A)\Delta_x U_* + 2\nabla_x A\nabla_x U_*] - 2(\Phi\cdot U_*)U_*\wedge\Delta_x U_*\Big\}\\
=\,& b\Big\{ (\Phi\wedge U_*)\left[(1+A)|U_*|^2 + \left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\right]^{-1}[\Phi\cdot\Delta_x\Phi + (1+A-\Phi\cdot U_*)\,(U_*\cdot\Delta_x\Phi)]\\
&- AU_*\wedge\Delta_x\Phi - \left(\Pi_{U_*^\perp}\Phi\right)\wedge\Delta_x\Phi\Big\}\\
&+ a\Bigg[\Big\{ |\nabla_x[(1+A)U_*]|^2 - |\nabla_x U_*|^2 + 2\nabla_x[(1+A)U_*]\cdot\nabla_x\left(\Pi_{U_*^\perp}\Phi\right) + \left|\nabla_x\left(\Pi_{U_*^\perp}\Phi\right)\right|^2\Big\}\Pi_{U_*^\perp}\Phi\\
&+ 2\nabla_x A\nabla_x U_* + (1+A-\Phi\cdot U_*)\,\Delta_x U_*\Bigg]\\
&- b\Bigg[-2^{-1}(\Phi\wedge U_*)\left[(1+A)|U_*|^2 + \left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\right]^{-1}\Big\{ 2\,(1+A-\Phi\cdot U_*)\,(2\nabla_x\Phi\cdot\nabla_x U_* + \Phi\cdot\Delta_x U_*)\\
&+ 2(|U_*|^2-2)|\nabla_x(\Phi\cdot U_*)|^2 + 2|\nabla_x\Phi|^2 + 4[(\Phi\cdot U_*) - (1+A)]\nabla_x(|U_*|^2)\cdot\nabla_x(\Phi\cdot U_*)\\
&+ 2|U_*|^2|\nabla_x A|^2 + 4\nabla_x(U_*\cdot\Pi_{U_*^\perp}\Phi)\cdot\nabla_x A + 4(1+A)\nabla_x(|U_*|^2)\cdot\nabla_x A\\
&+ [(\Phi\cdot U_*) - (1+A)]^2\Delta_x(|U_*|^2)\Big\}\\
&- (\Pi_{U_*^\perp}\Phi + AU_*)\wedge[2\nabla_x(\Phi\cdot U_*)\nabla_x U_*] + [A - (\Phi\cdot U_*)]\Phi\wedge\Delta_x U_*\\
&+ \Pi_{U_*^\perp}\Phi\wedge(2\nabla_x A\nabla_x U_*) + [(\Phi\cdot U_*)^2 - 2A(\Phi\cdot U_*)]U_*\wedge\Delta_x U_*\\
&+ (1+A)U_*\wedge[(1+A)\Delta_x U_* + 2\nabla_x A\nabla_x U_*] - 2(\Phi\cdot U_*)U_*\wedge\Delta_x U_*\Bigg]
\end{aligned} \tag{5.16}
$$

Since

$$
\begin{aligned}
&2\nabla_x[(1+A)U_*]\cdot\nabla_x\left(\Pi_{U_*^\perp}\Phi\right)\\
=\,& 2\sum_{k=1}^{2}[U_*\partial_{x_k}A + (1+A)\partial_{x_k}U_*]\cdot[\partial_{x_k}\Phi - U_*\partial_{x_k}(U_*\cdot\Phi) - (U_*\cdot\Phi)\partial_{x_k}U_*]\\
=\,& 2\sum_{k=1}^{2}\Big\{ [(\partial_{x_k}A)\,U_*\cdot\partial_{x_k}\Phi + (1+A)\partial_{x_k}U_*\cdot\partial_{x_k}\Phi] - \partial_{x_k}(U_*\cdot\Phi)\left[|U_*|^2\partial_{x_k}A + (1+A)U_*\cdot\partial_{x_k}U_*\right]\\
&- (U_*\cdot\Phi)\left[(\partial_{x_k}A)\,U_*\cdot\partial_{x_k}U_* + (1+A)\,|\partial_{x_k}U_*|^2\right]\Big\}
\end{aligned}
$$

and

$$\left|\nabla_x\left(\Pi_{U_*^\perp}\Phi\right)\right|^2=\sum_{k=1}^2|\partial_{x_k}\Phi-U_*\partial_{x_k}\left(\Phi\cdot U_*\right)-(\Phi\cdot U_*)\partial_{x_k}U_*|^2,$$

then we obtain

$$\begin{aligned}
&\mathcal{N}[\Phi]\\
=\ &b\Big\{(\Phi\wedge U_*)\left[(1+A)|U_*|^2+\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\right]^{-1}\left[\Phi\cdot\Delta_x\Phi+(1+A-\Phi\cdot U_*)\left(U_*\cdot\Delta_x\Phi\right)\right]\\
&-AU_*\wedge\Delta_x\Phi-\left(\Pi_{U_*^\perp}\Phi\right)\wedge\Delta_x\Phi\Big\}\\
&+a\Bigg\{\Bigg[|\nabla_xA|^2|U_*|^2+2(1+A)\nabla_xA\cdot(U_*\cdot\nabla_xU_*)+A(2+A)\left|\nabla_xU_*\right|^2\\
&+2\sum_{k=1}^2\Big\{\left[(\partial_{x_k}A)\,U_*\cdot\partial_{x_k}\Phi+A\partial_{x_k}U_*\cdot\partial_{x_k}\Phi\right]-\partial_{x_k}\left(U_*\cdot\Phi\right)\left[|U_*|^2\partial_{x_k}A+(1+A)U_*\cdot\partial_{x_k}U_*\right]\\
&-(U_*\cdot\Phi)\left[(\partial_{x_k}A)\,U_*\cdot\partial_{x_k}U_*+(1+A)\left|\partial_{x_k}U_*\right|^2\right]\Big\}\\
&+\sum_{k=1}^2|\partial_{x_k}\Phi-U_*\partial_{x_k}\left(\Phi\cdot U_*\right)-(\Phi\cdot U_*)\partial_{x_k}U_*|^2\Bigg]\Pi_{U_*^\perp}\Phi\\
&+2\left(\nabla_xA+U_*\cdot\nabla_xU_*+\Phi\cdot\nabla_x\Phi\right)\nabla_xU_*+\Delta_xU_*-2\left(U_*\cdot\nabla_xU_*\right)\nabla_xU_*+\left(A-\Phi\cdot U_*\right)\Delta_xU_*\Bigg\}\\
&+2a\left[\left(\nabla_xU_*\cdot\nabla_x\Phi\right)\Phi-\left(\Phi\cdot\nabla_x\Phi\right)\nabla_xU_*\right]-2a\left(\nabla_xU_*\cdot\nabla_x\Phi\right)(U_*\cdot\Phi)U_*\\
&-2bU_*\wedge\left[\left(\nabla_xU_*\cdot\nabla_x\Phi\right)\Phi-\left(\Phi\cdot\nabla_x\Phi\right)\nabla_xU_*\right]\\
&+b(\Phi\wedge U_*)\left[(1+A)|U_*|^2+\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\right]^{-1}\left(1+A-\Phi\cdot U_*\right)\left(2\nabla_x\Phi\cdot\nabla_xU_*\right)\\
&-b(\Phi\wedge U_*)\left(2\nabla_x\Phi\cdot\nabla_xU_*\right)\\
&-b\Bigg[-2^{-1}(\Phi\wedge U_*)\left[(1+A)|U_*|^2+\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\right]^{-1}\Bigg\{2\left(1+A-\Phi\cdot U_*\right)\left(\Phi\cdot\Delta_xU_*\right)\\
&+2(|U_*|^2-2)|\nabla_x\left(\Phi\cdot U_*\right)|^2+2|\nabla_x\Phi|^2+8[(\Phi\cdot U_*)-(1+A)](U_*\cdot\nabla_xU_*)\cdot\nabla_x\left(\Phi\cdot U_*\right)\\
&+2|U_*|^2|\nabla_xA|^2+4\left[-2(\Phi\cdot U_*)U_*\cdot\nabla_xU_*+(1-|U_*|^2)\nabla_x\left(\Phi\cdot U_*\right)\right]\cdot\nabla_xA\\
&+8(1+A)\left(U_*\cdot\nabla_xU_*\right)\cdot\nabla_xA+2\left[(\Phi\cdot U_*)-(1+A)\right]^2\left(|\nabla_xU_*|^2+U_*\cdot\Delta_xU_*\right)\Bigg\}\\
&-\left(\Pi_{U_*^\perp}\Phi+AU_*\right)\wedge\left[2\nabla_x\left(\Phi\cdot U_*\right)\nabla_xU_*\right]+\left[A-\left(\Phi\cdot U_*\right)\right]\Phi\wedge\Delta_xU_*\\
&+\Pi_{U_*^\perp}\Phi\wedge\left(2\nabla_xA\nabla_xU_*\right)+\left[(\Phi\cdot U_*)^2-2A(\Phi\cdot U_*)-2(\Phi\cdot U_*)\right]U_*\wedge\Delta_xU_*\\
&+(1+A)U_*\wedge\left[A\Delta_xU_*+2\left(\nabla_xA+U_*\cdot\nabla_xU_*+\Phi\cdot\nabla_x\Phi\right)\nabla_xU_*+\Delta_xU_*-2\left(U_*\cdot\nabla_xU_*\right)\nabla_xU_*\right]\Bigg]\\
&+2bAU_*\wedge\left[\left(\Phi\cdot\nabla_x\Phi\right)\nabla_xU_*\right].
\end{aligned}\tag{5.17}$$

5.3. **Inner-outer gluing system.** By U_*-operation (5.6), we put the terms in U_* direction into $\Xi(x,t)U_*$. Then using (5.17), a sufficient condition for a desired blow-up solution to exist is that $(\Phi_{\rm in}^{[j]},\Phi_{\rm out})$ solve the following *gluing system*

$$\begin{aligned}
&\partial_t\Phi_{\rm out}=\mathbf{B}_{\Phi,U_*}\Delta_x\Phi_{\rm out}+\mathcal{G}\quad \text{ in }\ \mathbb{R}^2\times(0,T),\\
&\Phi_{\rm out}(x,0)=Z_*(x)+\sum_{m=1}^N\sum_{n=1}^3c_{mn}\vartheta_{mn}(x)\quad \text{ in }\ \mathbb{R}^2.
\end{aligned}\tag{5.18}$$

$$\begin{aligned}
\lambda_j^2\partial_t\Phi_{\rm in}^{[j]}=\ &\left(a-bW^{[j]}\wedge\right)\Big[\Delta_{y^{[j]}}\Phi_{\rm in}^{[j]}+|\nabla_{y^{[j]}}W^{[j]}|^2\Phi_{\rm in}^{[j]}-2\nabla_{y^{[j]}}\left(W^{[j]}\cdot\Phi_{\rm in}^{[j]}\right)\nabla_{y^{[j]}}W^{[j]}\\
&+2\left(\nabla_{y^{[j]}}W^{[j]}\cdot\nabla_{y^{[j]}}\Phi_{\rm in}^{[j]}\right)W^{[j]}\Big]+\mathcal{H}^{[j]}\quad \text{ in }\ \mathsf{D}_{2R},
\end{aligned}\tag{5.19}$$

where

$$\mathcal{H}^{[j]} := \mathcal{H}_1^{[j]} + \mathcal{H}_{\rm in}^{[j]}, \tag{5.20}$$

$$\begin{aligned}
\mathcal{H}_1^{[j]} := \lambda_j^2 Q_{-\gamma_j} \Bigg\{ & \left(a - bU^{[j]}\wedge\right)\left[|\nabla_x U^{[j]}|^2 \Pi_{U^{[j]\perp}} \Phi_{\rm out} - 2\nabla_x \left(U^{[j]}\cdot \Phi_{\rm out}\right)\nabla_x U^{[j]}\right] \\
& + \Pi_{U^{[j]\perp}}\Big\{ -\partial_t(\Phi_0^{*[j]}) + \left(a - bU^{[j]}\wedge\right)\Big[\Delta_x \Phi_0^{*[j]} + |\nabla_x U^{[j]}|^2 \Phi_0^{*[j]} - 2\nabla_x \Big(U^{[j]}\cdot \Phi_0^{*[j]}\Big)\nabla_x U^{[j]}\Big] \\
& - \partial_t U^{[j]}\Big\} - \Big(e^{i\theta_j}\tilde{M}_1^{[j]} + e^{-i\theta_j} M_{-1}^{[j]}\Big)_{\mathcal{C}_j^{-1}} \Bigg\},
\end{aligned} \tag{5.21}$$

$$\begin{aligned}
\mathcal{H}_{\rm in}^{[j]} := \lambda_j^2 Q_{-\gamma_j} \Bigg\{ & 2\left(a - bU^{[j]}\wedge\right)\Big\{ \Big[\nabla_x U^{[j]} \cdot \nabla_x \Big(\eta_R^{[j]} Q_{\gamma_j}\Phi_{\rm in}^{[j]}\Big)\Big]\Big(Q_{\gamma_j}\Phi_{\rm in}^{[j]}\Big) \\
& - \Big[\Big(Q_{\gamma_j}\Phi_{\rm in}^{[j]}\Big)\cdot \nabla_x \Big(\eta_R^{[j]} Q_{\gamma_j}\Phi_{\rm in}^{[j]}\Big)\Big]\nabla_x U^{[j]}\Big\}\Bigg\} \\
= 2\left(a - bW^{[j]}\wedge\right) & \Big\{\Big[\nabla_{y^{[j]}} W^{[j]} \cdot \nabla_{y^{[j]}}\Big(\eta_R^{[j]}\Phi_{\rm in}^{[j]}\Big)\Big]\Phi_{\rm in}^{[j]} - \Big[\Phi_{\rm in}^{[j]}\cdot \nabla_{y^{[j]}}\Big(\eta_R^{[j]}\Phi_{\rm in}^{[j]}\Big)\Big]\nabla_{y^{[j]}} W^{[j]}\Big\};
\end{aligned} \tag{5.22}$$

$$\mathsf{D}_{2R} := \left\{(y,t) \mid |y| \le 2R(t),\ t \in (0,T)\right\}. \tag{5.23}$$

$$
\begin{aligned}
\mathcal{G} :=& \sum_{j=1}^{N} \left(1-\eta_R^{[j]}\right) \left(a - bU^{[j]}\wedge\right) \left[|\nabla_x U^{[j]}|^2 \Phi_{\rm out} - 2\nabla_x \left(U^{[j]}\cdot \Phi_{\rm out}\right) \nabla_x U^{[j]}\right] \\
&+ \sum_{j=1}^{N} \left(1-\eta_R^{[j]}\right) \Big\{ -\partial_t(\eta_{d_q}^{[j]}\Phi_0^{*[j]}) + \left(a - bU^{[j]}\wedge\right) \Big[\Delta_x(\eta_{d_q}^{[j]}\Phi_0^{*[j]}) + |\nabla_x U^{[j]}|^2 \eta_{d_q}^{[j]}\Phi_0^{*[j]} \\
&- 2\nabla_x \left(U^{[j]}\cdot \eta_{d_q}^{[j]}\Phi_0^{*[j]}\right)\nabla_x U^{[j]}\Big] - \partial_t U^{[j]}\Big\} \\
&+ \sum_{j=1}^{N} \eta_R^{[j]} \left(e^{i\theta_j}\tilde{M}_1^{[j]} + e^{-i\theta_j} M_{-1}^{[j]}\right)_{\mathcal{C}_j^{-1}} \\
&+ \sum_{j=1}^{N} \eta_R^{[j]} Q_{\gamma_j} \left[\left(\lambda_j^{-1}\dot{\lambda}_j y^{[j]} + \lambda_j^{-1}\dot{\xi}^{[j]}\right)\cdot \nabla_{y^{[j]}} \Phi_{\rm in}^{[j]} - \dot{\gamma}_j J \Phi_{\rm in}^{[j]}\right] \\
&+ \sum_{j=1}^{N} Q_{\gamma_j} \Big\{ -\Phi_{\rm in}^{[j]}\partial_t \eta_R^{[j]} \\
&+ \left(a - bW^{[j]}\wedge\right)\Big[\Phi_{\rm in}^{[j]}\Delta_x \eta_R^{[j]} + 2\nabla_x \eta_R^{[j]}\nabla_x \Phi_{\rm in}^{[j]} - \left(W^{[j]}\cdot \Phi_{\rm in}^{[j]}\right)\left(2\nabla_x \eta_R^{[j]}\nabla_x W^{[j]}\right)\Big]\Big\} \\
&- \sum_{j=1}^{N} b\left(U_* - U^{[j]}\right)\wedge \Big\{\Delta_x(\eta_{d_q}^{[j]}\Phi_0^{*[j]}) + \eta_R^{[j]} Q_{\gamma_j}\Delta_x \Phi_{\rm in}^{[j]} + Q_{\gamma_j}\left(\Phi_{\rm in}^{[j]}\Delta_x \eta_R^{[j]} + 2\nabla_x \eta_R^{[j]}\nabla_x \Phi_{\rm in}^{[j]}\right) \\
&- 2\nabla_x\left(U^{[j]}\cdot\Phi_{\rm out}\right)\nabla_x U^{[j]} - 2\nabla_x\left[U^{[j]}\cdot\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]} + \eta_{d_q}^{[j]}\Phi_0^{*[j]}\right)\right]\nabla_x U^{[j]}\Big\} \\
&+ (a - bU_*\wedge)\left\{-2\sum_{j=1}^{N}\nabla_x\left[\Phi\cdot\left(U_* - U^{[j]}\right)\right]\nabla_x U^{[j]}\right\} \\
&+ (a - bU_*\wedge)\left\{-2\sum_{j=1}^{N}\nabla_x\left[U^{[j]}\cdot\sum_{k\neq j}\left(\eta_R^{[k]}Q_{\gamma_k}\Phi_{\rm in}^{[k]} + \eta_{d_q}^{[k]}\Phi_0^{*[k]}\right)\right]\nabla_x U^{[j]}\right\} \\
&+ \sum_{j=1}^{N}|\nabla_x U^{[j]}|^2\left(a - bU^{[j]}\wedge\right)\sum_{k\neq j}\left(\eta_R^{[k]}Q_{\gamma_k}\Phi_{\rm in}^{[k]} + \eta_{d_q}^{[k]}\Phi_0^{*[k]}\right) \\
&+ a\Phi\sum_{j\neq k}\nabla_x U^{[j]}\cdot\nabla_x U^{[k]} + \left[(\Phi\cdot U_*) - A\right]\partial_t U_* \\
&+ \sum_{j=1}^{N}\eta_R^{[j]}\left(U^{[j]} - U_*\right)\Big[-2a\left(\nabla_x W^{[j]}\cdot\nabla_x \Phi_{\rm in}^{[j]}\right) + a|\nabla_x U^{[j]}|^2\left(U^{[j]}\cdot\Phi_{\rm out}\right) \\
&+ \Big\{-\partial_t(\Phi_0^{*[j]}) + \left(a - bU^{[j]}\wedge\right)\Big[\Delta_x\Phi_0^{*[j]} + |\nabla_x U^{[j]}|^2\Phi_0^{*[j]} - 2\nabla_x\left(U^{[j]}\cdot\Phi_0^{*[j]}\right)\nabla_x U^{[j]}\Big] \\
&- \partial_t U^{[j]}\Big\}\cdot U^{[j]}\Big]
\end{aligned}
\tag{5.24}
$$

$$
\begin{aligned}
&+ b\Big\{(\Phi\wedge U_*)\left[(1+A)|U_*|^2+\big(U_*\cdot\Pi_{U_*^\perp}\Phi\big)\right]^{-1}\left[\Phi+(1+A-\Phi\cdot U_*)\,U_*\right]\cdot\Delta_x\left(\Phi-\Phi_{\rm out}\right)\\
&-\big(AU_*+\Pi_{U_*^\perp}\Phi\big)\wedge\Delta_x\left(\Phi-\Phi_{\rm out}\right)\Big\}\\
&+ a\Bigg\{\Bigg[|\nabla_x A|^2|U_*|^2+2(1+A)\nabla_x A\cdot(U_*\cdot\nabla_x U_*)+A(2+A)\,|\nabla_x U_*|^2\\
&+2\sum_{k=1}^{2}\Big\{\left[(\partial_{x_k}A)\,U_*\cdot\partial_{x_k}\Phi+A\partial_{x_k}U_*\cdot\partial_{x_k}\Phi\right]-\partial_{x_k}\left(U_*\cdot\Phi\right)\left[|U_*|^2\partial_{x_k}A+(1+A)U_*\cdot\partial_{x_k}U_*\right]\\
&-(U_*\cdot\Phi)\left[(\partial_{x_k}A)\,U_*\cdot\partial_{x_k}U_*+(1+A)\,|\partial_{x_k}U_*|^2\right]\Big\}\\
&+\sum_{k=1}^{2}|\partial_{x_k}\Phi-U_*\partial_{x_k}\left(\Phi\cdot U_*\right)-(\Phi\cdot U_*)\partial_{x_k}U_*|^2\Bigg]\Pi_{U_*^\perp}\Phi\\
&+2\left(\nabla_x A+U_*\cdot\nabla_x U_*+\Phi\cdot\nabla_x\Phi\right)\nabla_x U_*+\Delta_x U_*-2\left(U_*\cdot\nabla_x U_*\right)\nabla_x U_*+\left(A-\Phi\cdot U_*\right)\Delta_x U_*\Bigg\}\\
&+2\left(a-bU_*\wedge\right)\left[\left(\nabla_x U_*\cdot\nabla_x\Phi\right)\Phi-\left(\Phi\cdot\nabla_x\Phi\right)\nabla_x U_*\right]-2a\left(\nabla_x U_*\cdot\nabla_x\Phi\right)(U_*\cdot\Phi)U_*\\
&-\sum_{j=1}^{N}2\left(a-bU^{[j]}\wedge\right)\Big\{\left[\nabla_x U^{[j]}\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\\
&-\left[\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\nabla_x U^{[j]}\Big\}\\
&+b(\Phi\wedge U_*)\left[(1+A)|U_*|^2+\big(U_*\cdot\Pi_{U_*^\perp}\Phi\big)\right]^{-1}(1+A-\Phi\cdot U_*)\left(2\nabla_x\Phi\cdot\nabla_x U_*\right)\\
&-b(\Phi\wedge U_*)\left(2\nabla_x\Phi\cdot\nabla_x U_*\right)\\
&-b\Bigg[-2^{-1}(\Phi\wedge U_*)\left[(1+A)|U_*|^2+\big(U_*\cdot\Pi_{U_*^\perp}\Phi\big)\right]^{-1}\Bigg\{2\left(1+A-\Phi\cdot U_*\right)\left(\Phi\cdot\Delta_x U_*\right)\\
&+2(|U_*|^2-2)|\nabla_x\left(\Phi\cdot U_*\right)|^2+2|\nabla_x\Phi|^2+8[(\Phi\cdot U_*)-(1+A)](U_*\cdot\nabla_x U_*)\cdot\nabla_x\left(\Phi\cdot U_*\right)\\
&+2|U_*|^2|\nabla_x A|^2+4\left[-2(\Phi\cdot U_*)U_*\cdot\nabla_x U_*+(1-|U_*|^2)\nabla_x\left(\Phi\cdot U_*\right)\right]\cdot\nabla_x A\\
&+8(1+A)\left(U_*\cdot\nabla_x U_*\right)\cdot\nabla_x A+2\left[(\Phi\cdot U_*)-(1+A)\right]^2\left(|\nabla_x U_*|^2+U_*\cdot\Delta_x U_*\right)\Bigg\}\\
&-\left(\Pi_{U_*^\perp}\Phi+AU_*\right)\wedge\left[2\nabla_x\left(\Phi\cdot U_*\right)\nabla_x U_*\right]+\left[A-(\Phi\cdot U_*)\right]\Phi\wedge\Delta_x U_*\\
&+\Pi_{U_*^\perp}\Phi\wedge\left(2\nabla_x A\nabla_x U_*\right)+\left[(\Phi\cdot U_*)^2-2A(\Phi\cdot U_*)-2(\Phi\cdot U_*)\right]U_*\wedge\Delta_x U_*\\
&+(1+A)U_*\wedge\left[A\Delta_x U_*+2\left(\nabla_x A+U_*\cdot\nabla_x U_*+\Phi\cdot\nabla_x\Phi\right)\nabla_x U_*+\Delta_x U_*-2\left(U_*\cdot\nabla_x U_*\right)\nabla_x U_*\right]\Bigg]\\
&+2bAU_*\wedge\left[\left(\Phi\cdot\nabla_x\Phi\right)\nabla_x U_*\right]+\Xi_{\mathcal{G}}(x,t)U_*,
\end{aligned}
\tag{5.25}
$$

where $\Xi_1(x,t)$ is given in (D.68) and $\Xi_{\mathcal{G}}(x,t)$ is some scalar function due to U_*-operation; $\tilde{M}_1, M_{-1}$ are given in (4.42), (4.44) respectively;

$$
\mathbf{B}_{\Phi,U_*}:=a\mathbf{I}_3-bU_*\wedge+\tilde{\mathbf{B}}_{\Phi,U_*},
\tag{5.26}
$$

$\mathbf{I}_3$ is 3×3 identity matrix,

$$
\tilde{\mathbf{B}}_{\Phi,U_*}:=b\left[(1+A)|U_*|^2+\big(U_*\cdot\Pi_{U_*^\perp}\Phi\big)\right]^{-1}\begin{bmatrix}(\Phi\wedge U_*)_1\left[\Phi+(1+A-\Phi\cdot U_*)\,U_*\right]\\(\Phi\wedge U_*)_2\left[\Phi+(1+A-\Phi\cdot U_*)\,U_*\right]\\(\Phi\wedge U_*)_3\left[\Phi+(1+A-\Phi\cdot U_*)\,U_*\right]\end{bmatrix}-b\left(AU_*+\Pi_{U_*^\perp}\Phi\right)\wedge,
\tag{5.27}
$$

$$
\begin{aligned}
\tilde{\mathbf{B}}_{\Phi,U_*}\Delta_x\Phi_{\rm out}=b\Big\{&(\Phi\wedge U_*)\left[(1+A)|U_*|^2+\big(U_*\cdot\Pi_{U_*^\perp}\Phi\big)\right]^{-1}\left[\Phi+(1+A-\Phi\cdot U_*)\,U_*\right]\cdot\Delta_x\Phi_{\rm out}\\
&-AU_*\wedge\Delta_x\Phi_{\rm out}-\Pi_{U_*^\perp}\Phi\wedge\Delta_x\Phi_{\rm out}\Big\};
\end{aligned}
\tag{5.28}
$$

$$
Z_*(x)\in C^3(\mathbb{R}^2),\ \|Z_*\|_{C^3(\mathbb{R}^2)}<C_{Z_*},\ \left[\partial_{x_1}Z_{*1}+\partial_{x_2}Z_{*2}+i\left(\partial_{x_1}Z_{*2}-\partial_{x_2}Z_{*1}\right)\right](q^{[j]})\neq 0
\tag{5.29}
$$

where $C_{Z_*} \ll 1$, $j = 1, 2, \ldots, N$;

$$\vartheta_{mn} \in C^3(\mathbb{R}^2),\ \|\vartheta_{mn}\|_{C^3(\mathbb{R}^2)} \le 2,\ \vartheta_{mn}(q_k) = \delta_{mk}\mathbf{e}_n \ \text{ for } \ m, k = 1, 2, \ldots, N,\ n = 1, 2, 3, \tag{5.30}$$
$$\mathbf{e}_1 = [1, 0, 0]^T,\ \mathbf{e}_2 = [0, 1, 0]^T,\ \mathbf{e}_3 = [0, 0, 1]^T.$$

Denote Γ_{Φ, U_*} as the fundamental solution of

$$\partial_t \mathbf{f} = \mathbf{B}_{\Phi, U_*} \Delta_x \mathbf{f} \ \text{ in } \ \mathbb{R}^2 \times (0, T). \tag{5.31}$$

c_{mn} will be chosen to such the following vanishings hold

$$\left(\Gamma_{\Phi, U_*} * * \mathcal{G}\right)(q_k, T) + \left(\Gamma_{\Phi, U_*} * Z_*\right)(q_k, T) + \sum_{m=1}^{N} \sum_{n=1}^{3} c_{mn} \left(\Gamma_{\Phi, U_*} * \vartheta_{mn}\right)(q_k, T) = 0 \ \text{ for } \ k = 1, 2, \ldots, N, \tag{5.32}$$

which will be useful in the gluing procedure.

For any $\mathbf{f} \in C^3(\mathbb{R}^2)$ satisfying $\|\mathbf{f}\|_{C^3(\mathbb{R}^2)} < \infty$, by (7.5), we have

$$|\Gamma_{\Phi, U_*} * \mathbf{f}| \lesssim \|\mathbf{f}\|_{C^3(\mathbb{R}^2)} \ \text{ in } \ \mathbb{R}^2 \times (0, T). \tag{5.33}$$

By [17, Theorem 1.2] and $\mathbf{B}_{\Phi, U_*} \in C(\mathbb{R}^2 \times (0, T)) \cap L^\infty(\mathbb{R}^2 \times (0, T))$, we have

$$\left|D_x \left(\Gamma_{\Phi, U_*} * \mathbf{f}\right)\right| + \left|D_x^2 \left(\Gamma_{\Phi, U_*} * \mathbf{f}\right)\right| + \left|\partial_t \left(\Gamma_{\Phi, U_*} * \mathbf{f}\right)\right| \lesssim \|\mathbf{f}\|_{C^3(\mathbb{R}^2)} \ \text{ in } \ \mathbb{R}^2 \times (0, T). \tag{5.34}$$

By $W_p^{1,2}$ estimate (see [17, Lemma 2.1]), we have

$$\frac{\left|D_x \left(\Gamma_{\Phi, U_*} * \mathbf{f}\right)(x, t) - D_x \left(\Gamma_{\Phi, U_*} * \mathbf{f}\right)(x_*, t_*)\right|}{\left(|x - x_*| + \sqrt{|t - t_*|}\right)^\alpha} \lesssim C(\alpha) \|\mathbf{f}\|_{C^3(\mathbb{R}^2)} \ \text{ for } \ 0 < \alpha < 1, \quad (x, t), (x_*, t_*) \in \mathbb{R}^2 \times (0, T). \tag{5.35}$$

By (5.34), for $m, k = 1, 2, \ldots, N$, $n = 1, 2, 3$, we have

$$\left|\left(\Gamma_{\Phi, U_*} * \vartheta_{mn}\right)(q_k, T) - \vartheta_{mn}(q_k)\right| = \left|\left(\Gamma_{\Phi, U_*} * \vartheta_{mn}\right)(q_k, T) - \delta_{mk}\mathbf{e}_n\right| \lesssim T.$$

Thus we can find unique $c_{mn} = c_{mn}[\Phi, U_*, \mathcal{G}, Z_*] = c_{mn1} + c_{mn2}$ satisfying (5.32), where $c_{mn1} = c_{mn1}[\Phi, U_*, \mathcal{G}]$, $c_{mn2} = c_{mn2}[\Phi, U_*, Z_*]$ satisfy

$$\left(\Gamma_{\Phi, U_*} * * \mathcal{G}\right)(q_k, T) + \sum_{m=1}^{N} \sum_{n=1}^{3} c_{mn1} \left(\Gamma_{\Phi, U_*} * \vartheta_{mn}\right)(q_k, T) = 0,$$
$$\left(\Gamma_{\Phi, U_*} * Z_*\right)(q_k, T) + \sum_{m=1}^{N} \sum_{n=1}^{3} c_{mn2} \left(\Gamma_{\Phi, U_*} * \vartheta_{mn}\right)(q_k, T) = 0 \ \text{ for } \ k = 1, 2, \ldots, N$$

with the following upper bounds

$$|c_{mn1}| \lesssim \sum_{k=1}^{N} \left|\left(\Gamma_{\Phi, U_*} * * \mathcal{G}\right)(q_k, T)\right|, \quad |c_{mn2}| \lesssim \sum_{k=1}^{N} \left|\left(\Gamma_{\Phi, U_*} * Z_*\right)(q_k, T)\right| \lesssim \|Z_*\|_{C^3(\mathbb{R}^2)} \tag{5.36}$$

for $m = 1, 2, \ldots, N$, $n = 1, 2, 3$.

In order to find a solution for (5.18), it suffices to solve the following fixed point problem:

$$\mathcal{T}_{\mathrm{o}}[\Phi_{\mathrm{out}}] := \Gamma_{\Phi, U_*} * * \mathcal{G}[\Phi_{\mathrm{out}}] + \Gamma_{\Phi, U_*} * Z_* + \sum_{m=1}^{N} \sum_{n=1}^{3} \left(c_{mn1}[\Phi, U_*, \mathcal{G}[\Phi_{\mathrm{out}}]] + c_{mn2}[\Phi, U_*, Z_*]\right)\left(\Gamma_{\Phi, U_*} * \vartheta_{mn}\right). \tag{5.37}$$

Denote

$$\Phi_{\mathrm{out}}^{(1)} := \Gamma_{\Phi, U_*} * Z_* + \sum_{m=1}^{N} \sum_{n=1}^{3} c_{mn2}[\Phi, U_*, Z_*] \left(\Gamma_{\Phi, U_*} * \vartheta_{mn}\right). \tag{5.38}$$

By (5.33), (5.34), (5.35) and (5.36), we have

$$\begin{aligned} &\sup_{\mathbb{R}^2 \times (0, T)} \left(\left|\Phi_{\mathrm{out}}^{(1)}\right| + \left|D_x \Phi_{\mathrm{out}}^{(1)}\right| + \left|D_x^2 \Phi_{\mathrm{out}}^{(1)}\right| + \left|\partial_t \Phi_{\mathrm{out}}^{(1)}\right|\right) \le 9^{-1} \Lambda_{\mathrm{o}1} \|Z_*\|_{C^3(\mathbb{R}^2)}, \\ &\sup_{\mathbb{R}^2 \times (0, T)} \frac{\left|D_x \Phi_{\mathrm{out}}^{(1)}(x, t) - D_x \Phi_{\mathrm{out}}^{(1)}(x_*, t_*)\right|}{\left(|x - x_*| + \sqrt{|t - t_*|}\right)^\alpha} \le 9^{-1} C(\alpha) \Lambda_{\mathrm{o}1} \|Z_*\|_{C^3(\mathbb{R}^2)} \end{aligned} \tag{5.39}$$

for a large constant $\Lambda_{\mathrm{o}1} \ge 1$.

5.4. **Weighted topologies for the inner and outer problems.** The topologies for the inner and outer problems are listed in this section. Recall (5.3) and the form of (5.19). It is natural to introduce the new time variable

$$\tau_j = \tau_j(t) := \int_0^t \lambda_j^{-2}(s)ds + C_\tau T\lambda_*^{-2}(0), \quad \tau_j(0) = \tau_0 := C_\tau T\lambda_*^{-2}(0) \tag{5.40}$$

with a constant $C_\tau > 0$ sufficiently large. Then $\tau_j(t) \sim |\ln T|^{-2}(T-t)^{-1}|\ln(T-t)|^4$, which implies

$$\ln(\tau_j(t)) \sim |\ln(T-t)|, \quad \lambda_*(t) \sim |\ln T|^{-1}\tau_j^{-1}|\ln\tau_j|^2.$$

• For the inner problems, we are going to measure their right hand sides and solutions under the following norms respectively

$$\|H\|_{\nu,2+l,\varsigma_H} := \sup_{(y,\tau_j)\in\mathcal{D}_{2R}}\left[\lambda_*^{-\nu}(\tau_j)\langle y\rangle^{2+l}\left(|H(y,\tau_j)| + \langle y\rangle^{\varsigma_H}[H]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}(Q^-((y,\tau_j),\frac{|y|}{2}))}\right)\right],$$

where $0 < \varsigma_H < 1$ is small and

$$[H]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}(Q^-((y,\tau_j),\frac{|y|}{2}))} := \sup_{(y_*,\tau_*)\in Q^-((y,\tau_j),\frac{|y|}{2})}\frac{|H(y,\tau_j)-H(y_*,\tau_*)|}{\left(|y-y_*|+|\tau_j-\tau_*|^{\frac12}\right)^{\varsigma_H}}$$

where the symbol $\lambda_*(\tau_j) = \lambda_*(t(\tau_j))$ is abused and

$$\begin{aligned}&\mathcal{D}_{2R} := \{(y,\tau_j) \mid |y| < 2R,\ \tau_j > \tau_0\},\\ &Q^-((y,\tau_j),\frac{|y|}{2}) := \left\{(z,s) \mid |z-y| < \frac{|y|}{2}, \quad \max\left\{\tau_j - \frac{|y|}{2},\tau_0\right\} < s < \tau_j\right\}.\end{aligned} \tag{5.41}$$

Denote

$$\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l} := \sup_{\mathcal{D}_{2R}}\left[\lambda_*^{-\nu+\delta_0}(\tau_j)\langle y\rangle^l\left(\left|\Phi_{\rm in}^{[j]}(y,\tau_j)\right| + \langle y\rangle\left|D_y\Phi_{\rm in}^{[j]}(y,\tau_j)\right| + \langle y\rangle^2\left|D_y^2\Phi_{\rm in}^{[j]}(y,\tau_j)\right|\right)\right], \tag{5.42}$$

$$\begin{aligned}&\left[\Phi_{\rm in}^{[j]}\right]_{{\rm in},\nu-\delta_0,l,\varsigma_{\rm in}}\\ &:= \sup_{\mathcal{D}_{2R}}\left[\lambda_*^{-\nu+\delta_0}(\tau_j)\langle y\rangle^l\left(\langle y\rangle^{\varsigma_{\rm in}}\left[\Phi_{\rm in}^{[j]}\right]_{C^{\varsigma_{\rm in},\frac{\varsigma_{\rm in}}{2}}(Q^-((y,\tau_j),\frac{|y|}{2}))} + \langle y\rangle^{\varsigma_{\rm in}+1}\left[D_y\Phi_{\rm in}^{[j]}\right]_{C^{\varsigma_{\rm in},\frac{\varsigma_{\rm in}}{2}}(Q^-((y,\tau_j),\frac{|y|}{2}))}\right)\right],\end{aligned} \tag{5.43}$$

$$\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l,\varsigma_{\rm in}} := \|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l} + \left[\Phi_{\rm in}^{[j]}\right]_{{\rm in},\nu-\delta_0,l,\varsigma_{\rm in}}, \tag{5.44}$$

where $0 < \varsigma_{\rm in} < 1$ and

$$0 < \delta_0 < \nu < 1. \tag{5.45}$$

Set $R_0(t) = \lambda_*^{-\delta_0/6}(t)$, which will be used in the inner problem and orthogonal equations.

The inner problem will be solved in the following space.

$$B_{\rm in}^{[j]} := \left\{\mathbf{f} \mid \|\mathbf{f}\|_{{\rm in},\nu-\delta_0,l,\varsigma_{\rm in}} \le \Lambda_{\rm in}, \quad \mathbf{f}\cdot W^{[j]} = 0\right\}. \tag{5.46}$$

• For the outer problem, we define the following weights to control the right hand side of the outer problem

$$\varrho_1^{[j]} := \lambda_*^{\Theta}(\lambda_* R)^{-1}\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}, \quad \varrho_2^{[j]} := T^{-\sigma_0}\frac{\lambda_*^{1-\sigma_0}}{|x-q^{[j]}|^2}\mathbf{1}_{\{\lambda_* R/2\le|x-q^{[j]}|\le d_q\}}, \quad \varrho_3 := T^{-\sigma_0}. \tag{5.47}$$

where

$$d_q := \frac19\min_{k\ne m}|q^{[k]}-q^{[m]}|, \quad \Theta+\beta-1<0, \quad 0<\Theta<1, \quad 0<\sigma_0<1. \tag{5.48}$$

For a function $f(x,t)$, we define the L^∞-weighted norm

$$\|f\|_{**} := \sup_{(x,t)\in\mathbb{R}^2\times(0,T)}\left[\sum_{j=1}^N\left(\varrho_1^{[j]}+\varrho_2^{[j]}\right)+\varrho_3\right]^{-1}|f(x,t)|. \tag{5.49}$$

Also, we define the L^∞-weighted norm for $\Phi_{\rm out}$:

$$
\begin{aligned}
&\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\\
:=\ &\left(|\ln T|\lambda_*^{\Theta+1}(0)R(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)^{-1}\|\Phi_{\rm out}\|_{L^\infty(\mathbb{R}^2\times(0,T))}\\
&+\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)^{-1}\|\nabla_x\Phi_{\rm out}\|_{L^\infty(\mathbb{R}^2\times(0,T))}\\
&+\sup_{\mathbb{R}^2\times(0,T)}\left[|\ln(T-t)|\lambda_*^{\Theta+1}(t)R(t)+(T-t)\|Z_*\|_{C^3(\mathbb{R}^2)}\right]^{-1}|\Phi_{\rm out}(x,t)-\Phi_{\rm out}(x,T)|\\
&+\sup_{\mathbb{R}^2\times(0,T)}C^{-1}(\alpha)\left[\lambda_*^{\Theta}(t)+(T-t)^{\frac{\alpha}{2}}\|Z_*\|_{C^3(\mathbb{R}^2)}\right]^{-1}|\nabla_x\Phi_{\rm out}(x,t)-\nabla_x\Phi_{\rm out}(x,T)|\\
&+\sup C^{-1}(\alpha)\left[\lambda_*^{\Theta}(t)(\lambda_*(t)R(t))^{-\alpha}+\|Z_*\|_{C^3(\mathbb{R}^2)}\right]^{-1}\frac{|\nabla_x\Phi_{\rm out}(x,t)-\nabla_x\Phi_{\rm out}(x_*,t_*)|}{\left(|x-x_*|+\sqrt{|t-t_*|}\right)^{\alpha}}
\end{aligned}
\tag{5.50}
$$

under assumptions (C.1) for the parameters, where $\alpha\in(0,1)$, $C(\alpha)$ could be unbounded as $\alpha\to1^-$ and the last supremum is taken in the region

$$
x,\,x_*\in\mathbb{R}^2,\quad t,\,t_*\in(0,T),\quad |t-t_*|<\frac14(T-t).
$$

The outer problem will be solved in

$$
B_{\rm out}:=\left\{\mathbf{f}\mid\|\mathbf{f}\|_{\sharp,\Theta,\alpha}\le\Lambda_{\rm o},\ \mathbf{f}(q^{[j]},T)=0\ \text{ for }\ j=1,2,\dots,N\right\}
\tag{5.51}
$$

where $\Lambda_{\rm o}\ge1$ will be determined later.

6. Orthogonal equations

In order to find inner solutions with sufficient space-time decay, we need to solve the orthogonal equations for λ_j, γ_j and $\xi^{[j]}$ such that orthogonalities hold.

6.1. **Mode 0.** The corresponding scalar form (3.7) is given in (9.18). Notice $\mathcal{Z}_{0,1}(\rho_j)=-\frac12\rho_j w_{\rho_j}$ and $\mathcal{Z}_{0,1}(\rho_j)\rho_j=\frac{\rho_j^2}{\rho_j^2+1}$. Then

$$
\begin{aligned}
&\int_0^{2R_0}M_0(\rho_j,t)\mathcal{Z}_{0,1}(\rho_j)\rho_jd\rho_j\\
=&\int_0^{2R_0}\Bigg\{\lambda_j^{-1}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left\{\left[\frac{-3\rho_j^2}{2(\rho_j^2+1)^{\frac72}}+\frac{O(\zeta_j)}{\langle\rho_j\rangle^3}\right]\mathbf{1}_{\{\zeta_j\le1\}}+\frac{O(\zeta_j^{-1})}{\langle\rho_j\rangle^3}\mathbf{1}_{\{\zeta_j>1\}}\right\}ds\right]\\
&-ib\lambda_j^{-1}\int_0^t\frac{\overline{\dot p_j(s)e^{-i\gamma_j(t)}}}{t-s}\left\{\left[\frac{3(a-ib)\rho_j^2}{(\rho_j^2+1)^{\frac92}}+\frac{O(\zeta_j)}{\langle\rho_j\rangle^3}\right]\mathbf{1}_{\{\zeta_j\le1\}}+\frac{O(\zeta_j^{-1})}{\langle\rho_j\rangle^3}\mathbf{1}_{\{\zeta_j>1\}}\right\}ds\\
&-(a-ib)\lambda_j^{-1}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\Bigg[\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\\
&\times\left\{\left[\frac{4(a+ib)\rho_j^4}{(\rho_j^2+1)^{\frac72}}+\frac{O(\zeta_j)}{\langle\rho_j\rangle^3}\right]\mathbf{1}_{\{\zeta_j\le1\}}+\frac{O(\zeta_j^{-1})}{\langle\rho_j\rangle^3}\mathbf{1}_{\{\zeta_j>1\}}\right\}ds\Bigg]\\
&-(a-ib)\lambda_j^{-1}\mathrm{Re}\left[\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left\{\left[\frac{12(a+ib)\rho_j^4}{(\rho_j^2+1)^{\frac92}}+\frac{O(\zeta_j)}{\langle\rho_j\rangle^3}\right]\mathbf{1}_{\{\zeta_j\le1\}}+\frac{O(\zeta_j^{-1})}{\langle\rho_j\rangle^3}\mathbf{1}_{\{\zeta_j>1\}}\right\}ds\right]\\
&+\dot\lambda_j\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[\int_0^t\frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left\{\left[\frac{-(a+ib)\rho_j^4}{2(\rho_j^2+1)^{\frac52}}+\frac{O(\zeta_j)}{\langle\rho_j\rangle}\right]\mathbf{1}_{\{\zeta_j\le1\}}+\frac{O(\zeta_j^{-1})}{\langle\rho_j\rangle}\mathbf{1}_{\{\zeta_j>1\}}\right\}ds\right]\\
&-\frac{2\lambda_j^{-1}\dot p_je^{-i\gamma_j}\rho_j^3}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^{\frac52}}-\lambda_j^{-1}\dot\lambda_j\frac{4\rho_j^4[\rho_j^2+\rho_j(\rho_j^2+1)^{\frac12}+1]}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^4}\Bigg\}d\rho_j
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda_j^{-1} \int_0^t \frac{\dot p_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{-3\rho_j^2 - 8\rho_j^4}{2(\rho_j^2+1)^{\frac{7}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \\
&+ \lambda_j^{-1} \mathrm{Re} \left[\int_0^t \frac{\dot p_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{3\rho_j^2}{(\rho_j^2+1)^{\frac{9}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^5} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^5} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \right] \\
&- ib\lambda_j^{-1} \int_0^t \frac{\overline{\dot p_j(s) e^{-i\gamma_j(t)}}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{3(a-ib)\rho_j^2}{(\rho_j^2+1)^{\frac{9}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \\
&- (a-ib)\lambda_j^{-1} \mathrm{Re} \left[\int_0^t \frac{\dot p_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{4(a+ib)\rho_j^4}{(\rho_j^2+1)^{\frac{9}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \right] \\
&+ \dot\lambda_j \int_0^t \frac{\dot p_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{-(a+ib)\rho_j^4}{2(\rho_j^2+1)^{\frac{5}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \\
&+ \dot\lambda_j \mathrm{Re} \left[\int_0^t \frac{\dot p_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{(a+ib)\rho_j^4}{(\rho_j^2+1)^{\frac{7}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \right] \\
&- \int_0^{2R_0} \left\{ \frac{2\lambda_j^{-1} \dot p_j e^{-i\gamma_j} \rho_j^3}{[\rho_j + (\rho_j^2+1)^{\frac{1}{2}}](\rho_j^2+1)^{\frac{5}{2}}} + \lambda_j^{-1}\dot\lambda_j \frac{4\rho_j^4[\rho_j^2 + \rho_j(\rho_j^2+1)^{\frac{1}{2}} + 1]}{[\rho_j + (\rho_j^2+1)^{\frac{1}{2}}](\rho_j^2+1)^4} \right\} d\rho_j
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda_j^{-1} \int_0^t \frac{\dot p_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{-3\rho_j^2 - 8\rho_j^4}{2(\rho_j^2+1)^{\frac{7}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \\
&+ \lambda_j^{-1} \mathrm{Re} \left[\int_0^t \frac{\dot p_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{3\rho_j^2}{(\rho_j^2+1)^{\frac{9}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^5} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^5} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \right] \\
&- ib\lambda_j^{-1} \mathrm{Re} \left[\int_0^t \frac{\dot p_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{3(a+ib)\rho_j^2}{(\rho_j^2+1)^{\frac{9}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \right] \\
&+ b\lambda_j^{-1} \mathrm{Re} \left[\int_0^t \frac{\dot p_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{3(ia-b)\rho_j^2}{(\rho_j^2+1)^{\frac{9}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \right] \\
&- a\lambda_j^{-1} \mathrm{Re} \left[\int_0^t \frac{\dot p_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{4(a+ib)\rho_j^4}{(\rho_j^2+1)^{\frac{9}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \right] \\
&+ ib\lambda_j^{-1} \mathrm{Re} \left[\int_0^t \frac{\dot p_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{4(a+ib)\rho_j^4}{(\rho_j^2+1)^{\frac{9}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \right] \\
&+ \dot\lambda_j \int_0^t \frac{\dot p_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{-(a+ib)\rho_j^4}{2(\rho_j^2+1)^{\frac{5}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \\
&+ \dot\lambda_j \mathrm{Re} \left[\int_0^t \frac{\dot p_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{(a+ib)\rho_j^4}{(\rho_j^2+1)^{\frac{7}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \right] \\
&- \int_0^{2R_0} \left\{ \frac{2\lambda_j^{-1} \dot p_j e^{-i\gamma_j} \rho_j^3}{[\rho_j + (\rho_j^2+1)^{\frac{1}{2}}](\rho_j^2+1)^{\frac{5}{2}}} + \lambda_j^{-1}\dot\lambda_j \frac{4\rho_j^4[\rho_j^2 + \rho_j(\rho_j^2+1)^{\frac{1}{2}} + 1]}{[\rho_j + (\rho_j^2+1)^{\frac{1}{2}}](\rho_j^2+1)^4} \right\} d\rho_j
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda_j^{-1} \int_0^t \frac{\dot{p}_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{-3\rho_j^2 - 8\rho_j^4}{2(\rho_j^2+1)^{\frac{7}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \\
&\quad + \lambda_j^{-1} \mathrm{Re} \left[\int_0^t \frac{\dot{p}_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{(a^2+iab)(3\rho_j^2 - 4\rho_j^4)}{(\rho_j^2+1)^{\frac{9}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \right] \\
&\quad + ib\lambda_j^{-1} \mathrm{Re} \left[\int_0^t \frac{\dot{p}_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{(a+ib)(4\rho_j^4 - 3\rho_j^2)}{(\rho_j^2+1)^{\frac{9}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} \right. \\
&\quad \left. d\rho_j ds \right] \\
&\quad + \dot{\lambda}_j \int_0^t \frac{\dot{p}_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{-(a+ib)\rho_j^4}{2(\rho_j^2+1)^{\frac{5}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \\
&\quad + \dot{\lambda}_j \mathrm{Re} \left[\int_0^t \frac{\dot{p}_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{(a+ib)\rho_j^4}{(\rho_j^2+1)^{\frac{7}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \right] \\
&\quad - \int_0^{2R_0} \left\{ \frac{2\lambda_j^{-1} \dot{p}_j e^{-i\gamma_j} \rho_j^3}{[\rho_j + (\rho_j^2+1)^{\frac{1}{2}}](\rho_j^2+1)^{\frac{5}{2}}} + \lambda_j^{-1} \dot{\lambda}_j \frac{4\rho_j^4 [\rho_j^2 + \rho_j(\rho_j^2+1)^{\frac{1}{2}} + 1]}{[\rho_j + (\rho_j^2+1)^{\frac{1}{2}}](\rho_j^2+1)^4} \right\} d\rho_j \\
&= \lambda_j^{-1} \int_0^t \frac{\dot{p}_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{-3\rho_j^2 - 8\rho_j^4}{2(\rho_j^2+1)^{\frac{7}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \\
&\quad + (a - ib)\lambda_j^{-1} \\
&\quad \times \mathrm{Re} \left[\int_0^t \frac{\dot{p}_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{(a+ib)(3\rho_j^2 - 4\rho_j^4)}{(\rho_j^2+1)^{\frac{9}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \right] \\
&\quad + \dot{\lambda}_j \int_0^t \frac{\dot{p}_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{-(a+ib)\rho_j^4}{2(\rho_j^2+1)^{\frac{5}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \\
&\quad + \dot{\lambda}_j \mathrm{Re} \left[\int_0^t \frac{\dot{p}_j(s) e^{-i\gamma_j(t)}}{t-s} \int_0^{2R_0} \left\{ \left[\frac{(a+ib)\rho_j^4}{(\rho_j^2+1)^{\frac{7}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j ds \right] \\
&\quad - \int_0^{2R_0} \left\{ \frac{2\lambda_j^{-1} \dot{p}_j e^{-i\gamma_j} \rho_j^3}{[\rho_j + (\rho_j^2+1)^{\frac{1}{2}}](\rho_j^2+1)^{\frac{5}{2}}} + \lambda_j^{-1} \dot{\lambda}_j \frac{4\rho_j^4 [\rho_j^2 + \rho_j(\rho_j^2+1)^{\frac{1}{2}} + 1]}{[\rho_j + (\rho_j^2+1)^{\frac{1}{2}}](\rho_j^2+1)^4} \right\} d\rho_j.
\end{aligned}
$$

Recall (4.14), $\zeta_j = \iota_j(\rho_j^2+1), \iota_j = \frac{\lambda_j^2(t)}{t-s}$. We will estimate spatial integral term by term.

Part 1:

$$
\begin{aligned}
&\int_0^{2R_0} \left\{ \left[\frac{-3\rho_j^2 - 8\rho_j^4}{2(\rho_j^2+1)^{\frac{7}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] \mathbf{1}_{\{\zeta_j \le 1\}} + \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} \mathbf{1}_{\{\zeta_j > 1\}} \right\} d\rho_j \\
&= \left(-1 + O(R_0^{-2} + \iota_j \langle \ln \iota_j \rangle) \mathbf{1}_{\{\iota_j \le 1\}} \right) + O(\iota_j^{-1}) \mathbf{1}_{\{\iota_j > 1\}}
\end{aligned}
$$

since for $\iota_j > 1$,

$$
\int_0^{2R_0} \frac{O(\zeta_j^{-1})}{\langle \rho_j \rangle^3} d\rho_j = O(\iota_j^{-1});
$$

for $\iota_j < (4R_0^2+1)^{-1}$,

$$
\int_0^{2R_0} \left[\frac{-3\rho_j^2 - 8\rho_j^4}{2(\rho_j^2+1)^{\frac{7}{2}}} + \frac{O(\zeta_j)}{\langle \rho_j \rangle^3} \right] d\rho_j = -1 + O(R_0^{-2} + \iota_j \ln R_0)
$$

where we used

$$
\int_0^\infty 2^{-1}(-3x^2 - 8x^4)(x^2+1)^{-\frac{7}{2}} dx = -1;
$$

for $(4R_0^2+1)^{-1} \le \iota_j \le 1$,

$$\int_0^{(2\iota_j^{-1}-1)^{\frac{1}{2}}} \left[\frac{-3\rho_j^2-8\rho_j^4}{2(\rho_j^2+1)^{\frac{7}{2}}} + \frac{O(\zeta_j)}{\langle\rho_j\rangle^3}\right] d\rho_j + \int_{(2\iota_j^{-1}-1)^{\frac{1}{2}}}^{2R_0} \frac{O\left(\zeta_j^{-1}\right)}{\langle\rho_j\rangle^3} d\rho_j = -1 + O(\iota_j\langle\ln\iota_j\rangle).$$

Part 2:

$$\int_0^{2R_0} \left\{\left[\frac{(a+ib)(3\rho_j^2-4\rho_j^4)}{(\rho_j^2+1)^{\frac{9}{2}}} + \frac{O(\zeta_j)}{\langle\rho_j\rangle^3}\right]\mathbf{1}_{\{\zeta_j\le 1\}} + \frac{O\left(\zeta_j^{-1}\right)}{\langle\rho_j\rangle^3}\mathbf{1}_{\{\zeta_j>1\}}\right\} d\rho_j$$
$$= O(R_0^{-4} + \iota_j\langle\ln\iota_j\rangle)\mathbf{1}_{\{\iota_j\le 1\}} + O(\iota_j^{-1})\mathbf{1}_{\{\iota_j>1\}}$$

since for $\iota_j > 1$,

$$\int_0^{2R_0} \frac{O\left(\zeta_j^{-1}\right)}{\langle\rho_j\rangle^3} d\rho_j = O(\iota_j^{-1});$$

for $\iota_j < (4R_0^2+1)^{-1}$,

$$\int_0^{2R_0} \left[\frac{(a+ib)(3\rho_j^2-4\rho_j^4)}{(\rho_j^2+1)^{\frac{9}{2}}} + \frac{O(\zeta_j)}{\langle\rho_j\rangle^3}\right] d\rho_j = O(R_0^{-4} + \iota_j\ln R_0)$$

where we used

$$\int_0^\infty (3x^2-4x^4)(x^2+1)^{-\frac{9}{2}}\,dx = 0;$$

for $(4R_0^2+1)^{-1} \le \iota_j \le 1$,

$$\int_0^{(2\iota_j^{-1}-1)^{\frac{1}{2}}} \left[\frac{(a+ib)(3\rho_j^2-4\rho_j^4)}{(\rho_j^2+1)^{\frac{9}{2}}} + \frac{O(\zeta_j)}{\langle\rho_j\rangle^3}\right] d\rho_j + \int_{(2\iota_j^{-1}-1)^{\frac{1}{2}}}^{2R_0} \frac{O\left(\zeta_j^{-1}\right)}{\langle\rho_j\rangle^3} d\rho_j = O(\iota_j\langle\ln\iota_j\rangle).$$

Part 3: Notice

$$\int_0^{2R_0} \left(\langle\rho_j\rangle^{-1}\mathbf{1}_{\{\zeta_j\le 1\}} + \langle\rho_j\rangle^{-1}\zeta_j^{-1}\mathbf{1}_{\{\zeta_j>1\}}\right) d\rho_j = O(\min\{\ln R_0, \langle\ln\iota_j\rangle\})\mathbf{1}_{\{\iota_j\le 1\}} + O(\iota_j^{-1})\mathbf{1}_{\{\iota_j>1\}} \tag{6.1}$$

since for $\iota_j \ge 1$,

$$\int_0^{2R_0} \langle\rho_j\rangle^{-1}\zeta_j^{-1} d\rho_j = O(\iota_j^{-1});$$

for $\iota_j \le (4R_0^2+1)^{-1}$,

$$\int_0^{2R_0} \langle\rho_j\rangle^{-1} d\rho_j = O(\ln R_0);$$

for $(4R_0^2+1)^{-1} < \iota_j < 1$,

$$\int_0^{(2\iota_j^{-1}-1)^{\frac{1}{2}}} \langle\rho_j\rangle^{-1} d\rho_j + \int_{(2\iota_j^{-1}-1)^{\frac{1}{2}}}^{2R_0} \langle\rho_j\rangle^{-1}\zeta_j^{-1} d\rho_j = O(\langle\ln\iota_j\rangle).$$

Thus

$$\int_0^{2R_0} \left\{\left[\frac{-(a+ib)\rho_j^4}{2(\rho_j^2+1)^{\frac{5}{2}}} + \frac{O(\zeta_j)}{\langle\rho_j\rangle}\right]\mathbf{1}_{\{\zeta_j\le 1\}} + \frac{O\left(\zeta_j^{-1}\right)}{\langle\rho_j\rangle}\mathbf{1}_{\{\zeta_j>1\}}\right\} d\rho_j = O(\ln R_0)\mathbf{1}_{\{\iota_j\le 1\}} + O(\iota_j^{-1})\mathbf{1}_{\{\iota_j>1\}}.$$

The counterpart in Re:

$$\int_0^{2R_0} \left\{\left[\frac{(a+ib)\rho_j^4}{(\rho_j^2+1)^{\frac{7}{2}}} + \frac{O(\zeta_j)}{\langle\rho_j\rangle^3}\right]\mathbf{1}_{\{\zeta_j\le 1\}} + \frac{O\left(\zeta_j^{-1}\right)}{\langle\rho_j\rangle^3}\mathbf{1}_{\{\zeta_j>1\}}\right\} d\rho_j = O(1)\mathbf{1}_{\{\iota_j\le 1\}} + O(\iota_j^{-1})\mathbf{1}_{\{\iota_j>1\}}$$

since for $\iota_j > 1$,

$$\int_0^{2R_0} \langle\rho_j\rangle^{-3} O\left(\zeta_j^{-1}\right) d\rho_j = O(\iota_j^{-1});$$

for $\iota_j < (4R_0^2+1)^{-1}$,

$$\int_0^{2R_0} \langle\rho_j\rangle^{-3} d\rho_j = O(1);$$

for $(4R_0^2+1)^{-1} \le \iota_j \le 1$,

$$\int_0^{(2\iota_j^{-1}-1)^{\frac12}} \langle \rho_j\rangle^{-3} d\rho_j + \int_{(2\iota_j^{-1}-1)^{\frac12}}^{2R_0} \langle \rho_j\rangle^{-3} O\left(\zeta_j^{-1}\right) d\rho_j = O(1).$$

Part 4:

$$\begin{aligned}
&-\int_0^{2R_0} \left\{ \frac{2\lambda_j^{-1}\dot p_j e^{-i\gamma_j}\rho_j^3}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^{\frac52}} + \lambda_j^{-1}\dot\lambda_j \frac{4\rho_j^4[\rho_j^2+\rho_j(\rho_j^2+1)^{\frac12}+1]}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^4} \right\} d\rho_j \\
=&\ -\lambda_j^{-1}\dot p_j e^{-i\gamma_j}\left(\frac53 - \ln 4 + O(R_0^{-2})\right) - \lambda_j^{-1}\dot\lambda_j\left(\frac45 + O(R_0^{-2})\right)
\end{aligned}$$

since $\int_0^\infty 2x^3[x+(x^2+1)^{\frac12}]^{-1}(x^2+1)^{-\frac52}dx = \frac53 - \ln 4$, $\int_0^\infty 4x^4[x^2+x(x^2+1)^{\frac12}+1][x+(x^2+1)^{\frac12}]^{-1}(x^2+1)^{-4}dx = 0.8$.

In sum, we obtain

$$\begin{aligned}
&\int_0^{2R_0} M_0(\rho_j,t)\mathcal{Z}_{0,1}(\rho_j)\rho_j d\rho_j \\
=&\ \lambda_j^{-1}\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left[\left(-1+O(R_0^{-2}+\iota_j\langle\ln\iota_j\rangle)\mathbf{1}_{\{\iota_j\le1\}}\right)+O(\iota_j^{-1})\mathbf{1}_{\{\iota_j>1\}}\right]ds \\
&+(a-ib)\lambda_j^{-1}\mathrm{Re}\left[\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left(O(R_0^{-4}+\iota_j\langle\ln\iota_j\rangle)\mathbf{1}_{\{\iota_j\le1\}}+O(\iota_j^{-1})\mathbf{1}_{\{\iota_j>1\}}\right)ds\right] \\
&+\dot\lambda_j\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left(O(\ln R_0)\mathbf{1}_{\{\iota_j\le1\}}+O(\iota_j^{-1})\mathbf{1}_{\{\iota_j>1\}}\right)ds \\
&+\dot\lambda_j\mathrm{Re}\left[\int_0^t \frac{\dot p_j(s)e^{-i\gamma_j(t)}}{t-s}\left(O(1)\mathbf{1}_{\{\iota_j\le1\}}+O(\iota_j^{-1})\mathbf{1}_{\{\iota_j>1\}}\right)ds\right] \\
&-\lambda_j^{-1}\dot p_j e^{-i\gamma_j}\left(\frac53-\ln4+O(R_0^{-2})\right)-\lambda_j^{-1}\dot\lambda_j\left(\frac45+O(R_0^{-2})\right).
\end{aligned} \tag{6.2}$$

Next, we handle the influence from the outer couplings in the orthogonal equation. By (3.21) and (3.20), we have

$$\begin{aligned}
&(a-ib)\lambda_j^{-1}e^{-i\gamma_j(t)}\left[\partial_{x_1}(\Phi_{\rm out})_1+\partial_{x_2}(\Phi_{\rm out})_2+i\left(\partial_{x_1}(\Phi_{\rm out})_2-\partial_{x_2}(\Phi_{\rm out})_1\right)\right](q^{[j]},0) \\
&\times\int_0^{2R_0}\rho_j w_{\rho_j}^2(\rho_j)\mathcal{Z}_{0,1}(\rho_j)\rho_j d\rho_j \\
=&\ (a-ib)\lambda_j^{-1}e^{-i\gamma_j(t)}(1+O(R_0^{-2}))\left[\partial_{x_1}(\Phi_{\rm out})_1+\partial_{x_2}(\Phi_{\rm out})_2+i\left(\partial_{x_1}(\Phi_{\rm out})_2-\partial_{x_2}(\Phi_{\rm out})_1\right)\right](q^{[j]},0)
\end{aligned} \tag{6.3}$$

since $\int_0^{2R_0}\rho_j w_{\rho_j}^2(\rho_j)\mathcal{Z}_{0,1}(\rho_j)\rho_j d\rho_j = \int_0^{2R_0}\frac{4\rho_j^3}{(\rho_j^2+1)^3}d\rho_j = 1+O(R_0^{-2})$.

6.2. **Mode** 1.

Notice $\mathcal{Z}_{1,1}(\rho_j) = -\frac12 w_{\rho_j}$ and $\mathcal{Z}_{1,1}(\rho_j)\rho_j = \frac{\rho_j}{\rho_j^2+1}$. Then

$$\begin{aligned}
&\int_0^{2R_0}\left(M_1(\rho_j,t)-\tilde M_1(\rho_j,t)\right)\mathcal{Z}_{1,1}(\rho_j)\rho_j d\rho_j = -(\dot\xi_1^{[j]}-i\dot\xi_2^{[j]})\lambda_j^{-1}\int_0^{2R_0}\frac{\rho_j}{\rho_j^2+1}\frac{2}{\rho_j^2+1}d\rho_j \\
=&\ -(\dot\xi_1^{[j]}-i\dot\xi_2^{[j]})\lambda_j^{-1}(1+O(R_0^{-2})).
\end{aligned} \tag{6.4}$$

For the influence from the outer part, by (3.21) and (3.20),

$$\begin{aligned}
&2(a-ib)\lambda_j^{-1}\left(-\partial_{x_1}(\Phi_{\rm out})_3+i\partial_{x_2}(\Phi_{\rm out})_3\right)(q^{[j]},0)\int_0^{2R_0}w_{\rho_j}(\rho_j)\cos w(\rho_j)\mathcal{Z}_{1,1}(\rho_j)\rho_j d\rho_j \\
=&\ 2(a-ib)\lambda_j^{-1}\left(-\partial_{x_1}(\Phi_{\rm out})_3+i\partial_{x_2}(\Phi_{\rm out})_3\right)(q^{[j]},0)\int_0^{2R_0}\frac{-2\rho_j(\rho_j^2-1)}{(\rho_j^2+1)^3}d\rho_j \\
=&\ O(R_0^{-2})(a-ib)\lambda_j^{-1}\left(-\partial_{x_1}(\Phi_{\rm out})_3+i\partial_{x_2}(\Phi_{\rm out})_3\right)(q^{[j]},0)
\end{aligned} \tag{6.5}$$

where we used $\int_0^\infty \frac{-2x(x^2-1)}{(x^2+1)^3}dx = 0$.

6.3. **Linear theory for the non-local equations.** For notational simplicity, we shall drop the indices in the parameters $p_j(t) = \lambda_j e^{i\gamma_j(t)}$ for $j = 1, \ldots, N$ and just write $p = \lambda e^{i\gamma}$ in this section.

To introduce the space for the parameter function $p(t)$, we recall the non-local operator $\mathcal{B}_0$ appears at mode 0 is of the approximate form

$$\mathcal{B}_0[p] = \int_{-T}^{t-\lambda^2} \frac{\dot{p}(s)}{t-s} ds \; + O\big(\|\dot{p}\|_\infty\big).$$

For $\Theta \in (0,1)$, $\varpi \in \mathbb{R}$ and a continuous function $g : [-T,T] \to \mathbb{C}$, we define the norm

$$\|g\|_{\Theta,\varpi} = \sup_{t\in[-T,T]} (T-t)^{-\Theta} |\log(T-t)|^{\varpi} |g(t)|,$$

and for $\alpha \in (0,1)$, $m,\ \varpi \in \mathbb{R}$, we define the semi-norm

$$[g]_{\frac{\alpha}{2},m,\varpi} = \sup (T-t)^{-m} |\log(T-t)|^{\varpi} \frac{|g(t)-g(s)|}{(t-s)^{\frac{\alpha}{2}}},$$

where the supremum is taken over $s \le t$ in $[-T,T]$ such that $t - s \le \frac{1}{10}(T-t)$.

The following result was proved in [12, Proposition 6.5, Proposition 6.6] concerning the solvability of the non-local operators.

Proposition 6.1. *Let $\alpha_0, \frac{\alpha}{2} \in (0, \frac{1}{2})$, $\varpi \in \mathbb{R}$. There is $\flat > 0$ such that if $\Theta \in (0,\flat)$ and $m \le \Theta - \frac{\alpha}{2}$, then for $a(t) : [0,T] \to \mathbb{C}$ satisfying*

$$\begin{cases} |a(T)| > 0, \\ T^{\Theta} |\ln T|^{1+\sigma-\varpi} \|a(\cdot) - a(T)\|_{\Theta,\varpi-1} + [a]_{\frac{\alpha}{2},m,\varpi-1} \le C_1, \end{cases} \tag{6.6}$$

for some $\sigma,\ C_1 > 0$, then, for $T > 0$ sufficiently small there exist two operators $\mathcal{P}$ and $\mathcal{R}_0$ so that $p = \mathcal{P}[a] : [-T,T] \to \mathbb{C}$ satisfies

$$\mathcal{B}_0[p](t) = a(t) + \mathcal{R}_0[a](t), \quad t \in [0,T]$$

with

$$|\mathcal{R}_0[a](t)| \le C\Big(T^{\sigma} + T^{\Theta} \frac{\ln|\ln T|}{|\ln T|} \|a(\cdot) - a(T)\|_{\Theta,\varpi-1} + [a]_{\frac{\alpha}{2},m,\varpi-1}\Big) \frac{(T-t)^{m+\frac{(1+\alpha_0)\alpha}{2}}}{|\ln(T-t)|^{\varpi}}.$$

Moreover,

$$\mathcal{P}[a] = p_{0,\kappa} + \mathcal{P}_1[a] + \mathcal{P}_2[a],$$

with

$$p_{0,\kappa}(t) = \kappa |\ln T| \int_t^T \frac{1}{|\ln(T-s)|^2} \, ds, \quad t \le T,$$

where $\kappa = \kappa[a]$. Denote $p_1 = \mathcal{P}_1[a]$, $p_2 = \mathcal{P}_2[a]$. Then the following bounds hold:

$$\begin{aligned} &\kappa = |a(T)| \left(1 + O(|\ln T|^{-1})\right), \\ &|\dot{p}_1(t) - \dot{p}_{0,\kappa}(t)| \le C \frac{|\ln T|^{1-\sigma} \left(\ln(|\ln T|)\right)^2}{|\ln(T-t)|^{3-\sigma}}, \\ &|\ddot{p}_1(t)| \le C \frac{|\ln T|}{|\ln(T-t)|^3 (T-t)}, \\ &\|\dot{p}_2\|_{\Theta,\varpi} \le C\left(T^{\frac{1}{2}+\sigma-\Theta} + \|a(\cdot) - a(T)\|_{\Theta,\varpi-1}\right), \\ &[\dot{p}]_{\frac{\alpha}{2},m,\varpi} \le C\left(|\ln T|^{\varpi-3} T^{\flat-m-\frac{\alpha}{2}} + T^{\Theta} \frac{\ln|\ln T|}{|\ln T|} \|a(\cdot) - a(T)\|_{\Theta,\varpi-1} + [a]_{\frac{\alpha}{2},m,\varpi-1}\right). \end{aligned} \tag{6.7}$$

Proposition 6.1 gives an approximate inverse $\mathcal{P}$ of the operator $\mathcal{B}_0$, so that given $a(t)$ satisfying (6.6), $p := \mathcal{P}[a]$ satisfies

$$\mathcal{B}_0[p] = a + \mathcal{R}_0[a], \quad \text{in} \quad [0,T],$$

for a small remainder $\mathcal{R}_0[a]$.

We now impose constraints on the parameters such that contraction property in the non-local problem can be obtained. Roughly speaking, when applying Proposition 6.1, the term $a(t)$ is essentially from the outer solution. The vanishing and Hölder properties are exactly the ones inherited from the weighted topology (5.50) for the outer problem, namely

$$|a(t)-a(T)| \lesssim \lambda_*^{\Theta}(t) + (T-t)^{\frac{\alpha}{2}} \|Z_*\|_{C^3(\mathbb{R}^2)},$$
$$\frac{|a(t)-a(s)|}{|t-s|^{\alpha/2}} \lesssim \lambda_*^{\Theta}(t)(\lambda_*(t)R(t))^{-\alpha} + \|Z_*\|_{C^3(\mathbb{R}^2)}.$$

So it is then natural to choose in the $[\cdot]_{\frac{\alpha}{2},m,\varpi-1}$-seminorm

$$m := \min\{\Theta - \alpha(1-\beta), 0\} = \Theta - \alpha(1-\beta), \tag{6.8}$$

where we need

$$\Theta < \frac{\alpha}{2}, \quad \Theta - \alpha(1-\beta) < 0. \tag{6.9}$$

In order for both $\|a(\cdot) - a(T)\|_{\Theta,\varpi-1}$, $[a]_{\frac{\alpha}{2},m,\varpi-1}$ to be finite, we need

$$\varpi - 1 - 2\Theta < 0, \quad \varpi - 1 - 2m < 0. \tag{6.10}$$

Also the assumption $m \le \Theta - \frac{\alpha}{2}$ in Proposition 6.1 implies

$$\beta \le 1/2, \tag{6.11}$$

which is in the desired self-similar regime as we require before. Recall the estimate of $\mathcal{R}_0[a]$. We require

$$m + (1+\alpha_0)\frac{\alpha}{2} > \Theta,$$

namely,

$$0 < \alpha_0 < 1/2, \quad 2\beta - 1 + \alpha_0 > 0, \tag{6.12}$$

so that the vanishing order of $\mathcal{R}_0[a]$ as $t \to T$ is faster than the leading part $a(t)$ itself. We then conclude that with above choices of $m,\ \alpha_0,\ \frac{\alpha}{2},\ \varpi$, the remainder gains smallness

$$|\mathcal{R}_0[a](t)| \lesssim \lambda_*^{\Theta+\sigma_1} \tag{6.13}$$

compare to the leading part $a(t)$ itself, where

$$0 < \sigma_1 < m + \frac{(1+\alpha_0)\alpha}{2} - \Theta. \tag{6.14}$$

We will put the remainder in another piece of inner problem where the orthogonality condition is not satisfied, where the extra smallness measured above by σ_1 is crucial to control the non-orthogonal part. Recall the linear theory for the inner problem without orthogonality. For the inner solution solved from the right hand side involving $\mathcal{R}_0[a]$ to be in the desired topology, we require

$$1 + \Theta - \alpha(1-\beta) + \frac{(1+\alpha_0)\alpha}{2} - 2\beta > \nu - \delta_0. \tag{6.15}$$

In summary, the restrictions on the constants needed when dealing with the non-local reduced problem are given by

$$\begin{aligned} &0 < \beta < \frac{1}{2}, \quad 2\Theta < \alpha, \quad 0 < \alpha_0 < \frac{1}{2}, \quad 2\beta - 1 + \alpha_0 > 0, \\ &1 + \Theta - \alpha(1-\beta) + \frac{(1+\alpha_0)\alpha}{2} - 2\beta > \nu - \delta_0. \end{aligned} \tag{6.16}$$

7. Linear theory for the outer problem

7.1. Fundamental solution for the outer problem.

Consider

$$Lu = \sum_{|\alpha|\le 1, |\beta| \le 1} A^{\alpha\beta} D^\alpha D^\beta u \ \text{ in } \ \mathbb{R}^d$$

where

$$D^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d}, \quad \alpha = (\alpha_1, \cdots, \alpha_d), \quad u = (u^1, \cdots, u^n)^T$$

and, for each α, β, $A^{\alpha\beta} = [A_{ij}^{\alpha\beta}(t,x)]_{i,j=1}^n$ is an $n \times n$ real matrix-valued function.

The parabolic systems that we study are

$$u_t = Lu. \tag{7.1}$$

Assume that $|A^{\alpha\beta}| \le \Lambda$ and satisfies Legendre-Hadamard ellipticity

$$\sum_{|\alpha|=|\beta|=1} \theta^T \xi^\alpha \xi^\beta A^{\alpha\beta}(t,x)\theta \ge \lambda |\xi|^2 |\theta|^2 \tag{7.2}$$

for all $(t,x) \in \mathbb{R}^{d+1}$, $\xi \in \mathbb{R}^d$, and $\theta \in \mathbb{R}^n$.

We use the symbol in [20] to give some basis definitions. Define the parabolic distance between $X = (t,x)$ and $Y = (s,y)$ in $\mathbb{R}^{d+1}$ by

$$|X - Y| = \max\left\{|x-y|, \sqrt{|t-s|}\right\}.$$

We define the $(d+1)$-dimensional cylinders $Q_r(X)$, $Q_r^+(X)$, and $Q_r^-(X)$, by

$$Q_r(X) = \{Y \in \mathbb{R}^{d+1} : |Y - X| < r\} = (s - r^2, s + r^2) \times B_r(x),$$
$$Q_r^+(X) = (s, s + r^2) \times B_r(x), \quad \text{and} \quad Q_r^-(X) = (s - r^2, s) \times B_r(x).$$

For $X = (t,x) \in \mathbb{R}^{d+1}$ and $r > 0$, we define

$$\omega_{\mathbf{A}}^{\mathsf{x}}(r,X) := \fint_{Q_r^-(X)} |\mathbf{A}(y,s) - \bar{\mathbf{A}}_{x,r}^{\mathsf{x}}(s)|\, dyds, \quad \text{where} \quad \bar{\mathbf{A}}_{x,r}^{\mathsf{x}}(s) := \fint_{B_r(x)} \mathbf{A}(z,s)\, dz.$$

Then for a subset Q of $\mathbb{R}^{d+1}$, we define

$$\omega_{\mathbf{A}}^{\mathsf{x}}(r,Q) := \sup\{\omega_{\mathbf{A}}^{\mathsf{x}}(r,X) : X \in Q\} \quad \text{and} \quad \omega_{\mathbf{A}}^{\mathsf{x}}(r) := \omega_{\mathbf{A}}^{\mathsf{x}}(r, \mathbb{R}^{d+1}).$$

We say that $\mathbf{A}$ is of **Dini mean oscillation in** x over Q and write $\mathbf{A} \in \mathsf{DMO}_{\mathsf{x}}(Q)$ if $\omega_{\mathbf{A}}^{\mathsf{x}}(r,Q)$ satisfies the Dini condition

$$\int_0^1 \frac{\omega_{\mathbf{A}}^{\mathsf{x}}(r,Q)}{r}\, dr < +\infty. \tag{7.3}$$

Similarly, for $X = (t,x) \in \mathbb{R}^{d+1}$ and $r > 0$, we define

$$|\omega|_{\mathbf{A}}^{\mathsf{x}}(r,X) := \fint_{Q_r^-(X)} \fint_{B_r(x)} |\mathbf{A}(y,s) - \mathbf{A}(z,s)|\, dzdyds$$

Then for a subset Q of $\mathbb{R}^{d+1}$, we define

$$|\omega|_{\mathbf{A}}^{\mathsf{x}}(r,Q) := \sup\{|\omega|_{\mathbf{A}}^{\mathsf{x}}(r,X) : X \in Q\} \quad \text{and} \quad |\omega|_{\mathbf{A}}^{\mathsf{x}}(r) := |\omega|_{\mathbf{A}}^{\mathsf{x}}(r, \mathbb{R}^{d+1}).$$

We say that $\mathbf{A}$ is of **Dini mean absolute oscillation in** x over Q and write $\mathbf{A} \in |\mathsf{DMO}|_{\mathsf{x}}(Q)$ if $|\omega|_{\mathbf{A}}^{\mathsf{x}}(r,Q)$ satisfies the Dini condition

$$\int_0^1 \frac{|\omega|_{\mathbf{A}}^{\mathsf{x}}(r,Q)}{r}\, dr < +\infty. \tag{7.4}$$

We present some basic properties about $\mathsf{DMO}_{\mathsf{x}}(Q)$ and $|\mathsf{DMO}|_{\mathsf{x}}(Q)$ in the following Lemma.

Lemma 7.1.

(1) $|\mathsf{DMO}|_{\mathsf{x}}(Q) \subset \mathsf{DMO}_{\mathsf{x}}(Q)$.
(2) *If* $|\nabla_x f(x,t)|$ *are uniformly bounded in* Q, *then* $f(x,t)$ *is in* $|\mathsf{DMO}|_{\mathsf{x}}(Q)$.
(3) *For all* $f, g \in \mathsf{DMO}_{\mathsf{x}}(Q)(|\mathsf{DMO}|_{\mathsf{x}}(Q))$, $c \in \mathbb{R}$, *then* $f + g, cf \in \mathsf{DMO}_{\mathsf{x}}(Q)(|\mathsf{DMO}|_{\mathsf{x}}(Q))$.
(4) *For* $f \in |\mathsf{DMO}|_{\mathsf{x}}(Q)$, *then* $|f| \in |\mathsf{DMO}|_{\mathsf{x}}(Q)$.
(5) *For* $f \in |\mathsf{DMO}|_{\mathsf{x}}(Q)$ *and* $|f| \ge \epsilon_0 > 0$, *then* $\frac{1}{f}, , |f|^\theta \in |\mathsf{DMO}|_{\mathsf{x}}(Q)$ *with* $0 < \theta < 1$.
(6) *For all* $f, g \in |\mathsf{DMO}|_{\mathsf{x}}(Q) \cap L^\infty(Q)$, *then* $fg, |f|^\theta \in |\mathsf{DMO}|_{\mathsf{x}}(Q) \cap L^\infty(Q)$ *with* $\theta > 1$.

Proof. The proof is straightforward by the definition. □

Assume the principal coefficients $A^{\alpha\beta} \in \mathsf{DMO}_{\mathsf{x}}(\mathbb{R}^{n+1})$. Then Theorem 1.3 in [20] can be generalized to parabolic systems (7.1) (see [20]). Indeed, $W^{2,p}$ estimate for parabolic systems is given in [18], which can be used to generalize Lemma 2.2 in [20] to parabolic systems. Lemma 2.3 in [20], which is first given in Theorem 3.3 in [17] can be also generalized to parabolic systems.

Claim: for any $\delta \in (0,1)$, there exists a constant $C = C(d, \lambda, \Lambda, \omega_{\mathbf{A}}^{\mathsf{x}}, \delta)$ and a universal constant $c > 0$ such that for $0 < t - s \le 1$,

$$\begin{aligned}&(t-s)\left(|\partial_t \Gamma(x,t,y,s)| + |D^2\Gamma(x,t,y,s)|\right) + (t-s)^{\frac{1}{2}}|D\Gamma(x,t,y,s)| + |\Gamma(x,t,y,s)| \\ &\le C(t-s)^{-\frac{d}{2}} e^{-c\left(\frac{|x-y|}{\sqrt{t-s}}\right)^{2-\delta}};\end{aligned} \tag{7.5}$$

for $s < t_1 < t_2 \le s+1$,

$$\frac{|\Gamma(x_1,t_1,y,s)-\Gamma(x_2,t_2,y,s)|}{\left(|x_1-x_2|+\sqrt{|t_1-t_2|}\right)^{\alpha}} \le C(\alpha)(t_2-s)^{-\frac{\alpha}{2}}\left[(t_1-s)^{-\frac{d}{2}}e^{-c\left(\frac{|x_1-y|}{\sqrt{t_1-s}}\right)^{2-\delta}}+(t_2-s)^{-\frac{d}{2}}e^{-c\left(\frac{|x_2-y|}{\sqrt{t_2-s}}\right)^{2-\delta}}\right]; \tag{7.6}$$

$$\begin{aligned}&\frac{|(D_x\Gamma)(x_1,t_1,y,s)-(D_x\Gamma)(x_2,t_2,y,s)|}{\left(|x_1-x_2|+\sqrt{|t_1-t_2|}\right)^{\alpha}}\\ \le{}& C(\alpha)(t_2-s)^{-\frac{\alpha}{2}}\left[(t_1-s)^{-\frac{d+1}{2}}e^{-c\left(\frac{|x_1-y|}{\sqrt{t_1-s}}\right)^{2-\delta}}+(t_2-s)^{-\frac{d+1}{2}}e^{-c\left(\frac{|x_2-y|}{\sqrt{t_2-s}}\right)^{2-\delta}}\right],\end{aligned} \tag{7.7}$$

where α is an arbitrary number in $(0,1)$.

For $|f|\lesssim 1$, by (B.2),

$$|\Gamma * f|\lesssim 1.$$

Combining [17, Theorem 1.2], we have

$$|D_x(\Gamma * f)|+\left|D_x^2(\Gamma * f)\right|+|\partial_t(\Gamma * f)|\lesssim 1.$$

Proof. By Theorem 3.2 in [17] and generalized version of Theorem 1.3 in [20], there exists a constant $C = C(d,\lambda,\Lambda,\omega_{\mathbf{A}}^{\times},\delta)$ and a universal constant $c>0$ such that for $0<t-s\le 1$,

$$\begin{aligned}&(t-s)(|\partial_t\Gamma(x,t,y,s)|+|D_x^2\Gamma(x,t,y,s)|)+(t-s)^{\frac{1}{2}}|D_x\Gamma(x,t,y,s)|+|\Gamma(x,t,y,s)|\\ \le{}& C(t-s)^{-\frac{d}{2}}e^{-c(\frac{|x-y|}{\sqrt{t-s}})^{2-\delta}}.\end{aligned} \tag{7.8}$$

$C,c>0$ will vary from line to line in the following calculation.

For any fixed x_*,t_*,y,s, set $\rho_*=(t_*-s)^{\frac{1}{2}}$. Consider $\Gamma(x_*+\rho_*z,t_*+\rho_*^2\tau,y,s)$ as a function of z,τ. For $p>n+2$, set $\alpha_1=1-\frac{n+2}{p}$. Then

$$\begin{aligned}&\rho_*^{-1-\alpha_1}\|\Gamma(x_*+\rho_*z,t_*+\rho_*^2\tau,y,s)\|_{L^p(B(0,\frac{1}{2})\times(-\frac{1}{4},0))}\\ \lesssim{}& (t_*-s)^{-\frac{1+\alpha_1}{2}}\|(t_*+\rho_*^2\tau-s)^{-\frac{d}{2}}e^{-c\left(\frac{|x_*+\rho_*z-y|}{\sqrt{t_*+\rho_*^2\tau-s}}\right)^{2-\delta}}\|_{L^p(B(0,\frac{1}{2})\times(-\frac{1}{4},0))}\\ \lesssim{}& \begin{cases}(t_*-s)^{-\frac{d+1+\alpha_1}{2}} & \text{if } |x_*-y|\le(t_*-s)^{\frac{1}{2}}\\ (t_*-s)^{-\frac{d+1+\alpha_1}{2}}e^{-c(\frac{|x_*-y|}{\sqrt{t_*-s}})^{2-\delta}} & \text{if } |x_*-y|>(t_*-s)^{\frac{1}{2}}\end{cases}\\ \sim{}& (t_*-s)^{-\frac{d+1+\alpha_1}{2}}e^{-c(\frac{|x_*-y|}{\sqrt{t_*-s}})^{2-\delta}},\end{aligned}$$

where we have used

$$\frac{3}{4}(t_*-s)\le t_*+\rho_*^2\tau-s\le t_*-s,$$

$$\frac{|x_*+\rho_*z-y|}{\sqrt{t_*+\rho_*^2\tau-s}}\begin{cases}\lesssim 1 & \text{if } |x_*-y|\le(t_*-s)^{\frac{1}{2}}\\ \sim\frac{|x_*-y|}{\sqrt{t_*-s}} & \text{if } |x_*-y|>(t_*-s)^{\frac{1}{2}}\end{cases}.$$

$$\begin{aligned}&\rho_*^{-\alpha_1}\|(D\Gamma)(x_*+\rho_*z,t_*+\rho_*^2\tau,y,s)\|_{L^p(B(0,\frac{1}{2})\times(-\frac{1}{4},0))}\\ \lesssim{}& (t_*-s)^{-\frac{\alpha_1}{2}}\|(t_*+\rho_*^2\tau-s)^{-\frac{d+1}{2}}e^{-c(\frac{|x_*+\rho_*z-y|}{\sqrt{t_*+\rho_*^2\tau-s}})^{2-\delta}}\|_{L^p(B(0,\frac{1}{2})\times(-\frac{1}{4},0))}\\ \lesssim{}& (t_*-s)^{-\frac{d+1+\alpha_1}{2}}e^{-c(\frac{|x_*-y|}{\sqrt{t_*-s}})^{2-\delta}}.\end{aligned}$$

$$\begin{aligned}&\rho_*^{1-\alpha_1}\|(D^2\Gamma)(x_*+\rho_*z,t_*+\rho_*^2\tau,y,s)\|_{L^p(B(0,\frac{1}{2})\times(-\frac{1}{4},0))}\\ \lesssim{}& (t_*-s)^{\frac{1-\alpha_1}{2}}\|(t_*+\rho_*^2\tau-s)^{-\frac{d+2}{2}}e^{-c(\frac{|x_*+\rho_*z-y|}{\sqrt{t_*+\rho_*^2\tau-s}})^{2-\delta}}\|_{L^p(B(0,\frac{1}{2})\times(-\frac{1}{4},0))}\\ \lesssim{}& (t_*-s)^{-\frac{d+1+\alpha_1}{2}}e^{-c(\frac{|x_*-y|}{\sqrt{t_*-s}})^{2-\delta}}.\end{aligned}$$

$$\sup_{x_1,x_2\in B(x_*,\frac{(t_*-s)^{\frac{1}{2}}}{2}),t_1,t_2\in(t_*-\frac{t_*-s}{4},t_*)}\frac{|(D_x\Gamma)(x_1,t_1,y,s)-(D_x\Gamma)(x_2,t_2,y,s)|}{(|x_1-x_2|+\sqrt{|t_1-t_2|})^{\alpha_1}}\lesssim(t_*-s)^{-\frac{d+1+\alpha_1}{2}}e^{-c(\frac{|x_*-y|}{\sqrt{t_*-s}})^{2-\delta}}$$

which implies that

$$\frac{|(D_x\Gamma)(x_1,t_1,y,s)-(D_x\Gamma)(x_*,t_*,y,s)|}{(|x_1-x_*|+\sqrt{|t_1-t_*|})^{\alpha_1}}\lesssim (t_*-s)^{-\frac{d+1+\alpha_1}{2}}e^{-c(\frac{|x_*-y|}{\sqrt{t_*-s}})^{2-\delta}}$$

for $(x_1,t_1)\in B(x_*,\frac{(t_*-s)^{\frac12}}{2})\times(t_*-\frac{t_*-s}{4},t_*)$.

For $(x_1,t_1)\notin B(x_*,\frac{(t_*-s)^{\frac12}}{2})\times(t_*-\frac{t_*-s}{4},t_*)$, by (7.8), we have

$$\begin{aligned}&\frac{|(D_x\Gamma)(x_1,t_1,y,s)-(D_x\Gamma)(x_*,t_*,y,s)|}{(|x_1-x_*|+\sqrt{|t_1-t_*|})^{\alpha_1}}\\ &\lesssim (t_*-s)^{-\frac{\alpha_1}{2}}\left\{(t_1-s)^{-\frac{d+1}{2}}e^{-c(\frac{|x_1-y|}{\sqrt{t_1-s}})^{2-\delta}}+(t_*-s)^{-\frac{d+1}{2}}e^{-c(\frac{|x_*-y|}{\sqrt{t_*-s}})^{2-\delta}}\right\}.\end{aligned}$$

Similarly, we have

$$\frac{|\Gamma(x_1,t_1,y,s)-\Gamma(x_*,t_*,y,s)|}{(|x_1-x_*|+\sqrt{|t_1-t_*|})^{\alpha_2}}\lesssim (t_*-s)^{-\frac{d+\alpha_2}{2}}e^{-c(\frac{|x_*-y|}{\sqrt{t_*-s}})^{2-\delta}}$$

for $(x_1,t_1)\in B(x_*,\frac{(t_*-s)^{\frac12}}{2})\times(t_*-\frac{t_*-s}{4},t_*)$.

For $(x_1,t_1)\notin B(x_*,\frac{(t_*-s)^{\frac12}}{2})\times(t_*-\frac{t_*-s}{4},t_*)$, by (7.8), we get

$$\frac{|\Gamma(x_1,t_1,y,s)-\Gamma(x_*,t_*,y,s)|}{(|x_1-x_*|+\sqrt{|t_1-t_*|})^{\alpha_2}}\lesssim (t_*-s)^{-\frac{\alpha_2}{2}}\left\{(t_1-s)^{-\frac d2}e^{-c(\frac{|x_1-y|}{\sqrt{t_1-s}})^{2-\delta}}+(t_*-s)^{-\frac d2}e^{-c(\frac{|x_*-y|}{\sqrt{t_*-s}})^{2-\delta}}\right\},$$

where

$$\alpha_2=\begin{cases}2-\frac{n+2}{q} & \text{if } q<n+2\\ 1-\epsilon \text{ for any } \epsilon\in(0,1), & \text{if } q\ge n+2.\end{cases}$$

□

7.2. $|\mathsf{DMO}|_{\mathsf{x}}$ **property for the leading coefficients.** Notice

$$A^{(1,0)(1,0)}=A^{(0,1)(0,1)}=aI-bU_*\wedge,$$

and for all other indices it is zero. Then

$$\sum_{|\alpha|=|\beta|=1}\xi^\alpha\xi^\beta A^{\alpha\beta}(t,x)=|\xi|^2(aI-bU_*\wedge).$$

Next

$$\begin{aligned}&\Re\left(\sum_{|\alpha|=|\beta|=1}\theta^T\xi^\alpha\xi^\beta A^{\alpha\beta}(t,x)\bar\theta\right)=\Re\left(\theta^T[|\xi|^2(aI-bU_*\wedge)\bar\theta]\right)\\ &=|\xi|^2\Re\left(a|\theta|^2-b\theta^T[U_*\wedge\bar\theta]\right)\\ &=|\xi|^2\Re\left(a|\theta|^2-b(\theta_1+i\theta_2)^T[U_*\wedge(\theta_1-i\theta_2)]\right)\\ &=|\xi|^2\left(a|\theta|^2-b\{\theta_1^T[U_*\wedge\theta_1]+\theta_2^T[U_*\wedge\theta_2]\}\right)\\ &=a|\xi|^2|\theta|^2.\end{aligned}$$

Recalling (5.26) and (5.27), we will prove the coefficients in $\mathbf{B}_{\Phi,U_*}$ belong to $|\mathsf{DMO}|_{\mathsf{x}}(\mathbb{R}^{n+1})\cap L^\infty(\mathbb{R}^{n+1})$.

Obviously, all components of U_* are in $L^\infty(\mathbb{R}^{n+1})$. We will prove that all components of U_* belong to $|\mathsf{DMO}|_{\mathsf{x}}(\mathbb{R}^{n+1})$. Then by Lemma 7.1, the multiplicity of finite many terms of the component of U_* also belong to $|\mathsf{DMO}|_{\mathsf{x}}(\mathbb{R}^{n+1})$ automatically. Recall

$$U^{[j]}(y)=\frac{1}{|y^{[j]}|^2+1}\begin{bmatrix}2y^{[j]}\\ |y^{[j]}|^2-1\end{bmatrix},$$

where

$$\frac{2y^{[j]}}{|y^{[j]}|^2+1}=\frac{2\lambda_j(t)(x-\xi^{[j]}(t))}{|x-\xi^{[j]}(t)|^2+\lambda_j^2(t)},\quad \frac{|y^{[j]}|^2-1}{|y^{[j]}|^2+1}=\frac{|x-\xi^{[j]}(t)|^2-\lambda_j^2(t)}{|x-\xi^{[j]}(t)|^2+\lambda_j^2(t)}=1-\frac{2\lambda_j^2(t)}{|x-\xi^{[j]}(t)|^2+\lambda_j^2(t)}.$$

It suffices to prove

$$\frac{\lambda_j^2(t)}{|x-\xi^{[j]}(t)|^2+\lambda_j^2(t)},\ \frac{\lambda_j(t)(x-\xi^{[j]}(t))}{|x-\xi^{[j]}(t)|^2+\lambda_j^2(t)}\in|\mathsf{DMO}|_{\mathsf{x}}. \tag{7.9}$$

Proof of (7.9).

$$\left|\frac{\lambda_j^2(s)}{|w-\xi^{[j]}(s)|^2+\lambda_j^2(s)}-\frac{\lambda_j^2(s)}{|z-\xi^{[j]}(s)|^2+\lambda_j^2(s)}\right| = \lambda_j^2(s)\frac{\left||z-\xi^{[j]}(s)|^2-|w-\xi^{[j]}(s)|^2\right|}{\left(|w-\xi^{[j]}(s)|^2+\lambda_j^2(s)\right)\left(|z-\xi^{[j]}(s)|^2+\lambda_j^2(s)\right)}$$
$$\le |w-z|\lambda_j^2(s)\frac{\left|z-\xi^{[j]}(s)\right|+\left|w-\xi^{[j]}(s)\right|}{\left(|w-\xi^{[j]}(s)|^2+\lambda_j^2(s)\right)\left(|z-\xi^{[j]}(s)|^2+\lambda_j^2(s)\right)}$$
$$\lesssim |w-z|\lambda_j(s)\left[\left(\left|w-\xi^{[j]}(s)\right|^2+\lambda_j^2(s)\right)^{-1}+\left(\left|z-\xi^{[j]}(s)\right|^2+\lambda_j^2(s)\right)^{-1}\right].$$

Then

$$\begin{aligned}
&\fint_{Q_r^-(X)}\fint_{B_r(x)}\left|\frac{\lambda_j^2(s)}{\left|w-\xi^{[j]}(s)\right|^2+\lambda_j^2(s)}-\frac{\lambda_j^2(s)}{\left|z-\xi^{[j]}(s)\right|^2+\lambda_j^2(s)}\right|dzdwds\\
&\lesssim |w-z|\fint_{Q_r^-(X)}\fint_{B_r(x)}\lambda_j(s)\left[\left(\left|w-\xi^{[j]}(s)\right|^2+\lambda_j^2(s)\right)^{-1}+\left(\left|z-\xi^{[j]}(s)\right|^2+\lambda_j^2(s)\right)^{-1}\right]dzdwds\\
&\sim |w-z|\fint_{t-r^2}^{t}\fint_{B_r(x)}\lambda_j(s)\left(\left|z-\xi^{[j]}(s)\right|^2+\lambda_j^2(s)\right)^{-1}dzds \qquad (7.10)\\
&\le |w-z|\fint_{t-r^2}^{t}\fint_{B_r(0)}\lambda_j(s)\left(|z|^2+\lambda_j^2(s)\right)^{-1}dzds \sim |w-z|r^{-4}\int_{t-r^2}^{t}\lambda_j(s)\int_0^r\left(v^2+\lambda_j^2(s)\right)^{-1}vdvds\\
&\sim |w-z|r^{-4}\int_{t-r^2}^{t}\lambda_j(s)\ln\left(1+\lambda_j^{-2}(s)r^2\right)ds.
\end{aligned}$$

In order to get $\frac{\lambda_j^2(t)}{|x-\xi^{[j]}(t)|^2+\lambda_j^2(t)}\in|\mathsf{DMO}|_{\mathsf{x}}$, it suffices to prove the following integral is bounded.

$$\begin{aligned}
&\int_0^1 r^{-4}\int_{t-r^2}^{t}\lambda_j(s)\ln\left(1+\lambda_j^{-2}(s)r^2\right)dsdr=\int_0^1 r^{-4}\int_0^t\lambda_j(s)\ln\left(1+\lambda_j^{-2}(s)r^2\right)\mathbf{1}_{\{s\ge t-r^2\}}dsdr\\
&=\int_0^t\int_0^1 r^{-4}\lambda_j(s)\ln\left(1+\lambda_j^{-2}(s)r^2\right)\mathbf{1}_{\{r\ge(t-s)^{\frac12}\}}drds=\int_0^t\lambda_j^{-2}(s)\int_{\frac{(t-s)^{\frac12}}{\lambda_j(s)}}^{\frac{1}{\lambda_j(s)}}z^{-4}\ln(1+z^2)dzds\\
&=\int_{t-(T-t)}^{t}+\int_0^{t-(T-t)}\cdots\lesssim 1.
\end{aligned}$$

For the last "$\lesssim$", we need the following estimate.

For the first part, since $T-t\le T-s\le 2(T-t)$,

$$\begin{aligned}
&\int_{t-(T-t)}^{t}\lambda_j^{-2}(s)\int_{\frac{(t-s)^{\frac12}}{\lambda_j(s)}}^{\frac{1}{\lambda_j(s)}}z^{-4}\ln(1+z^2)dzds\lesssim\int_{t-(T-t)}^{t}\lambda_j^{-2}(t)\int_{c_2\frac{(t-s)^{\frac12}}{\lambda_j(t)}}^{\frac{c_1}{\lambda_j(t)}}z^{-4}\ln(1+z^2)dzds\\
&=\int_{t-(T-t)}^{t-\lambda_j^2(t)}+\int_{t-\lambda_j^2(t)}^{t}\cdots\lesssim 1
\end{aligned}$$

where $c_1,c_2>0$ are some constants and for the last "$\lesssim$", we use the following estimate.

$$\begin{aligned}
&\int_{t-(T-t)}^{t-\lambda_j^2(t)}\lambda_j^{-2}(t)\int_{c_2\frac{(t-s)^{\frac12}}{\lambda_j(t)}}^{\frac{c_1}{\lambda_j(t)}}z^{-4}\ln(1+z^2)dzds\lesssim\int_{t-(T-t)}^{t-\lambda_j^2(t)}\lambda_j^{-2}(t)\left(\frac{(t-s)^{\frac12}}{\lambda_j(t)}\right)^{-3}\ln\left(1+\left(\frac{(t-s)^{\frac12}}{\lambda_j(t)}\right)^2\right)ds\\
&\sim\int_1^{\frac{(T-t)^{\frac12}}{\lambda_j(t)}}y^{-2}\ln(1+y^2)dy\lesssim 1,
\end{aligned}$$

$$\int_{t-\lambda_j^2(t)}^{t}\lambda_j^{-2}(t)\int_{c_2\frac{(t-s)^{\frac12}}{\lambda_j(t)}}^{\frac{c_1}{\lambda_j(t)}}z^{-4}\ln(1+z^2)dzds\lesssim 1.$$

For the second part, since $\frac{T-s}{2} \le t-s \le T-s$,

$$\int_0^{t-(T-t)} \lambda_j^{-2}(s) \int_{\frac{(t-s)^{\frac12}}{\lambda_j(s)}}^{\frac{1}{\lambda_j(s)}} z^{-4}\ln(1+z^2)dzds \le \int_0^{t-(T-t)} \lambda_j^{-2}(s) \int_{\frac{(T-s)^{\frac12}}{\sqrt2\lambda_j(s)}}^{\frac{1}{\lambda_j(s)}} z^{-4}\ln(1+z^2)dzds$$

$$\lesssim \int_0^{t-(T-t)} \lambda_j^{-2}(s)(\frac{(T-s)^{\frac12}}{\sqrt2\lambda_j(s)})^{-3}\ln\left(1+(\frac{(T-s)^{\frac12}}{\sqrt2\lambda_j(s)})^2\right)ds$$

$$\sim \int_0^{t-(T-t)} \lambda_j(s)(T-s)^{-\frac32}\ln\left(1+(\frac{(T-s)^{\frac12}}{\sqrt2\lambda_j(s)})^2\right)ds \lesssim 1.$$

Next for $i=1,2$,

$$\left|\frac{\lambda_j(s)\left(w_i-\xi_i^{[j]}(s)\right)}{\left|w-\xi^{[j]}(s)\right|^2+\lambda_j^2(s)} - \frac{\lambda_j(s)\left(z_i-\xi_i^{[j]}(s)\right)}{\left|z-\xi^{[j]}(s)\right|^2+\lambda_j^2(s)}\right|$$

$$= \lambda_j(s)\left|\frac{\left(w_i-\xi_i^{[j]}(s)\right)\left(\left|z-\xi^{[j]}(s)\right|^2+\lambda_j^2(s)\right) - \left(z_i-\xi_i^{[j]}(s)\right)\left(\left|w-\xi^{[j]}(s)\right|^2+\lambda_j^2(s)\right)}{\left(|w-\xi^{[j]}(s)|^2+\lambda_j^2(s)\right)\left(|z-\xi^{[j]}(s)|^2+\lambda_j^2(s)\right)}\right|$$

$$\le |w-z|\,\lambda_j(s)\frac{\left|z-\xi^{[j]}(s)\right|^2+\lambda_j^2(s)+\left|z-\xi^{[j]}(s)\right|\left(\left|w-\xi^{[j]}(s)\right|+\left|z-\xi^{[j]}(s)\right|\right)}{\left(|w-\xi^{[j]}(s)|^2+\lambda_j^2(s)\right)\left(|z-\xi^{[j]}(s)|^2+\lambda_j^2(s)\right)}$$

$$\lesssim |w-z|\,\lambda_j(s)\left[\left(\left|w-\xi^{[j]}(s)\right|^2+\lambda_j^2(s)\right)^{-1}+\left(\left|z-\xi^{[j]}(s)\right|^2+\lambda_j^2(s)\right)^{-1}\right].$$

Thus we conclude that $\frac{\lambda_j(t)(x-\xi^{[j]}(t))}{|x-\xi^{[j]}(t)|^2+\lambda_j^2(t)} \in |\mathsf{DMO}|_{\mathsf{x}}$ by the same reason as (7.10).

□

By Lemma 7.1(2), since (4.21), then $\sum_{j=1}^{N}\Phi_0^{*[j]} \in |\mathsf{DMO}|_x$; for $\Phi_{\rm out} \in B_{\rm out}$ defined in (5.51), then $\Phi_{\rm out} \in |\mathsf{DMO}|_x$; $\eta_{d_q}^{[j]}(x,t) \in |\mathsf{DMO}|_x$.

By (5.44), one has

$$\left|\eta\left(\frac{x-\xi^{[j]}(s)}{\lambda_j(s)R(s)}\right)Q_{\gamma_j}\Phi_{\rm in}^{[j]}\left(\frac{x-\xi^{[j]}(s)}{\lambda_j(s)},s\right) - \eta\left(\frac{z-\xi^{[j]}(s)}{\lambda_j(s)R(s)}\right)Q_{\gamma_j}\Phi_{\rm in}^{[j]}\left(\frac{z-\xi^{[j]}(s)}{\lambda_j(s)},s\right)\right|$$
$$\lesssim \lambda_*^{-1}(s)|x-z|\|\Phi_{\rm in}\|^{[j]}_{{\rm in},\nu-\delta_0,l}\lambda_*^{\nu-\delta_0}(s).$$

Then for $|x-z|\le r$,

$$\fint_{Q_r^-(X)}\fint_{B_r(x)} |\Phi_{\rm in}(s,x)-\Phi_{\rm in}(s,z)|\,dxdzds$$
$$\lesssim r^{-2}\int_{t-r^2}^t r\lambda_*^{\nu-\delta_0-1}(s)ds \sim r^{-1}|\ln T|^{\nu-\delta_0-1}\int_{t-r^2}^t \frac{(T-s)^{\nu-\delta_0-1}}{|\ln(T-s)|^{2\nu-2\delta_0-2}}ds$$
$$\lesssim r^{-1}|\ln T|^{-(\nu-\delta_0-1)}\int_{t-r^2}^t (T-s)^{\nu-\delta_0-1}ds$$
$$\sim r^{-1}|\ln T|^{-(\nu-\delta_0-1)}\left[(T-t+r^2)^{\nu-\delta_0}-(T-t)^{\nu-\delta_0}\right]$$
$$\lesssim |\ln T|^{-(\nu-\delta_0-1)}r^{2\nu-2\delta_0-1}$$

which is Dini function when

$$\nu-\delta_0-\frac12 > 0. \tag{7.11}$$

In this case, $\Phi_{\rm in} \in |\mathsf{DMO}|_{\mathsf{x}}$.

In sum, by Lemma 7.1 and (5.1), under the assumption, we have (7.11).

$$\Phi \in |\mathsf{DMO}|_{\mathsf{x}}. \tag{7.12}$$

Recalling the definition (5.4) for A, by (4.2), (7.12),$|\Phi| \ll 1$ in (5.3) and Lemma 7.1, we have $A \in |\mathsf{DMO}|_{\mathsf{x}}$. By (5.5) and (5.3), we have $|A| \ll 1$. By the similar argument, the coefficients of $\mathbf{B}_{\Phi,U_*}$ defined in (5.26) belong to $|\mathsf{DMO}|_{\mathsf{x}}$.

8. Solving the gluing system

• Recall the space (5.51) for solving the outer problem, we choose $\Lambda_{\rm o} = 6\Lambda_{\rm o1}$, where $\Lambda_{\rm o1}$ is given in (5.39). In order to find a solution for (5.18), it is equivalent to find a fixed point for (5.37). One part of (5.37) has been handled in (5.39). For the remaining part, we define

$$\Phi_{\rm out}^{(2)} := \Gamma_{\Phi,U_*} ** \mathcal{G}[\Phi_{\rm out}] + \sum_{m=1}^{N}\sum_{n=1}^{3} c_{mn1}[\Phi, U_*, \mathcal{G}[\Phi_{\rm out}]] \left(\Gamma_{\Phi,U_*} * \vartheta_{mn}\right). \tag{8.1}$$

For $\Phi_{\rm in}^{[j]} \in B_{\rm in}^{[j]}$, $j = 1, 2, \ldots, N$, by Lemma D.1, (5.36), (5.33), (5.34), (5.35) and Proposition C.1 as well as the previous estimate (5.39), we finally have

$$\mathcal{T}_{\rm o}[\Phi_{\rm out}] = \Phi_{\rm out}^{(1)} + \Phi_{\rm out}^{(2)} \in B_{\rm out}. \tag{8.2}$$

The regularity of parabolic system with $\mathsf{DMO}_{\mathsf{x}}$ coefficients are guaranteed by [17, Theorem 1.2]. Then Schauder fixed point theorem implies the existence of (5.18).

• Due to the non-local feature at mode 0, we will only solve the non-local problem at leading order and leave the remainder to another piece of inner problem without orthogonality at mode 0. The smallness in the remainder is crucial to control the solution to the non-orthogonal inner problem, and this is the reason for the parameter restriction (6.15). See also Proposition 9.3 and Proposition 6.1. In order to simplify the calculation, we also put the mode 0 part of $\mathcal{H}_{\rm in}^{[j]}$ into the non-orthogonal version of the inner problem. More precisely, for $j \in \{1, \ldots, N\}$, (5.19) is split into the following two parts:

$$\begin{aligned}
\lambda_j^2 \partial_t \Phi_{\rm in}^{[j1]} =& \left(a - bW^{[j]}\wedge\right) \Big[\Delta_{y^{[j]}} \Phi_{\rm in}^{[j1]} + |\nabla_{y^{[j]}} W^{[j]}|^2 \Phi_{\rm in}^{[j1]} - 2\nabla_{y^{[j]}} \left(W^{[j]} \cdot \Phi_{\rm in}^{[j1]}\right) \nabla_{y^{[j]}} W^{[j]} \\
&+ 2\left(\nabla_{y^{[j]}} W^{[j]} \cdot \nabla_{y^{[j]}} \Phi_{\rm in}^{[j1]}\right) W^{[j]}\Big] + \mathcal{H}_1^{[j]} + \mathcal{H}_{\rm in}^{[j]} - \left(\left(\mathcal{H}_{\rm in}^{[j]}\right)_{\mathbb{C}_j,0}\right)_{\mathbb{C}_j^{-1}} \\
&+ \lambda_j^2 \left(c_0^{[j]}(\tau_j(t))\eta(|y^{[j]}|)\mathcal{Z}_{0,1}(|y^{[j]}|) + e^{i\theta_j} c_1^{[j]}(\tau_j(t))\eta(|y^{[j]}|)\mathcal{Z}_{1,1}(|y^{[j]}|)\right)_{\mathbb{C}_j^{-1}} \\
&- \lambda_j \left[\mathcal{R}_0[\Phi_{\rm out}](t) \left(\int_{B_2} \eta(y)\mathcal{Z}_{0,1}^2(y)dy\right)^{-1} \eta(|y^{[j]}|)\mathcal{Z}_{0,1}(|y^{[j]}|)\right]_{\mathbb{C}_j^{-1}} \quad \text{in } \mathsf{D}_{2R},
\end{aligned} \tag{8.3}$$

$$\begin{aligned}
\lambda_j^2 \partial_t \Phi_{\rm in}^{[j2]} =& \left(a - bW^{[j]}\wedge\right) \Big[\Delta_{y^{[j]}} \Phi_{\rm in}^{[j2]} + |\nabla_{y^{[j]}} W^{[j]}|^2 \Phi_{\rm in}^{[j2]} - 2\nabla_{y^{[j]}} \left(W^{[j]} \cdot \Phi_{\rm in}^{[j2]}\right) \nabla_{y^{[j]}} W^{[j]} \\
&+ 2\left(\nabla_{y^{[j]}} W^{[j]} \cdot \nabla_{y^{[j]}} \Phi_{\rm in}^{[j2]}\right) W^{[j]}\Big] + \left(\left(\mathcal{H}_{\rm in}^{[j]}\right)_{\mathbb{C}_j,0}\right)_{\mathbb{C}_j^{-1}} \\
&+ \lambda_j \left[\mathcal{R}_0[\Phi_{\rm out}](t) \left(\int_{B_2} \eta(y)\mathcal{Z}_{0,1}^2(y)dy\right)^{-1} \eta(|y^{[j]}|)\mathcal{Z}_{0,1}(|y^{[j]}|)\right]_{\mathbb{C}_j^{-1}} \quad \text{in } \mathsf{D}_{2R},
\end{aligned} \tag{8.4}$$

where $\mathcal{R}_0[\Phi_{\rm out}](t)$ is given in Proposition 6.1 with $m = \Theta - \alpha(1-\beta)$ given in (6.8); $c_0^{[j]}$, $c_1^{[j]}$ are given by proposition 9.1:

$$\begin{aligned}
c_0^{[j]}(\tau_j) = c_0^{[j]}[\mathcal{H}_1^{[j]}](\tau_j) =& -\left(\int_{B_2} \eta(y)\mathcal{Z}_{0,1}^2(y)dy\right)^{-1} \int_{B_{2R_0}} \left(\mathcal{H}_1^{[j]}\right)_{\mathbb{C}_j,0}(y,\tau_j)\mathcal{Z}_{0,1}(y)dy \\
&+ R_0^{-\epsilon_0} O\left(\tau_j^{-\nu} \left\|\left(\mathcal{H}_1^{[j]}\right)_{\mathbb{C}_j,0}\right\|_{\nu,l+2}\right) \quad \text{for } 0 < \epsilon_0 < l+1,
\end{aligned}$$

$$c_1^{[j]}(\tau_j) = c_1^{[j]}[\mathcal{H}_1^{[j]} + \mathcal{H}_{\rm in}^{[j]}](\tau_j) = -\left(\int_{B_2} \eta(y)\mathcal{Z}_{1,1}^2(y)dy\right)^{-1} \int_{B_{2R_0}} \left(\mathcal{H}_1^{[j]} + \mathcal{H}_{\rm in}^{[j]}\right)_{\mathbb{C}_j,1} (y,\tau_j)\mathcal{Z}_{1,1}(y)dy$$
$$+ R_0^{-\epsilon_1} O\left(\tau_j^{-\nu}\left\|\left(\mathcal{H}_1^{[j]} + \mathcal{H}_{\rm in}^{[j]}\right)_{\mathbb{C}_j,1}\right\|_{v,l+2}\right) \quad \text{for} \quad 0<\epsilon_1<l+1.$$

By Proposition 6.1, we can find $\lambda_j, \gamma_j, \xi^{[j]}$ satisfying (5.3) to make

$$\lambda_j c_0^{[j]}(\tau_j(t)) - \mathcal{R}_0[\Phi_{\rm out}](t)\left(\int_{B_2} \eta(y)\mathcal{Z}_{0,1}^2(y)dy\right)^{-1} = 0, \quad c_1^{[j]}(\tau_j(t)) = 0.$$

Applying Proposition 9.1 to (8.3) and Lemma 9.3 to (8.4), under the parameter requirements (D.89), (D.91), (D.93), $2\beta+\delta_0-\nu<0$ and (6.16) , we have

$$\Phi_{\rm in}^{[j1]} + \Phi_{\rm in}^{[j2]} \in B_{\rm in}^{[j]}.$$

The compactness of this mapping guaranteed by the similar reason as solving the outer problem.

Combining restrictions for the outer problem (5.48), (7.11), (D.31), (C.1), for the inner and nonlocal problems (5.45), (D.95), (6.16), we need to solve the following parameter inequality:

$$\begin{cases} \nu-\delta_0-\frac{1}{2}>0, \quad \Theta+\beta+\delta_0-\nu<0, \quad 3\beta<1+\Theta, \\ 0<\delta_0<\beta<\frac{1}{2}, \quad \beta(l+1)-1+\nu-\delta_0-\Theta>0, \quad \Theta+2\beta-1<0, \quad 2\beta+\delta_0-\nu<0, \\ 0<\Theta<\beta, \ 0<\alpha<1, \ \Theta+\frac{1}{2}-\beta-\frac{\alpha}{2}<0, \quad 0<\sigma_0, \\ \beta-\sigma_0-\frac{\alpha}{2}<0, \ 1-\sigma_0-(1+\alpha)(1-\beta)<0, \quad \Theta+2\sigma_0-\beta<0, \\ 0<\nu<1, \quad 0<l<1, \quad \nu+\beta l-1<0, \\ 0<\alpha_0<\frac{1}{2}, \quad 2\beta-1+\alpha_0>0, \\ 1+\Theta-\alpha(1-\beta)+\frac{(1+\alpha_0)\alpha}{2}-2\beta>\nu-\delta_0. \end{cases} \tag{8.5}$$

This can be solved by using Mathematica. Indeed, sound choices satisfying all the restrictions are given by

$$\begin{aligned} &0<\Theta<\frac{1}{4}, \quad \frac{1}{4}<\beta<\frac{1+\Theta}{4}, \quad 0<\sigma_0<\frac{\beta-\Theta}{2}, \quad 0<\delta_0<\frac{1-4\Theta}{4}, \\ &1-2\beta+\delta_0+\Theta<\nu<\frac{1}{4}\left(3-4\beta+4\delta_0+4\Theta\right), \quad \frac{1-\beta+\delta_0-\nu+\Theta}{\beta}<l<1 \\ &\max\left\{1-2\beta, 2\nu-2\delta_0+2\beta-1-2\Theta\right\}<\alpha_0<\frac{1}{2}, \\ &\max\left\{2\Theta+1-2\beta, 2\beta-2\sigma_0, \frac{\beta-\sigma_0}{1-\beta}, \frac{2(\nu-\delta_0+2\beta-1-\Theta)}{2\beta+\alpha_0-1}\right\}<\alpha<1. \end{aligned} \tag{8.6}$$

Therefore, we have solved the inner-outer gluing system (5.18) and (5.19).

9. Linear theory for the inner problem

In this section, we study the key linear theory for inner problem (5.19). Since this section is pretty independent of the other parts, we abuse R for more general cases. Recall (5.40), in time variable τ_j, (5.19) is the usual parabolic system. Since the inner problems for $j=1,2,\ldots,N$ all have the same structure and they are "localized", we omit the subscripts or superscripts "$_j$", "$_{[j]}$" in this section for brevity and all spatial derivative is about y. Consider

$$\begin{cases} \partial_\tau \Psi = (a-bW\wedge)\left(L_{\rm in}\Psi\right) + H \quad \text{in} \quad \mathcal{D}_R \\ \Psi(y,\tau)\cdot W(y) = H(y,\tau)\cdot W(y) = 0 \quad \text{in} \quad \mathcal{D}_R, \end{cases} \tag{9.1}$$

where

$$L_{\rm in}\mathbf{f} := \Delta\mathbf{f} + |\nabla W|^2\mathbf{f} - 2\nabla\left(W\cdot\mathbf{f}\right)\nabla W + 2\left(\nabla W\cdot\nabla\mathbf{f}\right)W, \tag{9.2}$$
$$\mathcal{D}_R := \left\{(y,\tau) \mid \tau\in(\tau_0,\infty), y\in B_{R(\tau)}\right\}, \quad B_R := \left\{y \mid |y|\le R(\tau)\right\}.$$

Throughout this section, we assume that $v(\tau), R(\tau), R_0(\tau) \in C^1(\tau_0, \infty)$ with the form

$$\begin{aligned}
&v(\tau) = a_0\tau^{a_1}(\ln\tau)^{a_2}(\ln\ln\tau)^{a_3}\cdots, \quad R(\tau) = b_0\tau^{b_1}(\ln\tau)^{b_2}(\ln\ln\tau)^{b_3}\cdots,\\
&R_0(\tau) = c_0\tau^{c_1}(\ln\tau)^{c_2}(\ln\ln\tau)^{c_3}\cdots, \quad v(\tau) > 0, \quad 1 \ll R_0(\tau) \ll R(\tau) \ll \tau^{\frac{1}{2}},\\
&v' = O(\tau^{-1}v), \quad R' = O(\tau^{-1}R), \quad R_0' = O(\tau^{-1}R_0),\\
&C_v^{-1}v(\tau) \le v(s) \le C_v v(\tau) \text{ for all } \tau \le s \le 2\tau, \text{ with a constant } C_v > 1, \quad \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds = O(\tau v(\tau)).
\end{aligned} \tag{9.3}$$

where $a_0, b_0, c_0 > 0$, $a_i, b_i, c_i \in \mathbb{R}$, $i = 1, 2, \ldots$ and R_0 is non-decreasing.

For brevity, we write $v = v(\tau)$, $R = R(\tau)$, $R_0 = R_0(\tau)$.

Suppose that $\Psi_{\mathbb{C}}(y,\tau)$, $H_{\mathbb{C}}(y,\tau)$ have the following Fourier expansion,

$$\begin{aligned}
\Psi_{\mathbb{C}}(y,\tau) &= \sum_{k\in\mathbb{Z}} \psi_k(\rho,\tau)e^{ik\theta}, \quad \psi_k(\rho,\tau) = (2\pi)^{-1}\int_0^{2\pi} \Psi_{\mathbb{C}}(\rho e^{i\theta},\tau)e^{-ik\theta}d\theta,\\
H_{\mathbb{C}}(y,\tau) &= \sum_{k\in\mathbb{Z}} h_k(\rho,\tau)e^{ik\theta}, \quad h_k(\rho,\tau) = (2\pi)^{-1}\int_0^{2\pi} H_{\mathbb{C}}(\rho e^{i\theta},\tau)e^{-ik\theta}d\theta.
\end{aligned} \tag{9.4}$$

Denote

$$\Psi_k = \left(\psi_k(\rho,\tau)e^{ik\theta}\right)_{\mathbb{C}^{-1}}, \quad H_k = \left(h_k(\rho,\tau)e^{ik\theta}\right)_{\mathbb{C}^{-1}} \quad \text{for } k \in \mathbb{Z}; \quad h_\top(y,\tau) := \sum_{|k|\ge 2} h_k(\rho,\tau)e^{ik\theta}. \tag{9.5}$$

It is easy to see

$$|\Psi_k| = |\psi_k|, \quad |H_k| = |h_k|. \tag{9.6}$$

For $\ell \in \mathbb{R}$ and $v(\tau) > 0$ and vectorial complex-valued function $\mathbf{f}$, the weighted topology are defined by

$$\begin{aligned}
&\|\mathbf{f}\|_{v,\ell}^{\mathcal{R}} := \sup_{(y,\tau)\in\mathcal{D}_{\mathcal{R}}} v^{-1}(\tau)\langle y\rangle^{\ell}|\mathbf{f}(y,\tau)|,\\
&[\mathbf{f}]_{\varsigma,v,\ell}^{\mathcal{R}} := \sup_{(y,\tau)\in\mathcal{D}_{\mathcal{R}}} v^{-1}(\tau)\langle y\rangle^{\ell+\varsigma} \sup_{(y_*,\tau_*)\in Q^-((y,\tau),\frac{|y|}{2})} \frac{|\mathbf{f}(y,\tau)-\mathbf{f}(y_*,\tau_*)|}{\left(|y-y_*|+|\tau-\tau_*|^{\frac{1}{2}}\right)^{\varsigma}}, \quad 0<\varsigma<1
\end{aligned}$$

where $\mathcal{R}$ is a scalar function which could depend on τ. By (9.6), we have

$$\|\psi_k\|_{v,\ell}^{\mathcal{R}} = \|\Psi_k\|_{v,\ell}^{\mathcal{R}}, \quad \|h_k\|_{v,\ell}^{\mathcal{R}} = \|H_k\|_{v,\ell}^{\mathcal{R}}. \tag{9.7}$$

The key inner linear theory is stated as follows.

Proposition 9.1. *Consider*

$$\begin{cases}
\partial_\tau\Psi = (a - bW\wedge)(L_{\rm in}\Psi) + H + \left(\sum_{k=0}^{1} e^{ik\theta}c_k(\tau)\eta(|y|)\mathcal{Z}_{k,1}(|y|)\right)_{\mathbb{C}^{-1}} & \text{in } \mathcal{D}_R,\\
\Psi_{\rm in}(y,\tau)\cdot W(y) = H(y,\tau)\cdot W(y) = 0 \quad \text{in } \mathcal{D}_R,
\end{cases} \tag{9.8}$$

where the Fourier expansion of H is given in (9.4). Suppose $\|h_0\|_{v,\ell}^R, \|h_1\|_{v,\ell}^R, \|h_{-1}\|_{v,\ell_{-1}}^R$, $\|h_\top\|_{v,\ell}^R, [H]_{\varsigma,v,\ell_H}^R < \infty$ with $1 < \ell < 3$, $\ell_{-1} > \frac{3}{2}$, $0 < \varsigma < 1$, $\ell_H > 0$, then there exists linear mappings $(\Psi_{\rm in}, c_0(\tau), c_1(\tau)) = (\mathcal{T}_{\rm in}[H], \mathcal{T}_{c_0}[h_0](\tau), \mathcal{T}_{c_1}[h_1](\tau))$ solving (9.8), where

$$c_k(\tau) = -\left(\int_{B_2}\eta(y)\mathcal{Z}_{k,1}^2(y)dy\right)^{-1}\int_{B_{2R_0}} h_k(y,\tau)\mathcal{Z}_{k,1}(y)dy + c_{*k}[h_k] \quad \textit{for} \ \ k = 0, 1$$

and $c_{*k}[h_k]$ *depends on* h_k *linearly and satisfies* $c_{*k}[h_k] \lesssim R_0^{-\epsilon_0} v(\tau)\|h_k\|_{v,\ell_k}^R$ *with* $0 < \epsilon_0 < \ell - 1$ *sufficiently small;* $\Psi_{\rm in}$ *satisfies the following estimate in* $\mathcal{D}_{R/2}$

$$
\begin{aligned}
&\langle y\rangle^2 \left|D^2\Psi\right| + \langle y\rangle \left|D\Psi\right| + |\Psi| \\
\lesssim\ & R_0^5 v(\tau)\langle y\rangle^{2-\ell}\|h_0\|_{v,\ell}^R + R_0^6 v(\tau)\langle y\rangle^{2-\ell}\|h_1\|_{v,\ell}^R + v(\tau)\langle y\rangle^{2-\ell}\|h_\top\|_{v,\ell}^R \\
&+ \|h_{-1}\|_{v,\ell_{-1}}^R \begin{cases} v(\tau)\tau^{1-\frac{\ell_{-1}}{2}} + \tau^{-\frac{\ell_{-1}}{2}} \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \frac{3}{2} < \ell_{-1} < 2 \\ v(\tau)(\ln\tau)^2 + \tau^{-1}\ln\tau \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ell_{-1} = 2 \\ v(\tau)\ln\tau + \tau^{-1}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ell_{-1} > 2 \end{cases} \\
&+ v(\tau)\langle y\rangle^{2-\ell_H}[H]_{\varsigma,v,\ell_H}^R .
\end{aligned}
\tag{9.9}
$$

The rest of this section is devoted to the proof of Proposition 9.1.

9.1. **Complex-valued form of the inner linear equation.** The following Lemma is applied for transforming the inner problem from the parabolic system into the complex-scalar form.

Lemma 9.1. *For* $L_{\rm in}$ *defined in* (9.2) *and* $\Psi \cdot W = 0$*, we have*

$$[(a - bW\wedge)(L_{\rm in}\Psi)] \cdot W = 0 \tag{9.10}$$

and

$$[(a - bW\wedge)(L_{\rm in}\Psi)]_{\mathbb{C}} = (a - ib)\mathcal{L}_{\rm in}\Psi_{\mathbb{C}}, \tag{9.11}$$

where

$$\mathcal{L}_{\rm in}\Psi_{\mathbb{C}} := \left[\partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_{\theta\theta} - \frac{1}{\rho^2} + i\frac{2\cos w(\rho)}{\rho^2}\partial_\theta + \frac{8}{(\rho^2+1)^2}\right]\Psi_{\mathbb{C}}. \tag{9.12}$$

Then (9.1) *is equivalent to the complex equation*

$$\partial_\tau\Psi_{\mathbb{C}} = (a - ib)\mathcal{L}_{\rm in}\Psi_{\mathbb{C}} + H_{\mathbb{C}} \quad \text{in } \ \mathcal{D}_R. \tag{9.13}$$

Under the Fourier expansion (9.4)*, then*

$$\mathcal{L}_{\rm in}\left(e^{ik\theta}\psi_k\right) = e^{ik\theta}\mathcal{L}_k\psi_k, \tag{9.14}$$

where

$$\mathcal{L}_k f := \partial_{\rho\rho} f + \frac{\partial_\rho f}{\rho} + V_k(\rho) f, \quad V_k(\rho) := -\frac{(k+1)^2\rho^4 + (2k^2-6)\rho^2 + (k-1)^2}{(\rho^2+1)^2}\frac{1}{\rho^2}. \tag{9.15}$$

It follows that

$$\partial_\tau\Psi_k = (a - bW\wedge)(L_{\rm in}\Psi_k) + H_k \tag{9.16}$$

is equivalent to

$$\partial_\tau\psi_k = (a - ib)\mathcal{L}_k\psi_k + h_k. \tag{9.17}$$

Proof. Set

$$\Psi(y,\tau) = \varphi_1(y,\tau)E_1(y) + \varphi_2(y,\tau)E_2(y), \quad \text{that is,} \quad \Psi_{\mathbb{C}} = \varphi_1 + i\varphi_2.$$

By (3.5), one has

$$
\begin{aligned}
\Delta(\varphi_1 E_1) &= (\Delta\varphi_1)E_1 + 2\nabla\varphi_1\nabla E_1 + \varphi_1\Delta E_1 \\
&= (\Delta\varphi_1)E_1 + 2\left(\partial_\rho\varphi_1\partial_\rho E_1 + \frac{1}{\rho^2}\partial_\theta\varphi_1\partial_\theta E_1\right) + \varphi_1\left(\partial_{\rho\rho} + \frac{\partial_\rho}{\rho} + \frac{\partial_{\theta\theta}}{\rho^2}\right)E_1 \\
&= (\Delta\varphi_1)E_1 - 2\partial_\rho\varphi_1 w_\rho W + \frac{2}{\rho^2}\partial_\theta\varphi_1\cos w E_2 \\
&\quad + \varphi_1\left[-w_{\rho\rho}W - w_\rho^2 E_1 - \frac{1}{\rho}w_\rho W - \frac{1}{\rho^2}\cos w(\sin wW + \cos wE_1)\right] \\
&= \left[\Delta\varphi_1 - \varphi_1\left(w_\rho^2 + \frac{\cos^2 w}{\rho^2}\right)\right]E_1 + \frac{2\cos w}{\rho^2}\partial_\theta\varphi_1 E_2 \\
&\quad + \left[-2w_\rho\partial_\rho\varphi_1 - \varphi_1\left(w_{\rho\rho} + \frac{w_\rho}{\rho} + \frac{\sin w\cos w}{\rho^2}\right)\right]W \\
&= \left(\partial_{\rho\rho}\varphi_1 + \frac{1}{\rho}\partial_\rho\varphi_1 + \frac{1}{\rho^2}\partial_{\theta\theta}\varphi_1 - \frac{1}{\rho^2}\varphi_1\right)E_1 + \frac{2\cos w}{\rho^2}\partial_\theta\varphi_1 E_2 \\
&\quad + \left(-2w_\rho\partial_\rho\varphi_1 - \frac{2\sin w\cos w}{\rho^2}\varphi_1\right)W
\end{aligned}
$$

where we have used $w_{\rho\rho} + \frac{w_\rho}{\rho} - \frac{\sin w\cos w}{\rho^2} = 0$ in the last equality above. Similarly,

$$
\begin{aligned}
\Delta(\varphi_2 E_2) &= \Delta(\varphi_2)E_2 + 2\nabla\varphi_2\nabla E_2 + \varphi_2\Delta E_2 \\
&= \Delta(\varphi_2)E_2 - 2\frac{1}{\rho^2}\partial_\theta\varphi_2(\sin wW + \cos wE_1) - \varphi_2\frac{1}{\rho^2}E_2 \\
&= -\frac{2\cos w}{\rho^2}\partial_\theta\varphi_2 E_1 + \left(\partial_{\rho\rho}\varphi_2 + \frac{1}{\rho}\partial_\rho\varphi_2 + \frac{1}{\rho^2}\partial_{\theta\theta}\varphi_2 - \frac{1}{\rho^2}\varphi_2\right)E_2 - \frac{2\sin w}{\rho^2}\partial_\theta\varphi_2 W.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Delta\Psi = &\left(\partial_{\rho\rho}\varphi_1 + \frac{1}{\rho}\partial_\rho\varphi_1 + \frac{1}{\rho^2}\partial_{\theta\theta}\varphi_1 - \frac{1}{\rho^2}\varphi_1 - \frac{2\cos w}{\rho^2}\partial_\theta\varphi_2\right)E_1 \\
&+ \left(\partial_{\rho\rho}\varphi_2 + \frac{1}{\rho}\partial_\rho\varphi_2 + \frac{1}{\rho^2}\partial_{\theta\theta}\varphi_2 - \frac{1}{\rho^2}\varphi_2 + \frac{2\cos w}{\rho^2}\partial_\theta\varphi_1\right)E_2 \\
&+ \left(-2w_\rho\partial_\rho\varphi_1 - \frac{2\sin w}{\rho^2}\partial_\theta\varphi_2 - \frac{2\sin w\cos w}{\rho^2}\varphi_1\right)W.
\end{aligned}
$$

By

$$
\begin{aligned}
&\partial_\rho W = w_\rho E_1, \quad \partial_\theta W = \sin wE_2, \quad \partial_\rho\Psi = (\partial_\rho\varphi_1)E_1 + (\partial_\rho\varphi_2)E_2 - \varphi_1 w_\rho W, \\
&\partial_\theta\Psi = (\partial_\theta\varphi_1)E_1 + (\partial_\theta\varphi_2)E_2 + \varphi_1\cos wE_2 - \varphi_2(\sin wW + \cos wE_1),
\end{aligned}
$$

we have

$$
\begin{aligned}
2(\nabla W\cdot\nabla\Psi)W &= 2\left(\partial_\rho W\cdot\partial_\rho\Psi + \frac{1}{\rho^2}\partial_\theta W\cdot\partial_\theta\Psi\right)W \\
&= 2\Big\{w_\rho E_1\cdot(\partial_\rho\varphi_1 E_1 + \partial_\rho\varphi_2 E_2 - w_\rho\varphi_1 W) \\
&\quad + \frac{1}{\rho^2}\sin wE_2\cdot[\partial_\theta\varphi_1 E_1 + \partial_\theta\varphi_2 E_2 + \varphi_1\cos wE_2 - \varphi_2(\sin wW + \cos wE_1)]\Big\}W \\
&= 2\left[w_\rho\partial_\rho\varphi_1 + \frac{1}{\rho^2}(\sin w\partial_\theta\varphi_2 + \sin w\cos w\varphi_1)\right]W \\
&= \left(\frac{2\sin w\cos w}{\rho^2}\varphi_1 + 2w_\rho\partial_\rho\varphi_1 + \frac{2\sin w}{\rho^2}\partial_\theta\varphi_2\right)W.
\end{aligned}
$$

Then

$$
\begin{aligned}
& (a-bW\wedge)\left(L_{\text{in}}\Psi\right) \\
= \ & (a-bW\wedge)\left[\left(\partial_{\rho\rho}\varphi_1+\frac{1}{\rho}\partial_\rho\varphi_1+\frac{1}{\rho^2}\partial_{\theta\theta}\varphi_1-\frac{1}{\rho^2}\varphi_1-\frac{2\cos w}{\rho^2}\partial_\theta\varphi_2\right)E_1\right. \\
& +\left(\partial_{\rho\rho}\varphi_2+\frac{1}{\rho}\partial_\rho\varphi_2+\frac{1}{\rho^2}\partial_{\theta\theta}\varphi_2-\frac{1}{\rho^2}\varphi_2+\frac{2\cos w}{\rho^2}\partial_\theta\varphi_1\right)E_2 \\
& +\left(-\frac{2\varphi_1}{\rho^2}\cos w\sin w-2\partial_\rho\varphi_1 w_\rho-\frac{2}{\rho^2}\partial_\theta\varphi_2\sin w\right)W+\frac{8}{(\rho^2+1)^2}\left(\varphi_1E_1+\varphi_2E_2\right) \\
& \left.+\left(\frac{2\varphi_1}{\rho^2}\cos w\sin w+2\partial_\rho\varphi_1 w_\rho+\frac{2}{\rho^2}\partial_\theta\varphi_2\sin w\right)W\right] \\
= \ & \left\{\partial_{\rho\rho}\varphi_1+\frac{1}{\rho}\partial_\rho\varphi_1+\frac{1}{\rho^2}\partial_{\theta\theta}\varphi_1+\left[\frac{8}{(\rho^2+1)^2}-\frac{1}{\rho^2}\right]\varphi_1-\frac{2\cos w}{\rho^2}\partial_\theta\varphi_2\right\}(aE_1-bE_2) \\
& +\left\{\partial_{\rho\rho}\varphi_2+\frac{1}{\rho}\partial_\rho\varphi_2+\frac{1}{\rho^2}\partial_{\theta\theta}\varphi_2+\left[\frac{8}{(\rho^2+1)^2}-\frac{1}{\rho^2}\right]\varphi_2+\frac{2\cos w}{\rho^2}\partial_\theta\varphi_1\right\}(aE_2+bE_1).
\end{aligned}
$$

Thus, we have (9.10) and (9.11). Then it follows that (9.1) is equivalent to (9.13).

It is straightforward to get (9.14) and (9.17). □

The linearly independent kernels $\mathcal{Z}_{k,1},\mathcal{Z}_{k,2}$ of $\mathcal{L}_k$ in (9.15) satisfying the Wronskian $W[\mathcal{Z}_{k,1},\mathcal{Z}_{k,2}]=\rho^{-1}$ are given as follows:

$$
\begin{cases}
\mathcal{Z}_{-1,1}(\rho)=\frac{\rho^2}{\rho^2+1}, & \mathcal{Z}_{-1,2}(\rho)=\frac{4\rho^2(\rho^2\ln(\rho)-1)-1}{4\rho^2(\rho^2+1)}, \qquad k=-1,\\
\mathcal{Z}_{0,1}(\rho)=\frac{\rho}{\rho^2+1}, & \mathcal{Z}_{0,2}(\rho)=\frac{\rho^4+4\rho^2\ln(\rho)-1}{2\rho(\rho^2+1)}, \qquad k=0,\\
\mathcal{Z}_{1,1}(\rho)=\frac{1}{\rho^2+1}, & \mathcal{Z}_{1,2}(\rho)=\frac{\rho^4+4\rho^2+4\ln(\rho)}{4(\rho^2+1)}, \qquad k=1,\\
\mathcal{Z}_{k,1}(\rho)=\frac{\rho^{1-k}}{\rho^2+1}, & \mathcal{Z}_{k,2}(\rho)=\frac{\rho^{k-1}}{\rho^2+1}\left(\frac{\rho^4}{2k+2}+\frac{\rho^2}{k}+\frac{1}{2k-2}\right), \qquad k\neq-1,0,1.
\end{cases} \tag{9.18}
$$

It is easy to see for $k\neq-1,0,1$,

$$
\mathcal{Z}_{k,1}(\rho)\sim\begin{cases}\rho^{1-k} & \text{if } \rho\to0\\ \rho^{-1-k} & \text{if } \rho\to\infty\end{cases}, \quad \mathcal{Z}_{k,2}(\rho)\sim\begin{cases}\frac{\rho^{k-1}}{2k-2} & \text{if } \rho\to0\\ \frac{\rho^{k+1}}{2k+2} & \text{if } \rho\to\infty\end{cases}.
$$

Recall (3.7) and (9.4) and notice $\mathcal{Z}_{0,1}(\rho)=-\frac{1}{2}\rho w_\rho$. Then for mode 0,

$$
(h_0(\rho,\tau))_{\mathbb{C}^{-1}}\cdot Z_{0,1}+i\,(h_0(\rho,\tau))_{\mathbb{C}^{-1}}\cdot Z_{0,2}=\rho w_\rho h_0(\rho,\tau)=-2\mathcal{Z}_{0,1}(\rho)h_0(\rho,\tau).
$$

Notice $\mathcal{Z}_{1,1}(\rho)=-\frac{1}{2}w_\rho$. For mode 1,

$$
\begin{aligned}
\left(h_1(\rho,\tau)e^{i\theta}\right)_{\mathbb{C}^{-1}}\cdot Z_{1,1}&=\mathrm{Re}(h_1(\rho,\tau)e^{i\theta})w_\rho\cos\theta+\mathrm{Im}(h_1(\rho,\tau)e^{i\theta})w_\rho\sin\theta,\\
\left(h_1(\rho,\tau)e^{i\theta}\right)_{\mathbb{C}^{-1}}\cdot Z_{1,2}&=\mathrm{Re}(h_1(\rho,\tau)e^{i\theta})w_\rho\sin\theta-\mathrm{Im}(h_1(\rho,\tau)e^{i\theta})w_\rho\cos\theta,
\end{aligned}
$$

whose equivalent complex form is given by

$$
\left(h_1(\rho,\tau)e^{i\theta}\right)_{\mathbb{C}^{-1}}\cdot Z_{1,1}-i\left(h_1(\rho,\tau)e^{i\theta}\right)_{\mathbb{C}^{-1}}\cdot Z_{1,2}=w_\rho h_1(\rho,\tau)=-2\mathcal{Z}_{1,1}(\rho)h_1(\rho,\tau).
$$

The quadratic form corresponding to $\mathcal{L}_k$ in B_R is defined as

$$
Q_{R,k}(f,f)=2\pi\int_0^R\left[|\partial_\rho f|^2+\frac{(k+1)^2\rho^4+(2k^2-6)\rho^2+(k-1)^2}{(\rho^2+1)^2}\frac{|f|^2}{\rho^2}\right]\rho d\rho. \tag{9.19}
$$

Specially,

$$
\begin{aligned}
Q_{R,0}(f,f)&=2\pi\int_0^R\left[|\partial_\rho f|^2+\frac{\rho^4-6\rho^2+1}{(\rho^2+1)^2}\frac{|f|^2}{\rho^2}\right]\rho d\rho=2\pi\int_0^R\left[|\partial_\rho f|^2+\frac{|f|^2}{\rho^2}-\frac{8}{(\rho^2+1)^2}|f|^2\right]\rho d\rho,\\
Q_{R,1}(f,f)&=2\pi\int_0^R\left[|\partial_\rho f|^2+\frac{4(\rho^2-1)}{(\rho^2+1)^2}|f|^2\right]\rho d\rho,\\
Q_{R,-1}(f,f)&=2\pi\int_0^R\left[|\partial_\rho f|^2+\frac{-4\rho^2+4}{(\rho^2+1)^2}\frac{|f|^2}{\rho^2}\right]\rho d\rho=2\pi\int_0^R\left[|\partial_\rho f|^2+4\frac{|f|^2}{\rho^2}-\frac{4\rho^2+12}{(\rho^2+1)^2}|f|^2\right]\rho d\rho.
\end{aligned}
$$

Define the following norms

$$\|f\|_{X(B_R)} = \left[2\pi \int_0^R \left(|\partial_\rho f|^2 + \frac{|f|^2}{\rho^2}\right)\rho d\rho\right]^{\frac{1}{2}},$$

$$\|f\|_{H_0^1(B_R)} = \left(2\pi \int_0^R |\partial_\rho f|^2 \rho d\rho\right)^{\frac{1}{2}}, \ \|f\|_{L^2(B_R)} = \left(2\pi \int_0^R |f|^2 \rho d\rho\right)^{\frac{1}{2}}.$$

Set

$$X_0(B_R) = \left\{ f(\rho) \;\middle|\; f(R) = 0, \quad \|f\|_{X(B_R)} < \infty \right\}.$$

There exists the Sobolev embedding $X_0 \to L^\infty$ by Schwartz inequality from in [26, p 216]:

$$\|f\|^2_{L^\infty(B_R)} \le \|\frac{f}{\rho}\|_{L^2(B_R)} \|\partial_\rho f\|_{L^2(B_R)} \le \|f\|^2_{X(B_R)} \quad \text{for} \quad f \in X_0. \tag{9.20}$$

For $|k| \ge 2$, it is easy to see

$$\|f\|^2_{X(B_R)} \le Q_{R,k}(f, f). \tag{9.21}$$

For any complex function f, $f = f_1 + if_2$ where f_1 and f_2 are real and imaginary parts. $Q_{R,k}(f, f) = Q_{R,k}(f_1, f_1) + Q_{R,k}(f_2, f_2)$. Thus we only need to consider the case that f is a real-valued function.

By [53, Lemma 4.2], $Q_{R,k}(f, f) \ge 0$ for all $f \in C^2(B_R) \cap C(\bar{B}_R)$ with $f = 0$ on ∂B_R and $Q_{R,k}(f, f) = 0$ implies $f \equiv 0$.

9.2. **Energy estimates.** The analysis about the first eigenvalue of $Q_{R,k}$ is given in the following Lemma.

Lemma 9.2. *Let*

$$\lambda_{R,k} = \inf_{f \in X_0(B_R)\setminus\{0\}} \frac{Q_{R,k}(f, f)}{\|f\|^2_{L^2(B_R)}} \quad \textit{for} \quad k \ne 1, \quad \lambda_{R,1} = \inf_{f \in H_0^1(B_R)\setminus\{0\}} \frac{Q_{R,1}(f, f)}{\|f\|^2_{L^2(B_R)}} \quad \textit{for} \quad k = 1.$$

$\lambda_{R,k}$ is attained by a real-valued function in $X_0(B_R)$ for $k \ne 1$ and $H_0^1(B_R)$ for $k = 1$. When R is large,

$$\lambda_{R,0} \sim (R^2 \ln R)^{-1}, \quad \lambda_{R,1} \sim R^{-4}, \quad \lambda_{R,-1} \gtrsim (R^2 \ln R)^{-1}, \quad \lambda_{R,k} \gtrsim |k|^2 R^{-2} \quad \textit{for} \quad |k| \ge 2.$$

Proof. For any complex-valued function $f = f_1 + if_2$ where $f_1 = \mathrm{Re}f$, $f_2 = \mathrm{Im}f$,

$$\frac{Q_{R,k}(f, f)}{\|f\|^2_{L^2(B_R)}} = \frac{Q_{R,k}(f_1, f_1) + Q_{R,k}(f_2, f_2)}{\|f_1\|^2_{L^2(B_R)} + \|f_2\|^2_{L^2(B_R)}} \ge \min\left\{\frac{Q_{R,k}(f_1, f_1)}{\|f_1\|^2_{L^2(B_R)}}, \frac{Q_{R,k}(f_2, f_2)}{\|f_2\|^2_{L^2(B_R)}}\right\}.$$

Thus for $k \ne 1$,

$$\lambda_{R,k} = \inf\left\{\frac{Q_{R,k}(f, f)}{\|f\|^2_{L^2(B_R)}} \mid f \in X_0(B_R)\setminus\{0\}, \text{ f is real-valued}\right\}.$$

The same argument can be applied to $\lambda_{R,1}$.

Thus, we focus on real-valued functions in the next step. We choose a sequence $f_n \in X_0(B_R)$ ($f_n \in H_0^1(B_R)$ if $k = 1$) with $\|f_n\|_{L^2(B_R)} = 1$ and $Q_{R,k}(f_n, f_n) \to \lambda_{R,k}$.

Without loss of generality, we assume $Q_{R,k}(f_n, f_n) \le \lambda_{R,k} + 1$. By the definition (9.19),

$$\int_0^R (\partial_\rho f_n)^2 \rho d\rho \lesssim \lambda_{R,k} + 1.$$

The Sobolev compact embedding theorem implies $f_n \to f_\infty$ in $L^2(B_R)$ up to a subsequence.

For $k \ne 1$,

$$Q_{R,k}(f_n, f_n) = 2\pi \int_0^R \left[(\partial_\rho f_n)^2 + \frac{(k-1)^2}{(\rho^2+1)^2} \frac{f_n^2}{\rho^2} + \frac{(k+1)^2\rho^2 + (2k^2-6)}{(\rho^2+1)^2} f_n^2\right] \rho d\rho.$$

Up to a subsequence, we have

$$\int_0^R \left[(\partial_\rho f_\infty)^2 + \frac{(k-1)^2}{(\rho^2+1)^2} \frac{f_\infty^2}{\rho^2}\right] \rho d\rho \le \liminf_{n\to\infty} \int_0^R \left[(\partial_\rho f_n)^2 + \frac{(k-1)^2}{(\rho^2+1)^2} \frac{f_n^2}{\rho^2}\right] \rho d\rho,$$

$$\int_0^R \frac{(k+1)^2\rho^2 + (2k^2-6)}{(\rho^2+1)^2} f_\infty^2 \rho d\rho = \lim_{n\to\infty} \int_0^R \frac{(k+1)^2\rho^2 + (2k^2-6)}{(\rho^2+1)^2} f_n^2 \rho d\rho.$$

Moreover,

$$\int_0^R \left[(\partial_\rho f_\infty)^2 + \frac{(k-1)^2}{(\rho^2+1)^2}\frac{f_\infty^2}{\rho^2}\right]\rho d\rho \sim C(R,k)\int_0^R \left[(\partial_\rho f_\infty)^2 + \frac{f_\infty^2}{\rho^2}\right]\rho d\rho.$$

Thus

$$Q_{R,k}(f_\infty, f_\infty) \le \lambda_{R,k}, \quad \|f_\infty\|_{L^2(B_R)} = 1, \quad f_\infty \in X_0(B_R),$$

which implies the minima $\lambda_{R,k}$ is attained by f_∞.

For $k=1$, similarly, we choose a subsequence such that $f_n \rightharpoonup f_\infty$ in $H_0^1(B_R)$, $f_n \to f_\infty$ in $L^2(B_R)$.

$$\int_0^R (\partial_\rho f_\infty)^2 \rho d\rho \le \liminf_{n\to\infty}\int_0^R (\partial_\rho f_n)^2\rho d\rho, \quad \int_0^R \frac{4(\rho^2-1)}{(\rho^2+1)^2} f_\infty^2 \rho d\rho = \lim_{n\to\infty}\int_0^R \frac{4(\rho^2-1)}{(\rho^2+1)^2} f_n^2\rho d\rho.$$

Then

$$Q_{R,1}(f_\infty, f_\infty) \le \lambda_{R,1}, \quad \|f_\infty\|_{L^2(B_R)} = 1, \quad f_\infty \in H_0^1(B_R).$$

Thus f_∞ attains $\lambda_{R,1}$.

Next, we will use Lagrange multiplier for the real-valued minimum function f_∞ to estimate $\lambda_{R,k}$, $k=-1,0,1$. In order to avoid confusion, we denote w_k as the eigenfunction corresponding to the eigenvalue $\lambda_{R,k}$ for every mode k with the normalization $\|w_k\|_{L^2(B_R)} = 1$.

For $k=0$,

$$\mathcal{L}_0 w_0 = -\lambda_{R,0} w_0 \ \text{ in } \ B_R, \quad w_0 = 0 \ \text{ on } \ \partial B_R.$$

w_0 is given by

$$w_0(\rho) = \mathcal{Z}_{0,2}(\rho)\int_0^\rho (-\lambda_{R,0} f(s))\mathcal{Z}_{0,1}(s)sds + \mathcal{Z}_{0,1}(\rho)\int_\rho^R (-\lambda_{R,0} f(s))\mathcal{Z}_{0,2}(s)sds - A_{R,0}\mathcal{Z}_{0,1}(\rho)$$

where

$$A_{R,0} = (\mathcal{Z}_{0,1}(R))^{-1}\mathcal{Z}_{0,2}(R)\int_0^R (-\lambda_{R,0} w_0(s))\mathcal{Z}_{0,1}(s)sds.$$

For $0 \le \rho \le 1$,

$$|\mathcal{Z}_{0,2}(\rho)\int_0^\rho w_0(s)\mathcal{Z}_{0,1}(s)sds| \lesssim \rho^{-1}\|w_0\|_{L^2(B_\rho)}\|\mathcal{Z}_{0,1}\|_{L^2(B_\rho)} \lesssim 1,$$

$$|\mathcal{Z}_{0,1}(\rho)\int_\rho^R w_0(s)\mathcal{Z}_{0,2}(s)sds| \le |\mathcal{Z}_{0,1}(\rho)\int_1^R w_0(s)\mathcal{Z}_{0,2}(s)sds| + |\mathcal{Z}_{0,1}(\rho)\int_\rho^1 w_0(s)\mathcal{Z}_{0,2}(s)sds| \lesssim R^2,$$

$$|A_{R,0}\mathcal{Z}_{0,1}(\rho)| \lesssim |A_{R,0}| \lesssim \lambda_{R,0} R^2(\ln R)^{\frac{1}{2}}.$$

Thus $\|w_0\|_{L^2(B_1)} \lesssim \lambda_{R,0}R^2(\ln R)^{\frac{1}{2}}$.

For $\rho \ge 1$,

$$\|\mathcal{Z}_{0,2}(\rho)\int_0^\rho w_0(s)\mathcal{Z}_{0,1}(s)sds\|_{L^2(B_R\setminus B_1)} \le \|\mathcal{Z}_{0,2}\|_{L^2(B_R\setminus B_1)}\|w_0\|_{L^2(B_R)}\|\mathcal{Z}_{0,1}\|_{L^2(B_R)} \lesssim R^2(\ln R)^{\frac{1}{2}},$$

$$\|\mathcal{Z}_{0,1}(\rho)\int_\rho^R w_0(s)\mathcal{Z}_{0,2}(s)sds\|_{L^2(B_R\setminus B_1)} \lesssim \|\mathcal{Z}_{0,1}\|_{L^2(B_R\setminus B_1)}\|w_0\|_{L^2(B_R\setminus B_1)}\|\mathcal{Z}_{0,2}\|_{L^2(B_R\setminus B_1)} \lesssim R^2(\ln R)^{\frac{1}{2}},$$

$$\|A_{R,0}\mathcal{Z}_{0,1}(\rho)\|_{L^2(B_R\setminus B_1)} \lesssim \lambda_{R,0}R^2\ln R.$$

Thus when R is large, we have

$$1 = \|w_0\|_{L^2(B_R)} \lesssim \lambda_{R,0}R^2\ln R.$$

On the other hand, when R is large,

$$\|\eta_{\frac{R}{2}}\mathcal{Z}_{0,1}\|_{L^2(B_R)}^2 \sim \ln R,$$

$$\begin{aligned} Q_{R,0}(\eta_{\frac{R}{2}}\mathcal{Z}_{0,1}, \eta_{\frac{R}{2}}\mathcal{Z}_{0,1}) &= \left(\int_0^{\frac{R}{2}} + \int_{\frac{R}{2}}^R\right)\left[(\partial_\rho(\eta_{\frac{R}{2}}\mathcal{Z}_{0,1}))^2 + \frac{\rho^4-6\rho^2+1}{(\rho^2+1)^2}\frac{(\eta_{\frac{R}{2}}\mathcal{Z}_{0,1})^2}{\rho^2}\right]\rho d\rho \\ = \int_{\frac{R}{2}}^\infty &\left[(\partial_\rho \mathcal{Z}_{0,1})^2 + \frac{\rho^4-6\rho^2+1}{(\rho^2+1)^2}\frac{(\mathcal{Z}_{0,1})^2}{\rho^2}\right]\rho d\rho + \int_{\frac{R}{2}}^R\left[(\partial_\rho(\eta_{\frac{R}{2}}\mathcal{Z}_{0,1}))^2 + \frac{\rho^4-6\rho^2+1}{(\rho^2+1)^2}\frac{(\eta_{\frac{R}{2}}\mathcal{Z}_{0,1})^2}{\rho^2}\right]\rho d\rho \\ \sim R^{-2}, \end{aligned}$$

where we used $\mathcal{L}\mathcal{Z}_{0,1} = 0$. Then we have

$$\lambda_{R,0} \lesssim (R^2\ln R)^{-1}.$$

For $k=1$,

$$\mathcal{L}_1 w_1 = -\lambda_{R,1} w_1 \quad \text{in} \quad B_R, \quad w_1 = 0 \quad \text{on} \quad \partial B_R.$$

w_1 can reformulated as

$$w_1(\rho) = \mathcal{Z}_{1,2}(\rho)\int_0^\rho (-\lambda_{R,1} w_1(s))\mathcal{Z}_{1,1}(s)sds + \mathcal{Z}_{1,1}(\rho)\int_\rho^R (-\lambda_{R,1} w_1(s))\mathcal{Z}_{1,2}(s)sds - A_{R,1}\mathcal{Z}_{1,1}(\rho), \tag{9.22}$$

where

$$A_{R,1} = (\mathcal{Z}_{1,1}(R))^{-1}\mathcal{Z}_{1,2}(R)\int_0^R (-\lambda_{R,1} w_1(s))\mathcal{Z}_{1,1}(s)sds.$$

For $0 \le \rho \le 1$,

$$|\mathcal{Z}_{1,2}(\rho)\int_0^\rho w_1(s)\mathcal{Z}_{1,1}(s)sds| \lesssim (|\ln\rho|+1)\|w_1\|_{L^2(B_\rho)}\|\mathcal{Z}_{1,1}\|_{L^2(B_\rho)} \lesssim 1,$$

$$|\mathcal{Z}_{1,1}(\rho)\int_\rho^R w_1(s)\mathcal{Z}_{1,2}(s)sds| \le |\mathcal{Z}_{1,1}(\rho)\int_1^R w_1(s)\mathcal{Z}_{1,2}(s)sds| + |\mathcal{Z}_{1,1}(\rho)\int_\rho^1 w_1(s)\mathcal{Z}_{1,2}(s)sds| \lesssim R^3,$$

$$|A_{R,1}\mathcal{Z}_{1,1}(\rho)| \lesssim |A_{R,1}| \lesssim \lambda_{R,1}R^4.$$

Thus $\|w_1\|_{L^2(B_1)} \lesssim \lambda_{R,1}R^4$.

For $\rho \ge 1$,

$$\|\mathcal{Z}_{1,2}(\rho)\int_0^\rho w_1(s)\mathcal{Z}_{1,1}(s)sds\|_{L^2(B_R\setminus B_1)} \le \|\mathcal{Z}_{1,2}\|_{L^2(B_R\setminus B_1)}\|w_1\|_{L^2(B_R)}\|\mathcal{Z}_{1,1}\|_{L^2(B_R)} \lesssim R^3,$$

$$\|\mathcal{Z}_{1,1}(\rho)\int_\rho^R w_1(s)\mathcal{Z}_{1,2}(s)sds\|_{L^2(B_R\setminus B_1)} \lesssim \|\mathcal{Z}_{1,1}\|_{L^2(B_R\setminus B_1)}\|w_1\|_{L^2(B_R\setminus B_1)}\|\mathcal{Z}_{1,2}\|_{L^2(B_R\setminus B_1)} \lesssim R^3,$$

$$\|A_{R,1}\mathcal{Z}_{1,1}(\rho)\|_{L^2(B_R\setminus B_1)} \lesssim \lambda_{R,1}R^4.$$

Thus when R is large, we have

$$1 = \|w_1\|_{L^2(B_R)} \lesssim \lambda_{R,1}R^4.$$

On the other hand, when R is large,

$$\|\eta_{\frac{R}{2}}\mathcal{Z}_{1,1}\|^2_{L^2(B_R)} \sim 1,$$

$$\begin{aligned} Q_{R,1}(\eta_{\frac{R}{2}}\mathcal{Z}_{1,1}, \eta_{\frac{R}{2}}\mathcal{Z}_{1,1}) &\sim \left(\int_0^{\frac{R}{2}} + \int_{\frac{R}{2}}^R\right)\left[(\partial_\rho(\eta_{\frac{R}{2}}\mathcal{Z}_{1,1}))^2 + \frac{4(\rho^2-1)}{(\rho^2+1)^2}(\eta_{\frac{R}{2}}\mathcal{Z}_{1,1})^2\right]\rho d\rho \\ &= \int_{\frac{R}{2}}^\infty \left[(\partial_\rho\mathcal{Z}_{1,1})^2 + \frac{4(\rho^2-1)}{(\rho^2+1)^2}(\mathcal{Z}_{1,1})^2\right]\rho d\rho + \int_{\frac{R}{2}}^R \left[(\partial_\rho(\eta_{\frac{R}{2}}\mathcal{Z}_{1,1}))^2 + \frac{4(\rho^2-1)}{(\rho^2+1)^2}(\eta_{\frac{R}{2}}\mathcal{Z}_{1,1})^2\right]\rho d\rho \\ &\sim R^{-4}, \end{aligned}$$

where we used $\mathcal{L}_1\mathcal{Z}_{1,1} = 0$. Thus

$$\lambda_{R,1} \lesssim R^{-4}.$$

For $k=-1$,

$$\mathcal{L}_{-1} w_{-1} = -\lambda_{R,-1} w_{-1} \quad \text{in} \quad B_R, \quad w_{-1} = 0 \quad \text{on} \quad \partial B_R.$$

$w_{-1}(\rho)$ can be written as

$$\begin{aligned} w_{-1}(\rho) = {} & \mathcal{Z}_{-1,2}(\rho)\int_0^\rho (-\lambda_{R,-1} w_{-1}(s))\mathcal{Z}_{-1,1}(s)sds + \mathcal{Z}_{-1,1}(\rho)\int_\rho^R (-\lambda_{R,-1} w_{-1}(s))\mathcal{Z}_{-1,2}(s)sds \\ & - A_{R,-1}\mathcal{Z}_{-1,1}(\rho) \end{aligned}$$

where

$$A_{R,-1} = (\mathcal{Z}_{-1,1}(R))^{-1}\mathcal{Z}_{-1,2}(R)\int_0^R (-\lambda_{R,-1} w_{-1}(s))\mathcal{Z}_{-1,1}(s)sds.$$

For $0 \le \rho \le 1$,

$$|\mathcal{Z}_{-1,2}(\rho)\int_0^\rho w_{-1}(s)\mathcal{Z}_{-1,1}(s)sds| \lesssim \rho^{-2}\|w_{-1}\|_{L^2(B_\rho)}\|\mathcal{Z}_{-1,1}\|_{L^2(B_\rho)} \lesssim 1,$$

$$
\begin{aligned}
&|\mathcal{Z}_{-1,1}(\rho)\int_\rho^R w_{-1}(s)\mathcal{Z}_{-1,2}(s)sds| \\
&\le |\mathcal{Z}_{-1,1}(\rho)\int_1^R w_{-1}(s)\mathcal{Z}_{-1,2}(s)sds| + |\mathcal{Z}_{-1,1}(\rho)\int_\rho^1 w_{-1}(s)\mathcal{Z}_{-1,2}(s)sds| \lesssim R\ln R,
\end{aligned}
$$

$$
|A_{R,-1}\mathcal{Z}_{-1,1}(\rho)| \lesssim |A_R| \lesssim \lambda_{R,-1}R\ln R.
$$

Thus $\|w_{-1}\|_{L^2(B_1)} \lesssim \lambda_{R,-1}R\ln R$.

For $\rho \ge 1$,

$$
\|\mathcal{Z}_{-1,2}(\rho)\int_0^\rho w_{-1}(s)\mathcal{Z}_{-1,1}(s)sds\|_{L^2(B_R\setminus B_1)} \le \|\mathcal{Z}_{-1,2}\|_{L^2(B_R\setminus B_1)}\|w_{-1}\|_{L^2(B_R)}\|\mathcal{Z}_{-1,1}\|_{L^2(B_R)} \lesssim R^2\ln R,
$$

$$
\|\mathcal{Z}_{-1,1}(\rho)\int_\rho^R w_{-1}(s)\mathcal{Z}_{-1,2}(s)sds\|_{L^2(B_R\setminus B_1)} \lesssim \|\mathcal{Z}_{-1,1}\|_{L^2(B_R\setminus B_1)}\|w_{-1}\|_{L^2(B_R\setminus B_1)}\|\mathcal{Z}_{-1,2}\|_{L^2(B_R\setminus B_1)} \lesssim R^2\ln R,
$$

$$
\|A_R\mathcal{Z}_{-1,1}(\rho)\|_{L^2(B_R\setminus B_1)} \lesssim \lambda_{R,-1}R^2\ln R.
$$

Thus when R is large, we have

$$
1 = \|w_{-1}\|_{L^2(B_R)} \lesssim \lambda_{R,-1}R^2\ln R.
$$

For $|k| \ge 2$,

$$
Q_{R,k}(f,f) \gtrsim 2\pi|k|^2\int_0^R \frac{|f|^2}{\rho^2}\rho d\rho \ge |k|^2R^{-2}\|f\|^2_{L^2(B_R)}.
$$

□

Lemma 9.3. *Consider*

$$
\begin{cases}\partial_\tau\phi_k = (a-ib)\mathcal{L}_k\phi_k + h & \text{in } \mathcal{D}_R,\\ \phi_k = 0 \ \text{ on } \partial\mathcal{D}_R, \quad \phi_k(\cdot,\tau_0)=0 & \text{in } B_{R(\tau_0)},\end{cases} \tag{9.23}
$$

where $R = R(\tau) \ge 1$. Assume $\|h(\cdot,\tau)\|^2_{L^2(B_R)} \lesssim g(\tau)$,

$$
\int_{\tau_0}^\tau e^{c\int^s \tilde\lambda_{R,k}}(\tilde\lambda_{R,k})^{-1}g(s)ds \lesssim e^{c\int^\tau\tilde\lambda_{R,k}}\min\left\{\tau,(\tilde\lambda_{R,k})^{-1}\right\}(\tilde\lambda_{R,k})^{-1}g(\tau),
$$

for any fixed constant $c>0$, where

$$
\tilde\lambda_{R,0} = (R^2\ln R)^{-1}, \quad \tilde\lambda_{R,1} = R^{-4}, \quad \tilde\lambda_{R,-1} = (R^2\ln R)^{-1}, \quad \tilde\lambda_{R,k} = |k|^2R^{-2},
$$

for $|k|\ge 2$. Then we have the following estimates

$$
\|\phi_k(\cdot,\tau)\|_{L^\infty(B_R)} \lesssim \left[\min\left\{\tau,(\tilde\lambda_{R,k})^{-1}\right\}(\tilde\lambda_{R,k})^{-1}g(\tau)\right]^{1/2} \quad \text{for } k\ne 1, \tag{9.24}
$$

$$
\|\phi_1(\cdot,\tau)\|_{H_0^1(B_R)} \lesssim \left[\min\left\{\tau,(\tilde\lambda_{R,1})^{-1}\right\}(\tilde\lambda_{R,1})^{-1}g(\tau)\right]^{1/2}. \tag{9.25}
$$

Furthermore, with the assumption $\|h(\cdot,\tau)\|^2_{L^1(B_R)} \lesssim l(\tau)$,

$$
\int_{\tau_0}^\tau e^{c\int^s\tilde\lambda_{R,k}}l(s)ds \lesssim e^{c\int^\tau\tilde\lambda_{R,k}}\min\left\{\tau,(\tilde\lambda_{R,k})^{-1}\right\}l(\tau) \quad \text{for } |k|\ge 2.
$$

Then we have

$$
\|\phi_k(\cdot,\tau)\|_{L^\infty(B_R)} \lesssim \left[\min\left\{\tau,(\tilde\lambda_{R,k})^{-1}\right\}l(\tau)+g(\tau)\right]^{1/2} \quad \text{for } |k|\ge 2, \tag{9.26}
$$

In particular, by (9.24) and (9.25), we have

$$
\begin{aligned}
\|\phi_0(\cdot,\tau)\|_{L^\infty(B_R)} &\lesssim R^2\ln R\theta_{R,\ell}v(\tau)\|h\|^R_{v,\ell}\\
\|\phi_{-1}(\cdot,\tau)\|_{L^\infty(B_R)} &\lesssim R^2\ln R\theta_{R,\ell}v(\tau)\|h\|^R_{v,\ell}\\
\|\phi_k(\cdot,\tau)\|_{L^\infty(B_R)} &\lesssim |k|^{-2}R^2\theta_{R,\ell}v(\tau)\|h\|^R_{v,\ell} \quad \text{for } |k|\ge 2\\
\|\phi_1(\cdot,\tau)\|_{H_0^1(B_R)} &\lesssim \min\{\tau^{\frac12},R^2\}R^2\theta_{R,\ell}v(\tau)\|h\|^R_{v,\ell}
\end{aligned}
$$

where

$$
\theta_{R,\ell} := \begin{cases}1 & \text{if } \ell>1\\ (\ln R)^{\frac12} & \text{if } \ell=1\\ R^{1-\ell} & \text{if } \ell<1\end{cases}. \tag{9.27}
$$

Proof. Multiplying $\bar{\phi}_k$ and integrating by parts, we have

$$\int_{B_R} \partial_\tau \phi_k \bar{\phi}_k + (a-ib) Q_{R,k}(\phi_k, \phi_k) = \int_{B_R} h\bar{\phi}_k.$$

We take the real part for both parts.

$$\frac{1}{2}\partial_\tau \int_{B_R} |\phi_k|^2 + aQ_{R,k}(\phi_k,\phi_k) = \int_{B_R} \mathrm{Re}\left(h\bar{\phi}_k\right). \tag{9.28}$$

By Lemma 9.2, we have

$$\partial_\tau \int_{B_R} |\phi_k|^2 + c\tilde{\lambda}_{R,k} \int_{B_R} |\phi_k|^2 \le 2\int_{B_R} |h||\phi_k|.$$

By Young inequality, we have

$$\partial_\tau \int_{B_R} |\phi_k|^2 + c\tilde{\lambda}_{R,k} \int_{B_R} |\phi_k|^2 \lesssim (\tilde{\lambda}_{R,k})^{-1} \int_{B_R} |h|^2 \lesssim (\tilde{\lambda}_{R,k})^{-1} g(\tau).$$

Since $\phi_k(\cdot,\tau_0) = 0$ in $B_{R(\tau_0)}$, we have

$$\int_{B_R} |\phi_k|^2 \lesssim \min\left\{\tau, (\tilde{\lambda}_{R,k})^{-1}\right\} (\tilde{\lambda}_{R,k})^{-1} g(\tau), \tag{9.29}$$

due to $\int_{\tau_0}^{\tau} e^{c\int^s \tilde{\lambda}_{R,k}} (\tilde{\lambda}_{R,k})^{-1} g(s) ds \lesssim e^{c\int^\tau \tilde{\lambda}_{R,k}} \min\left\{\tau, (\tilde{\lambda}_{R,k})^{-1}\right\} (\tilde{\lambda}_{R,k})^{-1} g(\tau)$.

Next we will estimate $Q_{R,k}(\phi_k,\phi_k)$. Integrating (9.28) from τ to $\tau+1$, we have

$$\frac{1}{2}\left(\int_{B_R} |\phi_k|^2(\tau+1) - \int_{B_R} |\phi_k|^2(\tau)\right) + a\int_\tau^{\tau+1} Q_{R,k}(\phi_k,\phi_k) = \int_\tau^{\tau+1}\int_{B_R} Re\left(h\bar{\phi}_k\right),$$

which derives

$$\int_\tau^{\tau+1} Q_{R,k}(\phi_k,\phi_k) \lesssim \min\left\{\tau, (\tilde{\lambda}_{R,k})^{-1}\right\} (\tilde{\lambda}_{R,k})^{-1} g(\tau).$$

So there exists $\tau_0 \in (\tau, \tau+1)$ such that

$$Q_{R,k}(\phi_k,\phi_k)(\tau_0) \lesssim \min\left\{\tau, (\tilde{\lambda}_{R,k})^{-1}\right\} (\tilde{\lambda}_{R,k})^{-1} g(\tau). \tag{9.30}$$

We multiply $\overline{\mathcal{L}_k \phi_k}$ to (9.23) and integrate by parts.

$$\begin{aligned} &-\int_{B_R} \nabla\partial_\tau\phi_k \cdot \nabla\bar{\phi}_k - \int_{B_R} \frac{(k+1)^2|y|^4 + (2k^2-6)|y|^2 + (k-1)^2}{(|y|^2+1)^2} \frac{\bar{\phi}_k \partial_\tau \phi_k}{|y|^2} \\ &= (a-ib)\int_{B_R} |\mathcal{L}_k\phi_k|^2 + \int_{B_R} \overline{\mathcal{L}_k\phi_k} h. \end{aligned}$$

Taking the real part of the equation, we have

$$\begin{aligned} &-\frac{1}{2}\partial_\tau \int_{B_R} |\nabla\phi_k|^2 - \frac{1}{2}\partial_\tau \int_{B_R} \frac{(k+1)^2|y|^4 + (2k^2-6)|y|^2 + (k-1)^2}{(|y|^2+1)^2} \frac{|\phi_k|^2}{|y|^2} \\ &= a\int_{B_R} |\mathcal{L}_k\phi_k|^2 + \int_{B_R} \mathrm{Re}\left(\overline{\mathcal{L}_k\phi_k} h\right). \end{aligned}$$

That is

$$\partial_\tau Q_{R,k}(\phi_k,\phi_k) = -2a\int_{B_R} |\mathcal{L}_k\phi_k|^2 - 2\int_{B_R} \mathrm{Re}\left(\overline{\mathcal{L}_k\phi_k} h\right). \tag{9.31}$$

By Young inequality, we derive

$$\partial_\tau Q_{R,k}(\phi_k,\phi_k) \lesssim \int_{B_R} |h|^2 \lesssim g(\tau). \tag{9.32}$$

Combing (9.30) and (9.32), we have

$$Q_{R,k}(\phi_k,\phi_k)(\tau+1) \lesssim \min\left\{\tau, (\tilde{\lambda}_{R,k})^{-1}\right\} (\tilde{\lambda}_{R,k})^{-1} g(\tau).$$

Since τ is arbitrary here, we get

$$Q_{R,k}(\phi_k,\phi_k)(\tau) \lesssim \min\left\{\tau, (\tilde{\lambda}_{R,k})^{-1}\right\} (\tilde{\lambda}_{R,k})^{-1} g(\tau) \quad \text{if} \quad \tau > \tau_0 + 1. \tag{9.33}$$

For $\tau_0 \le \tau \le \tau_0 + 1$, by (9.32) and $\phi_k(\cdot, \tau_0) = 0$ in $B_{R(\tau_0)}$, we have

$$Q_{R,k}(\phi_k, \phi_k)(\tau) \lesssim g(\tau). \tag{9.34}$$

Combing (9.20), (9.29) and (9.33), we have

$$\|\phi_k(\cdot,\tau)\|_{L^\infty(B_R)} \lesssim \|\phi_k(\cdot,\tau)\|_{X(B_R)} \lesssim \left[\min\left\{\tau, (\tilde\lambda_{R,k})^{-1}\right\}(\tilde\lambda_{R,k})^{-1} g(\tau)\right]^{1/2} \quad \text{for} \quad k \neq 1,$$

$$\|\phi_1(\cdot,\tau)\|_{H_0^1(B_R)} \lesssim \left[\min\left\{\tau, (\tilde\lambda_{R,1})^{-1}\right\}(\tilde\lambda_{R,1})^{-1} g(\tau)\right]^{1/2}.$$

For higher modes $|k| \ge 2$, we have another energy estimate.

$$\frac{1}{2}\partial_\tau \int_{B_R} |\phi_k|^2 + aQ_{R,k}(\phi_k,\phi_k) = \int_{B_R} \mathrm{Re}\left(h\bar\phi_k\right) \le \|h\|_{L^1(B_R)} \|\phi_k\|_{L^\infty(B_R)}.$$

Thanks to (9.21), $\|f\|^2_{X(B_R)} \le Q_{R,k}(f,f)$ for $|k| \ge 2$, combining (9.20) and Young inequality, we have

$$\frac{1}{2}\partial_\tau \int_{B_R} |\phi_k|^2 + \frac{a}{2} Q_{R,k}(\phi_k,\phi_k) \lesssim \|h\|^2_{L^1(B_R)}.$$

Then we have

$$\int_{B_R} |\phi_k|^2 \lesssim \min\left\{\tau, (\tilde\lambda_{R,k})^{-1}\right\} l(\tau) \tag{9.35}$$

since $\int_{\tau_0}^{\tau} e^{c\int^s \tilde\lambda_{R,k}} l(s) ds \lesssim e^{c\int^\tau \tilde\lambda_{R,k}} \min\left\{\tau, (\tilde\lambda_{R,k})^{-1}\right\} l(\tau)$ for $|k| \ge 2$.

By the same argument, we have

$$\frac{1}{2}\left(\int_{B_R} |\phi_k|^2(\tau+1) - \int_{B_R} |\phi_k|^2(\tau)\right) + a\int_\tau^{\tau+1} Q_{R,k}(\phi_k,\phi_k)$$
$$= \int_\tau^{\tau+1}\int_{B_R} Re\left(h\bar\phi_k\right) \le \int_\tau^{\tau+1} \|h\|_{L^1(B_R)}\|\phi_k\|_{L^\infty(B_R)}$$

which derives

$$\int_\tau^{\tau+1} Q_{R,k}(\phi_k,\phi_k) \lesssim \min\left\{\tau, (\tilde\lambda_{R,k})^{-1}\right\} l(\tau).$$

So there exists $\tau_0 \in (\tau, \tau+1)$ such that

$$Q_{R,k}(\phi_k,\phi_k)(\tau_0) \lesssim \min\left\{\tau, (\tilde\lambda_{R,k})^{-1}\right\} l(\tau).$$

Combing (9.32), we have

$$Q_{R,k}(\phi_k,\phi_k)(\tau+1) \lesssim \min\left\{\tau, (\tilde\lambda_{R,k})^{-1}\right\} l(\tau) + g(\tau).$$

By the same argument above, we have

$$Q_{R,k}(\phi_k,\phi_k)(\tau) \lesssim \min\left\{\tau, (\tilde\lambda_{R,k})^{-1}\right\} l(\tau) + g(\tau) \quad \text{if} \quad \tau \ge \tau_0.$$

Thus, by (9.20) and (9.21), we have

$$\|\phi_k(\cdot,\tau)\|_{L^\infty(B_R)} \lesssim \|\phi_k(\cdot,\tau)\|_{X(B_R)} \lesssim \left[\min\left\{\tau, (\tilde\lambda_{R,k})^{-1}\right\} l(\tau) + g(\tau)\right]^{1/2} \quad \text{for} \quad |k| \ge 2.$$

□

Consider

$$\begin{cases} \partial_\tau u = (a - ib)\Delta u + f, & \text{in } \mathbb{R}^2 \times (\tau_0, \infty) \\ u(x, \tau_0) = 0, & \text{in } \mathbb{R}^2. \end{cases} \tag{9.36}$$

The solution is given by the Duhamel's formula

$$u(x,\tau) = \int_{\tau_0}^{\tau}\int_{\mathbb{R}^2} \Gamma_d^\natural(x-z, t-s) f(z,s) dz ds. \tag{9.37}$$

Lemma 9.4. *Consider*

$$\partial_\tau\phi = (a-ib)\left(\partial_{\rho\rho}\phi + \frac{1}{\rho}\partial_\rho\phi - \frac{k^2}{\rho^2}\phi\right) + h(\rho,\tau)$$

where $k \ge 0$. Then we can find a solution ϕ given by

$$\phi(\rho,\tau) = \rho^k \Gamma^\natural_{2k+2} ** \left(\frac{h(y,s)}{|y|^k}\right). \tag{9.38}$$

Proof. Set

$$\phi(\rho,\tau) = \rho^k\psi(\rho,\tau).$$

Then

$$\partial_\tau\psi = (a-ib)\left(\partial_{\rho\rho}\psi + \frac{2k+1}{\rho}\partial_\rho\psi\right) + \frac{h(\rho,\tau)}{\rho^k}, \tag{9.39}$$

which can be regarded as the heat equation in $\mathbb{R}^{2k+2}$. Then ψ is given by

$$\psi(x,\tau) = \Gamma^\natural_{2k+2} ** \left(\frac{h(y,s)}{|y|^k}\right),$$

which satisfies (9.39) in weak sense and pointwise sense except at $\rho = 0$. (9.38) follows. □

Lemma 9.5. *For $d > 2$, $2 < \ell_* < d$ and $v_1 \ge 0$,*

$$\begin{aligned}\left(|\Gamma^\natural_d| ** \left(v_1(s)\langle z\rangle^{-\ell_*}\right)\right)(y,\tau,\tau_0) \lesssim\ & \mathbf{1}_{\{|y|\le\tau^{\frac12}\}}\left(\sup_{\tau_1\in[\tau/2,\tau]} v_1(\tau_1)\langle y\rangle^{2-\ell_*} + \tau^{-\frac{\ell_*}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v_1(s)ds\right)\\ &+ \mathbf{1}_{\{|y|>\tau^{\frac12}\}}\left(\tau\sup_{\tau_1\in[\tau/2,\tau]} v_1(\tau_1) + \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v_1(s)ds\right)|y|^{-\ell_*}\end{aligned} \tag{9.40}$$

where $c > 0$ depends on a.

Proof. Notice

$$v_1(s)\langle z\rangle^{-\ell_*} \sim v_1(s)\left(\mathbf{1}_{\{|z|\le1\}} + |z|^{-\ell_*}\mathbf{1}_{\{1<|z|\le\tau^{\frac12}\}} + |z|^{-\ell_*}\mathbf{1}_{\{|z|>\tau^{\frac12}\}}\right).$$

By [30, Lemma A.1],

$$|\Gamma^\natural_d| ** \left(v_1(s)\mathbf{1}_{\{|z|\le1\}}\right) \lesssim \tau^{-\frac{d}{2}}e^{-c\frac{|y|^2}{\tau}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v_1(s)ds + \sup_{\tau_1\in[\tau/2,\tau]} v_1(\tau_1)\left(\mathbf{1}_{\{|y|\le1\}} + \mathbf{1}_{\{|y|>1\}}|y|^{2-d}e^{-c\frac{|y|^2}{\tau}}\right),$$

and

$$\begin{aligned}|\Gamma^\natural_d| ** \left(v_1(s)|z|^{-\ell_*}\mathbf{1}_{\{1<|z|\le s^{\frac12}\}}\right) &\lesssim \tau^{-\frac{d}{2}}e^{-c\frac{|y|^2}{\tau}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v_1(s)s^{\frac{d-\ell_*}{2}}ds\\ &+ \sup_{\tau_1\in[\tau/2,\tau]} v_1(\tau_1)\left(\mathbf{1}_{\{|y|\le\tau^{\frac12}\}}\langle y\rangle^{2-\ell_*} + \mathbf{1}_{\{|y|>\tau^{\frac12}\}}|y|^{2-d}e^{-c\frac{|y|^2}{\tau}}\tau^{\frac{d-\ell_*}{2}}\right)\\ &\sim \mathbf{1}_{\{|y|\le\tau^{\frac12}\}}\left(\sup_{\tau_1\in[\tau/2,\tau]} v_1(\tau_1)\langle y\rangle^{2-\ell_*} + \tau^{-\frac{d}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v_1(s)s^{\frac{d-\ell_*}{2}}ds\right)\\ &+ \mathbf{1}_{\{|y|>\tau^{\frac12}\}}\left(\sup_{\tau_1\in[\tau/2,\tau]} v_1(\tau_1)|y|^{2-d}\tau^{\frac{d-\ell_*}{2}} + \tau^{-\frac{d}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v_1(s)s^{\frac{d-\ell_*}{2}}ds\right)e^{-c\frac{|y|^2}{\tau}}.\end{aligned}$$

By [30, Lemma A.2], we have

$$|\Gamma^\natural_d| ** \left(v_1(s)\langle z\rangle^{-\ell_*}\mathbf{1}_{\{|z|>s^{\frac12}\}}\right) \lesssim \left(\tau\sup_{\tau_1\in[\tau/2,\tau]} v_1(\tau_1) + \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v_1(s)ds\right)\left(\mathbf{1}_{\{|y|\le\tau^{\frac12}\}}\tau^{-\frac{\ell_*}{2}} + \mathbf{1}_{\{|y|>\tau^{\frac12}\}}|y|^{-\ell_*}\right).$$

Collecting above estimates, we conclude the validity of this Lemma. □

9.3. **Higher modes $|k| \geq 2$.** In order to analyze the case that the right hand side of the equation has singularity at $y = 0$, given $\mathcal{R} = \mathcal{R}(\tau)$, we introduce the norm

$$\|h\|_{v,\ell_1,\ell}^{\mathcal{R}} := \sup_{\mathcal{D}_{\mathcal{R}}} v(\tau)^{-1} \left(\mathbf{1}_{\{|y|\leq 1\}}|y|^{\ell_1} + \mathbf{1}_{\{|y|>1\}}|y|^{\ell}\right) |h(y,\tau)|.$$

Specially, if $\mathcal{R}(\tau) = \infty$, we use the notation $\|h\|_{v,\ell_1,\ell}^{\infty}$. It is easy to have $\|h\|_{v,0,\ell}^{\mathcal{R}} = \|h\|_{v,\ell}^{\mathcal{R}}$; $\|h\|_{v,\ell_1,\ell}^{\mathcal{R}} \leq \|h\|_{v,\ell}^{\mathcal{R}}$ if $\ell_1 > 0$; $\|h\|_{v,\ell_1,\ell}^{\mathcal{R}} \geq \|h\|_{v,\ell}^{\mathcal{R}}$ if $\ell_1 < 0$.

Lemma 9.6. *Consider*

$$\partial_\tau \Psi_k = (a - bW\wedge)\left(L_{\rm in}\Psi_k\right) + H_k \quad in \quad \mathcal{D}_R, \qquad \Psi_k(\cdot, \tau_0) = 0 \quad in \quad B_{R(\tau_0)}$$

where $H_k = \left(h_k(\rho,\tau)e^{ik\theta}\right)_{\mathbb{C}^{-1}}$, $\|H_k\|_{v,\ell_1,\ell}^{R} < \infty$, $0 \leq \ell_1 \leq 1.9$, $1 < \ell < 3$. *Then there exists a solution* $\Psi_k = \mathcal{T}_{kr}^{R}[H_k]$ *which is a linear mapping about* H_k *with the following estimate*

$$\langle y\rangle|\nabla\Psi_k(y,\tau)| + |\Psi_k(y,\tau)| \lesssim C(\ell)|k|^{-2}\|H_k\|_{v,\ell_1,\ell}^{R} v(\tau)R^{5-\ell}(\tau)\langle y\rangle^{-3}\ln(|y|+2), \tag{9.41}$$

where "$\lesssim$" is independent of k and $C(\ell)$ could be unbounded as $\ell \to 1$ or 3. Moreover, $\Psi_k \cdot W = 0$ *and* $e^{-ik\theta}\left(\Psi_k\right)_{\mathbb{C}}$ *is radial in space.*

Proof. For brevity, denote $\|h_k\| = \|h_k\|_{v,\ell_1,\ell}^{R}$ in this proof. Assume $h_k(\rho,\tau) = 0$ in $\mathcal{D}_R^c$. Consider

$$(a - bW\wedge)\left(L_{\rm in}G_k\right) = H_k \quad \text{where} \quad G_k = \left(g_k(\rho,\tau)e^{ik\theta}\right)_{\mathbb{C}^{-1}}.$$

By Lemma 9.1, it is equivalent to considering

$$(a - ib)\,\mathcal{L}_k g_k = h_k,$$

where g_k is given by

$$g_k(\rho,\tau) = (a+ib)\begin{cases} \mathcal{Z}_{k,2}(\rho)\int_0^\rho \mathcal{Z}_{k,1}(r)h_k(r,\tau)rdr + \mathcal{Z}_{k,1}(\rho)\int_\rho^\infty \mathcal{Z}_{k,2}(r)h_k(r,\tau)rdr & \text{if} \quad k \leq -2 \\ -\mathcal{Z}_{k,2}(\rho)\int_\rho^\infty \mathcal{Z}_{k,1}(r)h_k(r,\tau)rdr - \mathcal{Z}_{k,1}(\rho)\int_0^\rho \mathcal{Z}_{k,2}(r)h_k(r,\tau)rdr & \text{if} \quad k \geq 2 \end{cases}.$$

We will estimate the upper bound of g_k.

For $k \leq -2$, $\rho \leq 1$,

$$\begin{aligned} &\left|\mathcal{Z}_{k,2}(\rho)\int_0^\rho \mathcal{Z}_{k,1}(r)h_k(r,\tau)rdr\right| \lesssim |k|^{-1}\|h_k\|v(\tau)\rho^{k-1}\int_0^\rho r^{2-k-\ell_1}dr \\ &= |k|^{-1}\|h_k\|v(\tau)\rho^{k-1}\frac{\rho^{3-k-\ell_1}}{3-k-\ell_1} \sim |k|^{-2}\|h_k\|v(\tau)\rho^{2-\ell_1} \end{aligned}$$

for $0 \leq \ell_1 \leq 4$.

$$\begin{aligned} &\left|\mathcal{Z}_{k,1}(\rho)\int_\rho^\infty \mathcal{Z}_{k,2}(r)h_k(r,\tau)rdr\right| \lesssim \rho^{1-k}\|h_k\|v(\tau)\left(\int_\rho^1 |k|^{-1}r^{k-\ell_1}dr + \int_1^\infty |k|^{-1}r^{k+2-\ell}dr\right) \\ &\lesssim |k|^{-1}\rho^{1-k}\|h_k\|v(\tau)\left[\frac{\rho^{k-\ell_1+1}}{\ell_1-(k+1)} + \frac{1}{\ell-(k+3)}\right] \lesssim C_1(\ell)|k|^{-2}\|h_k\|v(\tau)\rho^{2-\ell_1} \end{aligned}$$

for $0 \leq \ell_1 \leq 4$ and $1 < \ell \leq 5$ where $C_1(\ell) \to \infty$ as $\ell \to 1$.

For $k \leq -2$, $\rho \geq 1$,

$$\begin{aligned} &\left|\mathcal{Z}_{k,2}(\rho)\int_0^\rho \mathcal{Z}_{k,1}(r)h_k(r,\tau)rdr\right| \lesssim |k|^{-1}\|h_k\|v(\tau)\rho^{k+1}\left(\int_0^1 r^{2-k-\ell_1}dr + \int_1^\rho r^{-\ell-k}dr\right) \\ &\lesssim |k|^{-1}\|h_k\|v(\tau)\rho^{k+1}\left(\frac{1}{3-k-\ell_1} + \frac{\rho^{1-k-\ell}}{1-k-\ell}\right) \lesssim C_2(\ell)|k|^{-2}\|h_k\|v(\tau)\rho^{2-\ell} \end{aligned}$$

for $0 \leq \ell_1 \leq 4$ and $0 \leq \ell < 3$ where $C_2(\ell) \to \infty$ as $\ell \to 3$.

$$\begin{aligned} &|\mathcal{Z}_{k,1}(\rho)\int_\rho^\infty \mathcal{Z}_{k,2}(r)h_k(r,\tau)rdr| \lesssim \rho^{-1-k}\|h_k\|v(\tau)\int_\rho^\infty |k|^{-1}r^{k+2-\ell}dr \\ &= |k|^{-1}\|h_k\|v(\tau)\frac{\rho^{2-\ell}}{\ell-(k+3)} \lesssim C_3(\ell)|k|^{-2}\|h_k\|v(\tau)\rho^{2-\ell} \end{aligned}$$

for $1 < \ell \leq 4$ where $C_3(\ell) \to \infty$ as $\ell \to 1$.

In sum, for $0 \le \ell_1 \le 4$, $1 < l < 3$, $k \le -2$,

$$\|g_k\|^{\infty}_{v,\ell_1-2,\ell-2} \lesssim C_4(\ell)|k|^{-2}\|h_k\| \tag{9.42}$$

where $C_4(\ell) \to \infty$ as $\ell \to 1$ or 3.

For $k \ge 2$, $\rho \le 1$, $0 \le \ell_1 \le 3$, $0 \le \ell \le 4$,

$$\left|\mathcal{Z}_{k,2}(\rho)\int_\rho^\infty \mathcal{Z}_{k,1}(r)h_k(r,\tau)rdr\right| \lesssim k^{-1}\|h_k\|v(\tau)\rho^{k-1}\left(\int_1^\infty r^{-k-\ell}dr + \int_\rho^1 r^{2-k-\ell_1}dr\right)$$
$$= k^{-1}\|h_k\|v(\tau)\rho^{k-1}\left(\frac{1}{k+\ell-1} + \frac{1-\rho^{3-k-\ell_1}}{3-k-\ell_1}\mathbf{1}_{\{\ell_1\neq 3-k\}} + (-\ln\rho)\mathbf{1}_{\{\ell_1=3-k\}}\right).$$

Then for $k \ge 4$,

$$\left|\mathcal{Z}_{k,2}(\rho)\int_\rho^\infty \mathcal{Z}_{k,1}(r)h_k(r,\tau)rdr\right| \lesssim k^{-1}\|h_k\|v(\tau)\rho^{k-1}\left(k^{-1} + \frac{\rho^{3-k-\ell_1}}{k+\ell_1-3}\right) \sim k^{-2}\|h_k\|v(\tau)\rho^{2-\ell_1}.$$

For $k = 3$, $\ell_1 = 0$,

$$\left|\mathcal{Z}_{k,2}(\rho)\int_\rho^\infty \mathcal{Z}_{k,1}(r)h_k(r,\tau)rdr\right| \lesssim \|h_k\|v(\tau)\rho^2\langle\ln\rho\rangle.$$

For $k = 3$, $0 < \ell_1 \le 3$,

$$|\mathcal{Z}_{k,2}(\rho)\int_\rho^\infty \mathcal{Z}_{k,1}(r)h_k(r,\tau)rdr| \lesssim \|h_k\|v(\tau)\rho^2\left(3^{-1} + \frac{\rho^{-\ell_1}-1}{\ell_1}\right) \lesssim \|h_k\|v(\tau)\rho^{2-\ell_1}\langle\ln\rho\rangle$$

since $\rho^t - 1 = t\rho^{\theta t}\ln\rho$ for some $0 \le \theta \le 1$.

For $k = 2$, $0 \le \ell_1 < 1$,

$$|\mathcal{Z}_{k,2}(\rho)\int_\rho^\infty \mathcal{Z}_{k,1}(r)h_k(r,\tau)rdr| \lesssim \|h_k\|v(\tau)\rho\left(2^{-1} + \frac{1-\rho^{1-\ell_1}}{1-\ell_1}\right) \lesssim \|h_k\|v(\tau)\rho\langle\ln\rho\rangle.$$

For $k = 2$, $\ell_1 = 1$,

$$|\mathcal{Z}_{k,2}(\rho)\int_\rho^\infty \mathcal{Z}_{k,1}(r)h_k(r,\tau)rdr| \lesssim \|h_k\|v(\tau)\rho(2^{-1} - \ln\rho) \sim \|h_k\|v(\tau)\rho\langle\ln\rho\rangle.$$

For $k = 2$, $1 < \ell_1 \le 3$,

$$|\mathcal{Z}_{k,2}(\rho)\int_\rho^\infty \mathcal{Z}_{k,1}(r)h_k(r,\tau)rdr| \lesssim \|h_k\|v(\tau)\rho\left(2^{-1} + \frac{\rho^{1-\ell_1}-1}{\ell_1-1}\right) \lesssim \|h_k\|v(\tau)\rho^{2-\ell_1}\langle\ln\rho\rangle.$$

For the other part,

$$\left|\mathcal{Z}_{k,1}(\rho)\int_0^\rho \mathcal{Z}_{k,2}(r)h_k(r,\tau)rdr\right| \lesssim \|h_k\|v(\tau)\rho^{1-k}\int_0^\rho k^{-1}r^{k-\ell_1}dr$$
$$= k^{-1}\|h_k\|v(\tau)\frac{\rho^{2-\ell_1}}{k+1-\ell_1} \sim k^{-2}\|h_k\|v(\tau)\rho^{2-\ell_1}$$

for $0 \le \ell_1 \le 2.9$.

For $k \ge 2$, $\rho \ge 1$,

$$\left|\mathcal{Z}_{k,2}(\rho)\int_\rho^\infty \mathcal{Z}_{k,1}(r)h_k(r,\tau)rdr\right| \lesssim k^{-1}\|h_k\|v(\tau)\rho^{k+1}\int_\rho^\infty r^{-k-\ell}dr$$
$$= k^{-1}\|h_k\|v(\tau)\frac{\rho^{2-\ell}}{k+\ell-1} \sim k^{-2}\|h_k\|v(\tau)\rho^{2-\ell},$$

when $0 \le \ell \le 4$.

$$\left|\mathcal{Z}_{k,1}(\rho)\int_0^\rho \mathcal{Z}_{k,2}(r)h_k(r,\tau)rdr\right| \lesssim \|h_k\|v(\tau)\rho^{-1-k}\left(\int_0^1 k^{-1}r^{k-\ell_1}dr + \int_1^\rho k^{-1}r^{k+2-\ell}dr\right)$$
$$\lesssim k^{-1}\|h_k\|v(\tau)\rho^{-1-k}\left(\frac{1}{k+1-\ell_1} + \frac{\rho^{k+3-\ell}}{k+3-\ell}\right) \sim k^{-2}\|h_k\|v(\tau)\rho^{2-\ell},$$

when $0 \le \ell_1 \le 2.9$, $0 \le \ell \le 4$.

In sum, for $0 \le \ell_1 \le 2.9$, $0 \le \ell \le 4$, when $k \ge 4$,

$$\|g_k\|^{\infty}_{v,\ell_1-2,\ell-2} \lesssim k^{-2}\|h_k\|; \tag{9.43}$$

when $k=3$,

$$\|g_3\|^{\infty}_{v,\epsilon+\ell_1-2,\ell-2}\lesssim C(\epsilon)\|h_k\|;\tag{9.44}$$

when $k=2$,

$$\|g_2\|^{\infty}_{v,\epsilon+(\ell_1-1)_+-1,\ell-2}\lesssim C(\epsilon)\|h_k\|\tag{9.45}$$

where $\epsilon>0$ is arbitrarily small and $C(\epsilon)$ is a constant depending on ϵ.

Combining (9.42), (9.43), (9.44) and (9.45), for $0\le\ell_1\le 1.9$, $1<\ell<3$, we have

$$\|G_k\|^{\infty}_{v,\ell-2}=\|g_k\|^{\infty}_{v,\ell-2}\lesssim C(\ell)|k|^{-2}\|h_k\|\quad\text{for}\quad |k|\ge 2\tag{9.46}$$

where $C(\ell)\to\infty$ as $\ell\to 1$ or 3.

Consider

$$\begin{cases}\partial_\tau\Phi_k=(a-bW\wedge)\left(L_{\text{in}}\Phi_k\right)+G_k\quad\text{in}\quad\mathcal{D}_{2R},\\ \Phi_k=0\quad\text{on}\quad\partial\mathcal{D}_{2R},\quad\Phi_k(\cdot,\tau_0)=0\quad\text{in}\quad B_{2R(\tau_0)}.\end{cases}\tag{9.47}$$

In order to find a solution Φ_k with the form $\Phi_k=\left(\phi_k(\rho,\tau)e^{ik\theta}\right)_{\mathbb{C}^{-1}}$, by Lemma 9.1, it suffices to consider

$$\begin{cases}\partial_\tau\phi_k=(a-ib)\,\mathcal{L}_k\phi_k+g_k\quad\text{in}\quad\mathcal{D}_{2R},\\ \phi_k=0\quad\text{on}\quad\partial\mathcal{D}_{2R},\quad\phi_k(\cdot,\tau_0)=0\quad\text{in}\quad B_{2R(\tau_0)}.\end{cases}\tag{9.48}$$

Recall (9.15). Set $\phi_k(\rho,\tau)=\rho^{|k-1|}\tilde\phi_k(\rho,\tau)$. Then

$$\begin{cases}\partial_\tau\tilde\phi_k=(a-ib)\left[\partial_{\rho\rho}\tilde\phi_k+(2|k-1|+1)\,\frac{\partial_\rho\tilde\phi_k}{\rho}+\frac{-4k\rho^2+8-4k}{(\rho^2+1)^2}\tilde\phi_k\right]+\frac{g_k}{\rho^{|k-1|}}\quad\text{in}\quad\mathcal{D}_{2R},\\ \tilde\phi_k=0\quad\text{on}\quad\partial\mathcal{D}_{2R},\quad\tilde\phi_k(\cdot,\tau_0)=0\quad\text{in}\quad B_{2R(\tau_0)}.\end{cases}\tag{9.49}$$

By changing the variable, it is easy to transform (9.49) into a parabolic system in the parabolic cylinder for which the spatial domain is independent of time. Then the existence follows by classical parabolic theory.

Applying Lemma 9.3 to (9.48), we have

$$\|\phi_k(\cdot,\tau)\|_{L^\infty(B_{2R(\tau)})}\lesssim|k|^{-2}v(\tau)R^{5-\ell}(\tau)\|g_k\|^{\infty}_{v,\ell-2}.\tag{9.50}$$

In order to improve the spatial decay of ϕ_k, we reformulate the equation (9.48) into the following form

$$\begin{cases}\partial_\tau\phi_k=(a-ib)\left[\partial_{\rho\rho}\phi_k+\frac{\partial_\rho\phi_k}{\rho}-\frac{(k+1)^2}{\rho^2}\phi_k\right]+\tilde g_k\quad\text{in}\quad\mathcal{D}_{2R},\\ \phi_k=0\quad\text{on}\quad\partial\mathcal{D}_{2R},\quad\phi_k(\cdot,\tau_0)=0\quad\text{in}\quad B_{2R(\tau_0)},\end{cases}\tag{9.51}$$

where $\tilde g_k=\tilde g_k(\rho,\tau):=(a-ib)\left[V_k+\frac{(k+1)^2}{\rho^2}\right]\phi_k+g_k=(a-ib)\frac{(4k+8)\rho^2+4k}{(\rho^2+1)^2}\frac{1}{\rho^2}\phi_k+g_k$.

Set $\phi_{*k}(y,\tau)=e^{i(k+1)\theta}\phi_k(\rho,\tau)$. Then (9.51) is equivalent to

$$\begin{cases}\partial_\tau\phi_{*k}=(a-ib)\Delta_{\mathbb{R}^2}\phi_{*k}+e^{i(k+1)\theta}\tilde g_k\quad\text{in}\quad\mathcal{D}_{2R},\\ \phi_{*k}=0\quad\text{on}\quad\partial\mathcal{D}_{2R},\quad\phi_{*k}(\cdot,\tau_0)=0\quad\text{in}\quad B_{2R(\tau_0)}.\end{cases}\tag{9.52}$$

(9.52) can be regarded as a real-valued parabolic system in varying time domain. Combining [19, Theorem 3.2] and [43, Lemma 2.26 and Remark 2.27][2], there exists a fundamental solution $\Gamma_2(x,y,t,s)$ for the homogeneous part of (9.52) with the estimate

$$|\Gamma_2(x,y,t,s)|\le N\,(t-s)^{-1}\,e^{-\frac{\kappa|x-y|^2}{t-s}}$$

and the positive constants N, κ are independent of t,s. Then by scaling argument, we have

$$|\nabla_y\Gamma_2(x,y,\tau,s)|\lesssim(t-s)^{-\frac{3}{2}}\,e^{-\frac{\kappa|x-y|^2}{t-s}}.\tag{9.53}$$

and ϕ_{*k} can be written as

$$\phi_{*k}(y,\tau)=\int_{\tau_0}^{\tau}\int_{B_{2R(s)}}\Gamma_2(y,z,\tau,s)e^{i(k+1)\theta(z)}\tilde g_k(|z|,s)dzds\tag{9.54}$$

where $\theta(z)=\arctan(\frac{z_2}{z_1})$.

In order to utilize the special form of $e^{i(k+1)\theta}\tilde g_k$, we set $\tilde g_k=0$ in $\mathcal{D}^c_{2R}$ and want to find $\tilde P_k(y,\tau)$ satisfying

$$\Delta_{\mathbb{R}^2}\tilde P_k(y,\tau)=e^{i(k+1)\theta}\tilde g_k\quad\text{in}\quad\mathbb{R}^2.\tag{9.55}$$

Set $\tilde P_k(y,\tau)=e^{i(k+1)\theta}\tilde p_k(\rho,\tau)$.

$$\partial_{\rho\rho}\tilde p_k+\frac{1}{\rho}\partial_\rho\tilde p_k-\frac{(k+1)^2}{\rho^2}\tilde p_k=\tilde g_k.$$

Set $\tilde{p}_k = \rho^{|k+1|}\tilde{p}_{k,1}(\rho,\tau)$. It is equivalent to considering

$$\partial_{\rho\rho}\tilde{p}_{k,1} + (2|k+1|+1)\frac{\partial_\rho \tilde{p}_{k,1}}{\rho} = \rho^{-|k+1|}\tilde{g}_k.$$

$\tilde{p}_{k,1}$ is given by

$$\tilde{p}_{k,1}(\rho,\tau) = -\rho^{-2|k+1|}\int_0^\rho u^{2|k+1|-1}\int_u^\infty r r^{-|k+1|}\tilde{g}_k(r,\tau)drdu.$$

Notice

$$|\tilde{g}_k| \lesssim \mathbf{1}_{\{r\le 2R(\tau)\}}\left[|k|\left(\rho^{-2}\mathbf{1}_{\{\rho\le 1\}} + \rho^{-4}\mathbf{1}_{\{\rho>1\}}\right)|\phi_k| + v(\tau)\langle\rho\rangle^{2-\ell}\|g_k\|^\infty_{v,\ell-2}\right].$$

Then

$$\begin{aligned}&|\tilde{p}_{k,1}| \lesssim \rho^{-2|k+1|}\int_0^\rho u^{2|k+1|-1}\int_u^\infty r\mathbf{1}_{\{r\le 2R(\tau)\}}\\ &\times\left[|k|\left(r^{-2-|k+1|}\mathbf{1}_{\{r\le1\}} + r^{-4-|k+1|}\mathbf{1}_{\{r>1\}}\right)|\phi_k| + v(\tau)\left(r^{-|k+1|}\mathbf{1}_{\{r\le1\}} + r^{2-\ell-|k+1|}\mathbf{1}_{\{r>1\}}\right)\|g_k\|^\infty_{v,\ell-2}\right]drdu.\end{aligned}$$

We estimate by Lemma A.1 that

$$\begin{aligned}&\rho^{-2|k+1|}\int_0^\rho u^{2|k+1|-1}\int_u^\infty \mathbf{1}_{\{r\le 2R(\tau)\}} r v(\tau)\left(r^{-|k+1|}\mathbf{1}_{\{r\le1\}} + r^{2-\ell-|k+1|}\mathbf{1}_{\{r>1\}}\right)\|g_k\|^\infty_{v,\ell-2}drdu\\ &\lesssim C(\ell)v(\tau)\|g_k\|^\infty_{v,\ell-2}\begin{cases}|k|^{-2}\left(\rho^{2-|k+1|}\mathbf{1}_{\{\rho\le1\}} + \rho^{4-\ell-|k+1|}\mathbf{1}_{\{\rho>1\}}\right) & \text{for } k\le -4,\ k\ge 2\\ R(\tau)\left(\langle\ln\rho\rangle\mathbf{1}_{\{\rho\le1\}} + \rho^{1-\ell}\mathbf{1}_{\{\rho>1\}}\right) & \text{for } k=-3\\ R^2(\tau)\left(\mathbf{1}_{\{\rho\le1\}} + \rho^{1-\ell}\mathbf{1}_{\{\rho>1\}}\right) & \text{for } k=-2\end{cases}\end{aligned}$$

where for the cases $k=-3,-2$, we have used

$$\mathbf{1}_{\{r\le 2R(\tau)\}}r^{2-\ell-|k+1|}\mathbf{1}_{\{r>1\}} \lesssim \begin{cases}R(\tau)r^{-1-\ell} & \text{for } k=-3\\ R^2(\tau)r^{-1-\ell} & \text{for } k=-2.\end{cases}$$

By (9.50) and Lemma A.1,

$$\begin{aligned}&\rho^{-2|k+1|}\int_0^\rho u^{2|k+1|-1}\int_u^\infty r\mathbf{1}_{\{r\le 2R(\tau)\}}|k|\left(r^{-2-|k+1|}\mathbf{1}_{\{r\le1\}} + r^{-4-|k+1|}\mathbf{1}_{\{r>1\}}\right)|\phi_k|drdu\\ &\lesssim |k|^{-1}v(\tau)R^{5-\ell}(\tau)\|g_k\|^\infty_{v,\ell-2}\begin{cases}|k|^{-2}\left(\rho^{-|k+1|}\mathbf{1}_{\{\rho\le1\}} + \rho^{-2-|k+1|}\mathbf{1}_{\{\rho>1\}}\right) & \text{for } k\le -4,\ k\ge 2\\ \rho^{-2}\mathbf{1}_{\{\rho\le1\}} + \rho^{-4}\langle\ln\rho\rangle\mathbf{1}_{\{\rho>1\}} & \text{for } k=-3\\ \rho^{-1}\mathbf{1}_{\{\rho\le1\}} + \rho^{-2}\mathbf{1}_{\{\rho>1\}} & \text{for } k=-2.\end{cases}\end{aligned}$$

$|\partial_\rho\tilde{p}_{k,1}|$ can also be bounded by Lemma A.1 similarly. As a result, for $\rho\le 2R(\tau)$,

$$\begin{aligned}&|k|^{-1}\rho|\partial_\rho\tilde{p}_{k,1}| + |\tilde{p}_{k,1}|\\ &\lesssim C(\ell)v(\tau)R^{5-\ell}(\tau)\|g_k\|^\infty_{v,\ell-2}\begin{cases}|k|^{-2}\left(\rho^{-|k+1|}\mathbf{1}_{\{\rho\le1\}} + \rho^{-1-|k+1|}\mathbf{1}_{\{\rho>1\}}\right), & k\le -4,\ k\ge 2\\ \rho^{-2}\mathbf{1}_{\{\rho\le1\}} + \rho^{-3}\mathbf{1}_{\{\rho>1\}}, & k=-3\\ \rho^{-1}\mathbf{1}_{\{\rho\le1\}} + \rho^{-2}\mathbf{1}_{\{\rho>1\}}, & k=-2.\end{cases}\end{aligned}$$

Notice $\tilde{P}_k(y,\tau) = e^{i(k+1)\theta}\rho^{|k+1|}\tilde{p}_{k,1}(\rho,\tau)$. Then

$$\begin{aligned}&\left|\nabla\tilde{P}_k\right| = \left(\left|\partial_\rho\tilde{P}_k\right|^2 + \rho^{-2}\left|\partial_\theta\tilde{P}_k\right|^2\right)^{\frac12} = \left(\left||k+1|\rho^{|k+1|-1}\tilde{p}_{k,1} + \rho^{|k+1|}\partial_\rho\tilde{p}_{k,1}\right|^2 + \rho^{-2}|k+1|^2\left|\rho^{|k+1|}\tilde{p}_{k,1}\right|^2\right)^{\frac12}\\ &\lesssim |k+1|\rho^{|k+1|-1}\left(|\tilde{p}_{k,1}| + |k+1|^{-1}\rho|\partial_\rho\tilde{p}_{k,1}|\right)\\ &\lesssim C(\ell)v(\tau)R^{5-\ell}(\tau)\|g_k\|^\infty_{v,\ell-2}\begin{cases}|k|^{-1}\left(\rho^{-1}\mathbf{1}_{\{\rho\le1\}} + \rho^{-2}\mathbf{1}_{\{\rho>1\}}\right), & k\le -4,\ k\ge 2\\ \rho^{-1}\mathbf{1}_{\{\rho\le1\}} + \rho^{-2}\mathbf{1}_{\{\rho>1\}}, & k=-3\\ \rho^{-1}\mathbf{1}_{\{\rho\le1\}} + \rho^{-2}\mathbf{1}_{\{\rho>1\}}, & k=-2\end{cases}\\ &\lesssim |k|^{-1}C(\ell)v(\tau)R^{5-\ell}(\tau)\left(\rho^{-1}\mathbf{1}_{\{\rho\le1\}} + \rho^{-2}\mathbf{1}_{\{\rho>1\}}\right)\|g_k\|^\infty_{v,\ell-2}.\end{aligned}$$

By (9.54) and (9.55), we have

$$\phi_k(y,\tau) = -e^{-i(k+1)\theta}\int_{\tau_0}^\tau\int_{B_{2R(s)}}\nabla_z\Gamma_2(y,z,\tau,s)\cdot\nabla\tilde{P}_k(z,s)dzds.$$

Combining (9.53), then

$$
\begin{aligned}
|\phi_k| &\lesssim \int_{\tau_0}^{\tau}\int_{B_{2R(s)}} (t-s)^{-\frac{3}{2}}\, e^{-\frac{\kappa|y-z|^2}{t-s}} \left|\nabla \tilde{P}_k(z,s)\right| dzds\\
&\lesssim |k|^{-1}C(\ell)\|g_k\|_{v,\ell-2}^{\infty}\int_{\tau_0}^{\tau}\int_{B_{2R(s)}} (t-s)^{-\frac{3}{2}}\, e^{-\frac{\kappa|y-z|^2}{t-s}} v(s)R^{5-\ell}(s)\left(|z|^{-2}\mathbf{1}_{\{|z|\le 1\}}+|z|^{-3}\mathbf{1}_{\{|z|>1\}}\right)|z|dzds,
\end{aligned}
$$

which can be estimated by similar convolution estimate in $\mathbb{R}^3$. By the same argument in [30, Lemma A.1], we have

$$
|\phi_k| \lesssim |k|^{-1}C(\ell)v(\tau)R^{5-\ell}(\tau)|y|^{-1}\ln(|y|+2)\|g_k\|_{v,\ell-2}^{\infty} \quad \text{for} \quad 1\le |y|\le \tau^{\frac{1}{2}}. \tag{9.56}
$$

Combining (9.50) and (9.56), we have

$$
|\Phi_k| = |\phi_k| \lesssim |k|^{-1}C(\ell)v(\tau)R^{5-\ell}(\tau)\langle y\rangle^{-1}\ln(|y|+2)\|g_k\|_{v,\ell-2}^{\infty}.
$$

Applying [17, Theorem 1.2] and scaling argument to (9.47), we have

$$
\langle y\rangle^2|D^2\Phi_k| + \langle y\rangle|D\Phi_k| + |\Phi_k| \lesssim C(\ell)v(\tau)R^{5-\ell}(\tau)\langle y\rangle^{-1}\ln(|y|+2)\|G_k\|_{v,\ell-2}^{\infty} \quad \text{in} \quad \mathcal{D}_{3R/2}. \tag{9.57}
$$

We take $\Psi_k = (a-bW\wedge)(L_{\text{in}}\Phi_k)$ and manipulate $(a-bW\wedge)L_{\text{in}}$ to (9.47). Combining (9.57), (9.46) and then scaling argument, we conclude (9.41). Recalling $\Phi_k = \left(\phi_k(\rho,\tau)e^{ik\theta}\right)_{\mathbb{C}^{-1}}$ and applying Lemma 9.1, we have $(\Psi_k)_{\mathbb{C}} = e^{ik\theta}(a-ib)\mathcal{L}_k\phi_k$.

□

Obviously, the Ψ_k given in Lemma 9.6 loses some power of $R(\tau)$ when $|y|$ is small. We will construct a Ψ_k with better estimate by another gluing procedure.

Proposition 9.2. *Consider*

$$
\partial_\tau\Psi_k = (a-bW\wedge)(L_{\text{in}}\Psi_k) + H_k \quad \text{in} \quad \mathcal{D}_R, \quad \Psi_k(\cdot,\tau_0)=0 \quad \text{in} \quad B_{R(\tau_0)}
$$

where $H_k = \left(h_k(\rho,\tau)e^{ik\theta}\right)_{\mathbb{C}^{-1}}$, $\|H_k\|_{v,\ell}^{R} < \infty$, $1<\ell<3$. There exists a solution $\Psi_k = \mathcal{T}_k^R[H_k]$ as a linear mapping about H_k with the following estimate

$$
\langle y\rangle|\nabla\Psi_k(y,\tau)| + |\Psi_k(y,\tau)| \lesssim C(\ell)|k|^{-2}v(\tau)\langle y\rangle^{2-\ell}\|H_k\|_{v,\ell}^{R}
$$

where $C(\ell)$ is given in Lemma 9.6. Moreover, $\Psi_k\cdot W = 0$ and $e^{-ik\theta}(\Psi_k)_{\mathbb{C}}$ is radial in space.

Proof. Denote $\|h_k\| = \|h_k\|_{v,\ell}^{R}$ and take $h_k=0$ in $\mathcal{D}_R^c$. By Lemma 9.1, it is equivalent to considering

$$
\partial_\tau\psi_k = (a-ib)\mathcal{L}_k\psi_k + h_k \quad \text{in} \quad \mathcal{D}_R. \tag{9.58}
$$

Set $\psi_k(\rho,\tau) = \eta_{R_0}(\rho)\psi_{i,k}(\rho,\tau) + \psi_{o,k}(\rho,\tau)$, where $\eta_{R_0}(\rho) = \eta(\frac{\rho}{R_0})$ and R_0 is a large fixed constant independent of τ_0, τ, k. In order to find a solution for (9.58), it suffices to consider the following inner-outer system

$$
\begin{cases}
\partial_\tau\psi_{o,k} = (a-ib)\left[\partial_{\rho\rho}\psi_{o,k} + \frac{1}{\rho}\partial_\rho\psi_{o,k} - \frac{(k+1)^2}{\rho^2}\psi_{o,k}\right] + J[\psi_{o,k},\psi_{i,k}]\mathbf{1}_{\{\rho\le 4R(\tau)\}} \quad \text{in} \quad \mathbb{R}^2\times(\tau_0,\infty),\\
\psi_{o,k}(\cdot,\tau_0) = 0 \quad \text{in} \quad \mathbb{R}^2.
\end{cases} \tag{9.59}
$$

$$
\partial_\tau\psi_{i,k} = (a-ib)\mathcal{L}_k\psi_{i,k} + K[\psi_{o,k}] \quad \text{in} \quad \mathcal{D}_{2R_0}, \quad \psi_{i,k}(\cdot,\tau_0) = 0 \quad \text{in} \quad B_{2R(\tau_0)} \tag{9.60}
$$

where

$$
\begin{aligned}
J[\psi_{o,k},\psi_{i,k}] &:= (a-ib)(1-\eta_{R_0})\left[\frac{(k+1)^2}{\rho^2} + V_k(\rho)\right]\psi_{o,k} + A_0[\psi_{i,k}] + (1-\eta_{R_0})h_k\\
&= (a-ib)(1-\eta_{R_0})\frac{(4k+8)\rho^2+4k}{(\rho^2+1)^2\rho^2}\psi_{o,k} + A_0[\psi_{i,k}] + (1-\eta_{R_0})h_k,\\
K[\psi_{o,k}] &:= (a-ib)\left[\frac{(k+1)^2}{\rho^2} + V_k(\rho)\right]\psi_{o,k} + h_k = (a-ib)\frac{(4k+8)\rho^2+4k}{(\rho^2+1)^2\rho^2}\psi_{o,k} + h_k,\\
A_0[\psi_{i,k}] &:= (a-ib)\left[\left(\partial_{\rho\rho}\eta_{R_0} + \frac{1}{\rho}\partial_\rho\eta_{R_0}\right)\psi_{i,k} + 2\partial_\rho\eta_{R_0}\partial_\rho\psi_{i,k}\right].
\end{aligned}
$$

Set $\Psi_{i,k}(y,\tau) = \left(\psi_{i,k}(\rho,\tau)e^{ik\theta}\right)_{\mathbb{C}^{-1}}$, that is, $\psi_{i,k} = e^{-ik\theta}(\Psi_{i,k}\cdot E_1 + i\Psi_{i,k}\cdot E_2)$. By Lemma 9.1, (9.60) is equivalent to

$$
\partial_\tau\Psi_{i,k} = (a-bW\wedge)L_{\text{in}}\Psi_{i,k} + \left(K[\psi_{o,k}]e^{ik\theta}\right)_{\mathbb{C}^{-1}} \quad \text{in} \quad \mathcal{D}_{2R_0}, \quad \Psi_{i,k}(\cdot,\tau_0) = 0 \quad \text{in} \quad B_{2R(\tau_0)}. \tag{9.61}
$$

The linear theories of (9.59) and (9.61) are given by Lemma 9.4 and Lemma 9.6 respectively and we reformulate (9.59) and (9.61) into the following form

$$\begin{aligned}\psi_{o,k}(\rho,\tau) &= \rho^{|k+1|}\left[\Gamma^{\natural}_{2|k+1|+2} * \left(|z|^{-|k+1|}J[\psi_{o,k},\psi_{i,k}]\right)\right](\rho,\tau,\tau_0),\\ \Psi_{i,k}(y,\tau) &= \mathcal{T}^{2R_0}_{kr}\left[\left(K[\psi_{o,k}]e^{ik\theta}\right)_{\mathbb{C}^{-1}}\right].\end{aligned} \tag{9.62}$$

We will solve $(\psi_{o,k},\Psi_{i,k})$ for (9.62) by the contraction mapping theorem.

By Lemma 9.6,

$$\langle y\rangle\left|\nabla\mathcal{T}^{2R_0}_{kr}\left[\left(h_k e^{ik\theta}\right)_{\mathbb{C}^{-1}}\right]\right| + \left|\mathcal{T}^{2R_0}_{kr}\left[\left(h_k e^{ik\theta}\right)_{\mathbb{C}^{-1}}\right]\right| \le D_i w_{i,k}(\rho,\tau)\|h_k\|,$$

where the constant $D_i \ge 1$ is independent of k; for $C(\ell)$ given in Lemma 9.6,

$$w_{i,k}(\rho,\tau) := C(\ell)|k|^{-2}v(\tau)R_0^{5-\ell}\ln R_0\langle\rho\rangle^{-3}.$$

Denote

$$\begin{aligned}\mathcal{B}_{i,k} := \Big\{&F(y,\tau)\in C^1\left(B_{2R_0},\mathbb{R}^3\right)\ \mid\ F(y,\tau) = \left(e^{ik\theta}f(\rho,\tau)\right)_{\mathbb{C}^{-1}}\quad\text{for some radial scalar function}\\ &f(\rho,\tau)\ \text{ and }\ \langle y\rangle|\nabla_y F(y,\tau)| + |F(y,\tau)| \le 2D_i w_{i,k}(\rho,\tau)\|h_k\|\Big\}.\end{aligned}$$

For any $\tilde{\Psi}_{i,k}\in\mathcal{B}_{i,k}$, denote $\tilde{\psi}_{i,k} = e^{-ik\theta}\left(\tilde{\Psi}_{i,k}\cdot E_1 + i\tilde{\Psi}_{i,k}\cdot E_2\right)$. We will find a solution $\psi_{o,k} = \psi_{o,k}[\tilde{\psi}_{i,k}]$ of (9.59) by the contraction mapping theorem. Let us estimate $J[\psi_{o,k},\tilde{\psi}_{i,k}]$ term by term. Notice

$$\begin{aligned}\left|\partial_\rho\tilde{\psi}_{i,k}\right| &= \left|e^{-ik\theta}\left(\tilde{\Psi}_{i,k}\cdot\partial_\rho E_1 + E_1\cdot\partial_\rho\tilde{\Psi}_{i,k} + i\tilde{\Psi}_{i,k}\cdot\partial_\rho E_2 + iE_2\cdot\partial_\rho\tilde{\Psi}_{i,k}\right)\right|\\ &\lesssim |\tilde{\Psi}_{i,k}|\langle\rho\rangle^{-2} + |\partial_\rho\tilde{\Psi}_{i,k}| \lesssim D_i C(\ell)|k|^{-2}v(\tau)R_0^{5-\ell}\ln R_0\langle\rho\rangle^{-4}\|h_k\|.\end{aligned}$$

Then

$$\begin{aligned}|A_0[\tilde{\psi}_{i,k}]| &= \left|\left(\partial_{\rho\rho}\eta_{R_0} + \frac{1}{\rho}\partial_\rho\eta_{R_0}\right)\tilde{\psi}_{i,k}(\rho,\tau) + 2\partial_\rho\eta_{R_0}\partial_\rho\tilde{\psi}_{i,k}(\rho,\tau)\right|\\ &\lesssim D_iC(\ell)\mathbf{1}_{\{R_0\le\rho\le 2R_0\}}|k|^{-2}v(\tau)R_0^{-\ell}\ln R_0\|h_k\| \lesssim D_iC(\ell)|k|^{-2}R_0^{\tilde{\ell}-\ell}\ln R_0 v(\tau)\langle\rho\rangle^{-\tilde{\ell}}\|h_k\|,\end{aligned}$$

where $1<\tilde{\ell}<\ell$.

$$|(1-\eta_{R_0})h_k| \lesssim \mathbf{1}_{\{\rho\ge R_0\}}v(\tau)\langle\rho\rangle^{-\ell}\|h_k\| \lesssim R_0^{\tilde{\ell}-\ell}v(\tau)\langle\rho\rangle^{-\tilde{\ell}}\|h_k\|.$$

By Lemma A.2,

$$\begin{aligned}&\rho^{|k+1|}\left|\Gamma^{\natural}_{2|k+1|+2}\right| * \left(v(s)|z|^{-|k+1|}\langle z\rangle^{-\tilde{\ell}}\right)\\ &\lesssim w_{o,k}(\rho,\tau) := \min\left\{|k|^{-2}v(\tau)\left(\rho\mathbf{1}_{\{\rho\le 1\}} + \rho^{2-\tilde{\ell}}\mathbf{1}_{\{1<\rho\le C_1\tau^{\frac12}\}}\right),\rho^{-\tilde{\ell}}\int_{\tau_0}^{\tau}v(s)ds\right\}\end{aligned}$$

where C_1 is a constant independent of k. The spatial decay rate near $\rho=0$ is restricted by the case $k=-2$. It follows that

$$\left|\rho^{|k+1|}\Gamma^{\natural}_{2|k+1|+2} * \left\{|z|^{-|k+1|}\left[A_0[\tilde{\psi}_{i,k}] + (1-\eta_{R_0})h_k\right]\right\}\right| \le D_oD_iC(\ell)R_0^{(\tilde{\ell}-\ell)/2}w_{o,k}(\rho,\tau)\|h_k\|$$

where the constant $D_o\ge 1$ is independent of k.

Denote

$$\mathcal{B}_{o,k} := \left\{f(\rho,\tau)\ \mid\ |f(\rho,\tau)| \le 2D_oD_iC(\ell)R_0^{(\tilde{\ell}-\ell)/2}w_{o,k}(\rho,\tau)\|h_k\|\right\}.$$

For any $\tilde{\psi}_{o,k}\in\mathcal{B}_{o,k}$

$$\begin{aligned}&\left|(1-\eta_{R_0})\frac{(4k+8)\rho^2+4k}{(\rho^2+1)^2\rho^2}\tilde{\psi}_{o,k}\mathbf{1}_{\{\rho\le 4R(\tau)\}}\right| \lesssim |k|^{-1}D_oD_iC(\ell)R_0^{(\tilde{\ell}-\ell)/2}v(\tau)\langle\rho\rangle^{-2-\tilde{\ell}}\mathbf{1}_{\{R_0\le\rho\le 4R(\tau)\}}\|h_k\|\\ &\lesssim R_0^{-2}D_oD_iC(\ell)R_0^{(\tilde{\ell}-\ell)/2}v(\tau)\langle\rho\rangle^{-\tilde{\ell}}\|h_k\|.\end{aligned}$$

By the same convolution estimate above, with the small quantity R_0^{-2} when R_0 is large,

$$\rho^{|k+1|}\Gamma^{\natural}_{2|k+1|+2} * \left(|z|^{-|k+1|}J[\tilde{\psi}_{o,k},\tilde{\psi}_{i,k}]\right) \in \mathcal{B}_{o,k}.$$

We can deduce the contraction mapping property by the same way.

Now we have found a solution $\psi_{o,k} = \psi_{o,k}[\tilde{\psi}_{i,k}] \in \mathcal{B}_{o,k}$. Let us estimate the following term in $\mathcal{D}_{2R_0}$:

$$\left|\frac{(4k+8)\rho^2+4k}{(\rho^2+1)^2\rho^2}\psi_{o,k}\right| \lesssim |k|(1+\rho)^{-2}\rho^{-2}D_oD_iC(\ell)R_0^{(\tilde{\ell}-\ell)/2}|k|^{-2}v(\tau)\left(\rho\mathbf{1}_{\{\rho\le 1\}}+\rho^{2-\tilde{\ell}}\mathbf{1}_{\{1<\rho\le C_1\tau^{\frac12}\}}\right)\|h_k\|$$
$$\lesssim |k|^{-1}D_oD_iC(\ell)R_0^{(\tilde{\ell}-\ell)/2}v(\tau)\left(\rho^{-1}\mathbf{1}_{\{\rho\le 1\}}+\rho^{-\ell}\mathbf{1}_{\{1<\rho\le C_1\tau^{\frac12}\}}\right)\|h_k\|.$$

By Lemma 9.6,

$$\langle y\rangle\left|\nabla_y\mathcal{T}_{kr}^{2R_0}\left\{\left[e^{ik\theta}(a-ib)\frac{(4k+8)\rho^2+4k}{(\rho^2+1)^2\rho^2}\psi_{o,k}\right]_{\mathbb{C}^{-1}}\right\}\right|+\left|\mathcal{T}_{kr}^{2R_0}\left\{\left[e^{ik\theta}(a-ib)\frac{(4k+8)\rho^2+4k}{(\rho^2+1)^2\rho^2}\psi_{o,k}\right]_{\mathbb{C}^{-1}}\right\}\right|$$
$$\lesssim |k|^{-1}D_oD_iR_0^{(\tilde{\ell}-\ell)/2}w_{i,k}(\rho,\tau)\|h_k\|.$$

Since $R_0^{(\tilde{\ell}-\ell)/2}$ provides small quantity when R_0 is large, we have

$$\mathcal{T}_{kr}^{2R_0}\left[\left(e^{ik\theta}K[\psi_{o,k}[\tilde{\psi}_{i,k}]]\right)_{\mathbb{C}^{-1}}\right]\in\mathcal{B}_{i,k}.$$

The contraction property can be deduced by the same way. Thus we find a solution $\Psi_{i,k}=\Psi_{i,k}[h_k]\in\mathcal{B}_{i,k}$. Finally we find a solution $(\psi_{o,k},\Psi_{i,k})$ for (9.59) and (9.61).

From the construction process and the topology of $\mathcal{B}_{i,k}$, $\Psi_{i,k}[h_k]=0$ if $h_k=0$, which deduces that $\Psi_{i,k}[h_k]$ is a linear mapping about h_k. By the similar argument, $\psi_{o,k}[h_k]$ is also a linear mapping about h_k. So does ψ_k.

We will regard D_o, D_i and R_0 as general constants hereafter. Reviewing the calculation process, we have

$$|J[\psi_{o,k},\psi_{i,k}]|\mathbf{1}_{\{\rho\le 4R(\tau)\}}\lesssim C(\ell)v(\tau)\langle\rho\rangle^{-\ell}\|h_k\|.$$

Using (9.62) again, the upper bound of $\psi_{o,k}$ can be improved to

$$|\psi_{o,k}|\lesssim C(\ell)\min\left\{|k|^{-2}v(\tau)\left(\rho\mathbf{1}_{\{\rho\le1\}}+\rho^{2-\ell}\mathbf{1}_{\{1<\rho\le C_1\tau^{\frac12}\}}\right),\rho^{-\ell}\int_{\tau_0}^{\tau}v(s)ds\right\}\|h_k\|.$$

Combining the upper bound of $\psi_{o,k}$ and $\Psi_{i,k}$, we have

$$|\Psi_k|\lesssim C(\ell)|k|^{-2}v(\tau)\left(R_0^{5-\ell}\langle\rho\rangle^{-3}\mathbf{1}_{\{\rho\le 2R_0\}}+\rho^{2-\ell}\mathbf{1}_{\{2R_0<\rho\le 2R(\tau)\}}\right)\|h_k\|\lesssim C(\ell)|k|^{-2}v(\tau)\langle\rho\rangle^{2-\ell}\|h_k\|$$

in $\mathcal{D}_R$. By scaling argument, the proof of proposition is concluded. □

9.4. **Mode** 0.

Proposition 9.3. *Consider*

$$\begin{cases}\partial_\tau\Psi_0=(a-bW\wedge)\left(L_{\text{in}}\Psi_0\right)+H_0 & \text{in } \mathcal{D}_R,\\ \Psi_0=0 \ \text{ on } \partial\mathcal{D}_R, \quad \Psi_0(\cdot,\tau_0)=0 & \text{in } B_{R(\tau_0)},\end{cases}$$

where $H_0=(h_0(\rho,\tau))_{\mathbb{C}^{-1}}$, $\|H_0\|_{v,\ell}^R<\infty$. Then there exists a linear mapping $\Psi_0=\mathcal{T}_{00}^R[H_0]$ with the following estimate

$$|\Psi_0|\lesssim\|H_0\|_{v,\ell}^R v(\tau)\langle y\rangle^{-1}\begin{cases}R^2\ln R & \text{if } \ell>1\\ R^2(\ln R)^{\frac32} & \text{if } \ell=1\\ R^{3-\ell}\ln R & \text{if } \ell<1\end{cases}.$$

Moreover, $\Psi_0\cdot W=0$ and $(\Psi_0)_{\mathbb{C}}$ is radial in space.

Proof. Denote $\|h_0\|=\|h_0\|_{v,\ell}^R$. In order to find a solution with the form $\Psi_0=(\psi_0(\rho,\tau))_{\mathbb{C}^{-1}}$, by Lemma 9.1, it is equivalent to considering

$$\begin{cases}\partial_\tau\psi_0=(a-ib)\mathcal{L}_0\psi_0+h_0 & \text{in } \mathcal{D}_R,\\ \psi_0=0 \ \text{ on } \partial\mathcal{D}_R, \quad \psi_0(\cdot,\tau_0)=0 & \text{in } B_{R(\tau_0)}.\end{cases} \tag{9.63}$$

The existence of (9.63) is deduced by the same argument as (9.48). By Lemma 9.3,

$$\|\psi_0(\cdot,\tau)\|_{L^\infty(B_R)}\lesssim R^2\ln R\theta_{R,\ell}v(\tau)\|h_0\|.$$

In order to improve the spatial decay, we reformulate (9.63) into the following form

$$\begin{cases}\partial_\tau\psi_0=(a-ib)\left(\partial_{\rho\rho}\psi_0+\frac{1}{\rho}\partial_\rho\psi_0-\frac{1}{\rho^2}\psi_0\right)+\tilde{h}_0 & \text{in } \mathcal{D}_R,\\ \psi_0=0 \ \text{ on } \partial\mathcal{D}_R, \quad \psi_0(\cdot,\tau_0)=0 & \text{in } B_{R(\tau_0)},\end{cases} \tag{9.64}$$

where $\tilde{h}_0 := (a-ib)\frac{8}{(\rho^2+1)^2}\psi_0 + h_0$. Set $\psi_0 = \rho\psi_{*0}$. Then (9.64) is equivalent to

$$\begin{cases} \partial_\tau \psi_{*0} = (a-ib)\Delta_{\mathbb{R}^4}\psi_{*0} + |y|^{-1}\tilde{h}_0 \text{ in } \mathcal{D}_R, \\ \psi_{*0} = 0 \text{ on } \partial\mathcal{D}_R, \quad \psi_{*0}(\cdot,\tau_0) = 0 \text{ in } B_{R(\tau_0)} \end{cases} \tag{9.65}$$

where we abuse the symbol $\mathcal{D}_R = \left\{(y,\tau) \mid y \in \mathbb{R}^4,\ |y| \le R(\tau)\right\}$ as the corresponding time-varying domain in $\mathbb{R}^4$ and similarly $\partial\mathcal{D}_R$, $B_{R(\tau_0)}$. By the same argument for deducing (9.54), the fundamental solution for (9.65) is given by $\Gamma_4(x,y,t,s)$ with the bound

$$|\Gamma_4(x,y,t,s)| \lesssim (t-s)^{-2} e^{-\frac{\kappa|x-y|^2}{t-s}} \text{ for a constant } \kappa > 0.$$

Then

$$\begin{aligned} |\psi_0| = \rho|\psi_{*0}| &\lesssim \rho\left|\Gamma_4 ** \left(|z|^{-1}|\tilde{h}_0|\mathbf{1}_{\{|z|\le R(s)\}}\right)\right| \\ &\lesssim R^2\ln R\theta_{R,\ell}v(\tau)\langle\rho\rangle^{-1}\|h_0\| + \|h_0\|v(\tau)\begin{cases}\langle\rho\rangle^{-1} & \text{if } \ell > 1 \\ (R(\tau))^{1-\ell+\epsilon}\langle\rho\rangle^{1-\epsilon} & \text{if } \ell \le 1\end{cases} \\ &\sim \|h_0\|v(\tau)\langle\rho\rangle^{-1}\begin{cases} R^2\ln R & \text{if } \ell > 1 \\ R^2(\ln R)^{\frac{3}{2}} & \text{if } \ell = 1 \\ R^{3-\ell}\ln R & \text{if } \ell < 1\end{cases} \end{aligned} \tag{9.66}$$

where we used

$$|z|^{-1}\langle z\rangle^{-\ell}\mathbf{1}_{\{|z|\le R(s)\}} \lesssim (R(s))^{1-\ell+\epsilon}|z|^{-1}\langle z\rangle^{-1-\epsilon}$$

for $\ell \le 1$ with a small fixed constant $\epsilon > 0$.

□

Next, we will give the linear theory with the orthogonal condition.

Lemma 9.7. *Consider*

$$\partial_\tau\Psi_0 = (a - bW\wedge)(L_{\text{in}}\Psi_0) + H_0 \ \ in \ \ \mathcal{D}_R, \quad \Psi_0(\cdot,\tau_0) = 0 \ \ in \ \ B_{R(\tau_0)}$$

where $H_0 = (h_0(\rho,\tau))_{\mathbb{C}^{-1}}$, $\|H_0\|_{v,\ell}^R < \infty$ *with* $1 < \ell < 3$ *and the orthogonal condition*

$$\int_{B_{R(\tau)}} h_0(y,\tau)\mathcal{Z}_{0,1}(y)dy = 0 \ \ for\ all \ \ \tau \in (\tau_0,\infty). \tag{9.67}$$

Then there exists a solution $\Psi_0 = \mathcal{T}_{0r}^R[H_0]$ *as a linear mapping about* H_0 *with the following estimate*

$$\langle y\rangle|\nabla\Psi_0(y,\tau)| + |\Psi_0(y,\tau)| \lesssim C(\ell)v(\tau)R^{5-\ell}\ln R\langle y\rangle^{-3}\|H_0\|_{v,\ell}^R. \tag{9.68}$$

Moreover, $\Psi_0 \cdot W = 0$ *and* $(\Psi_0)_{\mathbb{C}}$ *is radial in space.*

Proof. This proof follows Lemma 9.6. Denote $\|h_0\| = \|h_0\|_{v,\ell}^R$ and assume $h_0 = 0$ in $\mathcal{D}_R^c$. We consider

$$(a - bW\wedge)(L_{\text{in}}G_0) = H_0 \text{ where } G_0 = (g_0(\rho,\tau))_{\mathbb{C}^{-1}}.$$

By Lemma 9.1, it is equivalent to considering

$$(a-ib)\,\mathcal{L}_0 g_0 = h_0,$$

where g_0 is given by

$$g_0(\rho,\tau) = (a+ib)\left(\mathcal{Z}_{0,2}(\rho)\int_0^\rho h_0(r,\tau)\mathcal{Z}_{0,1}(r)rdr - \mathcal{Z}_{0,1}(\rho)\int_0^\rho h_0(r,\tau)\mathcal{Z}_{0,2}(r)rdr\right).$$

Then under the orthogonal condition (9.67), if $1 < \ell < 3$, we have

$$\|G_0\|_{v,\ell-2}^\infty = \|g_0\|_{v,\ell-2}^\infty \lesssim C(\ell)\|h_0\|. \tag{9.69}$$

Next, let us consider

$$\begin{cases} \partial_\tau\Phi_0 = (a - bW\wedge)(L_{\text{in}}\Phi_0) + G_0 \text{ in } \mathcal{D}_{2R}, \\ \Phi_0 = 0 \text{ on } \partial\mathcal{D}_{2R}, \quad \Phi_0(\cdot,\tau_0) = 0 \text{ in } B_{2R(\tau_0)}. \end{cases} \tag{9.70}$$

By Lemma 9.3, there exists a solution $\Phi_0 = \Phi_0[G_0]$ with the form $\Phi_0 = (\phi_0(\rho,\tau))_{\mathbb{C}^{-1}}$ for some scalar function ϕ_0 and the estimate

$$|\Phi_0(y,\tau)| \lesssim v(\tau)R^{5-\ell}\ln R\langle y\rangle^{-1}\|G_0\|_{v,\ell-2}^\infty.$$

Applying [17, Theorem 1.2] and scaling argument to (9.70), we have

$$\langle y\rangle^2|D^2\Phi_0|+\langle y\rangle|D\Phi_0|+|\Phi_0|\lesssim v(\tau)R^{5-\ell}\ln R\langle y\rangle^{-1}\|G_0\|^{\infty}_{v,\ell-2}\quad\text{in}\quad\mathcal{D}_{3R/2}. \tag{9.71}$$

We take $\Psi_0=(a-bW\wedge)(L_{\rm in}\Phi_0)$ and manipulate $(a-bW\wedge)L_{\rm in}$ to (9.70). Combining (9.71), (9.69) and then scaling argument, we conclude (9.68). Applying Lemma 9.1, we have $(\Psi_0)_{\mathbb{C}}=(a-ib)\mathcal{L}_0\phi_0$.

□

Proposition 9.4. *Consider*

$$\partial_\tau\Psi_0=(a-bW\wedge)(L_{\rm in}\Psi_0)+H_0+(c_0(\tau)\eta(\rho)\mathcal{Z}_{0,1}(\rho))_{\mathbb{C}^{-1}}\quad in\ \ \mathcal{D}_R,\quad \Psi_0(\cdot,\tau_0)=0\quad in\ \ B_{R(\tau_0)}$$

where $H_0=(h_0(\rho,\tau))_{\mathbb{C}^{-1}}$, $\|H_0\|^R_{v,\ell}<\infty$ *with* $1<\ell<3$. *Then there exists a solution* $(\Psi_0,c_0)=(\mathcal{T}^R_0[H_0],c_0[H_0](\tau))$ *which is a linear mapping about* H_0 *with the following estimate*

$$\langle y\rangle|\nabla\Psi_0|+|\Psi_0|\lesssim C(\ell)\ln R_0 v(\tau)\left(R_0^{5-\ell}\langle y\rangle^{-3}\mathbf{1}_{\{|y|\le 2R_0\}}+\langle y\rangle^{2-\ell}\mathbf{1}_{\{|y|>2R_0\}}\right)\|H_0\|^R_{v,\ell},$$

$$c_0[H_0](\tau)=-\left(\int_{B_2}\eta(y)\mathcal{Z}^2_{0,1}(y)dy\right)^{-1}\int_{B_{2R_0}}h(y,\tau)\mathcal{Z}_{0,1}(y)dy+c_{*0}[H_0],$$

where R_0 *is given in* (9.3); $c_{*0}[H_0]$ *is a scalar function linearly depending on* H_0 *and satisfies* $|c_{*0}[H_0]|\lesssim R_0^{-\epsilon}v(\tau)\|H_0\|^R_{v,\ell}$ *and* $0<\epsilon<\ell-1$ *is a small constant independent of* τ_0.

Proof. Denote $\|h_0\|=\|h_0\|^R_{v,\ell}$ and take $h_0=0$ in $\mathcal{D}^c_R$. By Lemma 9.1, in order to find a solution Ψ_0 with the form $\Psi_0=(\psi_0(\rho,\tau))_{\mathbb{C}^{-1}}$, it is equivalent to considering

$$\partial_\tau\psi_0=(a-ib)\mathcal{L}_0\psi_0+h_0+c_0(\tau)\eta(\rho)\mathcal{Z}_{0,1}(\rho)\quad\text{in}\quad\mathcal{D}_R,\quad \psi_0(\cdot,\tau_0)=0\quad\text{in}\quad B_{R(\tau_0)}$$

Set $\psi_0=\eta_{R_0}(\rho)\psi_{i,0}(\rho,\tau)+\psi_{o,0}(\rho,\tau)$, where $\eta_{R_0}(\rho)=\eta(\frac{\rho}{R_0})$. In order to find a solution ψ_0, it suffices to consider the following inner-outer system

$$\begin{cases}\partial_\tau\psi_{o,0}=(a-ib)\left(\partial_{\rho\rho}\psi_{o,0}+\frac{1}{\rho}\partial_\rho\psi_{o,0}-\frac{1}{\rho^2}\psi_{o,0}\right)+J[\psi_{o,0},\psi_{i,0}]\mathbf{1}_{\{\rho\le 4R(\tau)\}}\quad\text{in}\quad\mathbb{R}^2\times(\tau_0,\infty),\\ \psi_{o,0}(\cdot,\tau_0)=0\quad\text{in}\quad\mathbb{R}^2,\end{cases} \tag{9.72}$$

$$\partial_\tau\psi_{i,0}=(a-ib)\mathcal{L}_0\psi_{i,0}+K[\psi_{o,0}]+c_0(\tau)\eta(\rho)\mathcal{Z}_{0,1}(\rho)\quad\text{in}\quad\mathcal{D}_{2R_0},\quad\psi_{i,0}(\cdot,\tau_0)=0\quad\text{in}\quad B_{2R_0(\tau_0)} \tag{9.73}$$

where

$$\begin{aligned}J[\psi_{o,0},\psi_{i,0}]&=(1-\eta_{R_0})\frac{8(a-ib)\psi_{o,0}}{(1+\rho^2)^2}+A_0[\psi_{i,0}]+(1-\eta_{R_0})h_0,\\ K[\psi_{o,0}]&=\frac{8(a-ib)\psi_{o,0}}{(1+\rho^2)^2}+h_0,\\ A_0[\psi_{i,0}]&=(a-ib)\left[\left(\partial_{\rho\rho}\eta_{R_0}+\frac{1}{\rho}\partial_\rho\eta_{R_0}\right)\psi_{i,0}+2\partial_\rho\eta_{R_0}\partial_\rho\psi_{i,0}\right]-\psi_{i,0}\partial_\tau\eta_{R_0}.\end{aligned}$$

Denote $\Psi_{i,0}(y,\tau)=(\psi_{i,0})_{\mathbb{C}^{-1}}$, that is, $\psi_{i,0}=\Psi_{i,0}\cdot E_1+i\Psi_{i,0}\cdot E_2$. By Lemma 9.1, (9.73) is equivalent to

$$\partial_\tau\Psi_{i,0}=(a-bW\wedge)L_{\rm in}\Psi_{i,0}+(K[\psi_{o,0}]+c_0(\tau)\eta(\rho)\mathcal{Z}_{0,1}(\rho))_{\mathbb{C}^{-1}}\quad\text{in}\quad\mathcal{D}_{2R_0},\quad\Psi_{i,0}(\cdot,\tau_0)=0\quad\text{in}\quad B_{2R_0(\tau_0)} \tag{9.74}$$

In order to meet the orthogonal condition (9.67) in $\mathcal{D}_{2R_0}$, we take

$$\begin{aligned}c_0(\tau)=c_0[\psi_{o,0}](\tau)&:=C_{0,1}\int_{B_{2R_0}}K[\psi_{o,0}](y,\tau)\mathcal{Z}_{0,1}(y)dy\\ &=C_{0,1}\int_{B_{2R_0}}\left[\frac{8(a-ib)\psi_{o,0}(y,\tau)}{(1+|y|^2)^2}+h_0(y,\tau)\right]\mathcal{Z}_{0,1}(y)dy\end{aligned}$$

where $C_{0,1}:=-(\int_{B_2}\eta(y)\mathcal{Z}^2_{0,1}(y)dy)^{-1}$.

The linear theories of (9.72) and (9.74) are given by Lemma 9.4 and Lemma 9.7 respectively and we reformulate (9.72) and (9.74) into the following form

$$\begin{aligned}\psi_{o,0}(\rho,\tau)&=\rho\left[\Gamma_4^\natural**\left(|z|^{-1}J[\psi_{o,0},\psi_{i,0}]\right)\right](\rho,\tau,\tau_0),\\ \Psi_{i,0}(y,\tau)&=\mathcal{T}^{2R_0}_{0r}\left[(K[\psi_{o,0}]+c_0[\psi_{o,0}](\tau)\eta(\rho)\mathcal{Z}_{0,1}(\rho))_{\mathbb{C}^{-1}}\right].\end{aligned} \tag{9.75}$$

We will solve this system by the contraction mapping theorem.

Denote $H_I := \left[h_0 + C_{0,1}\left(\int_{B_{2R_0}} h_0(y,\tau)\mathcal{Z}_{0,1}(y)dy\right)\eta(\rho)\mathcal{Z}_{0,1}(\rho)\right]_{\mathbb{C}^{-1}}$. It is easy to have $\|H_I\|_{v,\ell}^{2R_0} \lesssim \|h_0\|$. Inspired Lemma 9.7, if $(H_I)_{\mathbb{C}}$ satisfies the orthogonal condition (9.67) in $\mathcal{D}_{2R_0}$, we have the following estimate

$$\langle y\rangle|\nabla\mathcal{T}_{0r}^{2R_0}[H_I](y,\tau)| + |\mathcal{T}_{0r}^{2R_0}[H_I](y,\tau)| \le D_i w_{i,0}(\rho,\tau)\|h_0\|,$$

where $D_i \ge 1$ is a constant and

$$w_{i,0}(\rho,\tau) := v(\tau)R_0^{5-\ell}\ln R_0\langle\rho\rangle^{-3}.$$

Denote

$$\mathcal{B}_{i,0} := \Big\{F(y,\tau)\in C^1\left(B_{2R_0},\mathbb{R}^3\right) \ | \ F(y,\tau) = (f(\rho,\tau))_{\mathbb{C}^{-1}} \quad \text{for some radial scalar function}$$
$$f(\rho,\tau) \ \text{ and } \ \langle y\rangle|\nabla_y F(y,\tau)| + |F(y,\tau)| \le 2D_i w_{i,0}(\rho,\tau)\|h_0\|\Big\}.$$

For any $\tilde{\Psi}_{i,0}\in\mathcal{B}_{i,0}$, denote $\tilde{\psi}_{i,0} = \tilde{\Psi}_{i,0}\cdot E_1 + i\tilde{\Psi}_{i,0}\cdot E_2$. We will find a solution $\psi_{o,0} = \psi_{o,0}[\tilde{\psi}_{i,0}]$ of (9.59) by the contraction mapping theorem. Let us estimate $J[\psi_{o,0},\tilde{\psi}_{i,0}]$ term by term. By (3.5),

$$\begin{aligned}\left|\partial_\rho\tilde{\psi}_{i,0}\right| &= \left|\tilde{\Psi}_{i,0}\cdot\partial_\rho E_1 + E_1\cdot\partial_\rho\tilde{\Psi}_{i,0} + i\tilde{\Psi}_{i,0}\cdot\partial_\rho E_2 + iE_2\cdot\partial_\rho\tilde{\Psi}_{i,0}\right|\\ &\lesssim |\tilde{\Psi}_{i,0}|\langle\rho\rangle^{-2} + |\nabla_y\tilde{\Psi}_{i,0}| \lesssim D_i v(\tau)R_0^{5-\ell}\ln R_0\langle\rho\rangle^{-4}\|h_0\|.\end{aligned}$$

Then by the assumption $|\partial_\tau R_0| = O(R_0^{-1})$,

$$\begin{aligned}|A_0[\tilde{\psi}_{i,0}]| \le& \left|\left(\partial_{\rho\rho}\eta_{R_0} + \frac{1}{\rho}\partial_\rho\eta_{R_0}\right)\tilde{\psi}_{i,0} + 2\partial_\rho\eta_{R_0}\partial_\rho\tilde{\psi}_{i,0}\right| + |\tilde{\psi}_{i,0}||\partial_\tau\eta_{R_0}|\\ &\lesssim D_i\mathbf{1}_{\{R_0\le\rho\le 2R_0\}}v(\tau)R_0^{-\ell}\ln R_0\|h_0\|\\ &\lesssim D_i R_0^{\tilde{\ell}-\ell}\ln R_0 v(\tau)\langle\rho\rangle^{-\tilde{\ell}}\|h_0\|,\end{aligned}$$

where $1<\tilde{\ell}<\ell$.

$$|(1-\eta_{R_0})h_0| \lesssim \mathbf{1}_{\{\rho\ge R_0\}}v(\tau)\langle\rho\rangle^{-\ell}\|h_0\| \lesssim R_0^{\tilde{\ell}-\ell}v(\tau)\langle\rho\rangle^{-\tilde{\ell}}\|h_0\|.$$

Notice

$$\rho\left|\Gamma_4^\natural\right| ** (v(s)|z|^{-1}\langle z\rangle^{-\tilde{\ell}}) \lesssim w_{o,0}(\rho,\tau) := v(\tau)\langle\rho\rangle^{2-\tilde{\ell}}\mathbf{1}_{\{\rho\le\tau^{\frac{1}{2}}\}} + \rho^{-\tilde{\ell}}\int_{\tau_0}^{\tau} v(s)ds\mathbf{1}_{\{\rho>\tau^{\frac{1}{2}}\}}.$$

It follows that

$$\left|\rho\Gamma_4^\natural ** \left\{|z|^{-1}\left[A_0[\tilde{\psi}_{i,0}] + (1-\eta_{R_0})h_0\right]\right\}\right| \le D_o D_i R_0^{\tilde{\ell}-\ell}\ln R_0 w_{o,0}(\rho,\tau)\|h_0\|,$$

where $D_o\ge 1$ is a constant.

Denote

$$\mathcal{B}_{o,0} := \left\{f(\rho,\tau) \ | \ |f(\rho,\tau)| \le 2D_o D_i R_0^{\tilde{\ell}-\ell}\ln R_0 w_{o,0}(\rho,\tau)\|h_0\|\right\}.$$

For any $\tilde{\psi}_{o,0}\in\mathcal{B}_{o,0}$

$$\begin{aligned}&\left|(1-\eta_{R_0})\frac{8}{(\rho^2+1)^2}\tilde{\psi}_{o,0}\mathbf{1}_{\{\rho\le 4R(\tau)\}}\right| \lesssim D_o D_i R_0^{\tilde{\ell}-\ell}\ln R_0 v(\tau)\langle\rho\rangle^{-2-\tilde{\ell}}\mathbf{1}_{\{R_0\le\rho\le 4R(\tau)\}}\|h_0\|\\ \lesssim& R_0^{-2}D_o D_i R_0^{\tilde{\ell}-\ell}\ln R_0 v(\tau)\langle\rho\rangle^{-\tilde{\ell}}\|h_0\|.\end{aligned}$$

By the same convolution estimate above, with the small quantity R_0^{-2} when R_0 is large,

$$\rho\Gamma_4^\natural ** \left(|z|^{-1}J[\tilde{\psi}_{o,0},\tilde{\psi}_{i,0}]\right) \in \mathcal{B}_{o,0}.$$

We can deduce the contraction mapping property by the same way.

Now we have found a solution $\psi_{o,0} = \psi_{o,0}[\tilde{\psi}_{i,0}]\in\mathcal{B}_{o,0}$. It follows that

$$\|8(1+\rho^2)^{-2}\psi_{o,0}[\tilde{\psi}_{i,0}] + C_{0,1}\int_{B_{2R_0}}\frac{8\psi_{o,0}[\tilde{\psi}_{i,0}](y,\tau)}{(1+|y|^2)^2}\mathcal{Z}_{0,1}(y)dy\eta(\rho)\mathcal{Z}_{0,1}(\rho)\|_{v,\ell}^{2R_0} \lesssim D_o D_i R_0^{\tilde{\ell}-\ell}\ln R_0\|h\|.$$

Due to the choice of $c_0(\tau)$, $h_{II} := K[\psi_{o,0}[\tilde{\psi}_{i,0}]] + c_0[\psi_{o,0}[\tilde{\psi}_{i,0}]](\tau)\eta(\rho)Z_{0,1}(\rho)$ satisfies the orthogonal condition (9.67) in $\mathcal{D}_{2R_0}$. By Lemma 9.7, we have

$$\mathcal{T}_{0r}^{2R_0}[h_{II}]\in\mathcal{B}_{i,0}$$

since $R_0^{\tilde{\ell}-\ell}\ln R_0$ provides small quantity when R_0 is large.

The contraction property can be deduced by the same way. Thus we find a solution $\Psi_{i,0} = \Psi_{i,0}[h_0] \in \mathcal{B}_{i,0}$. Finally we find a solution $(\psi_{o,0}, \Psi_{i,0})$ for (9.72) and (9.74).

From the construction process and the topology of $\mathcal{B}_{i,0}$, $\Psi_{i,0}[h_0] = 0$ if $h_0 = 0$, which deduces that $\psi_{i,0}[h_0]$ is a linear mapping about h_0. By the similar argument, $\psi_{o,0}[h_0]$ and $c_0[h_0]$ are also linear mappings about h_0. So does ψ_0.

We will regard D_o, D_i as general constants hereafter. Since $\psi_{o,0}[h_0] \in \mathcal{B}_{o,0}$, then

$$c_0[h_0](\tau) = C_{0,1}\int_{B_{2R_0}} h_0(y,\tau)\mathcal{Z}_{0,1}(y)dy + c_{*0}[h_0].$$

where $c_{*0}[h_0](\tau)$ is a linear mapping about h_0 and $|c_{*0}[h_0]| \lesssim R_0^{\tilde{\ell}-\ell}\ln R_0 v(\tau)\|h_0\|$.

Reviewing the calculation process, we have

$$|J[\psi_{o,0},\psi_{i,0}]|\mathbf{1}_{\{\rho\le 4R(\tau)\}} \lesssim \ln R_0 v(\tau)\langle\rho\rangle^{-\ell}\|h_0\|.$$

Using (9.75) again, the upper bound of $\psi_{o,0}$ can be improved to

$$|\psi_{o,0}| \lesssim \ln R_0\left(v(\tau)\langle\rho\rangle^{2-\ell}\mathbf{1}_{\{\rho\le\tau^{\frac12}\}} + \rho^{-\ell}\int_{\tau_0}^{\tau} v(s)ds\mathbf{1}_{\{\rho>\tau^{\frac12}\}}\right)\|h_0\|.$$

Combining the upper bound of $\psi_{o,0}$ and $\Psi_{i,0}$, we have

$$|\Psi_0(y,\tau)| \lesssim \ln R_0 v(\tau)\left(R_0^{5-\ell}\langle y\rangle^{-3}\mathbf{1}_{\{|y|\le 2R_0\}} + \langle y\rangle^{2-\ell}\mathbf{1}_{\{|y|>2R_0\}}\right)\|h_0\| \ \text{ in } \ \mathcal{D}_R.$$

By scaling argument again, the proof of proposition is concluded.

□

9.5. **Mode** 1.

Proposition 9.5. *Consider*

$$\begin{cases}\partial_\tau\Psi_1 = (a-bW\wedge)(L_{\text{in}}\Psi_1) + H_1 & \text{in } \ \mathcal{D}_R,\\ \Psi_1 = 0 \ \text{ on } \ \partial\mathcal{D}_R, \quad \Psi_1(\cdot,\tau_0) = 0 & \text{in } \ B_{R(\tau_0)},\end{cases}$$

where $H_1 = \left(h_1(\rho,\tau)e^{i\theta}\right)_{\mathbb{C}^{-1}}$, $\|H_1\|_{v,\ell}^R < \infty$. *Then there exists a linear mapping* $\Psi_1 = \mathcal{T}_{10}^R[H_1]$ *with the following estimate*

$$|\Psi_1(y,\tau)| \lesssim \min\{\tau^{\frac12}, R^2\}R^2\theta_{R,\ell}v(\tau)\langle\rho\rangle^{-2}\|H_1\|_{v,\ell}^R.$$

Moreover, $\Psi_1\cdot W = 0$ *and* $e^{-i\theta}\left(\Psi_1\right)_{\mathbb{C}}$ *is radial in space.*

Proof. In order to find a solution Ψ_1 with the form $\Psi_1 = \left(\psi_1(\rho,\tau)e^{i\theta}\right)_{\mathbb{C}^{-1}}$, by Lemma 9.1, it is equivalent to considering

$$\begin{cases}\partial_\tau\psi_1 = (a-ib)\mathcal{L}_1\psi_1 + h_1 & \text{in } \ \mathcal{D}_R,\\ \psi_1 = 0 \ \text{ on } \ \partial\mathcal{D}_R, \quad \psi_1(\cdot,\tau_0) = 0 & \text{in } \ B_{R(\tau_0)}.\end{cases}$$

For brevity, denote $\|h_1\| = \|h_1\|_{v,\ell}^R$. By Lemma 9.3,

$$\|\psi_1(\cdot,\tau)\|_{H_0^1(B_R)} \lesssim \min\{\tau^{\frac12}, R^2\}R^2\theta_{R,\ell}v(\tau)\|h_1\|. \tag{9.76}$$

In order to get the L^∞ estimate, we reformulate the equation into the following form:

$$\begin{cases}\partial_\tau\psi_1 = (a-ib)\left(\partial_{\rho\rho}\psi_1 + \frac{1}{\rho}\partial_\rho\psi_1\right) + \hat{h}_1 & \text{in } \ \mathcal{D}_R,\\ \psi_1 = 0 \ \text{ on } \ \partial\mathcal{D}_R, \quad \psi_1(\cdot,\tau_0) = 0 & \text{in } \ B_{R(\tau_0)},\end{cases} \tag{9.77}$$

where $\hat{h}_1 = \frac{-4\rho^2+4}{(\rho^2+1)^2}\psi_1 + h_1$. Then by [19, Corollary 6 and Remark 6],

By the similar argument for deducing (9.54), denote $\Gamma_2(x,y,t,s)$ as the fundamental solution of the homogeneous part of (9.77) with the estimate

$$|\Gamma_2(x,y,t,s)| \le N(t-s)^{-1}e^{-\frac{\kappa|x-y|^2}{t-s}}$$

and the positive constants N, κ are independent of t,s.

$$\psi_1(y,\tau) = \int_{B_{R(\tau-1)}}\Gamma_2(y,z,\tau,\tau-1)\psi_1(z,\tau-1)dz + \int_{\tau-1}^{\tau}\int_{B_{R(s)}}\Gamma_2(y,z,\tau,s)\hat{h}_1(z,s)dzds$$

Then

$$|\int_{B_{R(\tau-1)}} \Gamma_2(y,z,\tau,\tau-1)\psi_1(z,\tau-1)dz| \lesssim \int_{B_{R(\tau-1)}} e^{-\kappa|y-z|^2}|\psi_1(z,\tau-1)|dz \lesssim \min\{\tau^{\frac12},R^2\}R^2\theta_{R,\ell}v(\tau)\|h_1\|$$

by (9.76). Since $\|\hat{h}_1(\cdot,\tau)\|_{L^2(B_R)} \lesssim \min\{\tau^{\frac12},R^2\}R^2\theta_{R,\ell}v(\tau)\|h_1\|$,

$$\left|\int_{\tau-1}^{\tau}\int_{B_{R(s)}} \Gamma_2(y,z,\tau,s)\hat{h}(z,s)dzds\right| \lesssim \int_{\tau-1}^{\tau}\int_{B_{R(s)}} (\tau-s)^{-1}e^{-\frac{\kappa|y-z|^2}{\tau-s}}\left|\hat{h}_1(z,s)\right|dzds$$

$$\le \int_{\tau-1}^{\tau}(\tau-s)^{-1}\left(\int_{B_{R(s)}} e^{-\frac{2\kappa|y-z|^2}{\tau-s}}dz\right)^{\frac12}\|\hat{h}_1(\cdot,s)\|_{L^2(B_{R(s)})}ds$$

$$\lesssim \min\{\tau^{\frac12},R^2\}R^2\theta_{R,\ell}v(\tau)\|h_1\|\int_{\tau-1}^{\tau}(\tau-s)^{-\frac12} \sim \min\{\tau^{\frac12},R^2\}R^2\theta_{R,\ell}v(\tau)\|h_1\|.$$

Therefore, we get

$$\|\psi_1(\cdot,\tau)\|_{L^\infty} \lesssim \min\{\tau^{\frac12},R^2\}R^2\theta_{R,\ell}v(\tau)\|h_1\|.$$

In order to get the spatial decay, we rewrite the equation into the following form:

$$\begin{cases}(a+ib)\partial_\tau\psi_1 = \partial_{\rho\rho}\psi_1 + \frac{1}{\rho}\partial_\rho\psi_1 - \frac{4}{\rho^2}\psi_1 + \tilde{h}_1 \quad \text{in} \quad \mathcal{D}_R,\\ \psi_1 = 0 \quad \text{on} \quad \partial\mathcal{D}_R, \quad \psi_1(\cdot,\tau_0) = 0 \quad \text{in} \quad B_{R(\tau_0)},\end{cases}$$

where $\tilde{h}_1 = \frac{12\rho^2+4}{(\rho^2+1)^2}\frac{1}{\rho^2}\psi_1 + h_1$. By similar argument in (9.66),

$$|\psi_1(\rho,\tau)| \lesssim \rho^2\left|\Gamma_6 ** \left(|y|^{-2}|\tilde{h}_1|\mathbf{1}_{\{|y|\le R(s)\}}\right)\right|.$$

where

$$|\Gamma_6(x,y,t,s)| \lesssim (t-s)^{-3}e^{-\frac{\kappa|x-y|^2}{t-s}} \quad \text{for a constant} \quad \kappa>0.$$

Since

$$\left|\Gamma_6 ** \left(|z|^{-2}\frac{12|z|^2+4}{(|z|^2+1)^2}\frac{1}{|z|^2}|\psi_1(z,s)|\mathbf{1}_{\{|z|\le R(s)\}}\right)\right| \lesssim (\rho^{-2}\mathbf{1}_{\{\rho\le1\}} + \rho^{-3.9}1_{\{\rho>1\}})\min\{\tau^{\frac12},R^2\}R^2\theta_{R,\ell}v(\tau)\|h_1\|,$$

and

$$\left|\Gamma_6 ** \left(|z|^{-2}|h_1(z,s)|\mathbf{1}_{\{|z|\le R(s)\}}\right)\right| \lesssim v(\tau)\|h_1\|\left(\langle\ln\rho\rangle\mathbf{1}_{\{\rho\le1\}} + \begin{cases}\langle\rho\rangle^{-4}\langle\ln\rho\rangle & \text{if} \quad \ell\ge4\\ \langle\rho\rangle^{-\ell} & \text{if} \quad 0<\ell<4\\ R^{-\ell+\epsilon}\langle\rho\rangle^{-\epsilon} & \text{if} \quad \ell\le0\end{cases}\right)$$

where we used $|y|^{-2}\langle|y|\rangle^{-\ell}\mathbf{1}_{\{1<|y|\le R(s)\}} \lesssim R^{-\ell+\epsilon}\langle|y|\rangle^{-2-\epsilon}$ for $\ell\le0$ where the constant $\epsilon>0$ can be chosen arbitrarily small. Then we have

$$|\psi_1(\rho,\tau)| \lesssim (\mathbf{1}_{\{\rho\le1\}} + \rho^{-1.9}1_{\{\rho>1\}})\min\{\tau^{\frac12},R^2\}R^2\theta_{R,\ell}v(\tau)\|h_1\|.$$

By the iteration of the above estimate, we gain

$$|\psi_1(\rho,\tau)| \lesssim \min\{\tau^{\frac12},R^2\}R^2\theta_{R,\ell}v(\tau)\langle\rho\rangle^{-2}\|h_1\|.$$

□

Lemma 9.8. *Consider*

$$\partial_\tau\Psi_1 = (a-bW\wedge)(L_{\text{in}}\Psi_1) + H_1 \quad \textit{in} \quad \mathcal{D}_R, \quad \Psi_1(\cdot,\tau_0)=0 \quad \textit{in} \quad B_{R(\tau_0)}$$

where $H_1 = \left(h_1(\rho,\tau)e^{i\theta}\right)_{\mathbb{C}^{-1}}$, $\|H_1\|_{v,\ell}^R < \infty$ *with* $0<\ell<3$ *and the orthogonal condition*

$$\int_{B_{R(\tau)}} h_1(y,\tau)\mathcal{Z}_{1,1}(y)dy = 0 \quad \textit{for all} \quad \tau\ge\tau_0. \tag{9.78}$$

Then there exists a solution $\Psi_1 = \mathcal{T}_{1r}^R[H_1]$ *which is a linear mapping about* H_1 *with the following estimate*

$$\langle y\rangle|\nabla\Psi_1(y,\tau)| + |\Psi_1(y,\tau)| \lesssim C(\ell)\min\{\tau^{\frac12},R^2\}R^{5-\ell}v(\tau)\langle\rho\rangle^{-4}\|H_1\|_{v,\ell}^R \quad \textit{in} \quad \mathcal{D}_R.$$

Moreover, $\Psi_1\cdot W = 0$ *and* $e^{-i\theta}(\Psi_1)_{\mathbb{C}}$ *is radial in space.*

Proof. Denote $\|h_1\| = \|h_1\|_{v,\ell}^R$ and set $h_1 = 0$ in $\mathcal{D}_R^c$. Consider

$$(a - bW\wedge)\,(L_{\rm in} G_1) = H_1 \quad \text{where} \quad G_1 = \big(g_1(\rho,\tau)e^{i\theta}\big)_{\mathbb{C}^{-1}}.$$

By Lemma 9.1, it is equivalent to considering

$$(a - ib)\,\mathcal{L}_1 g_1 = h_1,$$

where g_1 is given by

$$g_1(\rho,\tau) = (a+ib)\left(\mathcal{Z}_{1,2}(\rho)\int_0^\rho h_1(r,\tau)\mathcal{Z}_{1,1}(r)rdr - \mathcal{Z}_{1,1}(\rho)\int_0^\rho h_1(r,\tau)\mathcal{Z}_{1,2}(r)rdr\right),$$

with the following estimate

$$\|g_1\|_{v,\ell-2}^{\infty} \lesssim \|h_1\| \quad \text{for} \quad 0<\ell<4. \tag{9.79}$$

In fact,

$$\mathcal{Z}_{1,1}(\rho)\int_0^\rho h_1(r,\tau)\mathcal{Z}_{1,2}(r)rdr \lesssim v(\tau)\langle\rho\rangle^{2-\ell}\|h_1\| \quad \text{if} \quad \ell<4.$$

For $\rho \le 1$,

$$|\mathcal{Z}_{1,2}(\rho)\int_0^\rho h_1(r,\tau)\mathcal{Z}_{1,1}(r)rdr| \lesssim v(\tau)\|h_1\|.$$

For $\rho \ge 1$, by the orthogonal condition (9.78),

$$|\mathcal{Z}_{1,2}(\rho)\int_0^\rho h_1(r,\tau)\mathcal{Z}_{1,1}(r)rdr| = |\mathcal{Z}_{1,2}(\rho)\int_\rho^\infty h_1(r,\tau)\mathcal{Z}_{1,1}(r)rdr| \lesssim v(\tau)\langle\rho\rangle^{2-\ell}\|h_1\| \quad \text{for} \quad \ell>0.$$

Next, let us consider

$$\begin{cases}\partial_\tau\Phi_1 = (a-bW\wedge)\,(L_{\rm in}\Phi_1) + G_1 & \text{in } \mathcal{D}_{2R},\\ \Phi_1 = 0 \ \text{ on } \partial\mathcal{D}_{2R}, \quad \Phi_1(\cdot,\tau_0) = 0 & \text{in } B_{2R(\tau_0)}.\end{cases}$$

Here Φ_1 is given by proposition 9.5 and has the estimate

$$|\Phi_1(\rho,\tau)| \lesssim \min\{\tau^{\frac12}, R^2\}R^{5-\ell}v(\tau)\langle\rho\rangle^{-2}\|g_1\|_{v,\ell-2}^{\infty}$$

when $\ell<3$. Take $\Psi_1 = (a-bW\wedge)\,(L_{\rm in}\Phi_1)$. Combining scaling argument and (9.79), we have

$$\langle y\rangle|\nabla\Psi_1(y,\tau)| + |\Psi_1(y,\tau)| \lesssim \min\{\tau^{\frac12}, R^2\}R^{5-\ell}v(\tau)\langle\rho\rangle^{-4}\|h_1\| \quad \text{in} \quad \mathcal{D}_R.$$

□

Proposition 9.6. *Consider*

$$\partial_\tau\Psi_1 = (a-bW\wedge)\,(L_{\rm in}\Psi_1) + H_1 + \big(c_1(\tau)\eta(\rho)\mathcal{Z}_{1,1}(\rho)e^{i\theta}\big)_{\mathbb{C}^{-1}} \quad \text{in} \ \mathcal{D}_R, \quad \Psi_1(\cdot,\tau_0) = 0 \quad \text{in} \ B_{R(\tau_0)}$$

where $H_1 = \big(h_1(\rho,\tau)e^{i\theta}\big)_{\mathbb{C}^{-1}}$, $\|H_1\|_{v,\ell}^R < \infty$ *with* $1<\ell<3$. *Then there exists a solution* $(\Psi_1, c_1) = (\mathcal{T}_1^R[H_1], c_1[H_1](\tau))$ *which is a linear mapping about* H_1 *with the following estimate*

$$\langle y\rangle|\nabla_y\Psi_1(y,\tau)| + |\Psi_1(y,\tau)| \lesssim C(\ell)R_0 v(\tau)(R_0^{6-\ell}\langle\rho\rangle^{-4}\mathbf{1}_{\{\rho\le 2R_0\}} + \langle\rho\rangle^{2-\ell}\mathbf{1}_{\{\rho>2R_0\}})\|H_1\|_{v,\ell}^R,$$

$$c_1[H_1](\tau) = -\left(\int_{B_2}\eta(y)\mathcal{Z}_{1,1}^2(y)dy\right)^{-1}\int_{B_{2R_0}} h_1(y,\tau)\mathcal{Z}_{1,1}(y)dy + c_{*1}[H_1](\tau)$$

where R_0 *is given in* (9.3)*;* $c_{*1}[H_1]$ *is a scalar function linearly depending on* H_1 *and satisfies* $|c_{*1}[H_1]| \lesssim R_0^{-\epsilon}v(\tau)\|H_1\|_{v,\ell}^R$ *where* $0<\epsilon<\ell-1$ *is a small constant independent of* τ_0.

Moreover, $\Psi_1\cdot W = 0$ *and* $e^{-i\theta}(\Psi_1)_{\mathbb{C}}$ *is radial in space.*

Proof. In order to find a solution Ψ_1 with the form $\Psi_1 = \big(\psi_1(\rho,\tau)e^{i\theta}\big)_{\mathbb{C}^{-1}}$, by Lemma 9.1, it is equivalent to considering

$$\partial_\tau\psi_1 = (a-ib)\mathcal{L}_1\psi_1 + h_1 + c_1(\tau)\eta(\rho)\mathcal{Z}_{1,1}(\rho) \quad \text{in} \quad \mathcal{D}_R, \quad \psi_1(\cdot,\tau_0) = 0 \quad \text{in} \quad B_{R(\tau_0)}.$$

Denote $\|h_1\| = \|h_1\|_{v,\ell}$ and take $h_1 = 0$ in $\mathcal{D}_R^c$. Set $\psi_1(\rho,\tau) = \eta_{R_0}\psi_{i,1}(\rho,\tau) + \psi_{o,1}(\rho,\tau)$, where $\eta_{R_0} = \eta(\frac{\rho}{R_0})$. In order to find a solution ψ_1, it suffices to consider the following inner-outer system

$$\begin{cases}\partial_\tau\psi_{o,1} = (a-ib)\left(\partial_{\rho\rho}\psi_{o,1} + \frac{1}{\rho}\partial_\rho\psi_{o,1} - \frac{4}{\rho^2}\psi_{o,1}\right) + J[\psi_{o,1},\psi_{i,1}]\mathbf{1}_{\{\rho\le 4R(\tau)\}} & \text{in } \mathbb{R}^2\times(\tau_0,\infty),\\ \psi_{o,1}(\cdot,\tau_0) = 0 \quad \text{in } \mathbb{R}^2.\end{cases} \tag{9.80}$$

$$\partial_\tau\psi_{i,1} = (a-ib)\mathcal{L}_1\psi_{i,1} + K[\psi_{o,1}] + c_1(\tau)\eta(\rho)\mathcal{Z}_{1,1}(\rho) \quad \text{in} \quad \mathcal{D}_{2R_0}, \quad \psi_{i,1}(\cdot,\tau_0) = 0 \quad \text{in} \quad B_{2R_0(\tau_0)} \tag{9.81}$$

where

$$J[\psi_{o,1},\psi_{i,1}]=(a-ib)\,(1-\eta_{R_0})\left(\frac{4}{\rho^2}+V_1(\rho)\right)\psi_{o,1}+A_0[\psi_{i,1}]+(1-\eta_{R_0})h_1$$
$$=(a-ib)\,(1-\eta_{R_0})\,\frac{12\rho^2+4}{(\rho^2+1)^2\rho^2}\psi_{o,1}+A_0[\psi_{i,1}]+(1-\eta_{R_0})h_1,$$
$$A_0[\psi_{i,1}]=(a-ib)\left[\left(\partial_{\rho\rho}\eta_{R_0}+\frac{1}{\rho}\partial_\rho\eta_{R_0}\right)\psi_{i,1}(\rho,\tau)+2\partial_\rho\eta_{R_0}\partial_\rho\psi_{i,1}(\rho,\tau)\right]-\psi_{i,1}\partial_\tau\eta_{R_0},$$
$$K[\psi_{o,1}]=(a-ib)\left(\frac{4}{\rho^2}+V_1(\rho)\right)\psi_{o,1}+h_1=(a-ib)\frac{12\rho^2+4}{(\rho^2+1)^2\rho^2}\psi_{o,1}+h_1.$$

Set $\Psi_{i,1}(y,\tau)=\left(e^{i\theta}\psi_{i,1}(\rho,\tau)\right)_{\mathbb{C}^{-1}}$, that is, $\psi_{i,1}=e^{-i\theta}\left(\Psi_{i,1}\cdot E_1+i\Psi_{i,1}\cdot E_2\right)$. By Lemma 9.1, (9.81) is equivalent to

$$\begin{cases}\partial_\tau\Psi_{i,1}=(a-bW\wedge)\,L_{\rm in}\Psi_{i,1}+\left[\left(K[\psi_{o,1}]+c_1(\tau)\eta(\rho)\mathcal{Z}_{1,1}(\rho)\right)e^{i\theta}\right]_{\mathbb{C}^{-1}} \quad \text{in } \mathcal{D}_{2R_0},\\ \Psi_{i,1}(\cdot,\tau_0)=0 \quad \text{in } B_{2R_0(\tau_0)}\end{cases}\tag{9.82}$$

In order to meet the orthogonal condition (9.78) in $\mathcal{D}_{2R_0}$, we take

$$c_1(\tau)=c_1[\psi_{o,1}](\tau):=C_{1,1}\int_{B_{2R_0}}\left[(a-ib)\frac{12|y|^2+4}{(|y|^2+1)^2|y|^2}\psi_{o,1}(y,\tau)+h_1(y,\tau)\right]\mathcal{Z}_{1,1}(y)dy$$

where $C_{1,1}:=-(\int_{B_2}\eta(y)\mathcal{Z}_{1,1}^2(y)dy)^{-1}$.

The linear theories of (9.80) and (9.82) are given by Lemma 9.4 and Lemma 9.8 respectively and we reformulate (9.80) and (9.82) into the following form

$$\begin{aligned}\psi_{o,1}(\rho,\tau)&=\rho^2\left[\Gamma_6^\natural**\left(|z|^{-2}J[\psi_{o,1},\psi_{i,1}]\right)\right](\rho,\tau,\tau_0),\\ \Psi_{i,1}(y,\tau)&=\mathcal{T}_{1r}^{2R_0}\left[\left[\left(K[\psi_{o,1}]+c_1(\tau)\eta(\rho)\mathcal{Z}_{1,1}(\rho)\right)e^{i\theta}\right]_{\mathbb{C}^{-1}}\right].\end{aligned}\tag{9.83}$$

We will solve $(\psi_{o,1},\Psi_{i,1})$ by the contraction mapping theorem.

Denote $H_I:=\left\{\left[h_1+C_{1,1}\left(\int_{B_{2R_0}}h(y,\tau)\mathcal{Z}_{1,1}(y)dy\right)\eta(\rho)\mathcal{Z}_{1,1}(\rho)\right]e^{i\theta}\right\}_{\mathbb{C}^{-1}}$. It is easy to have $\|H_I\|_{v,\ell}^{2R_0}\lesssim\|h_1\|$. Inspired Lemma 9.8, if $e^{-i\theta}(H_I)_{\mathbb{C}}$ satisfies the orthogonal condition (9.78) in $\mathcal{D}_{2R_0}$, we have the following estimate

$$\langle y\rangle|\nabla_y\left(\mathcal{T}_{1r}^{2R_0}[H_I]\right)(y,\tau)|+|\mathcal{T}_{1r}^{2R_0}[H_I](y,\tau)|\le D_iw_{i,1}(\rho,\tau)\|h_1\|,$$

where the constant $D_i\ge 1$.

$$w_{i,1}(\rho,\tau):=R_0^{7-\ell}v(\tau)\langle\rho\rangle^{-4}.$$

Denote

$$\mathcal{B}_{i,1}:=\Big\{F(y,\tau)\in C^1\left(B_{2R_0},\mathbb{R}^3\right)\ |\ F(y,\tau)=\left(e^{i\theta}f(\rho,\tau)\right)_{\mathbb{C}^{-1}}\quad\text{for some radial scalar function}$$
$$f(\rho,\tau)\quad\text{and}\quad\langle y\rangle|\nabla_yF(y,\tau)|+|F(y,\tau)|\le 2D_iw_{i,1}(\rho,\tau)\|h_1\|\Big\}.$$

For any $\tilde\Psi_{i,1}\in\mathcal{B}_{i,1}$, denote $\tilde\psi_{i,1}=e^{-i\theta}\left(\tilde\Psi_{i,1}\cdot E_1+i\tilde\Psi_{i,1}\cdot E_2\right)$. We will find a solution $\psi_{o,1}=\psi_{o,1}[\tilde\Psi_{i,1}]$ of (9.80) by the contraction mapping theorem.

Let us estimate $J[\psi_{o,1},\tilde\psi_{i,1}]$ term by term. By (3.5),

$$\left|\partial_\rho\tilde\psi_{i,1}\right|=\left|e^{-i\theta}\left(\tilde\Psi_{i,1}\cdot\partial_\rho E_1+E_1\cdot\partial_\rho\tilde\Psi_{i,1}+i\tilde\Psi_{i,1}\cdot\partial_\rho E_2+iE_2\cdot\partial_\rho\tilde\Psi_{i,1}\right)\right|$$
$$\lesssim|\tilde\Psi_{i,1}|\langle\rho\rangle^{-2}+|\nabla_y\tilde\Psi_{i,1}|\lesssim D_iR_0^{7-\ell}v(\tau)\langle\rho\rangle^{-5}\|h_1\|.$$

Then by the assumption $|\partial_\tau R_0|=O(R_0^{-1})$,

$$|A_0[\tilde\psi_{i,1}]|\le\left|\left(\partial_{\rho\rho}\eta_{R_0}+\frac{1}{\rho}\partial_\rho\eta_{R_0}\right)\tilde\psi_{i,1}(\rho,\tau)+2\partial_\rho\eta_{R_0}\partial_\rho\tilde\psi_{i,1}(\rho,\tau)\right|+|\psi_{i,1}\partial_\tau\eta_{R_0}|$$
$$\lesssim D_i\mathbf{1}_{\{R_0\le\rho\le 2R_0\}}v(\tau)R_0^{1-\ell}\|h_1\|\lesssim\mathbf{1}_{\{\rho\ge R_0\}}D_iR_0^{\tilde\ell-\ell}v(\tau)\langle\rho\rangle^{1-\tilde\ell}\|h_1\|,$$

where $\max\{1,\ell-1\}<\tilde\ell<\ell$.

$$|(1-\eta_{R_0})h_1|\lesssim\mathbf{1}_{\{\rho\ge R_0\}}v(t)\langle\rho\rangle^{-\ell}\|h_1\|\lesssim\mathbf{1}_{\{\rho\ge R_0\}}R_0^{\tilde\ell-\ell}v(t)\langle\rho\rangle^{-\tilde\ell}\|h_1\|.$$

By Lemma 9.5, for $1 < \tilde{\ell} < 5$,

$$\rho^2 \left|\Gamma_6^\natural\right| ** \left(v(s)|z|^{-2}\langle z\rangle^{1-\tilde{\ell}}\mathbf{1}_{\{|z|\geq R_0(s)\}}\right) \lesssim \rho^2 \left|\Gamma_6^\natural\right| ** \left(v(s)\langle z\rangle^{-1-\tilde{\ell}}\right)$$
$$\lesssim w_{o,1}(\rho,\tau) := v(\tau)\rho^2\mathbf{1}_{\{\rho\leq 1\}} + v(\tau)\rho^{3-\tilde{\ell}}\mathbf{1}_{\{1<\rho\leq\tau^{\frac{1}{2}}\}} + \tau v(\tau)\rho^{1-\tilde{\ell}}\mathbf{1}_{\{\rho>\tau^{\frac{1}{2}}\}}.$$

It follows that

$$\left|\rho^2\Gamma_6^\natural ** \left[|y|^{-2}(A_0[\tilde{\psi}_{i,1}] + (1-\eta_{R_0})h_1)\right]\right| \leq D_o D_i R_0^{\tilde{\ell}-\ell}(\tau_0) w_{o,1}(\rho,\tau)\|h_1\|$$

where the constant $D_o \geq 1$. Then we denote

$$\mathcal{B}_{o,1} := \left\{ f(\rho,\tau) \mid |f(\rho,\tau)| \leq 2D_o D_i R_0^{\tilde{\ell}-\ell}(\tau_0) w_{o,1}(\rho,\tau)\|h_1\| \right\}.$$

For any $\tilde{\psi}_{o,1} \in \mathcal{B}_{o,1}$,

$$\left|(1-\eta_{R_0})\frac{12\rho^2+4}{(\rho^2+1)^2\rho^2}\tilde{\psi}_{o,1}\mathbf{1}_{\{\rho\leq 4R(\tau)\}}\right| \lesssim D_o D_i R_0^{\tilde{\ell}-\ell} v(\tau)\langle\rho\rangle^{-1-\tilde{\ell}}\mathbf{1}_{\{R_0\leq\rho\leq 4R(\tau)\}}\|h_1\|$$
$$\lesssim R_0^{-2} D_o D_i R_0^{\tilde{\ell}-\ell} v(\tau)\langle\rho\rangle^{1-\tilde{\ell}}\|h_1\|.$$

By Lemma 9.5, due to the small quantity provided by R_0^{-2},

$$\rho^2\Gamma_6^\natural ** \left(|z|^{-2}J[\tilde{\psi}_{o,1},\tilde{\psi}_{i,1}]\right) \in \mathcal{B}_{o,1}.$$

We can deduce the contraction mapping property by the same way. Thus we find a solution $\psi_{o,1} = \psi_{o,1}[\tilde{\psi}_{i,1}] \in \mathcal{B}_{o,1}$. Then in $\mathcal{D}_{2R_0}$, we have the following estimate:

$$\left|\frac{12\rho^2+4}{(\rho^2+1)^2\rho^2}\psi_{o,1}\right| \lesssim (1+\rho)^{-2}\rho^{-2}D_o D_i R_0^{\tilde{\ell}-\ell}(\tau_0)v(\tau)\left(\rho^2\mathbf{1}_{\{\rho\leq 1\}} + \rho^{3-\tilde{\ell}}\mathbf{1}_{\{1<\rho\leq\tau^{\frac{1}{2}}\}}\right)\|h_1\|$$
$$\lesssim D_o D_i R_0^{\tilde{\ell}-\ell}(\tau_0)v(\tau)\langle\rho\rangle^{-\ell}\|h_1\|$$

when $\ell \leq 1+\tilde{l}$;

$$\left|C_{1,1}\left(\int_{B_{2R_0}}\frac{12|y|^2+4}{(|y|^2+1)^2|y|^2}\psi_{o,1}[\tilde{\psi}_{i,1}](y,\tau)\mathcal{Z}_{1,1}(y)dy\right)\eta(\rho)\mathcal{Z}_{1,1}(\rho)\right| \lesssim D_o D_i R_0^{\tilde{\ell}-\ell}(\tau_0)v(\tau)\langle\rho\rangle^{-\ell}\|h_1\|.$$

Due to the choice of $c_1(\tau)$, $h_{II} := K[\psi_{o,1}[\tilde{\psi}_{i,1}]] + c_1[\psi_{o,1}[\tilde{\psi}_{i,1}]](\tau)\eta(\rho)\mathcal{Z}_{1,1}(\rho)$ satisfies the orthogonal condition (9.78) in $\mathcal{D}_{2R_0}$. By Lemma 9.8, we have

$$\mathcal{T}_{1r}^{2R_0}[h_{II}] \in \mathcal{B}_{i,1}$$

due to the small quantity provided by $R_0^{\tilde{\ell}-\ell}(\tau_0)$. The contraction property can be deduced by the same way.

Thus we find a solution $\Psi_{i,1} = \Psi_{i,1}[h_1] \in \mathcal{B}_{i,1}$. Finally we find a solution $(\psi_{o,1}, \Psi_{i,1})$ for (9.80) and (9.82).

The linear dependence on h can be achieved similarly as the higher mode case.

We will regard D_o, D_i as general constants hereafter. Since $\psi_{o,1}[h_1] \in \mathcal{B}_{o,1}$, then

$$c_1[h_1](\tau) = C_{1,1}\int_{B_{2R_0}} h_1(y,\tau)\mathcal{Z}_{1,1}(y)dy + c_{*1}[h_1]$$

where $c_{*1}[h_1]$ depends on h_1 linearly and $|c_{*1}[h_1]| \lesssim R_0^{\tilde{\ell}-\ell}v(\tau)\|h_1\|$.

Reviewing the calculation process, we have

$$|J[\psi_{o,1},\psi_{i,1}]|\,\mathbf{1}_{\{\rho\leq 4R(\tau)\}} \lesssim R_0 v(\tau)\langle\rho\rangle^{-\ell}\|h_1\|.$$

Iterating Lemma 9.5, the upper bound of $\psi_{o,1}$ can be improved to

$$|\psi_{o,1}| \lesssim R_0(v(\tau)\langle\rho\rangle^{2-\ell}\mathbf{1}_{\{\rho\leq\tau^{\frac{1}{2}}\}} + \tau v(\tau)\rho^{-\ell}\mathbf{1}_{\{\rho>\tau^{\frac{1}{2}}\}})\|h_1\|.$$

Combining the upper bound of $\psi_{o,1}$ and $\Psi_{i,1}$, we have

$$|\Psi_1(y,\tau)| \lesssim R_0 v(\tau)\left(R_0^{6-\ell}\langle\rho\rangle^{-4}\mathbf{1}_{\{\rho\leq 2R_0\}} + \langle\rho\rangle^{2-\ell}\mathbf{1}_{\{\rho>2R_0\}}\right)\|h_1\| \text{ in } \mathcal{D}_R.$$

By scaling argument, the proof of the proposition is concluded.

□

9.6. **Mode -1.** Consider

$$\begin{cases}(a+ib)\partial_\tau\phi_n(\rho,\tau)=\mathcal{L}_n\phi_n(\rho,\tau),\\ \phi_n(\rho,\tau_0)=g(\rho),\end{cases}$$

where $\tau_0\geq 1$,

$$\mathcal{L}_n=\partial_{\rho\rho}+\frac{1}{\rho}\partial_\rho-\frac{(n-1)^2}{\rho^2}-\frac{4n}{\rho^2+1}+\frac{8}{(\rho^2+1)^2},$$

Assume g is a Schwartz function.

Set $\phi_n(\rho,\tau)=\rho^{-\frac{1}{2}}A_n(\rho,\tau)$, then

$$\begin{cases}(a+ib)\partial_\tau A_n(\rho,\tau)=\tilde{\mathcal{L}}_n A_n(\rho,\tau),\\ A_n(\rho,\tau_0)=\rho^{\frac{1}{2}}g(\rho).\end{cases}$$

where $\tilde{\mathcal{L}}_n=\partial_{\rho\rho}+\frac{1}{4}\rho^{-2}-\frac{(n-1)^2}{\rho^2}-\frac{4n}{\rho^2+1}+\frac{8}{(\rho^2+1)^2}$.

Recall the generalized eigenfunctions $\Phi^n(\rho,\xi)$ with respect to $-\tilde{\mathcal{L}}_n$ is given by

$$-\tilde{\mathcal{L}}_n\Phi^n(\rho,\xi)=\xi\Phi^n(\rho,\xi).$$

We multiply $\Phi^n(\rho,\xi)$ and integrate by parts. Then

$$\begin{cases}(a+ib)\partial_\tau\hat{A}_n(\xi,\tau)=-\xi\hat{A}_n(\xi,\tau),\\ \hat{A}_n(\xi,\tau_0)=\int_0^\infty\rho^{\frac{1}{2}}g(\rho)\Phi^n(\rho,\xi)d\rho,\end{cases}$$

where $\hat{A}_n(\xi,\tau)=\int_0^\infty A_{-1}(\rho,\tau)\Phi^n(\rho,\xi)d\rho$. Thus

$$\hat{A}_n(\xi,\tau)=e^{-(a-ib)\xi\tau}\hat{A}_n(\xi,\tau_0).$$

By the distorted Fourier transform,

$$\begin{aligned}A_n(\rho,\tau)&=\int_0^\infty\hat{A}_n(\xi,\tau)\Phi^n(\rho,\xi)\rho_n(d\xi)\\&=\int_0^\infty e^{-(a-ib)\xi\tau}\hat{A}_n(\xi,0)\Phi^n(\rho,\xi)\rho_n(d\xi)\\&=\int_0^\infty e^{-(a-ib)\xi\tau}\Phi^n(\rho,\xi)\int_0^\infty x^{\frac{1}{2}}g(x)\Phi^n(x,\xi)dx\rho_n(d\xi)\\&=\int_0^\infty\int_0^\infty e^{-(a-ib)\xi\tau}\Phi^n(\rho,\xi)\Phi^n(x,\xi)\rho_n(d\xi)x^{\frac{1}{2}}g(x)dx.\end{aligned}$$

By Duhamel's principle,

$$\phi_n(\rho,\tau)=\int_{\tau_0}^\tau\int_0^\infty\int_0^\infty e^{-(a-ib)\xi(\tau-s)}\rho^{-\frac{1}{2}}\Phi^n(\rho,\xi)\Phi^n(x,\xi)x^{\frac{1}{2}}h_n(x,s)\rho_n(d\xi)dxds. \tag{9.84}$$

For $n=-1$, we summarize the results in [33, Section 4.3.2].

Proposition 9.7 ([33]). *For all $\rho\geq 0$, $\xi\geq 0$, we have*

$$\left|\Phi^{-1}(\rho,\xi)\right|\lesssim\begin{cases}\rho^{\frac{5}{2}}\langle\rho\rangle^{-2} & \text{if }\ \rho^2\xi\leq 1\\ \xi^{-\frac{1}{4}}\langle\xi\rangle^{-1} & \text{if }\ \rho^2\xi>1\end{cases}.$$

$\Phi^{-1}(\rho,\xi)$ has the following expansion:

$$\Phi^{-1}(\rho,\xi)=\Phi_0^{-1}(\rho)+\rho^{\frac{1}{2}}\sum_{j=1}^\infty(-\rho^2\xi)^j\Phi_j(\rho^2),$$

which converges absolutely, where $\Phi_0^{-1}(\rho)=\frac{\rho^{\frac{5}{2}}}{1+\rho^2}$. It converges uniformly if $\rho\xi^{\frac{1}{2}}$ remains bounded. Here $\Phi_j(u)\geq 0$ are smooth functions of $u\geq 0$ satisfying

$$\Phi_j(u)\leq\frac{1}{j!}\frac{u}{1+u},\quad \text{for all }\ u\geq 0, j\geq 1,$$

and $\Phi_1(u)\geq c_1\frac{u}{1+u}$ for all $u\geq 0$ with some absolute constant $c_1>0$.

The spectrum measure $\rho_{-1}(d\xi)$ of $-\tilde{\mathcal{L}}_{-1}$ is absolutely continuous on $\xi \geq 0$ with density

$$\frac{d\rho_{-1}(\xi)}{d\xi} \sim \langle\xi\rangle^2.$$

Proposition 9.8. *Consider*

$$\begin{cases}(a+ib)\partial_\tau\phi_{-1}(\rho,\tau) = \mathcal{L}_{-1}\phi_{-1}(\rho,\tau) + h(\rho,\tau) \quad in \quad (0,\infty)\times(\tau_0,\infty),\\ \phi_{-1}(\rho,\tau_0) = 0 \quad in \quad (0,\infty).\end{cases}$$

where $\tau_0 \geq 1$, $\|h\|_{v,\ell}^{\infty} < \infty$, where $\ell > \frac{3}{2}$. Then the solution $\phi_{-1} = \mathcal{T}_{-1}[h]$, where $\mathcal{T}_{-1}[h]$ is given by the linear mapping (9.84) with $n=-1$, satisfies the following estimate

$$\begin{aligned}|\phi_{-1}(\rho,\tau)| \lesssim{}& \|h\|_{v,\ell}^{\infty}\mathbf{1}_{\{\rho\leq\tau^{\frac12}\}}\begin{cases}v(\tau)\tau^{1-\frac{\ell}{2}} + \tau^{-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & if \ \ \ell<2\\ v(\tau)(\ln\tau)^2 + \tau^{-1}\ln\tau\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & if \ \ \ell=2\\ v(\tau)\ln\tau + \tau^{-1}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & if \ \ \ell>2\end{cases}\\ &+\|h\|_{v,\ell}^{\infty}\mathbf{1}_{\{\rho>\tau^{\frac12}\}}\rho^{-\frac12}\begin{cases}v(\tau)\tau^{\frac54-\frac{\ell}{2}} + \tau^{\frac14-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & if \ \ \ell<2\\ v(\tau)\tau^{\frac14}\langle\ln\tau\rangle + \tau^{-\frac34}\langle\ln\tau\rangle\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & if \ \ \ell=2\\ v(\tau)\tau^{\frac14} + \tau^{-\frac34}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & if \ \ \ell>2\end{cases}.\end{aligned}$$

Moreover, assuming $2<\ell<\frac52$ and the orthogonality condition

$$\int_{\mathbb{R}^2} h(y,\tau)\mathcal{Z}_{-1,1}(y)dy = 0 \quad for\ all \quad \tau>\tau_0, \tag{9.85}$$

we have the following estimate

$$|\phi_{-1}(\rho,\tau)| \lesssim \|h\|_{v,\ell}^{\infty}\begin{cases}v(\tau)\langle\rho\rangle^{2-\ell} + \tau^{-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & if \ \ \rho\leq\tau^{\frac12}\\ \rho^{-\frac12}(v(\tau)\tau^{\frac54-\frac{\ell}{2}} + \tau^{\frac14-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds) & if \ \ \rho>\tau^{\frac12}\end{cases}.$$

Proof. Without loss of generality, we assume $\|h\|_{v,\ell}^{\infty} = 1$.

Estimate without orthogonality.

$$|\phi_{-1}(\rho,\tau)| \lesssim \rho^{-\frac12}\int_{\tau_0}^{\tau} v(s)\int_0^\infty\int_0^\infty e^{-a\xi(\tau-s)}|\Phi^{-1}(\rho,\xi)||\Phi^{-1}(x,\xi)|x^{\frac12}\langle x\rangle^{-\ell}\langle\xi\rangle^2 dxd\xi ds.$$

First, we consider

$$F(\xi) := \int_0^\infty |\Phi^{-1}(x,\xi)|x^{\frac12}\langle x\rangle^{-\ell}dx = \int_0^{\xi^{-\frac12}} + \int_{\xi^{-\frac12}}^\infty \cdots := F_1 + F_2.$$

For F_1,

$$F_1 \lesssim \int_0^{\xi^{-\frac12}} x^{\frac52}\langle x\rangle^{-2}x^{\frac12}\langle x\rangle^{-\ell}dx \lesssim \begin{cases}\begin{cases}\xi^{\frac{\ell}{2}-1} & \text{if} \ \ \ell<2\\ \langle\ln\xi\rangle & \text{if} \ \ \ell=2\\ 1 & \text{if} \ \ \ell>2\end{cases} & \text{for} \ \ \xi\leq 1\\ \xi^{-2} & \text{for} \ \ \xi>1\end{cases}.$$

For F_2,

$$F_2 \lesssim \int_{\xi^{-\frac12}}^\infty \xi^{-\frac14}\langle\xi\rangle^{-1}x^{\frac12}\langle x\rangle^{-\ell}dx \lesssim \begin{cases}\xi^{\frac{\ell}{2}-1} & \text{if} \ \ \xi\leq 1\\ \xi^{-\frac54} & \text{if} \ \ \xi>1\end{cases},$$

where we used $\ell > \frac32$.

Thus

$$F(\xi) \lesssim \begin{cases}\begin{cases}\xi^{\frac{\ell}{2}-1} & \text{if} \ \ \ell<2\\ \langle\ln\xi\rangle & \text{if} \ \ \ell=2\\ 1 & \text{if} \ \ \ell>2\end{cases} & \text{for} \ \ \xi\leq 1\\ \xi^{-\frac54} & \text{for} \ \ \xi>1\end{cases}.$$

Next, let us estimate

$$P(\rho,\tau,s) := \int_0^\infty e^{-a\xi(\tau-s)}|\Phi^{-1}(\rho,\xi)|F(\xi)\langle\xi\rangle^2 d\xi = \int_0^{\frac{1}{\rho^2}} + \int_{\frac{1}{\rho^2}}^\infty \cdots := P_1 + P_2.$$

First, let us estimate P_1,

$$P_1 \lesssim \rho^{\frac{5}{2}}\langle\rho\rangle^{-2}\int_0^{\frac{1}{\rho^2}} e^{-a\xi(\tau-s)}F(\xi)\langle\xi\rangle^2 d\xi.$$

For $\rho \geq 1$,

$$P_1 \lesssim \rho^{\frac{1}{2}}\int_0^{\frac{1}{\rho^2}} e^{-a\xi(\tau-s)}\begin{cases}\xi^{\frac{\ell}{2}-1} & \text{if } \ell<2\\ \langle\ln\xi\rangle & \text{if } \ell=2\ d\xi\\ 1 & \text{if } \ell>2\end{cases}$$

$$\lesssim \begin{cases}\begin{cases}\rho^{\frac{1}{2}-\ell} & \text{if } \tau-s\leq\rho^2\\ \rho^{\frac{1}{2}}(\tau-s)^{-\frac{\ell}{2}} & \text{if } \tau-s>\rho^2\end{cases} & \text{if } \ell<2\\ \begin{cases}\rho^{-\frac{3}{2}}\langle\ln\rho\rangle & \text{if } \tau-s\leq\rho^2\\ \rho^{\frac{1}{2}}(\tau-s)^{-1}\langle\ln(a(\tau-s))\rangle & \text{if } \tau-s>\rho^2\end{cases} & \text{if } \ell=2\\ \begin{cases}\rho^{-\frac{3}{2}} & \text{if } \tau-s\leq\rho^2\\ \rho^{\frac{1}{2}}(\tau-s)^{-1} & \text{if } \tau-s>\rho^2\end{cases} & \text{if } \ell>2\end{cases}$$

by Lemma A.3.

For $\rho<1$,

$$P_1 \lesssim \rho^{\frac{5}{2}}\Big(\int_0^1 + \int_1^{\frac{1}{\rho^2}}\Big)e^{-a\xi(\tau-s)}F(\xi)\langle\xi\rangle^2 d\xi,$$

where

$$\int_0^1 e^{-a\xi(\tau-s)}F(\xi)\langle\xi\rangle^2 d\xi \lesssim \begin{cases}\begin{cases}1 & \text{if } \tau-s\leq 1\\ (\tau-s)^{-\frac{\ell}{2}} & \text{if } \tau-s>1\end{cases} & \text{for } \ell<2\\ \begin{cases}1 & \text{if } \tau-s\leq 1\\ (\tau-s)^{-1}\langle\ln(a(\tau-s))\rangle & \text{if } \tau-s>1\end{cases} & \text{for } \ell=2\\ \begin{cases}1 & \text{if } \tau-s\leq 1\\ (\tau-s)^{-1} & \text{if } \tau-s>1\end{cases} & \text{for } \ell>2\end{cases}$$

by the same estimate above.

$$\int_1^{\frac{1}{\rho^2}} e^{-a\xi(\tau-s)}F(\xi)\langle\xi\rangle^2 d\xi \lesssim \int_1^{\frac{1}{\rho^2}} e^{-a\xi(\tau-s)}\xi^{\frac{3}{4}}d\xi \sim (\tau-s)^{-\frac{7}{4}}\int_{a(\tau-s)}^{\frac{a(\tau-s)}{\rho^2}} e^{-z}z^{\frac{3}{4}}dz$$

$$\lesssim \begin{cases}\rho^{-\frac{7}{2}} & \text{if } \tau-s\leq\rho^2\\ (\tau-s)^{-\frac{7}{4}} & \text{if } \rho^2<\tau-s\leq 1\\ (\tau-s)^{-\frac{7}{4}}e^{-\frac{a(\tau-s)}{2}} & \text{if } \tau-s>1\end{cases}.$$

Thus for $\rho<1$,

$$P_1 \lesssim \begin{cases}\begin{cases}\rho^{-1} & \text{if } \tau-s\leq\rho^2\\ \rho^{\frac{5}{2}}(\tau-s)^{-\frac{7}{4}} & \text{if } \rho^2<\tau-s\leq 1\\ \rho^{\frac{5}{2}}(\tau-s)^{-\frac{\ell}{2}} & \text{if } \tau-s>1\end{cases} & \text{for } \ell<2\\ \begin{cases}\rho^{-1} & \text{if } \tau-s\leq\rho^2\\ \rho^{\frac{5}{2}}(\tau-s)^{-\frac{7}{4}} & \text{if } \rho^2<\tau-s\leq 1\\ \rho^{\frac{5}{2}}(\tau-s)^{-1}\langle\ln(a(\tau-s))\rangle & \text{if } \tau-s>1\end{cases} & \text{for } \ell=2\\ \begin{cases}\rho^{-1} & \text{if } \tau-s\leq\rho^2\\ \rho^{\frac{5}{2}}(\tau-s)^{-\frac{7}{4}} & \text{if } \rho^2<\tau-s\leq 1\\ \rho^{\frac{5}{2}}(\tau-s)^{-1} & \text{if } \tau-s>1\end{cases} & \text{for } \ell>2\end{cases}.$$

Next, let us estimate P_2.

$$P_2 \lesssim \int_{\frac{1}{\rho^2}}^{\infty} e^{-a\xi(\tau-s)}\xi^{-\frac{1}{4}}\langle\xi\rangle^{-1}F(\xi)\langle\xi\rangle^2 d\xi.$$

For $\rho \le 1$,

$$\begin{aligned} P_2 &\lesssim \int_{\frac{1}{\rho^2}}^{\infty} e^{-a\xi(\tau-s)}\xi^{-\frac{1}{4}}\langle\xi\rangle^{-1}\xi^{-\frac{5}{4}}\langle\xi\rangle^2 d\xi \sim \int_{\frac{1}{\rho^2}}^{\infty} e^{-a\xi(\tau-s)}\xi^{-\frac{1}{2}}d\xi \sim (\tau-s)^{-\frac{1}{2}}\int_{\frac{a(\tau-s)}{\rho^2}}^{\infty} e^{-z}z^{-\frac{1}{2}}dz \\ &\lesssim \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \tau-s\le\rho^2 \\ (\tau-s)^{-\frac{1}{2}}e^{-\frac{a(\tau-s)}{2\rho^2}} & \text{if } \tau-s>\rho^2 \end{cases}. \end{aligned}$$

For $\rho > 1$,

$$P_2 \lesssim \left(\int_1^{\infty} + \int_{\frac{1}{\rho^2}}^{1}\right) e^{-a\xi(\tau-s)}\xi^{-\frac{1}{4}}\langle\xi\rangle^{-1}F(\xi)\langle\xi\rangle^2 d\xi.$$

We estimate

$$\int_1^{\infty} e^{-a\xi(\tau-s)}\xi^{-\frac{1}{4}}\langle\xi\rangle^{-1}F(\xi)\langle\xi\rangle^2 d\xi \lesssim \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \tau-s\le 1 \\ (\tau-s)^{-\frac{1}{2}}e^{-\frac{a(\tau-s)}{2}} & \text{if } \tau-s>1 \end{cases}$$

by the same reason as above.

$$\begin{aligned} &\int_{\frac{1}{\rho^2}}^{1} e^{-a\xi(\tau-s)}\xi^{-\frac{1}{4}}\langle\xi\rangle^{-1}F(\xi)\langle\xi\rangle^2 d\xi \\ \lesssim &\int_{\frac{1}{\rho^2}}^{1} e^{-a\xi(\tau-s)}\xi^{-\frac{1}{4}} \begin{cases} \xi^{\frac{\ell}{2}-1} & \text{if } \ell<2 \\ \langle\ln\xi\rangle & \text{if } \ell=2 \\ 1 & \text{if } \ell>2 \end{cases} d\xi \\ \lesssim &\begin{cases} \begin{cases} 1 & \text{if } \tau-s\le 1 \\ (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } 1<\tau-s\le\rho^2 \\ (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}}e^{-\frac{a(\tau-s)}{2\rho^2}} & \text{if } \tau-s>\rho^2 \end{cases} & \text{for } \ell<2 \\ \begin{cases} 1 & \text{if } \tau-s\le 1 \\ (\tau-s)^{-\frac{3}{4}}\langle\ln(a(\tau-s))\rangle & \text{if } 1<\tau-s\le\rho^2 \\ (\tau-s)^{-\frac{3}{4}}\langle\ln(a(\tau-s))\rangle e^{-\frac{a(\tau-s)}{2\rho^2}} & \text{if } \tau-s>\rho^2 \end{cases} & \text{for } \ell=2 \\ \begin{cases} 1 & \text{if } \tau-s\le 1 \\ (\tau-s)^{-\frac{3}{4}} & \text{if } 1<\tau-s\le\rho^2 \\ (\tau-s)^{-\frac{3}{4}}e^{-\frac{a(\tau-s)}{2\rho^2}} & \text{if } \tau-s>\rho^2 \end{cases} & \text{for } \ell>2 \end{cases} \end{aligned}$$

by Lemma A.3.

Thus, for $\rho > 1$,

$$P_2 \lesssim \begin{cases} \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \tau-s\le 1 \\ (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } 1<\tau-s\le\rho^2 \\ (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}}e^{-\frac{a(\tau-s)}{4\rho^2}} & \text{if } \tau-s>\rho^2 \end{cases} & \text{for } \ell<2 \\ \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \tau-s\le 1 \\ (\tau-s)^{-\frac{3}{4}}\langle\ln(a(\tau-s))\rangle & \text{if } 1<\tau-s\le\rho^2 \\ (\tau-s)^{-\frac{3}{4}}\langle\ln(a(\tau-s))\rangle e^{-\frac{a(\tau-s)}{4\rho^2}} & \text{if } \tau-s>\rho^2 \end{cases} & \text{for } \ell=2 \\ \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \tau-s\le 1 \\ (\tau-s)^{-\frac{3}{4}} & \text{if } 1<\tau-s\le\rho^2 \\ (\tau-s)^{-\frac{3}{4}}e^{-\frac{a(\tau-s)}{4\rho^2}} & \text{if } \tau-s>\rho^2 \end{cases} & \text{for } \ell>2 \end{cases}.$$

In sum, for $\rho \le 1$,

$$P \lesssim \begin{cases} \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \tau - s \le \rho^2 \\ \rho^{\frac{5}{2}}(\tau-s)^{-\frac{7}{4}} & \text{if } \rho^2 < \tau - s \le 1 \\ \rho^{\frac{5}{2}}(\tau-s)^{-\frac{\ell}{2}} & \text{if } \tau - s > 1 \end{cases} & \text{for } \ell < 2 \\ \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \tau - s \le \rho^2 \\ \rho^{\frac{5}{2}}(\tau-s)^{-\frac{7}{4}} & \text{if } \rho^2 < \tau - s \le 1 \\ \rho^{\frac{5}{2}}(\tau-s)^{-1}\langle \ln(a(\tau-s))\rangle & \text{if } \tau - s > 1 \end{cases} & \text{for } \ell = 2 \\ \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \tau - s \le \rho^2 \\ \rho^{\frac{5}{2}}(\tau-s)^{-\frac{7}{4}} & \text{if } \rho^2 < \tau - s \le 1 \\ \rho^{\frac{5}{2}}(\tau-s)^{-1} & \text{if } \tau - s > 1 \end{cases} & \text{for } \ell > 2 \end{cases} .$$

For $\rho > 1$,

$$P \lesssim \begin{cases} \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \tau - s \le 1 \\ (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } 1 < \tau - s \le \rho^2 \\ \rho^{\frac{1}{2}}(\tau-s)^{-\frac{\ell}{2}} & \text{if } \tau - s > \rho^2 \end{cases} & \text{for } \ell < 2 \\ \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \tau - s \le 1 \\ (\tau-s)^{-\frac{3}{4}}\langle \ln(a(\tau-s))\rangle & \text{if } 1 < \tau - s \le \rho^2 \\ \rho^{\frac{1}{2}}(\tau-s)^{-1}\langle \ln(a(\tau-s))\rangle & \text{if } \tau - s > \rho^2 \end{cases} & \text{for } \ell = 2 \\ \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \tau - s \le 1 \\ (\tau-s)^{-\frac{3}{4}} & \text{if } 1 < \tau - s \le \rho^2 \\ \rho^{\frac{1}{2}}(\tau-s)^{-1} & \text{if } \tau - s > \rho^2 \end{cases} & \text{for } \ell > 2 \end{cases} .$$

Now let us estimate ϕ_{-1}. For $\rho \le 1$,

$$|\phi_{-1}(\rho,\tau)| \lesssim \rho^{-\frac{1}{2}}\Big(\int_{\tau-\rho^2}^{\tau} + \int_{\tau-1}^{\tau-\rho^2} + \int_{\frac{\tau_0}{2}}^{\tau-1}\Big) v(s)P(\rho,\tau,s)ds$$

$$\lesssim \rho^{-\frac{1}{2}}\Big[v(\tau)\int_{\tau-\rho^2}^{\tau}(\tau-s)^{-\frac{1}{2}}ds + v(\tau)\rho^{\frac{5}{2}}\int_{\tau-1}^{\tau-\rho^2}(\tau-s)^{-\frac{7}{4}}ds$$

$$+ \rho^{\frac{5}{2}}\begin{cases} \int_{\frac{\tau_0}{2}}^{\tau-1} v(s)(\tau-s)^{-\frac{\ell}{2}}ds & \text{if } \ell < 2 \\ \int_{\frac{\tau_0}{2}}^{\tau-1} v(s)(\tau-s)^{-1}\langle \ln(a(\tau-s))\rangle ds & \text{if } \ell = 2\Big] \\ \int_{\frac{\tau_0}{2}}^{\tau-1} v(s)(\tau-s)^{-1}ds & \text{if } \ell > 2 \end{cases}$$

$$\lesssim v(\tau)\rho^{\frac{1}{2}} + \rho^2\begin{cases} v(\tau)\tau^{1-\frac{\ell}{2}} + \tau^{-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ell < 2 \\ v(\tau)(\ln\tau)^2 + \tau^{-1}\ln\tau\int_{\frac{\tau_0}{2}}^{\frac{\tau_0}{2}} v(s)ds & \text{if } \ell = 2 \\ v(\tau)\ln\tau + \tau^{-1}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ell > 2 \end{cases} .$$

For $1<\rho\leq(\frac{\tau}{2})^{\frac12}$,

$$|\phi_{-1}(\rho,\tau)|\lesssim\rho^{-\frac12}(\int_{\tau-1}^{\tau}+\int_{\tau-\rho^2}^{\tau-1}+\int_{\frac{\tau_0}{2}}^{\tau-\rho^2})v(s)P(s)ds$$

$$\lesssim\rho^{-\frac12}[v(\tau)\int_{\tau-1}^{\tau}(\tau-s)^{-\frac12}ds+v(\tau)\int_{\tau-\rho^2}^{\tau-1}\begin{cases}(\tau-s)^{\frac14-\frac{\ell}{2}} & \text{if } \ \ell<2\\ (\tau-s)^{-\frac34}\langle\ln(a(\tau-s))\rangle & \text{if } \ \ell=2\ ds\\ (\tau-s)^{-\frac34} & \text{if } \ \ell>2\end{cases}$$

$$+\rho^{\frac12}\int_{\frac{\tau_0}{2}}^{\tau-\rho^2}v(s)\begin{cases}(\tau-s)^{-\frac{\ell}{2}} & \text{if } \ \ell<2\\ (\tau-s)^{-1}\langle\ln(a(\tau-s))\rangle & \text{if } \ \ell=2\ ds]\\ (\tau-s)^{-1} & \text{if } \ \ell>2\end{cases}$$

$$\lesssim\rho^{-\frac12}v(\tau)+v(\tau)\begin{cases}\rho^{2-\ell} & \text{if } \ \ell<2\\ \langle\ln\rho\rangle & \text{if } \ \ell=2\\ 1 & \text{if } \ \ell>2\end{cases}+\begin{cases}v(\tau)\tau^{1-\frac{\ell}{2}}+\tau^{-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ \ell<2\\ v(\tau)\int_{\rho^2}^{\frac{\tau}{2}}\langle\ln z\rangle z^{-1}dz+\tau^{-1}\ln\tau\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ \ell=2\\ v(\tau)\ln(\frac{\tau}{2\rho^2})+\tau^{-1}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ \ell>2\end{cases}$$

$$\lesssim\begin{cases}v(\tau)\tau^{1-\frac{\ell}{2}}+\tau^{-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ \ell<2\\ v(\tau)(\langle\ln\rho\rangle+\int_{\rho^2}^{\frac{\tau}{2}}\langle\ln z\rangle z^{-1}dz)+\tau^{-1}\ln\tau\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ \ell=2\ .\\ v(\tau)\langle\ln(\frac{\tau}{2\rho^2})\rangle+\tau^{-1}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ \ell>2\end{cases}$$

For $(\frac{\tau}{2})^{\frac12}\leq\rho\leq\tau^{\frac12}$,

$$|\phi_{-1}(\rho,\tau)|\lesssim\rho^{-\frac12}(\int_{\tau-1}^{\tau}+\int_{\tau-\rho^2}^{\tau-1}+\int_{\frac{\tau_0}{2}}^{\tau-\rho^2})v(s)P(s)ds$$

$$\lesssim\rho^{-\frac12}[v(\tau)\int_{\tau-1}^{\tau}(\tau-s)^{-\frac12}ds+\int_{\tau-\rho^2}^{\tau-1}v(s)\begin{cases}(\tau-s)^{\frac14-\frac{\ell}{2}} & \text{if } \ \ell<2\\ (\tau-s)^{-\frac34}\langle\ln(a(\tau-s))\rangle & \text{if } \ \ell=2\ ds\\ (\tau-s)^{-\frac34} & \text{if } \ \ell>2\end{cases}$$

$$+\rho^{\frac12}\int_{\frac{\tau_0}{2}}^{\tau-\rho^2}v(s)\begin{cases}(\tau-s)^{-\frac{\ell}{2}} & \text{if } \ \ell<2\\ (\tau-s)^{-1}\langle\ln(a(\tau-s))\rangle & \text{if } \ \ell=2\ ds]\\ (\tau-s)^{-1} & \text{if } \ \ell>2\end{cases}$$

$$\lesssim\rho^{-\frac12}[v(\tau)+v(\tau)\begin{cases}\tau^{\frac54-\frac{\ell}{2}} & \text{if } \ \ell<2\\ \tau^{\frac14}\langle\ln\tau\rangle & \text{if } \ \ell=2\\ \tau^{\frac14} & \text{if } \ \ell>2\end{cases}+\int_{\tau-\rho^2}^{\frac{\tau}{2}}v(s)ds\begin{cases}\tau^{\frac14-\frac{\ell}{2}} & \text{if } \ \ell<2\\ \tau^{-\frac34}\langle\ln\tau\rangle & \text{if } \ \ell=2\\ \tau^{-\frac34} & \text{if } \ \ell>2\end{cases}$$

$$+\rho^{\frac12}\int_{\frac{\tau_0}{2}}^{\tau-\rho^2}v(s)ds\begin{cases}\tau^{-\frac{\ell}{2}} & \text{if } \ \ell<2\\ \tau^{-1}\langle\ln\tau\rangle & \text{if } \ \ell=2]\\ \tau^{-1} & \text{if } \ \ell>2\end{cases}$$

$$\lesssim\rho^{-\frac12}[v(\tau)\begin{cases}\tau^{\frac54-\frac{\ell}{2}} & \text{if } \ \ell<2\\ \tau^{\frac14}\langle\ln\tau\rangle & \text{if } \ \ell=2\\ \tau^{\frac14} & \text{if } \ \ell>2\end{cases}+\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds\begin{cases}\tau^{\frac14-\frac{\ell}{2}} & \text{if } \ \ell<2\\ \tau^{-\frac34}\langle\ln\tau\rangle & \text{if } \ \ell=2]\\ \tau^{-\frac34} & \text{if } \ \ell>2\end{cases}$$

$$\sim\begin{cases}\tau^{1-\frac{\ell}{2}}v(\tau)+\tau^{-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ \ell<2\\ \langle\ln\tau\rangle v(\tau)+\tau^{-1}\langle\ln\tau\rangle\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ \ell=2\ .\\ v(\tau)+\tau^{-1}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ \ell>2\end{cases}$$

For $\rho \geq \tau^{\frac{1}{2}}$,

$$|\phi_{-1}(\rho,\tau)| \lesssim \rho^{-\frac{1}{2}}(\int_{\tau-1}^{\tau} + \int_{\frac{\tau_0}{2}}^{\tau-1})v(s)P(s)ds$$

$$\lesssim \rho^{-\frac{1}{2}}v(\tau)\int_{\tau-1}^{\tau}(\tau-s)^{-\frac{1}{2}}ds + \rho^{-\frac{1}{2}}\int_{\frac{\tau_0}{2}}^{\tau-1}v(s)\begin{cases}(\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } \ell<2\\ (\tau-s)^{-\frac{3}{4}}\langle\ln(a(\tau-s))\rangle & \text{if } \ell=2\ ds\\ (\tau-s)^{-\frac{3}{4}} & \text{if } \ell>2\end{cases}$$

$$\lesssim v(\tau)\rho^{-\frac{1}{2}} + \rho^{-\frac{1}{2}}\begin{cases}v(\tau)\tau^{\frac{5}{4}-\frac{\ell}{2}} + \tau^{\frac{1}{4}-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ell<2\\ v(\tau)\tau^{\frac{1}{4}}\langle\ln\tau\rangle + \tau^{-\frac{3}{4}}\langle\ln\tau\rangle\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ell=2\\ v(\tau)\tau^{\frac{1}{4}} + \tau^{-\frac{3}{4}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ell>2\end{cases}$$

$$\lesssim \rho^{-\frac{1}{2}}\begin{cases}v(\tau)\tau^{\frac{5}{4}-\frac{\ell}{2}} + \tau^{\frac{1}{4}-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ell<2\\ v(\tau)\tau^{\frac{1}{4}}\langle\ln\tau\rangle + \tau^{-\frac{3}{4}}\langle\ln\tau\rangle\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ell=2\\ v(\tau)\tau^{\frac{1}{4}} + \tau^{-\frac{3}{4}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ell>2\end{cases}.$$

Estimate with orthogonality.

For one part,

$$\rho^{-\frac{1}{2}}|\int_{\tau-1}^{\tau}\int_0^{\infty}\int_0^{\infty}e^{-(a-ib)\xi(\tau-s)}\Phi^{-1}(\rho,\xi)\Phi^{-1}(x,\xi)x^{\frac{1}{2}}h(x,s)\rho_{-1}(d\xi)dxds| \lesssim v(\tau)(\rho^{\frac{1}{2}}\mathbf{1}_{\{\rho\leq1\}} + \rho^{-\frac{1}{2}}\mathbf{1}_{\{\rho>1\}}). \tag{9.86}$$

For the other part,

$$\tilde{\phi}_{-1} := \rho^{-\frac{1}{2}}|\int_{\tau_0}^{\tau-1}\int_0^{\infty}\int_0^{\infty}e^{-(a-ib)\xi(\tau-s)}\Phi^{-1}(\rho,\xi)\Phi^{-1}(x,\xi)x^{\frac{1}{2}}h(x,s)\rho_{-1}(d\xi)dxds|.$$

By the orthogonal condition (9.85), we have

$$F(\xi,s) := |\int_0^{\infty}\Phi^{-1}(x,\xi)x^{\frac{1}{2}}h(x,s)dx| = |(\int_0^{\xi^{-\frac{1}{2}}} + \int_{\xi^{-\frac{1}{2}}}^{\infty})(\Phi^{-1}(x,\xi) - \frac{x^{\frac{5}{2}}}{1+x^2})x^{\frac{1}{2}}h(x,s)dx|$$

Firstly, by proposition 9.7, we have

$$|\int_0^{\xi^{-\frac{1}{2}}}(\Phi^{-1}(x,\xi) - \frac{x^{\frac{5}{2}}}{1+x^2})x^{\frac{1}{2}}h(x,s)dx| \lesssim v(s)\int_0^{\xi^{-\frac{1}{2}}}\frac{x^{\frac{5}{2}}}{1+x^2}x^2\xi x^{\frac{1}{2}}\langle x\rangle^{-\ell}dx \lesssim v(s)\begin{cases}\xi^{\frac{\ell}{2}-1} & \text{if } \xi\leq1\\ \xi^{-2} & \text{if } \xi\geq1\end{cases}$$

when $\ell<4$.

Secondly,

$$|\int_{\xi^{-\frac{1}{2}}}^{\infty}\Phi^{-1}(x,\xi)x^{\frac{1}{2}}h(x,s)dx| \lesssim v(s)\xi^{-\frac{1}{4}}\langle\xi\rangle^{-1}\int_{\xi^{-\frac{1}{2}}}^{\infty}x^{\frac{1}{2}}\langle x\rangle^{-\ell}dx \sim v(s)\begin{cases}\xi^{\frac{\ell}{2}-1} & \text{if } \xi\leq1\\ \xi^{-\frac{5}{4}} & \text{if } \xi\geq1\end{cases}$$

where we require $\ell>\frac{3}{2}$ to guarantee the integrability.

Thirdly, by (9.85),

$$\int_{\xi^{-\frac{1}{2}}}^{\infty}\frac{x^{\frac{5}{2}}}{1+x^2}x^{\frac{1}{2}}h(x,s)dx = -\int_0^{\xi^{-\frac{1}{2}}}\frac{x^{\frac{5}{2}}}{1+x^2}x^{\frac{1}{2}}h(x,s)dx,$$

where we require $\ell>2$. Then we have

$$\left|\int_{\xi^{-\frac{1}{2}}}^{\infty}\frac{x^{\frac{5}{2}}}{1+x^2}x^{\frac{1}{2}}h(x,s)dx\right| \lesssim v(s)\begin{cases}\xi^{\frac{\ell}{2}-1} & \text{if } \xi\leq1\\ \xi^{-2} & \text{if } \xi\geq1\end{cases}.$$

Thus

$$F(\xi,s) \lesssim v(s)(\xi^{\frac{\ell}{2}-1}\mathbf{1}_{\{\xi\leq1\}} + \xi^{-\frac{5}{4}}\mathbf{1}_{\{\xi>1\}}).$$

Next, let us estimate

$$P(\rho,\tau,s) := \int_0^{\infty}e^{-a\xi(\tau-s)}|\Phi^{-1}(\rho,\xi)|F(\xi,s)\langle\xi\rangle^2d\xi = \int_0^{\frac{1}{\rho^2}} + \int_{\frac{1}{\rho^2}}^{\infty}\cdots := P_1 + P_2.$$

Let us estimate P_1. For $\rho \geq 1$,

$$P_1 \lesssim v(s) \int_0^{\frac{1}{\rho^2}} e^{-a\xi(\tau-s)} \rho^{\frac{5}{2}} \langle\rho\rangle^{-2} \xi^{\frac{\ell}{2}-1} \langle\xi\rangle^2 d\xi \sim v(s)\rho^{\frac{1}{2}} \int_0^{\frac{1}{\rho^2}} e^{-a\xi(\tau-s)} \xi^{\frac{\ell}{2}-1} d\xi$$
$$\lesssim v(s) \begin{cases} \rho^{\frac{1}{2}-\ell} & \text{if } \tau - s \leq \rho^2 \\ \rho^{\frac{1}{2}} (\tau-s)^{-\frac{\ell}{2}} & \text{if } \tau - s > \rho^2 \end{cases}$$

by Lemma A.3.

For $\rho < 1$,

$$\begin{aligned} P_1 &\lesssim \int_0^{\frac{1}{\rho^2}} e^{-a\xi(\tau-s)} \rho^{\frac{5}{2}} \langle\rho\rangle^{-2} F(\xi,s) \langle\xi\rangle^2 d\xi \\ &\sim \rho^{\frac{5}{2}} \Big(\int_0^1 + \int_1^{\frac{1}{\rho^2}}\Big) e^{-a\xi(\tau-s)} F(\xi,s)\langle\xi\rangle^2 d\xi \\ &\lesssim v(s)\rho^{\frac{5}{2}} \Big(\int_0^1 e^{-a\xi(\tau-s)} \xi^{\frac{\ell}{2}-1} d\xi + \int_1^{\frac{1}{\rho^2}} e^{-a\xi(\tau-s)} \xi^{\frac{3}{4}} d\xi\Big) \\ &\lesssim v(s)\rho^{\frac{5}{2}} \begin{cases} \rho^{-\frac{7}{2}} & \text{if } \tau - s \leq \rho^2 \\ (\tau-s)^{-\frac{7}{4}} & \text{if } \rho^2 < \tau - s \leq 1 \\ (\tau-s)^{-\frac{\ell}{2}} & \text{if } \tau - s > 1 \end{cases} \end{aligned}$$

since

$$\int_0^1 e^{-a\xi(\tau-s)} \xi^{\frac{\ell}{2}-1} d\xi \lesssim \begin{cases} 1 & \text{if } \tau - s \leq 1 \\ (\tau-s)^{-\frac{\ell}{2}} & \text{if } \tau - s > 1 \end{cases}$$

by Lemma A.3 and

$$\int_1^{\frac{1}{\rho^2}} e^{-a\xi(\tau-s)} \xi^{\frac{3}{4}} d\xi \sim (\tau-s)^{-\frac{7}{4}} \int_{a(\tau-s)}^{\frac{a(\tau-s)}{\rho^2}} e^{-z} z^{\frac{3}{4}} dz \sim \begin{cases} \rho^{-\frac{7}{2}} & \text{if } \tau - s \leq \rho^2 \\ (\tau-s)^{-\frac{7}{4}} & \text{if } \rho^2 < \tau - s \leq 1 \\ (\tau-s)^{-\frac{7}{4}} e^{-\frac{a(\tau-s)}{2}} & \text{if } \tau - s > 1 \end{cases}.$$

Next, we will estimate P_2. For $\rho \leq 1$,

$$P_2 \lesssim v(s) \int_{\frac{1}{\rho^2}}^{\infty} e^{-a\xi(\tau-s)} \xi^{-\frac{1}{4}} \langle\xi\rangle^{-1} \xi^{-\frac{5}{4}} \langle\xi\rangle^2 d\xi \sim v(s) \int_{\frac{1}{\rho^2}}^{\infty} e^{-a\xi(\tau-s)} \xi^{-\frac{1}{2}} d\xi$$
$$\sim v(s)(\tau-s)^{-\frac{1}{2}} \int_{\frac{a(\tau-s)}{\rho^2}}^{\infty} e^{-z} z^{-\frac{1}{2}} dz \lesssim \begin{cases} v(s)(\tau-s)^{-\frac{1}{2}} & \text{if } \tau - s \leq \rho^2 \\ v(s)(\tau-s)^{-\frac{1}{2}} e^{-\frac{a(\tau-s)}{2\rho^2}} & \text{if } \tau - s > \rho^2 \end{cases}.$$

For $\rho > 1$,

$$\begin{aligned} P_2 &\lesssim \int_{\frac{1}{\rho^2}}^{\infty} e^{-a\xi(\tau-s)} \xi^{-\frac{1}{4}} \langle\xi\rangle^{-1} F(\xi,s) \langle\xi\rangle^2 d\xi = \int_{\frac{1}{\rho^2}}^1 + \int_1^{\infty} \cdots \\ &\lesssim v(s)\Big(\int_{\frac{1}{\rho^2}}^1 e^{-a\xi(\tau-s)} \xi^{\frac{\ell}{2}-\frac{5}{4}} d\xi + \int_1^{\infty} e^{-a\xi(\tau-s)} \xi^{-\frac{1}{2}} d\xi\Big) \\ &\lesssim v(s) \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \tau - s \leq 1 \\ (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } 1 < \tau - s \leq \rho^2 \\ (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} e^{-\frac{a(\tau-s)}{4\rho^2}} & \text{if } \tau - s > \rho^2 \end{cases} \end{aligned}$$

since

$$\int_1^{\infty} e^{-a\xi(\tau-s)} \xi^{-\frac{1}{2}} d\xi \lesssim \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \tau - s \leq 1 \\ (\tau-s)^{-\frac{1}{2}} e^{-\frac{a(\tau-s)}{2}} & \text{if } \tau - s > 1 \end{cases},$$

and

$$\int_{\frac{1}{\rho^2}}^1 e^{-a\xi(\tau-s)} \xi^{\frac{\ell}{2}-\frac{5}{4}} d\xi \lesssim \begin{cases} 1 & \text{if } \tau - s \leq 1 \\ (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } 1 < \tau - s \leq \rho^2 \\ (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} e^{-\frac{a(\tau-s)}{2\rho^2}} & \text{if } \tau - s > \rho^2 \end{cases}$$

by Lemma A.3.

In sum, for $\rho \leq 1$,

$$P(\rho,\tau,s) \lesssim \begin{cases} v(s)(\tau-s)^{-\frac{1}{2}} & \text{if } \tau-s \leq \rho^2 \\ v(s)\rho^{\frac{5}{2}}(\tau-s)^{-\frac{7}{4}} & \text{if } \rho^2 < \tau-s \leq 1 \\ v(s)\rho^{\frac{5}{2}}(\tau-s)^{-\frac{\ell}{2}} & \text{if } \tau-s > 1 \end{cases}.$$

For $\rho > 1$,

$$P(\rho,\tau,s) \lesssim v(s) \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \tau-s \leq 1 \\ (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } 1 < \tau-s \leq \rho^2 \\ \rho^{\frac{1}{2}}(\tau-s)^{-\frac{\ell}{2}} & \text{if } \tau-s > \rho^2 \end{cases}.$$

Finally, we will estimate $\tilde{\phi}_{-1}$. Obviously,

$$\tilde{\phi}_{-1} \lesssim \rho^{-\frac{1}{2}} \int_{\frac{\tau_0}{2}}^{\tau-1} P(\tau,s,\rho)ds.$$

For $\rho \leq 1$,

$$\tilde{\phi}_{-1} \lesssim \rho^{-\frac{1}{2}} \int_{\frac{\tau_0}{2}}^{\tau-1} v(s)\rho^{\frac{5}{2}}(\tau-s)^{-\frac{\ell}{2}}ds \lesssim \rho^2(v(\tau) + \tau^{-\frac{\ell}{2}} \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds).$$

For $1 < \rho \leq (\frac{\tau}{2})^{\frac{1}{2}}$,

$$\begin{aligned} \tilde{\phi}_{-1} &\lesssim \rho^{-\frac{1}{2}}(\int_{\tau-\rho^2}^{\tau-1} + \int_{\frac{\tau_0}{2}}^{\tau-\rho^2})P(\tau,s,\rho)ds \\ &\lesssim \rho^{-\frac{1}{2}}(v(\tau)\int_{\tau-\rho^2}^{\tau-1}(\tau-s)^{\frac{1}{4}-\frac{\ell}{2}}ds + (\int_{\frac{\tau}{2}}^{\tau-\rho^2} + \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}})v(s)\rho^{\frac{1}{2}}(\tau-s)^{-\frac{\ell}{2}}ds) \\ &\lesssim v(\tau)\rho^{2-\ell} + \tau^{-\frac{\ell}{2}} \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds \end{aligned}$$

where we used $\ell < \frac{5}{2}$.

For $(\frac{\tau}{2})^{\frac{1}{2}} < \rho \leq \tau^{\frac{1}{2}}$,

$$\begin{aligned} \tilde{\phi}_{-1} &\lesssim \rho^{-\frac{1}{2}}(\int_{\tau-\rho^2}^{\tau-1} + \int_{\frac{\tau_0}{2}}^{\tau-\rho^2})P(\tau,s,\rho)ds \\ &\lesssim \rho^{-\frac{1}{2}}(\int_{\tau-\rho^2}^{\tau-1} v(s)(\tau-s)^{\frac{1}{4}-\frac{\ell}{2}}ds + \int_{\frac{\tau_0}{2}}^{\tau-\rho^2} v(s)\rho^{\frac{1}{2}}(\tau-s)^{-\frac{\ell}{2}}ds) \\ &\lesssim \rho^{-\frac{1}{2}}(v(\tau)\tau^{\frac{5}{4}-\frac{\ell}{2}} + \tau^{\frac{1}{4}-\frac{\ell}{2}} \int_{\tau-\rho^2}^{\frac{\tau}{2}} v(s)ds + \rho^{\frac{1}{2}}\tau^{-\frac{\ell}{2}} \int_{\frac{\tau_0}{2}}^{\tau-\rho^2} v(s)ds) \\ &\sim v(\tau)\rho^{2-\ell} + \tau^{-\frac{\ell}{2}} \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds \end{aligned}$$

where we used $\ell < \frac{5}{2}$.

For $\rho > \tau^{\frac{1}{2}}$,

$$\tilde{\phi}_{-1} \lesssim \rho^{-\frac{1}{2}} \int_{\frac{\tau_0}{2}}^{\tau-1} v(s)(\tau-s)^{\frac{1}{4}-\frac{\ell}{2}}ds \lesssim \rho^{-\frac{1}{2}}(v(\tau)\tau^{\frac{5}{4}-\frac{\ell}{2}} + \tau^{\frac{1}{4}-\frac{\ell}{2}} \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds)$$

where we used $\ell < \frac{5}{2}$.

Combining (9.86), we conclude the estimate.

□

Appendix A. Some useful estimates

Lemma A.1. *Consider*

$$-\Delta u = f(x) \quad in \quad \mathbb{R}^n,$$

where $n \geq 3$, $f(x) = f(|x|)$ is radial with the upper bound $|f(x)| \lesssim |x|^{-l_1}\mathbf{1}_{\{|x|\leq 1\}} + |x|^{-l}\mathbf{1}_{\{|x|>1\}}$, $l_1 < n$, $l > 2$. u is given by

$$u(x) = \frac{1}{(n-2)|S^{n-1}|}\int_{\mathbb{R}^n} |x-y|^{2-n} f(y)dy,$$

where $|S^{n-1}|$ is the volume of the unit sphere S^{n-1}. Then

$$u(x) = u(|x|) = |x|^{2-n}\int_0^{|x|} a^{n-3}\int_a^\infty bf(b)dbda.$$

$$\partial_{|x|}u = -|x|^{1-n}\int_0^{|x|} f(a)a^{n-1}da.$$

Moreover,

$$|\partial_{|x|}u| \lesssim \frac{|x|^{1-l_1}}{n-l_1}\mathbf{1}_{\{|x|\leq 1\}} + \left(\frac{|x|^{1-n}}{n-l_1} + \frac{|x|^{1-l}}{n-l}\right)\mathbf{1}_{\{|x|>1\}}. \tag{A.1}$$

Specially, if $f(x) = |x|^{-l_1}\mathbf{1}_{\{|x|\leq 1\}} + |x|^{-l}\mathbf{1}_{\{|x|>1\}}$, then for $|x| \leq 1$,

$$u(x) = \frac{1}{(l-2)(n-2)} + \begin{cases} \frac{1}{(2-l_1)(n-2)} - \frac{|x|^{2-l_1}}{(2-l_1)(n-l_1)} & if \;\; l_1 < 2 \\ \frac{-\ln|x|}{n-2} + \frac{1}{(n-2)^2} & if \;\; l_1 = 2 \\ \frac{|x|^{2-l_1}}{(l_1-2)(n-l_1)} - \frac{1}{(l_1-2)(n-2)} & if \;\; 2 < l_1 < n \end{cases},$$

for $|x| \geq 1$,

$$\begin{aligned} u(x) &= \frac{|x|^{2-n}}{n-2}\left(\frac{1}{l-2} + \frac{1}{n-l_1}\right) + \begin{cases} \frac{|x|^{2-l}-|x|^{2-n}}{(l-2)(n-l)} & if \;\; 2 < l < n \\ \frac{|x|^{2-n}\ln|x|}{n-2} & if \;\; l = n \\ \frac{|x|^{2-n}-|x|^{2-l}}{(l-2)(l-n)} & if \;\; l > n \end{cases} \\ &= \begin{cases} \frac{|x|^{2-l}}{(l-2)(n-l)} + \frac{|x|^{2-n}}{(n-2)(n-l_1)} - \frac{|x|^{2-n}}{(n-2)(n-l)} & if \;\; 2 < l < n \\ \frac{|x|^{2-n}\ln|x|}{n-2} + \frac{|x|^{2-n}}{n-2}\left(\frac{1}{n-2} + \frac{1}{n-l_1}\right) & if \;\; l = n \\ \frac{|x|^{2-n}}{(n-2)(l-n)} + \frac{|x|^{2-n}}{(n-2)(n-l_1)} - \frac{|x|^{2-l}}{(l-2)(l-n)} & if \;\; l > n \end{cases} \\ &= \begin{cases} \frac{|x|^{2-l}}{(l-2)(n-l)} + \frac{(l_1-l)|x|^{2-n}}{(n-2)(n-l_1)(n-l)} & if \;\; 2 < l < n \\ \frac{|x|^{2-n}\ln|x|}{n-2} + \frac{|x|^{2-n}}{n-2}\left(\frac{1}{n-2} + \frac{1}{n-l_1}\right) & if \;\; l = n \\ \frac{(l-l_1)|x|^{2-n}}{(n-2)(n-l_1)(l-n)} - \frac{|x|^{2-l}}{(l-2)(l-n)} & if \;\; l > n \end{cases}. \end{aligned}$$

Roughly speaking, for $|x| \leq 1$,

$$u(x) \leq \frac{1}{(l-2)(n-2)} + \begin{cases} \frac{1}{(2-l_1)(n-2)} & if \;\; l_1 < 2 \\ \frac{-\ln|x|}{n-2} + \frac{1}{(n-2)^2} & if \;\; l_1 = 2 \\ \frac{|x|^{2-l_1}}{(l_1-2)(n-l_1)} & if \;\; 2 < l_1 < n; \end{cases}$$

for $|x| \geq 1$,

$$u(x) \leq \frac{|x|^{2-n}}{n-2}\left(\frac{1}{l-2} + \frac{1}{n-l_1}\right) + \begin{cases} \frac{|x|^{2-l}}{(l-2)(n-l)} & if \;\; 2 < l < n \\ \frac{|x|^{2-n}\ln|x|}{n-2} & if \;\; l = n \\ \frac{|x|^{2-n}}{(l-2)(l-n)} & if \;\; l > n, \end{cases}$$

$$u(x) \geq \begin{cases} \frac{|x|^{2-l}}{(l-2)(n-2)} + \frac{|x|^{2-n}}{(n-2)(n-l_1)} & if \;\; 2 < l < n \\ \frac{|x|^{2-n}\ln|x|}{n-2} + \frac{|x|^{2-n}}{n-2}\left(\frac{1}{n-2} + \frac{1}{n-l_1}\right) & if \;\; l = n \\ \frac{|x|^{2-n}}{(l-2)(n-2)} + \frac{|x|^{2-n}}{(n-2)(n-l_1)} & if \;\; l > n. \end{cases}$$

Proof. It is easy to see that u is radial. By the removable singularity theorem for harmonic function (It is used for the case $2 \le l_1 < n$) and maximum principle, we have

$$u(x) = u(|x|) = |x|^{2-n} \int_0^{|x|} a^{n-3} \int_a^\infty bf(b)dbda.$$

Denote $r = |x|$. It is straightforward to have

$$\partial_r u = (2-n)r^{1-n} \int_0^r a^{n-3} \int_a^\infty bf(b)dbda + r^{-1} \int_r^\infty bf(b)db = -r^{1-n} \int_0^r f(a)a^{n-1}da.$$

for $|f(x)| \lesssim |x|^{-l_1}\mathbf{1}_{\{|x|\le 1\}} + |x|^{-l}\mathbf{1}_{\{|x|>1\}}$, $l_1 < n$, $l > 2$.

For $r \le 1$,

$$\left|\int_0^r f(a)a^{n-1}da\right| \lesssim \frac{r^{n-l_1}}{n-l_1}.$$

For $r > 1$,

$$\left|\int_0^r f(a)a^{n-1}da\right| = \left|\left(\int_0^1 + \int_1^r\right) f(a)a^{n-1}da\right| \lesssim \frac{1}{n-l_1} + \frac{r^{n-l}}{n-l}.$$

Thus we have (A.1).

Hereafter, we assume $f(r) = r^{-l_1}\mathbf{1}_{\{r\le 1\}} + r^{-l}\mathbf{1}_{\{r>1\}}$. For $a \ge 1$,

$$\int_a^\infty bf(b)db = \int_a^\infty b^{1-l}db = \frac{a^{2-l}}{l-2},$$

where $l > 2$ is required to ensure the integrability here.

For $a \le 1$,

$$\int_a^\infty bf(b)db = \int_1^\infty b^{1-l}db + \int_a^1 b^{1-l_1}db = \frac{1}{l-2} + \begin{cases} \frac{1}{2-l_1}(1-a^{2-l_1}) & \text{if } l_1 < 2 \\ -\ln a & \text{if } l_1 = 2 \\ \frac{1}{l_1-2}(a^{2-l_1}-1) & \text{if } l_1 > 2 \end{cases}.$$

For $r \le 1$,

$$\begin{aligned} &\int_0^r a^{n-3} \int_a^\infty bf(b)dbda \\ = &\int_0^r \frac{1}{l-2}a^{n-3} + a^{n-3} \begin{cases} \frac{1}{2-l_1}(1-a^{2-l_1}) & \text{if } l_1 < 2 \\ -\ln a & \text{if } l_1 = 2 \ da \\ \frac{1}{l_1-2}(a^{2-l_1}-1) & \text{if } l_1 > 2 \end{cases} \\ = &\frac{r^{n-2}}{(l-2)(n-2)} + \begin{cases} \frac{r^{n-2}}{(2-l_1)(n-2)} - \frac{r^{n-l_1}}{(2-l_1)(n-l_1)} & \text{if } l_1 < 2 \\ \frac{r^{n-2}(-\ln r)}{n-2} + \frac{r^{n-2}}{(n-2)^2} & \text{if } l_1 = 2 \\ \frac{r^{n-l_1}}{(l_1-2)(n-l_1)} - \frac{r^{n-2}}{(l_1-2)(n-2)} & \text{if } 2 < l_1 < n \end{cases}, \end{aligned}$$

where we require $l_1 < n$ to guarantee the integrability.

For $r \ge 1$, since

$$\int_1^r a^{n-3} \int_a^\infty bf(b)dbda = \int_1^r \frac{a^{n-1-l}}{l-2}da = \begin{cases} \frac{r^{n-l}-1}{(l-2)(n-l)} & \text{if } 2 < l < n \\ \frac{\ln r}{n-2} & \text{if } l = n \\ \frac{1-r^{n-l}}{(l-2)(l-n)} & \text{if } l > n \end{cases},$$

then

$$
\begin{aligned}
&\int_0^r a^{n-3}\int_a^\infty bf(b)dbda\\
&=(\int_0^1+\int_1^r)a^{n-3}\int_a^\infty bf(b)dbda\\
&=\frac{1}{(l-2)(n-2)}+\begin{cases}\frac{1}{(2-l_1)(n-2)}-\frac{1}{(2-l_1)(n-l_1)} & \text{if } \ l_1<2\\ \frac{1}{(n-2)^2} & \text{if } \ l_1=2\\ \frac{1}{(l_1-2)(n-l_1)}-\frac{1}{(l_1-2)(n-2)} & \text{if } \ 2<l_1<n\end{cases}\\
&\quad+\begin{cases}\frac{r^{n-l}-1}{(l-2)(n-l)} & \text{if } \ 2<l<n\\ \frac{\ln r}{n-2} & \text{if } \ l=n\\ \frac{1-r^{n-l}}{(l-2)(l-n)} & \text{if } \ l>n\end{cases}\\
&=\frac{1}{n-2}(\frac{1}{l-2}+\frac{1}{n-l_1})+\begin{cases}\frac{r^{n-l}-1}{(l-2)(n-l)} & \text{if } \ 2<l<n\\ \frac{\ln r}{n-2} & \text{if } \ l=n\\ \frac{1-r^{n-l}}{(l-2)(l-n)} & \text{if } \ l>n.\end{cases}
\end{aligned}
$$

□

Lemma A.2. *Consider*

$$
\begin{cases}\partial_t u=\Delta u+f(x,t) & in \ \ \mathbb{R}^n\times(t_0,\infty)\\ u(\cdot,t_0)=0 & in \ \ \mathbb{R}^n\end{cases}
$$

where $n>4$, $|f(x,t)|\le C_f v(t)(|x|^{-l_1}\mathbf{1}_{\{|x|\le 1\}}+|x|^{-l}\mathbf{1}_{\{|x|>1\}})$, $l_1<n$, $l>2$. *u is given by*

$$
u(x,t)=\int_{t_0}^t\int_{\mathbb{R}^n}(4\pi(t-s))^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}f(y,s)dyds.
$$

Suppose that $0<v(t)\in C^1(t_0,\infty)$, *then for* $2<l\le n-2$, $l_1\le l$, $v'(t)\ge 0$, *we have*

$$
u(r,t)\le C_f\min\left\{2v(t)g(r),2r^{-l}\int_{t_0}^t v(s)ds\right\}\quad in \ \ \mathbb{R}^n\times(t_0,\infty);
$$

for $2<l\le n-2$, $0\le l_1\le l$, $v'(t)<0$, $-(v(t))^{-1}v'(t)\le C_v t^{-1}$, $(v(t))^{-1}\int_{t_0}^t v(s)ds\le C_v t$, $t_0\ge 2C_v\max\{C_1(n,l),C_2(n,l_1,l)\}$, *we have*

$$
u(r,t)\le C_f\min\left\{\max\left\{2,4(C_v)^2\frac{n-2}{n-l}\right\}v(t)g(r)\mathbf{1}_{\{r\le(2C_vC_1(n,l))^{-\frac12}t^{\frac12}\}},2r^{-l}\int_{t_0}^t v(s)ds\right\}.
$$

where

$$
C_1(n,l):=\frac{1}{(l-2)(n-l)},
$$

$$
C_2(n,l_1,l):=\frac{1}{(l-2)(n-2)}+\begin{cases}\frac{1}{(2-l_1)(n-2)} & if \ \ l_1<2\\ \frac{1}{n-2}+\frac{1}{(n-2)^2} & if \ \ l_1=2\\ \frac{1}{(l_1-2)(n-l_1)} & if \ \ 2<l_1<n\end{cases}.
$$

$-\Delta g=r^{-l_1}\mathbf{1}_{\{r\le 1\}}+r^{-l}\mathbf{1}_{\{r>1\}}$ *is given by Lemma A.1.*

Proof. Since

$$
|u(x,t)|\le C_f\int_{t_0}^t\int_{\mathbb{R}^n}[4\pi(t-s)]^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}v(s)\left(|y|^{-l_1}\mathbf{1}_{\{|y|\le 1\}}+|y|^{-l}\mathbf{1}_{\{|y|>1\}}\right)dyds,
$$

without loss of generality, we only need to consider the case $f(x,t)=v(t)(|x|^{-l_1}\mathbf{1}_{\{|x|\le 1\}}+|x|^{-l}\mathbf{1}_{\{|x|>1\}})$. With this radial right hand side, it is ready to get that $u(x,t)=u(|x|,t)$ is radial about the spatial variable.

Denote $r = |x|$, $Lg = \Delta g - \partial_t g + v(t)(r^{-l_1}\mathbf{1}_{\{r\le 1\}} + r^{-l}\mathbf{1}_{\{r>1\}})$. First we want to find a supersolution $\phi_1(r,t) = D_1 r^{-l}\int_{t_0}^t v(s)ds$ with $D_1 \ge 2$ in $\mathbb{R}^n$. Since $\Delta(r^{-l}) = l(l+2-n)r^{-l-2}$, then

$$\begin{aligned} L\phi_1 &= D_1 l(l+2-n)r^{-l-2}\int_{t_0}^t v(s)ds - D_1 r^{-l}v(t) + v(t)\left(r^{-l_1}\mathbf{1}_{\{r\le 1\}} + r^{-l}\mathbf{1}_{\{r>1\}}\right) \\ &\le D_1 l(l+2-n)r^{-l-2}\int_{t_0}^t v(s)ds - \frac{D_1}{2}r^{-l}v(t), \end{aligned}$$

when $l_1 \le l$. Take $D_1 = 2$. If $0 \le l \le n-2$, then $L\phi_1 \le 0$ in $\mathbb{R}^n \times (t_0,\infty)$. It follows

$$u(r,t) \le 2r^{-l}\int_{t_0}^t v(s)ds \quad \text{in} \quad \mathbb{R}^n \times (t_0,\infty).$$

Next, we want to improve the estimate of u in the region $|x| \lesssim t^{\frac{1}{2}}$. Set $-\Delta g = r^{-l_1}\mathbf{1}_{\{r\le 1\}} + r^{-l}\mathbf{1}_{\{r>1\}}$, where $g > 0$ is given by Lemma A.1. For $r \ge 1$, $l_1 \le l$, we have

$$g(r) \le C_1(n,l)r^{2-l}, \quad C_1(n,l) := \frac{1}{(l-2)(n-l)}.$$

For $r \le 1$,

$$g(r) \le C_2(n,l_1,l)\begin{cases} 1 & \text{if } \ l_1 < 2 \\ 1 - \ln r & \text{if } \ l_1 = 2 \\ r^{2-l_1} & \text{if } \ 2 < l_1 < n \end{cases}$$

where

$$C_2(n,l_1,l) := \frac{1}{(l-2)(n-2)} + \begin{cases} \frac{1}{(2-l_1)(n-2)} & \text{if } \ l_1 < 2 \\ \frac{1}{n-2} + \frac{1}{(n-2)^2} & \text{if } \ l_1 = 2 \\ \frac{1}{(l_1-2)(n-l_1)} & \text{if } \ 2 < l_1 < n \end{cases}.$$

Set $\phi_2 = D_2 v(t)g(r)$ with $D_2 \ge 2$. Then

$$\begin{aligned} L\phi_2 =& -D_2 v(t)(r^{-l_1}\mathbf{1}_{\{r\le 1\}} + r^{-l}\mathbf{1}_{\{r>1\}}) - D_2 v'(t)g(r) + v(t)(r^{-l_1}\mathbf{1}_{\{r\le 1\}} + r^{-l}\mathbf{1}_{\{r>1\}}) \\ \le& -\frac{D_2}{2}v(t)(r^{-l_1}\mathbf{1}_{\{r\le 1\}} + r^{-l}\mathbf{1}_{\{r>1\}}) - D_2 v'(t)g(r) \\ =& D_2 g(r)v(t)\left[-\frac{1}{2g(r)}(r^{-l_1}\mathbf{1}_{\{r\le 1\}} + r^{-l}\mathbf{1}_{\{r>1\}}) - \frac{v'(t)}{v(t)}\right]. \end{aligned}$$

If $v'(t) \ge 0$, then $L\phi_2 \le 0$ in $\mathbb{R}^n \times (t_0,\infty)$. Take $D_2 = 2$. Then

$$u(r,t) \le 2v(t)g(r) \quad \text{in} \quad \mathbb{R}^n \times (t_0,\infty).$$

Thus for $2 < l \le n-2$, $l_1 \le l$, $v'(t) \ge 0$, we have

$$u(r,t) \le \min\{2v(t)g(r), 2r^{-l}\int_{t_0}^t v(s)ds\} \quad \text{in} \quad \mathbb{R}^n \times (t_0,\infty).$$

For $v'(t) < 0$, we assume $-\frac{v'(t)}{v(t)} \le C_v t^{-1}$. Then, for $r > 1$, $\frac{r^{-l}}{2g(r)} \ge (2C_1(n,l))^{-1}r^{-2}$. So in $r \le r_0 := (2C_v C_1(n,l))^{-\frac{1}{2}}t^{\frac{1}{2}}$,

$$-\frac{r^{-l}}{2g(r)} - \frac{v'(t)}{v(t)} \le -(2C_1(n,l))^{-1}r^{-2} + C_v t^{-1} = (2C_1(n,l))^{-1}r^{-2}(-1 + 2C_1(n,l)C_v r^2 t^{-1}) \le 0.$$

For $r \le 1$, when $l_1 \ge 0$, $t_0 \ge 2C_v C_2(n,l_1,l)$, we have

$$-\frac{r^{-l_1}}{2g(r)} - \frac{v'(t)}{v(t)} \le -(2C_2(n,l_1,l))^{-1} + C_v t^{-1} \le 0.$$

We still need to find D_2 to guarantee $\phi_2(r_0,t) \ge u(r_0,t)$. It suffices to ensure $\phi_2(r_0,t) \ge \phi_1(r_0,t)$. That is $D_2 r_0^l g(r_0) \ge 2(v(t))^{-1}\int_{t_0}^t v(s)ds$.

By Lemma A.1, we know

$$g(r) \ge \frac{r^{2-l}}{(l-2)(n-2)} \quad \text{in} \quad r \ge 1.$$

Since $(v(t))^{-1}\int_{t_0}^t v(s)ds \le C_v t$, when $t_0 \ge 2C_vC_1(n,l)$, i.e. $r_0 \ge 1$, it suffices to ensure

$$D_2 r_0^l \frac{r_0^{2-l}}{(l-2)(n-2)} \ge 2C_v t.$$

That is

$$D_2 \ge 4(C(v))^2(l-2)(n-2)C_1(n,l) = 4(C_v)^2(n-l)^{-1}(n-2).$$

Take $D_2 = \max\{2, 4(C_v)^2(n-l)^{-1}(n-2)\}$. So ϕ_2 is a supersolution of u in $r \le (2C_vC_1(n,l))^{-\frac{1}{2}}t^{\frac{1}{2}}$.

Thus for $2 < l \le n-2$, $0 \le l_1 \le l$, $v'(t) < 0$, $-(v(t))^{-1}v'(t) \le C_v t^{-1}$, $(v(t))^{-1}\int_{t_0}^t v(s)ds \le C_v t$, $t_0 \ge 2C_v \max\{C_1(n,l), C_2(n,l_1,l)\}$,

$$u(r,t) \le \min\left\{\max\left\{2, 4(C_v)^2\frac{n-2}{n-l}\right\} v(t)g(r)\mathbf{1}_{\{r\le(2C_vC_1(n,l))^{-\frac{1}{2}}t^{\frac{1}{2}}\}}, 2r^{-l}\int_{t_0}^t v(s)ds\right\}.$$

□

Lemma A.3. *Assume $\int_0^1 x^a\langle\ln x\rangle^b dx < \infty$, that is, a, b satisfy either $a > -1$ or $a = -1$ and $b < -1$. For $0 \le x_0 \le x_1 \le \frac{1}{2}$, we have*

$$\int_{x_0}^{x_1} e^{-\lambda x}x^a(-\ln x)^b dx \lesssim \begin{cases} \begin{cases} x_1^{a+1}(-\ln x_1)^b & \text{if } a > -1 \\ (-\ln x_1)^{b+1} - (-\ln x_0)^{b+1} & \text{if } a = -1, b < -1 \end{cases} & \text{for } 0 \le \lambda \le x_1^{-1} \\ \frac{(\ln\lambda)^b}{\lambda^{a+1}} + \begin{cases} 0 & \text{if } a > -1 \\ (\ln\lambda)^{b+1} - (-\ln x_0)^{b+1} & \text{if } a = -1, b < -1 \end{cases} & \text{for } x_1^{-1} \le \lambda \le x_0^{-1} \\ \frac{(\ln\lambda)^b}{\lambda^{a+1}}e^{-\frac{x_0\lambda}{2}} & \text{for } \lambda \ge x_0^{-1} \end{cases}. \tag{A.2}$$

Specially, for a constant $C_* > 0$, $0 \le x_0 \le c_1$ where $c_1 = \min\{\frac{1}{2}, C_*\}$, we have

$$\int_{x_0}^{C_*} e^{-\lambda x}x^a\langle\ln x\rangle^b dx \lesssim \begin{cases} 1 & \text{for } 0 \le \lambda \le 2 \\ \frac{(\ln\lambda)^b}{\lambda^{a+1}} + \begin{cases} 0 & \text{if } a > -1 \\ (\ln\lambda)^{b+1} - (-\ln x_0)^{b+1} & \text{if } a = -1, b < -1 \end{cases} & \text{for } 2 \le \lambda \le x_0^{-1} \\ \frac{(\ln\lambda)^b}{\lambda^{a+1}}e^{-\frac{x_0\lambda}{2}} & \text{for } \lambda \ge x_0^{-1} \end{cases}.$$

For $x_0 = 0$,

$$\int_0^{C_*} e^{-\lambda x}x^a\langle\ln x\rangle^b dx \lesssim \begin{cases} 1 & \text{for } 0 \le \lambda \le 2 \\ \begin{cases} \frac{(\ln\lambda)^b}{\lambda^{a+1}} & \text{if } a > -1 \\ (\ln\lambda)^{b+1} & \text{if } a = -1, b < -1 \end{cases} & \text{for } 2 \le \lambda < \infty \end{cases}.$$

Proof. For $0 \le \lambda \le x_1^{-1}$,

$$\begin{aligned} \int_{x_0}^{x_1} e^{-\lambda x}x^a(-\ln x)^b dx &\sim \int_{x_0}^{x_1} x^a(-\ln x)^b dx \\ &\lesssim \begin{cases} x_1^{a+1}(-\ln x_1)^b & \text{if } a > -1 \\ (-\ln x_1)^{b+1} - (-\ln x_0)^{b+1} & \text{if } a = -1, b < -1 \end{cases}. \end{aligned} \tag{A.3}$$

For $\lambda \ge x_0^{-1}$,

$$\begin{aligned} \int_{x_0}^{x_1} e^{-\lambda x}x^a(-\ln x)^b dx &= \frac{1}{\lambda^{a+1}}\int_{x_0\lambda}^{x_1\lambda} e^{-z}z^a(\ln\lambda - \ln z)^b dz \\ &\lesssim \begin{cases} \frac{(\ln\lambda)^b}{\lambda^{a+1}}e^{-\frac{x_0\lambda}{2}} & \text{if } x_0^{-1} \le \lambda \le x_0^{-2} \\ \frac{1}{\lambda^{a+1}}e^{-\frac{3x_0\lambda}{4}} & \text{if } \lambda \ge x_0^{-2} \end{cases} \\ &\lesssim \frac{(\ln\lambda)^b}{\lambda^{a+1}}e^{-\frac{x_0\lambda}{2}}. \end{aligned} \tag{A.4}$$

In order to get the first "$\lesssim$", one needs the following estimate.

If $x_1\lambda \le \lambda^{\frac{1}{2}}$, that is $\lambda \le x_1^{-2}$,

$$\frac{1}{\lambda^{a+1}}\int_{x_0\lambda}^{x_1\lambda} e^{-z}z^a(\ln\lambda - \ln z)^b dz \sim \frac{(\ln\lambda)^b}{\lambda^{a+1}}\int_{x_0\lambda}^{x_1\lambda} e^{-z}z^a dz \lesssim \frac{(\ln\lambda)^b}{\lambda^{a+1}}e^{-\frac{x_0\lambda}{2}}. \tag{A.5}$$

If $x_0\lambda \geq \lambda^{\frac{1}{2}}$, that is $\lambda \geq x_0^{-2}$,

$$\frac{1}{\lambda^{a+1}} \int_{x_0\lambda}^{x_1\lambda} e^{-z} z^a (\ln\lambda - \ln z)^b dz \lesssim \frac{1}{\lambda^{a+1}} e^{-\frac{3x_0\lambda}{4}}, \tag{A.6}$$

since $-\ln x_1 \leq \ln\lambda - \ln z \leq \ln\lambda - \ln(x_0\lambda) \leq \frac{\ln\lambda}{2}$,

$$z^a \lesssim \begin{cases} 1 & \text{if } \ a \leq 0 \\ \lambda^a & \text{if } \ a > 0 \end{cases}, \quad (\ln\lambda - \ln z)^b \lesssim \begin{cases} 1 & \text{if } \ b \leq 0 \\ (\ln\lambda)^b & \text{if } \ b > 0 \end{cases}.$$

If $x_0\lambda \leq \lambda^{\frac{1}{2}} \leq x_1\lambda$, that is $x_1^{-2} \leq \lambda \leq x_0^{-2}$,

$$\frac{1}{\lambda^{a+1}} \int_{x_0\lambda}^{x_1\lambda} e^{-z} z^a (\ln\lambda - \ln z)^b dz \lesssim \frac{(\ln\lambda)^b}{\lambda^{a+1}} e^{-\frac{x_0\lambda}{2}},$$

since by (A.5),

$$\frac{1}{\lambda^{a+1}} \int_{x_0\lambda}^{\lambda^{\frac{1}{2}}} e^{-z} z^a (\ln\lambda - \ln z)^b dz \lesssim \frac{(\ln\lambda)^b}{\lambda^{a+1}} e^{-\frac{x_0\lambda}{2}},$$

and by (A.6),

$$\frac{1}{\lambda^{a+1}} \int_{\lambda^{\frac{1}{2}}}^{x_1\lambda} e^{-z} z^a (\ln\lambda - \ln z)^b dz \lesssim \frac{1}{\lambda^{a+1}} e^{-\frac{3\lambda^{\frac{1}{2}}}{4}}.$$

With the restriction $x_0^{-1} \leq \lambda \leq x_0^{-2}$, one has $\frac{1}{\lambda^{a+1}} e^{-\frac{3\lambda^{\frac{1}{2}}}{4}} \lesssim \frac{(\ln\lambda)^b}{\lambda^{a+1}} e^{-\frac{x_0\lambda}{2}}$.

For $x_1^{-1} \leq \lambda \leq x_0^{-1}$, we have

$$\int_{x_0}^{x_1} e^{-\lambda x} x^a (-\ln x)^b dx \lesssim \frac{(\ln\lambda)^b}{\lambda^{a+1}} + \begin{cases} 0 & \text{if } \ a > -1 \\ (\ln\lambda)^{b+1} - (-\ln x_0)^{b+1} & \text{if } \ a = -1, b < -1 \end{cases}.$$

Indeed, by (A.3),

$$\int_{x_0}^{\frac{1}{\lambda}} e^{-\lambda x} x^a (-\ln x)^b dx \lesssim \begin{cases} \frac{(\ln\lambda)^b}{\lambda^{a+1}} & \text{if } \ a > -1 \\ (\ln\lambda)^{1+b} - (-\ln x_0)^{1+b} & \text{if } \ a = -1, b < -1 \end{cases},$$

and by (A.4),

$$\int_{\frac{1}{\lambda}}^{x_1} e^{-\lambda x} x^a (-\ln x)^b dx \lesssim \frac{(\ln\lambda)^b}{\lambda^{a+1}}.$$

This complete the proof of general case (A.2).

For the special cases, if $C_* \leq \frac{1}{2}$, by (A.2),

$$\begin{aligned} \int_{x_0}^{C_*} e^{-\lambda x} x^a \langle \ln x\rangle^b dx &\lesssim \begin{cases} 1 & \text{for } \ 0 \leq \lambda \leq C_*^{-1} \\ \frac{(\ln\lambda)^b}{\lambda^{a+1}} + \begin{cases} 0 & \text{if } \ a > -1 \\ (\ln\lambda)^{b+1} - (-\ln x_0)^{b+1} & \text{if } \ a = -1, b < -1 \end{cases} & \text{for } \ C_*^{-1} \leq \lambda \leq x_0^{-1} \\ \frac{(\ln\lambda)^b}{\lambda^{a+1}} e^{-\frac{x_0\lambda}{2}} & \text{for } \ \lambda \geq x_0^{-1} \end{cases} \\ &\lesssim \begin{cases} 1 & \text{for } \ 0 \leq \lambda \leq 2 \\ \frac{(\ln\lambda)^b}{\lambda^{a+1}} + \begin{cases} 0 & \text{if } \ a > -1 \\ (\ln\lambda)^{b+1} - (-\ln x_0)^{b+1} & \text{if } \ a = -1, b < -1 \end{cases} & \text{for } \ 2 \leq \lambda \leq x_0^{-1} \\ \frac{(\ln\lambda)^b}{\lambda^{a+1}} e^{-\frac{x_0\lambda}{2}} & \text{for } \ \lambda \geq x_0^{-1} \end{cases}. \end{aligned}$$

If $C_* \geq \frac{1}{2}$,

$$\int_{x_0}^{C_*} e^{-\lambda x} x^a \langle \ln x\rangle^b dx = \left(\int_{x_0}^{\frac{1}{2}} + \int_{\frac{1}{2}}^{C_*}\right) e^{-\lambda x} x^a \langle \ln x\rangle^b dx.$$

The first part can be estimated by the same way as above. For the second part,

$$\int_{\frac{1}{2}}^{C_*} e^{-\lambda x} x^a \langle \ln x\rangle^b dx \lesssim e^{-\frac{\lambda}{2}}.$$

This concludes the proof.

$\square$

Appendix B. Convolution estimates in finite time

B.1. **Preliminary.** For $s \le t$ and $t \le t_* \le T$,

$$\begin{cases} \frac{T-s}{2} \le t-s \le T-s & \text{for } s \le t-(T-t) \\ T-t \le T-s \le 2(T-t) & \text{for } s \ge t-(T-t) \\ \frac{t_*-s}{2} \le t-s \le t_*-s & \text{for } s \le t-(t_*-t) \\ t_*-t \le t_*-s \le 2(t_*-t) & \text{for } s \ge t-(t_*-t). \end{cases} \tag{B.1}$$

For any $x \in \mathbb{R}^d$, $p > 0$, $b \ge 0$ and $L > 0$, $0 \le L_1 \le L_2 \le \infty$,

$$\begin{aligned} &\int_{\mathbb{R}^d} e^{-c(\frac{|x-y|}{\sqrt{L}})^p} |y-q|^{-b} \mathbf{1}_{\{L_1 \le |y-q| \le L_2\}} dy \\ &= L^{\frac{d}{2}-\frac{b}{2}} \int_{\mathbb{R}^d} e^{-c|\tilde{x}-z|^p} |z|^{-b} \mathbf{1}_{\{L_1 L^{-\frac12} \le |z| \le L_2 L^{-\frac12}\}} dz \\ &\le L^{\frac{d}{2}-\frac{b}{2}} \int_{\mathbb{R}^d} e^{-c|\tilde{x}-z|^p} \min\{|z|^{-b}, (L_1 L^{-\frac12})^{-b}\} \mathbf{1}_{\{|z| \le L_2 L^{-\frac12}\}} dz \\ &\le L^{\frac{d}{2}-\frac{b}{2}} \int_{\mathbb{R}^d} e^{-c|z|^p} \min\{|z|^{-b}, (L_1 L^{-\frac12})^{-b}\} \mathbf{1}_{\{|z| \le L_2 L^{-\frac12}\}} dz \\ &= L^{\frac{d}{2}-\frac{b}{2}} \left\{ \int_{\mathbb{R}^d} e^{-c|z|^p} \left[(L_1 L^{-\frac12})^{-b} \mathbf{1}_{\{|z| \le L_1 L^{-\frac12}\}} + |z|^{-b} \mathbf{1}_{\{L_1 L^{-\frac12} < |z| \le L_2 L^{-\frac12}\}} \right] dz \right\} \\ &\lesssim \begin{cases} L^{\frac{d}{2}} L_1^{-b} & \text{if } L \le L_1^2 \\ \begin{cases} L^{\frac{d}{2}-\frac{b}{2}} & \text{if } b < d \\ \langle \ln(\frac{L}{L_1^2}) \rangle & \text{if } b = d \\ L_1^{d-b} & \text{if } b > d \end{cases} & \text{if } L_1^2 < L \le L_2^2 \\ \begin{cases} L_2^{d-b} & \text{if } b < d \\ \langle \ln(\frac{L_2}{L_1}) \rangle & \text{if } b = d \\ L_1^{d-b} & \text{if } b > d \end{cases} & \text{if } L > L_2^2 \end{cases} \end{aligned} \tag{B.2}$$

where $\tilde{x} = (x-q)L^{-\frac12}$.

Specially, for $b \ge 0$, $L_3 \ge CL > 0$ where $C > 0$ is a constant,

$$\int_{\mathbb{R}^d} e^{-c(\frac{|x-y|}{\sqrt{L}})^p} |y-q|^{-b} \mathbf{1}_{\{|y-q| \ge \sqrt{L_3}\}} dy \lesssim L^{\frac{d}{2}} L_3^{-\frac{b}{2}}. \tag{B.3}$$

Next we want to establish the basic calculation about time variable. Set

$$g(s) = \begin{cases} (t-s)^{\frac{d}{2}-d_*} L_1^{-b} & \text{if } t-s \le L_1^2 \\ \begin{cases} (t-s)^{\frac{d}{2}-\frac{b}{2}-d_*} & \text{if } b < d \\ (t-s)^{-d_*} \langle \ln(\frac{t-s}{L_1^2}) \rangle & \text{if } b = d \\ (t-s)^{-d_*} L_1^{d-b} & \text{if } b > d \end{cases} & \text{if } L_1^2 < t-s \le L_2^2 \\ \begin{cases} (t-s)^{-d_*} L_2^{d-b} & \text{if } b < d \\ (t-s)^{-d_*} \langle \ln(\frac{L_2}{L_1}) \rangle & \text{if } b = d \\ (t-s)^{-d_*} L_1^{d-b} & \text{if } b > d \end{cases} & \text{if } t-s > L_2^2. \end{cases}$$

Claim: for $\delta > 0$, if $\frac{d}{2} + 1 > d_* > \frac{d}{2} + 1 - \delta$,

$$x^{-\delta} \int_{t-x}^{t} g(s) ds \le \infty,$$

and for $\delta = 0$, $d_* < \frac{d}{2}+1$,

$$\int_{t-x}^{t} g(s)ds \lesssim \begin{cases} \begin{cases} (\max\{x, L_2^2\})^{1-d_*} L_2^{d-b} & \text{if } d_* < 1 \\ \langle \ln(\frac{\max\{x,L_2^2\}}{L_2^2})\rangle L_2^{d-b} & \text{if } d_* = 1 \\ L_2^{d+2-b-2d_*} & \text{if } 1 < d_* < 1+\frac{d-b}{2} \\ \langle \ln(\frac{L_2}{L_1})\rangle & \text{if } d_* = 1+\frac{d-b}{2} \\ L_1^{d+2-b-2d_*} & \text{if } d_* > 1+\frac{d-b}{2} \end{cases} & \text{if } b < d \\ \begin{cases} (\max\{x, L_2^2\})^{1-d_*} \langle \ln(\frac{L_2}{L_1})\rangle & \text{if } d_* < 1 \\ \langle \ln(\frac{\max\{x,L_2^2\}}{L_1^2})\rangle \langle \ln(\frac{L_2}{L_1})\rangle & \text{if } d_* = 1 \\ L_1^{2-2d_*} & \text{if } d_* > 1 \end{cases} & \text{if } b = d \\ \begin{cases} (\max\{x, L_2^2\})^{1-d_*} L_1^{d-b} & \text{if } d_* < 1 \\ \langle \ln(\frac{\max\{x,L_2^2\}}{L_1^2})\rangle L_1^{d-b} & \text{if } d_* = 1 \\ L_1^{d+2-b-2d_*} & \text{if } d_* > 1 \end{cases} & \text{if } b > d. \end{cases} \tag{B.4}$$

and if $\delta > 0$, $d_* \le \frac{d}{2}+1-\delta$,

$$x^{-\delta}\int_{t-x}^{t} g(s)ds \lesssim \begin{cases} \begin{cases} (\max\{x, L_2^2\})^{1-d_*-\delta} L_2^{d-b} & \text{if } d_* \le 1-\delta \\ L_2^{d+2-b-2d_*-2\delta} & \text{if } 1-\delta < d_* \le 1+\frac{d-b}{2}-\delta \\ L_1^{d+2-b-2d_*-2\delta} & \text{if } d_* > 1+\frac{d-b}{2}-\delta \end{cases} & \text{if } b < d \\ \begin{cases} (\max\{x, L_2^2\})^{1-d_*-\delta} \langle \ln(\frac{L_2}{L_1})\rangle & \text{if } d_* \le 1-\delta \\ L_1^{2-2d_*-2\delta} & \text{if } d_* > 1-\delta \end{cases} & \text{if } b = d \\ \begin{cases} (\max\{x, L_2^2\})^{1-d_*-\delta} L_1^{d-b} & \text{if } d_* \le 1-\delta \\ L_1^{d+2-b-2d_*-2\delta} & \text{if } d_* > 1-\delta \end{cases} & \text{if } b > d. \end{cases} \tag{B.5}$$

Proof. For $x \le L_1^2$,

$$\int_{t-x}^{t} g(s)ds \lesssim x^{\frac{d}{2}+1-d_*} L_1^{-b}$$

under the assumption $d_* < \frac{d}{2}+1$.

For $L_1^2 < x \le L_2^2$,

$$\begin{aligned}
\int_{t-x}^{t} g(s)ds &= \Big(\int_{t-L_1^2}^{t} + \int_{t-x}^{t-L_1^2}\Big)g(s)ds \\
&\lesssim L_1^{d+2-b-2d_*} + \begin{cases}
\begin{cases} x^{\frac{d}{2}+1-\frac{b}{2}-d_*} & b < d+2-2d_* \\ \ln(\frac{x}{L_1^2}) & b = d+2-2d_* \\ L_1^{d+2-b-2d_*} & b > d+2-2d_* \end{cases} & \text{if } \ b < d \\
\begin{cases} x^{1-d_*}\langle \ln(\frac{x}{L_1^2})\rangle & \text{if } \ d_* < 1 \\ \langle \ln(\frac{x}{L_1^2})\rangle^2 & \text{if } \ d_* = 1 \\ L_1^{2-2d_*} & \text{if } \ d_* > 1 \end{cases} & \text{if } \ b = d \\
\begin{cases} x^{1-d_*}L_1^{d-b} & \text{if } \ d_* < 1 \\ L_1^{d-b}\ln(\frac{x}{L_1^2}) & \text{if } \ d_* = 1 \\ L_1^{d+2-b-2d_*} & \text{if } \ d_* > 1 \end{cases} & \text{if } \ b > d
\end{cases} \\
&\sim \begin{cases}
\begin{cases} x^{\frac{d}{2}+1-\frac{b}{2}-d_*} & b < d+2-2d_* \\ \langle\ln(\frac{x}{L_1^2})\rangle & b = d+2-2d_* \\ L_1^{d+2-b-2d_*} & b > d+2-2d_* \end{cases} & \text{if } \ b < d \\
\begin{cases} x^{1-d_*}\langle \ln(\frac{x}{L_1^2})\rangle & \text{if } \ d_* < 1 \\ \langle \ln(\frac{x}{L_1^2})\rangle^2 & \text{if } \ d_* = 1 \\ L_1^{2-2d_*} & \text{if } \ d_* > 1 \end{cases} & \text{if } \ b = d \\
\begin{cases} x^{1-d_*}L_1^{d-b} & \text{if } \ d_* < 1 \\ L_1^{d-b}\langle\ln(\frac{x}{L_1^2})\rangle & \text{if } \ d_* = 1 \\ L_1^{d+2-b-2d_*} & \text{if } \ d_* > 1 \end{cases} & \text{if } \ b > d
\end{cases}
\end{aligned}$$

where for $b = d$, we used

$$\int_{t-x}^{t-L_1^2} (t-s)^{-d_*}\langle \ln(\frac{t-s}{L_1^2})\rangle ds = L_1^{2-2d_*}\int_1^{\frac{x}{L_1^2}} z^{-d_*}\langle \ln z\rangle dz \lesssim \begin{cases} x^{1-d_*}\langle \ln(\frac{x}{L_1^2})\rangle & \text{if } \ d_* < 1 \\ \langle \ln(\frac{x}{L_1^2})\rangle^2 & \text{if } \ d_* = 1 \\ L_1^{2-2d_*} & \text{if } \ d_* > 1. \end{cases}$$

For $x > L_2^2$,

$$\int_{t-x}^{t} g(s)ds = (\int_{t-L_2^2}^{t} + \int_{t-x}^{t-L_2^2})g(s)ds$$

$$\lesssim \begin{cases} \begin{cases} L_2^{d+2-b-2d_*} & b < d+2-2d_* \\ \langle \ln(\frac{L_2}{L_1})\rangle & b = d+2-2d_* \\ L_1^{d+2-b-2d_*} & b > d+2-2d_* \end{cases} & \text{if} \;\; b < d \\ \begin{cases} L_2^{2-2d_*}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if} \;\; d_* < 1 \\ \langle \ln(\frac{L_2}{L_1})\rangle^2 & \text{if} \;\; d_* = 1 \\ L_1^{2-2d_*} & \text{if} \;\; d_* > 1 \end{cases} & \text{if} \;\; b = d \\ \begin{cases} L_2^{2-2d_*}L_1^{d-b} & \text{if} \;\; d_* < 1 \\ L_1^{d-b}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if} \;\; d_* = 1 \\ L_1^{d+2-b-2d_*} & \text{if} \;\; d_* > 1 \end{cases} & \text{if} \;\; b > d \end{cases} + \begin{cases} \begin{cases} x^{1-d_*}L_2^{d-b} & \text{if} \;\; d_* < 1 \\ \ln(\frac{x}{L_2^2})L_2^{d-b} & \text{if} \;\; d_* = 1 \\ L_2^{d+2-b-2d_*} & \text{if} \;\; d_* > 1 \end{cases} & \text{if} \;\; b < d \\ \begin{cases} x^{1-d_*}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if} \;\; d_* < 1 \\ \ln(\frac{x}{L_2^2})\langle \ln(\frac{L_2}{L_1})\rangle & \text{if} \;\; d_* = 1 \\ L_2^{2-2d_*}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if} \;\; d_* > 1 \end{cases} & \text{if} \;\; b = d \\ \begin{cases} x^{1-d_*}L_1^{d-b} & \text{if} \;\; d_* < 1 \\ \ln(\frac{x}{L_2^2})L_1^{d-b} & \text{if} \;\; d_* = 1 \\ L_2^{2-2d_*}L_1^{d-b} & \text{if} \;\; d_* > 1 \end{cases} & \text{if} \;\; b > d \end{cases}$$

$$\sim \begin{cases} \begin{cases} x^{1-d_*}L_2^{d-b} & \text{if} \;\; d_* < 1 \\ L_2^{d-b}\langle \ln(\frac{x}{L_2^2})\rangle & \text{if} \;\; d_* = 1 \\ L_2^{d+2-b-2d_*} & \text{if} \;\; 1 < d_* < 1+\frac{d-b}{2} \\ \langle \ln(\frac{L_2}{L_1})\rangle & \text{if} \;\; d_* = 1+\frac{d-b}{2} \\ L_1^{d+2-b-2d_*} & \text{if} \;\; d_* > 1+\frac{d-b}{2} \end{cases} & \text{if} \;\; b < d \\ \begin{cases} x^{1-d_*}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if} \;\; d_* < 1 \\ \langle \ln(\frac{x}{L_1^2})\rangle\langle \ln(\frac{L_2}{L_1})\rangle & \text{if} \;\; d_* = 1 \\ L_1^{2-2d_*} & \text{if} \;\; d_* > 1 \end{cases} & \text{if} \;\; b = d \\ \begin{cases} x^{1-d_*}L_1^{d-b} & \text{if} \;\; d_* < 1 \\ L_1^{d-b}\langle \ln(\frac{x}{L_1^2})\rangle & \text{if} \;\; d_* = 1 \\ L_1^{d+2-b-2d_*} & \text{if} \;\; d_* > 1 \end{cases} & \text{if} \;\; b > d. \end{cases}$$

Thus, for $d_* < \frac{d}{2}+1$,

$$\int_{t-x}^{t} g(s)ds \lesssim \begin{cases} x^{\frac{d}{2}+1-d_*}L_1^{-b} & \text{if } \ x \le L_1^2 \\ \begin{cases} \begin{cases} x^{\frac{d}{2}+1-\frac{b}{2}-d_*} & b < d+2-2d_* \\ \langle \ln(\frac{x}{L_1^2})\rangle & b = d+2-2d_* \\ L_1^{d+2-b-2d_*} & b > d+2-2d_* \end{cases} & \text{if } \ b < d \\ \begin{cases} x^{1-d_*}\langle \ln(\frac{x}{L_1^2})\rangle & \text{if } \ d_* < 1 \\ \langle \ln(\frac{x}{L_1^2})\rangle^2 & \text{if } \ d_* = 1 \\ L_1^{2-2d_*} & \text{if } \ d_* > 1 \end{cases} & \text{if } \ b = d \\ \begin{cases} x^{1-d_*}L_1^{d-b} & \text{if } \ d_* < 1 \\ L_1^{d-b}\langle \ln(\frac{x}{L_1^2})\rangle & \text{if } \ d_* = 1 \\ L_1^{d+2-b-2d_*} & \text{if } \ d_* > 1 \end{cases} & \text{if } \ b > d \end{cases} & \text{if } \ L_1^2 < x \le L_2^2 \\ \begin{cases} \begin{cases} x^{1-d_*}L_2^{d-b} & \text{if } \ d_* < 1 \\ L_2^{d-b}\langle \ln(\frac{x}{L_2^2})\rangle & \text{if } \ d_* = 1 \\ L_2^{d+2-b-2d_*} & \text{if } \ 1 < d_* < 1+\frac{d-b}{2} \\ \langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \ d_* = 1+\frac{d-b}{2} \\ L_1^{d+2-b-2d_*} & \text{if } \ d_* > 1+\frac{d-b}{2} \end{cases} & \text{if } \ b < d \\ \begin{cases} x^{1-d_*}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \ d_* < 1 \\ \langle \ln(\frac{x}{L_1^2})\rangle\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \ d_* = 1 \\ L_1^{2-2d_*} & \text{if } \ d_* > 1 \end{cases} & \text{if } \ b = d \\ \begin{cases} x^{1-d_*}L_1^{d-b} & \text{if } \ d_* < 1 \\ L_1^{d-b}\langle \ln(\frac{x}{L_1^2})\rangle & \text{if } \ d_* = 1 \\ L_1^{d+2-b-2d_*} & \text{if } \ d_* > 1 \end{cases} & \text{if } \ b > d \end{cases} & \text{if } \ x > L_2^2. \end{cases}$$

Then

$$
\begin{aligned}
&x^{-\delta}\int_{t-x}^{t} g(s)ds\\
\gtrsim&\begin{cases}
\begin{cases} L_1^{d+2-b-2d_*-2\delta} & \text{if } \ \delta \le \frac{d}{2}+1-d_* \\ \infty & \text{if } \ \delta > \frac{d}{2}+1-d_* \end{cases} & \text{if } \ x \le L_1^2 \\
\begin{cases}
\begin{cases}
\begin{cases} L_2^{d+2-b-2d_*-2\delta} & \text{if } \ b \le d+2-2d_*-2\delta \\ L_1^{d+2-b-2d_*-2\delta} & \text{if } \ b > d+2-2d_*-2\delta \end{cases} & b < d+2-2d_* \\
\begin{cases} L_2^{-2\delta}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \ \delta \le 0 \\ L_1^{-2\delta} & \text{if } \ \delta > 0 \end{cases} & b = d+2-2d_* \\
\begin{cases} L_2^{-2\delta}L_1^{d+2-b-2d_*} & \text{if } \ \delta \le 0 \\ L_1^{d+2-b-2d_*-2\delta} & \text{if } \ \delta > 0 \end{cases} & b > d+2-2d_*
\end{cases} & \text{if } \ b < d \\
\begin{cases}
\begin{cases} L_2^{2-2d_*-2\delta}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \ \delta \le 1-d_* \\ L_1^{2-2d_*-2\delta} & \text{if } \ \delta > 1-d_* \end{cases} & \text{if } \ d_* < 1 \\
\begin{cases} L_2^{-2\delta}\langle \ln(\frac{L_2}{L_1})\rangle^2 & \text{if } \ \delta \le 0 \\ L_1^{-2\delta} & \text{if } \ \delta > 0 \end{cases} & \text{if } \ d_* = 1 \\
\begin{cases} L_2^{-2\delta}L_1^{2-2d_*} & \text{if } \ \delta \le 0 \\ L_1^{2-2d_*-2\delta} & \text{if } \ \delta > 0 \end{cases} & \text{if } \ d_* > 1
\end{cases} & \text{if } \ b = d \\
\begin{cases}
\begin{cases} L_2^{2-2d_*-2\delta}L_1^{d-b} & \text{if } \ \delta \le 1-d_* \\ L_1^{d+2-b-2d_*-2\delta} & \text{if } \ \delta > 1-d_* \end{cases} & \text{if } \ d_* < 1 \\
\begin{cases} L_2^{-2\delta}L_1^{d-b}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \ \delta \le 0 \\ L_1^{d-b-2\delta} & \text{if } \ \delta > 0 \end{cases} & \text{if } \ d_* = 1 \\
\begin{cases} L_2^{-2\delta}L_1^{d+2-b-2d_*} & \text{if } \ \delta \le 0 \\ L_1^{d+2-b-2d_*-2\delta} & \text{if } \ \delta > 0 \end{cases} & \text{if } \ d_* > 1
\end{cases} & \text{if } \ b > d
\end{cases} & \text{if } \ L_1^2 < x \le L_2^2.
\end{cases}
\end{aligned}
$$

For $x > L_2^2$,

$$
x^{-\delta}\int_{t-x}^{t} g(s)ds \lesssim \begin{cases}
\begin{cases}
\begin{cases} x^{1-d_*-\delta}L_2^{d-b} & \text{if } \delta \le 1-d_* \\ L_2^{d+2-b-2d_*-2\delta} & \text{if } \delta > 1-d_* \end{cases} & \text{if } d_* < 1 \\
\begin{cases} x^{-\delta}\langle \ln(\frac{x}{L_2^2})\rangle L_2^{d-b} & \text{if } \delta \le 0 \\ L_2^{d-b-2\delta} & \text{if } \delta > 0 \end{cases} & \text{if } d_* = 1 \\
\begin{cases} x^{-\delta}L_2^{d+2-b-2d_*} & \text{if } \delta \le 0 \\ L_2^{d+2-b-2d_*-2\delta} & \text{if } \delta > 0 \end{cases} & \text{if } 1 < d_* < 1+\frac{d-b}{2} \\
\begin{cases} x^{-\delta}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \delta \le 0 \\ L_2^{-2\delta}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \delta > 0 \end{cases} & \text{if } d_* = 1+\frac{d-b}{2} \\
\begin{cases} x^{-\delta}L_1^{d+2-b-2d_*} & \text{if } \delta \le 0 \\ L_2^{-2\delta}L_1^{d+2-b-2d_*} & \text{if } \delta > 0 \end{cases} & \text{if } d_* > 1+\frac{d-b}{2}
\end{cases} & \text{if } b < d \\
\begin{cases}
\begin{cases} x^{1-d_*-\delta}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \delta \le 1-d_* \\ L_2^{2-2d_*-2\delta}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \delta > 1-d_* \end{cases} & \text{if } d_* < 1 \\
\begin{cases} x^{-\delta}\langle \ln(\frac{x}{L_1^2})\rangle\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \delta \le 0 \\ L_2^{-2\delta}\langle \ln(\frac{L_2}{L_1})\rangle^2 & \text{if } \delta > 0 \end{cases} & \text{if } d_* = 1 \\
\begin{cases} x^{-\delta}L_1^{2-2d_*} & \text{if } \delta \le 0 \\ L_2^{-2\delta}L_1^{2-2d_*} & \text{if } \delta > 0 \end{cases} & \text{if } d_* > 1
\end{cases} & \text{if } b = d \\
\begin{cases}
\begin{cases} x^{1-d_*-\delta}L_1^{d-b} & \text{if } \delta \le 1-d_* \\ L_2^{2-2d_*-2\delta}L_1^{d-b} & \text{if } \delta > 1-d_* \end{cases} & \text{if } d_* < 1 \\
\begin{cases} x^{-\delta}\langle \ln(\frac{x}{L_1^2})\rangle L_1^{d-b} & \text{if } \delta \le 0 \\ L_2^{-2\delta}L_1^{d-b}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \delta > 0 \end{cases} & \text{if } d_* = 1 \\
\begin{cases} x^{-\delta}L_1^{d+2-b-2d_*} & \text{if } \delta \le 0 \\ L_2^{-2\delta}L_1^{d+2-b-2d_*} & \text{if } \delta > 0 \end{cases} & \text{if } d_* > 1
\end{cases} & \text{if } b > d.
\end{cases}
$$

Specially, for $\delta = 0$,

$$
\int_{t-x}^{t} g(s)ds \lesssim \begin{cases}
\begin{cases} L_1^{d+2-b-2d_*} & \text{if } d_* \le \frac{d}{2}+1 \\ \infty & \text{if } d_* > \frac{d}{2}+1 \end{cases} & \text{if } x \le L_1^2 \\
\begin{cases}
\begin{cases} L_2^{d+2-b-2d_*} & \text{if } d_* < 1+\frac{d-b}{2} \\ \langle \ln(\frac{L_2}{L_1})\rangle & \text{if } d_* = 1+\frac{d-b}{2} \\ L_1^{d+2-b-2d_*} & \text{if } d_* > 1+\frac{d-b}{2} \end{cases} & \text{if } b < d \\
\begin{cases} L_2^{2-2d_*}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } d_* < 1 \\ \langle \ln(\frac{L_2}{L_1})\rangle^2 & \text{if } d_* = 1 \\ L_1^{2-2d_*} & \text{if } d_* > 1 \end{cases} & \text{if } b = d \\
\begin{cases} L_2^{2-2d_*}L_1^{d-b} & \text{if } d_* < 1 \\ L_1^{d-b}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } d_* = 1 \\ L_1^{d+2-b-2d_*} & \text{if } d_* > 1 \end{cases} & \text{if } b > d
\end{cases} & \text{if } L_1^2 < x \le L_2^2,
\end{cases}
$$

For $x > L_2^2$,

$$\int_{t-x}^{t} g(s)ds \lesssim \begin{cases} \begin{cases} x^{1-d_*}L_2^{d-b} & \text{if } \ d_* < 1 \\ \langle \ln(\frac{x}{L_2^2})\rangle L_2^{d-b} & \text{if } \ d_* = 1 \\ L_2^{d+2-b-2d_*} & \text{if } \ 1 < d_* < 1 + \frac{d-b}{2} \\ \langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \ d_* = 1 + \frac{d-b}{2} \\ L_1^{d+2-b-2d_*} & \text{if } \ d_* > 1 + \frac{d-b}{2} \end{cases} & \text{if } \ b < d \\ \begin{cases} x^{1-d_*}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \ d_* < 1 \\ \langle \ln(\frac{x}{L_1^2})\rangle \langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \ d_* = 1 \\ L_1^{2-2d_*} & \text{if } \ d_* > 1 \end{cases} & \text{if } \ b = d \\ \begin{cases} x^{1-d_*}L_1^{d-b} & \text{if } \ d_* < 1 \\ \langle \ln(\frac{x}{L_1^2})\rangle L_1^{d-b} & \text{if } \ d_* = 1 \\ L_1^{d+2-b-2d_*} & \text{if } \ d_* > 1 \end{cases} & \text{if } \ b > d. \end{cases}$$

For $\delta > 0$,

$$x^{-\delta}\int_{t-x}^{t} g(s)ds \lesssim \begin{cases} \begin{cases} L_1^{d+2-b-2d_*-2\delta} & \text{if } \ \delta \le \frac{d}{2} + 1 - d_* \\ \infty & \text{if } \ \delta > \frac{d}{2} + 1 - d_* \end{cases} & \text{if } \ x \le L_1^2 \\ \begin{cases} \begin{cases} L_2^{d+2-b-2d_*-2\delta} & \text{if } \ d_* \le 1 + \frac{d-b}{2} - \delta \\ L_1^{d+2-b-2d_*-2\delta} & \text{if } \ d_* > 1 + \frac{d-b}{2} - \delta \end{cases} & \text{if } \ b < d \\ \begin{cases} L_2^{2-2d_*-2\delta}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \ \delta \le 1 - d_* \\ L_1^{2-2d_*-2\delta} & \text{if } \ \delta > 1 - d_* \end{cases} & \text{if } \ b = d \\ \begin{cases} L_2^{2-2d_*-2\delta}L_1^{d-b} & \text{if } \ \delta \le 1 - d_* \\ L_1^{d+2-b-2d_*-2\delta} & \text{if } \ \delta > 1 - d_* \end{cases} & \text{if } \ b > d \end{cases} & \text{if } \ L_1^2 < x \le L_2^2. \end{cases}$$

For $x > L_2^2$,

$$x^{-\delta}\int_{t-x}^{t} g(s)ds \lesssim \begin{cases} \begin{cases} \begin{cases} x^{1-d_*-\delta}L_2^{d-b} & \text{if } \ \delta \le 1 - d_* \\ L_2^{d+2-b-2d_*-2\delta} & \text{if } \ \delta > 1 - d_* \end{cases} & \text{if } \ d_* < 1 + \frac{d-b}{2} \\ L_2^{-2\delta}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \ d_* = 1 + \frac{d-b}{2} \\ L_2^{-2\delta}L_1^{d+2-b-2d_*} & \text{if } \ d_* > 1 + \frac{d-b}{2} \end{cases} & \text{if } \ b < d \\ \begin{cases} \begin{cases} x^{1-d_*-\delta}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \ \delta \le 1 - d_* \\ L_2^{2-2d_*-2\delta}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \ \delta > 1 - d_* \end{cases} & \text{if } \ d_* < 1 \\ L_2^{-2\delta}\langle \ln(\frac{L_2}{L_1})\rangle^2 & \text{if } \ d_* = 1 \\ L_2^{-2\delta}L_1^{2-2d_*} & \text{if } \ d_* > 1 \end{cases} & \text{if } \ b = d \\ \begin{cases} \begin{cases} x^{1-d_*-\delta}L_1^{d-b} & \text{if } \ \delta \le 1 - d_* \\ L_2^{2-2d_*-2\delta}L_1^{d-b} & \text{if } \ \delta > 1 - d_* \end{cases} & \text{if } \ d_* < 1 \\ L_2^{-2\delta}L_1^{d-b}\langle \ln(\frac{L_2}{L_1})\rangle & \text{if } \ d_* = 1 \\ L_2^{-2\delta}L_1^{d+2-b-2d_*} & \text{if } \ d_* > 1 \end{cases} & \text{if } \ b > d. \end{cases}$$

□

• Throughout this section, we assume

$$v(t) = C_v(T)(T-t)^{m_1}(\ln(T-t))^{m_2}(\ln\ln(T-t))^{m_3}\cdots, \ l_i(t) = C_2(T)(T-t)^{k_{i1}}(\ln(T-t))^{k_{i2}}(\ln\ln(T-t))^{k_{i3}}\cdots$$

with finite terms multiplication and $m_j, k_{ij} \in \mathbb{R}$ for $i = 1, 2,\ j = 1, 2, \ldots$. And $l_2(t) \le C(T-t)^{\frac{1}{2}}$ where the general constant C is independent of T.

Set

$$\psi(x,t) = \int_0^t v(s)(t-s)^{-d_*}\int_{\mathbb{R}^d} e^{-c(\frac{|x-y|}{\sqrt{t-s}})^{2-\delta}}|y-q|^{-b}\mathbf{1}_{\{l_1(s)\le|y-q|\le l_2(s)\}}dyds.$$

Claim: for $b \geq 0$, $d_* < \frac{d}{2} + 1$,

$$\psi(x,t) \lesssim \int_0^{t-(T-t)} v(s)(T-s)^{-d_*} \begin{cases} l_2^{d-b}(s) & \text{if } b < d \\ \langle \ln(\frac{l_2(s)}{l_1(s)}) \rangle & \text{if } b = d \\ l_1^{d-b}(s) & \text{if } b > d \end{cases} ds$$

$$+ v(t) \begin{cases} \begin{cases} (T-t)^{1-d_*} l_2^{d-b}(t) & \text{if } d_* < 1 \\ \langle \ln(\frac{T-t}{l_2^2(t)}) \rangle l_2^{d-b}(t) & \text{if } d_* = 1 \\ l_2^{d+2-b-2d_*}(t) & \text{if } 1 < d_* < 1 + \frac{d-b}{2} \\ \langle \ln(\frac{l_2(t)}{l_1(t)}) \rangle & \text{if } d_* = 1 + \frac{d-b}{2} \\ l_1^{d+2-b-2d_*}(t) & \text{if } d_* > 1 + \frac{d-b}{2} \end{cases} & \text{if } b < d \\ \begin{cases} (T-t)^{1-d_*} \langle \ln(\frac{l_2(t)}{l_1(t)}) \rangle & \text{if } d_* < 1 \\ \langle \ln(\frac{T-t}{l_1^2(t)}) \rangle \langle \ln(\frac{l_2(t)}{l_1(t)}) \rangle & \text{if } d_* = 1 \\ l_1^{2-2d_*}(t) & \text{if } d_* > 1 \end{cases} & \text{if } b = d \\ \begin{cases} (T-t)^{1-d_*} l_1^{d-b}(t) & \text{if } d_* < 1 \\ \langle \ln(\frac{T-t}{l_1^2(t)}) \rangle l_1^{d-b}(t) & \text{if } d_* = 1 \\ l_1^{d+2-b-2d_*}(t) & \text{if } d_* > 1 \end{cases} & \text{if } b > d \end{cases}. \tag{B.6}$$

Remark B.1. *When $b = 0 < d$, the cases $d_* = \frac{d}{2} - \frac{b}{2} + 1$ and $d_* > \frac{d}{2} - \frac{b}{2} + 1$ are vacuum.*

Proof. Since $b \geq 0$, by (B.2),

$$\begin{aligned} \psi(x,t) &\lesssim \int_0^t v(s) \begin{cases} (t-s)^{\frac{d}{2}-d_*} l_1^{-b}(s) & \text{if } t-s \leq l_1^2(s) \\ \begin{cases} (t-s)^{\frac{d}{2}-\frac{b}{2}-d_*} & \text{if } b < d \\ (t-s)^{-d_*} \langle \ln(\frac{t-s}{l_1^2(s)}) \rangle & \text{if } b = d \\ (t-s)^{-d_*} l_1^{d-b}(s) & \text{if } b > d \end{cases} & \text{if } l_1^2(s) < t-s \leq l_2^2(s) \\ \begin{cases} (t-s)^{-d_*} l_2^{d-b}(s) & \text{if } b < d \\ (t-s)^{-d_*} \langle \ln(\frac{l_2(s)}{l_1(s)}) \rangle & \text{if } b = d \\ (t-s)^{-d_*} l_1^{d-b}(s) & \text{if } b > d \end{cases} & \text{if } t-s > l_2^2(s) \end{cases} ds \\ &= \int_0^{t-(T-t)} + \int_{t-(T-t)}^t \cdots =: P_1 + P_2 \end{aligned}$$

where we used $b \geq 0$ in the first "$\leq$".

For P_1, since $\frac{T-s}{2} \leq t-s \leq T-s$, $l_2^2(s) \leq C(T-s)$,

$$P_1 \sim \int_0^{t-(T-t)} v(s)(T-s)^{-d_*} \begin{cases} l_2^{d-b}(s) & \text{if } b < d \\ \langle \ln(\frac{l_2(s)}{l_1(s)}) \rangle & \text{if } b = d \\ l_1^{d-b}(s) & \text{if } b > d \end{cases} ds. \tag{B.7}$$

For P_2, since $T-t \le T-s \le 2(T-t)$,

$$P_2 \sim v(t)\int_{t-(T-t)}^{t} \begin{cases} (t-s)^{\frac{d}{2}-d_*} l_1^{-b}(t) & \text{if } t-s \le l_1^2(t) \\ \begin{cases} (t-s)^{\frac{d}{2}-\frac{b}{2}-d_*} & \text{if } b<d \\ (t-s)^{-d_*}\langle \ln(\frac{t-s}{l_1^2(t)})\rangle & \text{if } b=d \\ (t-s)^{-d_*} l_1^{d-b}(t) & \text{if } b>d \end{cases} & \text{if } l_1^2(t) < t-s \le l_2^2(t) \\ \begin{cases} (t-s)^{-d_*} l_2^{d-b}(t) & \text{if } b<d \\ (t-s)^{-d_*}\langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } b=d \\ (t-s)^{-d_*} l_1^{d-b}(t) & \text{if } b>d \end{cases} & \text{if } t-s > l_2^2(t) \end{cases} ds$$

$$\lesssim v(t) \begin{cases} \begin{cases} (T-t)^{1-d_*} l_2^{d-b}(t) & \text{if } d_*<1 \\ \langle \ln(\frac{T-t}{l_2^2(t)})\rangle l_2^{d-b}(t) & \text{if } d_*=1 \\ l_2^{d+2-b-2d_*}(t) & \text{if } 1<d_*<1+\frac{d-b}{2} \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } d_*=1+\frac{d-b}{2} \\ l_1^{d+2-b-2d_*}(t) & \text{if } d_*>1+\frac{d-b}{2} \end{cases} & \text{if } b<d \\ \begin{cases} (T-t)^{1-d_*}\langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } d_*<1 \\ \langle \ln(\frac{T-t}{l_1^2(t)})\rangle \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } d_*=1 \\ l_1^{2-2d_*}(t) & \text{if } d_*>1 \end{cases} & \text{if } b=d \\ \begin{cases} (T-t)^{1-d_*} l_1^{d-b}(t) & \text{if } d_*<1 \\ \langle \ln(\frac{T-t}{l_1^2(t)})\rangle l_1^{d-b}(t) & \text{if } d_*=1 \\ l_1^{d+2-b-2d_*}(t) & \text{if } d_*>1 \end{cases} & \text{if } b>d \end{cases}$$

by (B.4) and $l_2^2(t) \le C(T-t)$ when $d_* < \frac{d}{2}+1$.

□

- For $b \ge 0$, $d_* < \frac{d}{2}+1$, by (B.3),

$$\begin{aligned} &\int_0^t v(s)(t-s)^{-d_*}\int_{\mathbb{R}^d} e^{-c(\frac{|x-y|}{\sqrt{t-s}})^{2-\delta}} |y-q|^{-b}\mathbf{1}_{\{|y-q|\ge (T-s)^{\frac{1}{2}}\}} dyds \\ \lesssim &\int_0^t v(s)(t-s)^{\frac{d}{2}-d_*}(T-s)^{-\frac{b}{2}} ds = \int_0^{t-(T-t)} + \int_{t-(T-t)}^t \cdots \\ \lesssim &\int_0^{t-(T-t)} v(s)(T-s)^{\frac{d}{2}-d_*-\frac{b}{2}} ds + v(t)(T-t)^{-\frac{b}{2}}\int_{t-(T-t)}^t (t-s)^{\frac{d}{2}-d_*} ds \\ \sim &\int_0^{t-(T-t)} v(s)(T-s)^{\frac{d}{2}-d_*-\frac{b}{2}} ds + v(t)(T-t)^{1-d_*+\frac{d}{2}-\frac{b}{2}} \\ \sim &\int_0^t v(s)(T-s)^{\frac{d}{2}-d_*-\frac{b}{2}} ds \end{aligned} \tag{B.8}$$

for $d_* < \frac{d}{2}+1$.

B.2. **Convolution about $v(t)|x-q|^{-b}\mathbf{1}_{\{l_1(t)\le|x-q|\le l_2(t)\}}$.** For

$$|f(x,t)| \le v(t)|x-q|^{-b}\mathbf{1}_{\{l_1(t)\le|x-q|\le l_2(t)\}},$$

consider

$$\psi(x,t) = \mathcal{T}_d^{\text{out}}[f] := \int_0^t \int_{\mathbb{R}^d} \Gamma(x,t,y,s) f(y,s) dyds.$$

For $d \geq 1$,

$$
\begin{aligned}
|\psi(x,t)| \lesssim \int_0^{t-(T-t)} v(s)(T-s)^{-\frac{d}{2}} &\begin{cases} l_2^{d-b}(s) & \text{if } b<d \\ \langle \ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } b=d \\ l_1^{d-b}(s) & \text{if } b>d \end{cases} ds \\
+ v(t) &\begin{cases} \begin{cases} (T-t)^{1-\frac{d}{2}} l_2^{d-b}(t) & \text{if } d<2 \\ l_2^{2-b}(t)\langle \ln(\frac{T-t}{l_2^2(t)})\rangle & \text{if } d=2 \\ l_2^{2-b}(t) & \text{if } d>2, b<2 \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } b=2 \\ l_1^{2-b}(t) & \text{if } b>2 \end{cases} & \text{if } b<d \\ \begin{cases} (T-t)^{1-\frac{d}{2}} \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } d<2 \\ \langle \ln(\frac{T-t}{l_1^2(t)})\rangle \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } d=2 \\ l_1^{2-d}(t) & \text{if } d>2 \end{cases} & \text{if } b=d \\ \begin{cases} (T-t)^{1-\frac{d}{2}} l_1^{d-b}(t) & \text{if } d<2 \\ l_1^{2-b}(t)\langle \ln(\frac{T-t}{l_1^2(t)})\rangle & \text{if } d=2 \\ l_1^{2-b}(t) & \text{if } d>2 \end{cases} & \text{if } b>d. \end{cases}
\end{aligned} \tag{B.9}
$$

$$
\begin{aligned}
|\nabla\psi(x,t)| \lesssim \int_0^{t-(T-t)} v(s)(T-s)^{-\frac{d+1}{2}} &\begin{cases} l_2^{d-b}(s) & \text{if } b<d \\ \langle \ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } b=d \\ l_1^{d-b}(s) & \text{if } b>d \end{cases} ds \\
+ v(t) &\begin{cases} \begin{cases} l_2^{1-b}(t)\langle \ln(\frac{T-t}{l_2^2(t)})\rangle & \text{if } d=1 \\ l_2^{1-b}(t) & \text{if } d>1, b<1 \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } b=1 \\ l_1^{1-b}(t) & \text{if } b>1 \end{cases} & \text{if } b<d \\ \begin{cases} \langle \ln(\frac{T-t}{l_1^2(t)})\rangle \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } d=1 \\ l_1^{1-d}(t) & \text{if } d>1 \end{cases} & \text{if } b=d \\ \begin{cases} l_1^{1-b}(t)\langle \ln(\frac{T-t}{l_1^2(t)})\rangle & \text{if } d=1 \\ l_1^{1-b}(t) & \text{if } d>1 \end{cases} & \text{if } b>d. \end{cases}
\end{aligned} \tag{B.10}
$$

$$
\begin{aligned}
&|\psi(x,t)-\psi(x,T)| \\
&\lesssim (T-t)\int_0^{t-(T-t)} v(s)(T-s)^{-1-\frac{d}{2}} \begin{cases} l_2^{d-b}(s) & \text{if } b<d \\ \langle \ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } b=d \\ l_1^{d-b}(s) & \text{if } b>d \end{cases} ds \\
&+ v(t)\int_{t-(T-t)}^{t} \begin{cases} l_1^{-b}(t) & \text{if } t-s \leq l_1^2(t) \\ \begin{cases} (t-s)^{-\frac{b}{2}} & \text{if } b<d \\ (t-s)^{-\frac{d}{2}}\langle \ln(\frac{t-s}{l_1^2(t)})\rangle & \text{if } b=d \\ l_1^{d-b}(t)(t-s)^{-\frac{d}{2}} & \text{if } b>d \end{cases} & \text{if } l_1^2(t) < t-s \leq l_2^2(t) \\ \begin{cases} l_2^{d-b}(t)(t-s)^{-\frac{d}{2}} & \text{if } b<d \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle (t-s)^{-\frac{d}{2}} & \text{if } b=d \\ l_1^{d-b}(t)(t-s)^{-\frac{d}{2}} & \text{if } b>d \end{cases} & \text{if } t-s > l_2^2(t) \end{cases} ds \\
&+ v(t)(T-t)^{1-\frac{d}{2}} \begin{cases} l_2^{d-b}(t) & \text{if } b<d \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } b=d \\ l_1^{d-b}(t) & \text{if } b>d \end{cases} \\
&+ \int_t^T (T-s)^{-\frac{d}{2}} v(s) \begin{cases} l_2^{d-b}(s) & \text{if } b<d \\ \langle \ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } b=d \\ l_1^{d-b}(s) & \text{if } b>d \end{cases} ds.
\end{aligned} \tag{B.11}
$$

For $b \geq 0$, $0 < \alpha < 1$,

$$
\begin{aligned}
&|\nabla\psi(x,t) - \nabla\psi(x,T)| \\
&\lesssim C(\alpha)\Bigg[(T-t)^{\frac{\alpha}{2}}\int_0^{t-(T-t)} v(s)(T-s)^{-\frac{d+1+\alpha}{2}}\begin{cases} l_2^{d-b}(s) & \text{if } \ b<d \\ \langle\ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } \ b=d \ ds \\ l_1^{d-b}(s) & \text{if } \ b>d \end{cases} \\
&+ v(t)\begin{cases} \begin{cases} \langle\ln(\frac{T-t}{l_2^2(t)})\rangle l_2^{1-b}(t) & \text{if } \ d=1 \\ l_2^{1-b}(t) & \text{if } \ d>1, b<1 \\ \langle\ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } \ b=1 \\ l_1^{1-b}(t) & \text{if } \ b>1 \end{cases} & \text{if } \ b<d \\ \begin{cases} \langle\ln(\frac{T-t}{l_1^2(t)})\rangle\langle\ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } \ d=1 \\ l_1^{1-d}(t) & \text{if } \ d>1 \end{cases} & \text{if } \ b=d \\ \begin{cases} \langle\ln(\frac{T-t}{l_1^2(t)})\rangle l_1^{1-b}(t) & \text{if } \ d=1 \\ l_1^{1-b}(t) & \text{if } \ d>1 \end{cases} & \text{if } \ b>d. \end{cases}\Bigg] \\
&+ \int_t^T v(s)(T-s)^{-\frac{d+1}{2}}\begin{cases} l_2^{d-b}(s) & \text{if } \ b<d \\ \langle\ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } \ b=d \ ds. \\ l_1^{d-b}(s) & \text{if } \ b>d \end{cases}
\end{aligned} \tag{B.12}
$$

For $b \geq 0$, $0 < \alpha < 1$ and $t < t_* \leq T$,

$$
\begin{aligned}
&|\nabla\psi(x,t) - \nabla\psi(x_*,t_*)| \\
&\lesssim C(\alpha)\left(|x-x_*| + \sqrt{|t-t_*|}\right)^\alpha\Bigg[\int_0^{t-(T-t)} v(s)(T-s)^{-\frac{d+1+\alpha}{2}}\begin{cases} l_2^{d-b}(s) & \text{if } \ b<d \\ \langle\ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } \ b=d \ ds \\ l_1^{d-b}(s) & \text{if } \ b>d \end{cases} \\
&+ v(t)\begin{cases} l_2^{1-\alpha-b}(t) & \text{if } \ b\leq 1-\alpha \\ l_1^{1-\alpha-b}(t) & \text{if } \ b>1-\alpha \end{cases}\Bigg] \\
&+ \mathbf{1}_{\{t_*>\frac{T+t}{2}\}}\Bigg[(T-t_*)^{\frac{\alpha}{2}}v(t_*)\begin{cases} l_2^{1-\alpha-b}(t_*) & \text{if } \ b\leq 1-\alpha \\ l_1^{1-\alpha-b}(t_*) & \text{if } \ b>1-\alpha \end{cases} \\
&+ \int_t^{t_*-(T-t_*)} v(s)(T-s)^{-\frac{d+1}{2}}\begin{cases} l_2^{d-b}(s) & \text{if } \ b<d \\ \langle\ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } \ b=d \ ds \\ l_1^{d-b}(s) & \text{if } \ b>d \end{cases}\Bigg].
\end{aligned} \tag{B.13}
$$

(B.9) and (B.10) are derived by (7.5) and (B.6).

Proof of (B.11).

$$
\begin{aligned}
\psi(x,t) - \psi(x,T) &= \int_0^{t-(T-t)}\int_{\mathbb{R}^d}(\Gamma(x,t,y,s) - \Gamma(x,T,y,s))f(y,s)dyds \\
&+ \int_{t-(T-t)}^t\int_{\mathbb{R}^d}(\Gamma(x,t,y,s) - \Gamma(x,T,y,s))f(y,s)dyds - \int_t^T\int_{\mathbb{R}^d}\Gamma(x,T,y,s)f(y,s)dyds := I_1 + I_2 + I_3.
\end{aligned}
$$

By (7.5),

$$
\begin{aligned}
|I_1| &\lesssim (T-t)\int_0^{t-(T-t)}\int_{\mathbb{R}^d}\int_0^1 |(\partial_t\Gamma)(x,\theta t+(1-\theta)T,y,s)|\,|f(y,s)|dyds\\
&\lesssim (T-t)\int_0^{t-(T-t)}\int_{\mathbb{R}^d}\int_0^1 [\theta t+(1-\theta)T-s]^{-1-\frac{d}{2}}e^{-c\left(\frac{|x-y|}{\sqrt{\theta t+(1-\theta)T-s}}\right)^{2-\delta}} v(s)|y-q|^{-b}\mathbf{1}_{\{l_1(s)\le|y-q|\le l_2(s)\}}d\theta dyds\\
&\lesssim (T-t)\int_0^{t-(T-t)} v(s)(T-s)^{-1-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-c\left(\frac{|x-y|}{\sqrt{T-s}}\right)^{2-\delta}}|y-q|^{-b}\mathbf{1}_{\{l_1(s)\le|y-q|\le l_2(s)\}}dyds\\
&\lesssim (T-t)\int_0^{t-(T-t)} v(s)(T-s)^{-1-\frac{d}{2}}\begin{cases} l_2^{d-b}(s) & \text{if } b<d\\ \langle\ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } b=d\; ds\\ l_1^{d-b}(s) & \text{if } b>d\end{cases}
\end{aligned}
$$

where we used (B.2) in the last "$\lesssim$" and $l_2^2(s)\le C(T-s)$.

By (7.5),

$$
\begin{aligned}
|I_2| &\lesssim \int_{t-(T-t)}^t\int_{\mathbb{R}^d}(|\Gamma(x,t,y,s)|+|\Gamma(x,T,y,s)|)|f(y,s)|dyds\\
&\lesssim \int_{t-(T-t)}^t v(s)(t-s)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-c\left(\frac{|x-y|}{\sqrt{t-s}}\right)^{2-\delta}}|y-q|^{-b}\mathbf{1}_{\{l_1(s)\le|y-q|\le l_2(s)\}}dyds\\
&\quad+\int_{t-(T-t)}^t v(s)(T-s)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-c\left(\frac{|x-y|}{\sqrt{T-s}}\right)^{2-\delta}}|y-q|^{-b}\mathbf{1}_{\{l_1(s)\le|y-q|\le l_2(s)\}}dyds\\
&\lesssim \int_{t-(T-t)}^t v(s)(t-s)^{-\frac{d}{2}}\begin{cases}(t-s)^{\frac{d}{2}}l_1^{-b}(s) & \text{if } t-s\le l_1^2(s)\\ \begin{cases}(t-s)^{\frac{d}{2}-\frac{b}{2}} & \text{if } b<d\\ \langle\ln(\frac{t-s}{l_1^2(s)})\rangle & \text{if } b=d\\ l_1^{d-b}(s) & \text{if } b>d\end{cases} & \text{if } l_1^2(s)<t-s\le l_2^2(s)\\ \begin{cases} l_2^{d-b}(s) & \text{if } b<d\\ \langle\ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } b=d\\ l_1^{d-b}(s) & \text{if } b>d\end{cases} & \text{if } t-s>l_2^2(s)\end{cases} ds\\
&\quad+\int_{t-(T-t)}^t v(s)(T-s)^{-\frac{d}{2}}\begin{cases} l_2^{d-b}(s) & \text{if } b<d\\ \langle\ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } b=d\; ds\\ l_1^{d-b}(s) & \text{if } b>d\end{cases}\\
&\lesssim v(t)\int_{t-(T-t)}^t \begin{cases} l_1^{-b}(t) & \text{if } t-s\le l_1^2(t)\\ \begin{cases}(t-s)^{-\frac{b}{2}} & \text{if } b<d\\ (t-s)^{-\frac{d}{2}}\langle\ln(\frac{t-s}{l_1^2(t)})\rangle & \text{if } b=d\\ l_1^{d-b}(t)(t-s)^{-\frac{d}{2}} & \text{if } b>d\end{cases} & \text{if } l_1^2(t)<t-s\le l_2^2(t)\\ \begin{cases} l_2^{d-b}(t)(t-s)^{-\frac{d}{2}} & \text{if } b<d\\ \langle\ln(\frac{l_2(t)}{l_1(t)})\rangle(t-s)^{-\frac{d}{2}} & \text{if } b=d\\ l_1^{d-b}(t)(t-s)^{-\frac{d}{2}} & \text{if } b>d\end{cases} & \text{if } t-s>l_2^2(t)\end{cases} ds\\
&\quad+v(t)(T-t)^{1-\frac{d}{2}}\begin{cases} l_2^{d-b}(t) & \text{if } b<d\\ \langle\ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } b=d\\ l_1^{d-b}(t) & \text{if } b>d\end{cases}
\end{aligned}
$$

where we used (B.2) and $l_2^2(s)\le C(T-s)$.

By (7.5) and (B.2),

$$
\begin{aligned}
|I_3| &\lesssim \int_t^T \int_{\mathbb{R}^d} (T-s)^{-\frac{d}{2}} e^{-c\left(\frac{|x-y|}{\sqrt{T-s}}\right)^{2-\delta}} v(s)|y-q|^{-b}\mathbf{1}_{\{l_1(s)\le|y-q|\le l_2(s)\}}dyds\\
&\lesssim \int_t^T (T-s)^{-\frac{d}{2}} v(s)\begin{cases} l_2^{d-b}(s) & \text{if } \ b<d\\ \langle \ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } \ b=d \ ds\\ l_1^{d-b}(s) & \text{if } \ b>d\end{cases}
\end{aligned}
$$

where we used $l_2^2(s) \le C(T-s)$.

□

Proof of (B.12).

$$
\begin{aligned}
&\partial_{x_i}\psi(x,t) - \partial_{x_i}\psi(x,T)\\
&= \int_0^t\int_{\mathbb{R}^d} (\partial_{x_i}\Gamma(x,t,y,s) - \partial_{x_i}\Gamma(x,T,y,s))f(y,s)dyds - \int_t^T\int_{\mathbb{R}^d}\partial_{x_i}\Gamma(x,T,y,s)f(y,s)dyds\\
&=: I_1 + I_2.
\end{aligned}
$$

For I_1, by (7.7) and (B.2),

$$
\begin{aligned}
|I_1| &\lesssim (T-t)^{\frac{\alpha}{2}}\int_0^t\int_{\mathbb{R}^d}(T-s)^{-\frac{\alpha}{2}}\left[(t-s)^{-\frac{d+1}{2}}e^{-\beta(\frac{|x-y|}{\sqrt{t-s}})^{2-\delta}} + (T-s)^{-\frac{d+1}{2}}e^{-\beta(\frac{|x-y|}{\sqrt{T-s}})^{2-\delta}}\right]\\
&\quad * v(s)|y-q|^{-b}\mathbf{1}_{\{l_1(s)\le|y-q|\le l_2(s)\}}dyds\\
&\lesssim C(\alpha)(T-t)^{\frac{\alpha}{2}}\\
&\quad \left[\int_0^t v(s)(T-s)^{-\frac{\alpha}{2}}\begin{cases}(t-s)^{-\frac12}l_1^{-b}(s) & \text{if } \ t-s\le l_1^2(s)\\ \begin{cases}(t-s)^{-\frac{b+1}{2}} & \text{if } \ b<d\\ (t-s)^{-\frac{d+1}{2}}\langle\ln(\frac{t-s}{l_1^2(s)})\rangle & \text{if } \ b=d\\ (t-s)^{-\frac{d+1}{2}}l_1^{d-b}(s) & \text{if } \ b>d\end{cases} & \text{if } \ l_1^2(s)<t-s\le l_2^2(s)\\ \begin{cases}(t-s)^{-\frac{d+1}{2}}l_2^{d-b}(s) & \text{if } \ b<d\\ (t-s)^{-\frac{d+1}{2}}\langle\ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } \ b=d\\ (t-s)^{-\frac{d+1}{2}}l_1^{d-b}(s) & \text{if } \ b>d\end{cases} & \text{if } \ t-s>l_2^2(s)\end{cases} ds\right.\\
&\quad \left. + \int_0^t v(s)\begin{cases}(T-s)^{-\frac{d+1+\alpha}{2}}l_2^{d-b}(s) & \text{if } \ b<d\\ (T-s)^{-\frac{d+1+\alpha}{2}}\langle\ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } \ b=d \ ds\\ (T-s)^{-\frac{d+1+\alpha}{2}}l_1^{d-b}(s) & \text{if } \ b>d\end{cases}\right]\\
&= C(\alpha)(T-t)^{\frac{\alpha}{2}}\Big(\int_0^{t-(T-t)} + \int_{t-(T-t)}^t \cdots\Big)\\
&=: C(\alpha)(T-t)^{\frac{\alpha}{2}}(I_{11}+I_{12}).
\end{aligned}
$$

For I_{11}, by (B.1),

$$
I_{11} \lesssim \int_0^{t-(T-t)} v(s)(T-s)^{-\frac{d+1+\alpha}{2}}\begin{cases} l_2^{d-b}(s) & \text{if } \ b<d\\ \langle\ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } \ b=d \ ds\\ l_1^{d-b}(s) & \text{if } \ b>d\end{cases}
$$

since $t-s \sim t_*-s \sim T-s \gtrsim l_2^2(s)$.

For I_{12},

$$I_{12} \lesssim v(t)(T-t)^{-\frac{\alpha}{2}} \int_{t-(T-t)}^{t} \begin{cases} (t-s)^{-\frac{1}{2}} l_1^{-b}(t) & \text{if } \ t-s \le l_1^2(t) \\ \begin{cases} (t-s)^{-\frac{1}{2}-\frac{b}{2}} & \text{if } \ b<d \\ (t-s)^{-\frac{1}{2}-\frac{b}{2}} \langle \ln(\frac{t-s}{l_1^2(t)}) \rangle & \text{if } \ b=d \\ (t-s)^{-\frac{1}{2}-\frac{d}{2}} l_1^{d-b}(t) & \text{if } \ b>d \end{cases} & \text{if } \ l_1^2(t) < t-s \le l_2^2(t) \\ \begin{cases} (t-s)^{-\frac{1}{2}-\frac{d}{2}} l_2^{d-b}(t) & \text{if } \ b<d \\ (t-s)^{-\frac{1}{2}-\frac{d}{2}} \langle \ln(\frac{l_2(t)}{l_1(t)}) \rangle & \text{if } \ b=d \\ (t-s)^{-\frac{1}{2}-\frac{d}{2}} l_1^{d-b}(t) & \text{if } \ b>d \end{cases} & \text{if } \ t-s > l_2^2(t) \end{cases} ds$$

$$+ v(t)(T-t)^{-\frac{\alpha}{2}} \begin{cases} (T-t)^{\frac{1}{2}-\frac{d}{2}} l_2^{d-b}(t) & \text{if } \ b<d \\ (T-t)^{\frac{1}{2}-\frac{d}{2}} \langle \ln(\frac{l_2(t)}{l_1(t)}) \rangle & \text{if } \ b=d \\ (T-t)^{\frac{1}{2}-\frac{d}{2}} l_1^{d-b}(t) & \text{if } \ b>d \end{cases}$$

$$\lesssim v(t)(T-t)^{-\frac{\alpha}{2}} \begin{cases} \begin{cases} \langle \ln(\frac{T-t}{l_2^2(t)}) \rangle l_2^{1-b}(t) & \text{if } \ d=1 \\ l_2^{1-b}(t) & \text{if } \ d>1, b<1 \\ \langle \ln(\frac{l_2(t)}{l_1(t)}) \rangle & \text{if } \ b=1 \\ l_1^{1-b}(t) & \text{if } \ b>1 \end{cases} & \text{if } \ b<d \\ \begin{cases} \langle \ln(\frac{T-t}{l_1^2(t)}) \rangle \langle \ln(\frac{l_2(t)}{l_1(t)}) \rangle & \text{if } \ d=1 \\ l_1^{1-d}(t) & \text{if } \ d>1 \end{cases} & \text{if } \ b=d \\ \begin{cases} \langle \ln(\frac{T-t}{l_1^2(t)}) \rangle l_1^{1-b}(t) & \text{if } \ d=1 \\ l_1^{1-b}(t) & \text{if } \ d>1 \end{cases} & \text{if } \ b>d. \end{cases}$$

where we used (B.4) in the last "$\lesssim$".

For I_2,

$$|I_2| \lesssim \int_t^T v(s)(T-s)^{-\frac{d+1}{2}} \int_{\mathbb{R}^d} e^{-\beta(\frac{|x_*-y|}{\sqrt{T-s}})^{2-\delta}} |y-q|^{-b} \mathbf{1}_{\{l_1(s)\le|y-q|\le l_2(s)\}} dyds$$

$$\lesssim \int_t^T v(s) \begin{cases} (T-s)^{-\frac{d+1}{2}} l_2^{d-b}(s) & \text{if } \ b<d \\ (T-s)^{-\frac{d+1}{2}} \langle \ln(\frac{l_2(s)}{l_1(s)}) \rangle & \text{if } \ b=d \ ds \\ (T-s)^{-\frac{d+1}{2}} l_1^{d-b}(s) & \text{if } \ b>d \end{cases}$$

by (B.2).

□

Proof of (B.13).

$$\begin{aligned} &\partial_{x_i}\psi(x,t) - \partial_{x_i}\psi(x_*,t_*) \\ =& \int_0^t \int_{\mathbb{R}^d} (\partial_{x_i}\Gamma(x,t,y,s) - \partial_{x_i}\Gamma(x_*,t_*,y,s)) f(y,s) dyds - \int_t^{t_*} \int_{\mathbb{R}^d} \partial_{x_i}\Gamma(x_*,t_*,y,s) f(y,s) dyds \\ =:& I_1 + I_2. \end{aligned}$$

For I_1, by (7.7) and (B.2),

$$
\begin{aligned}
|I_1| &\lesssim (|x-x_*|+\sqrt{|t-t_*|})^\alpha \int_0^t \int_{\mathbb{R}^d} (t_*-s)^{-\frac{\alpha}{2}} \big[(t-s)^{-\frac{d+1}{2}} e^{-\beta(\frac{|x-y|}{\sqrt{t-s}})^{2-\delta}} + (t_*-s)^{-\frac{d+1}{2}} e^{-\beta(\frac{|x_*-y|}{\sqrt{t_*-s}})^{2-\delta}}\big] \\
&\quad * v(s)|y-q|^{-b}\mathbf{1}_{\{l_1(s)\le|y-q|\le l_2(s)\}} dyds \\
&\lesssim C(\alpha)(|x-x_*|+\sqrt{|t-t_*|})^\alpha \Big[\\
&\quad \int_0^t v(s)(t_*-s)^{-\frac{\alpha}{2}} \begin{cases} (t-s)^{-\frac{1}{2}} l_1^{-b}(s) & \text{if } t-s\le l_1^2(s) \\ \begin{cases} (t-s)^{-\frac{b+1}{2}} & \text{if } b<d \\ (t-s)^{-\frac{d+1}{2}} \langle \ln(\frac{t-s}{l_1^2(s)})\rangle & \text{if } b=d \\ (t-s)^{-\frac{d+1}{2}} l_1^{d-b}(s) & \text{if } b>d \end{cases} & \text{if } l_1^2(s)<t-s\le l_2^2(s) \\ \begin{cases} (t-s)^{-\frac{d+1}{2}} l_2^{d-b}(s) & \text{if } b<d \\ (t-s)^{-\frac{d+1}{2}} \langle \ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } b=d \\ (t-s)^{-\frac{d+1}{2}} l_1^{d-b}(s) & \text{if } b>d \end{cases} & \text{if } t-s>l_2^2(s) \end{cases} ds \\
&\quad + \int_0^t v(s) \begin{cases} (t_*-s)^{-\frac{1+\alpha}{2}} l_1^{-b}(s) & \text{if } t_*-s\le l_1^2(s) \\ \begin{cases} (t_*-s)^{-\frac{b+1+\alpha}{2}} & \text{if } b<d \\ (t_*-s)^{-\frac{d+1+\alpha}{2}} \langle \ln(\frac{t_*-s}{l_1^2(s)})\rangle & \text{if } b=d \\ (t_*-s)^{-\frac{d+1+\alpha}{2}} l_1^{d-b}(s) & \text{if } b>d \end{cases} & \text{if } l_1^2(s)<t_*-s\le l_2^2(s) \\ \begin{cases} (t_*-s)^{-\frac{d+1+\alpha}{2}} l_2^{d-b}(s) & \text{if } b<d \\ (t_*-s)^{-\frac{d+1+\alpha}{2}} \langle \ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } b=d \\ (t_*-s)^{-\frac{d+1+\alpha}{2}} l_1^{d-b}(s) & \text{if } b>d \end{cases} & \text{if } t_*-s>l_2^2(s) \end{cases} ds\Big] \\
&= C(\alpha)(|x-x_*|+\sqrt{|t-t_*|})^\alpha \Big(\int_0^{t-(T-t)} + \int_{t-(T-t)}^{t-(t_*-t)} + \int_{t-(t_*-t)}^t \cdots\Big) \\
&=: C(\alpha)(|x-x_*|+\sqrt{|t-t_*|})^\alpha (I_{11}+I_{12}+I_{13}).
\end{aligned}
$$

For I_{11},

$$
I_{11} \lesssim \int_0^{t-(T-t)} v(s)(T-s)^{-\frac{d+1+\alpha}{2}} \begin{cases} l_2^{d-b}(s) & \text{if } b<d \\ \langle \ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } b=d \; ds \\ l_1^{d-b}(s) & \text{if } b>d \end{cases}
$$

since $t-s\sim t_*-s\sim T-s\gtrsim l_2^2(s)$.

For I_{12}, by (B.2),

$$
\begin{aligned}
I_{12} &\lesssim v(t)\int_{t-(T-t)}^{t-(t_*-t)} \begin{cases} (t-s)^{-\frac{1+\alpha}{2}} l_1^{-b}(t) & \text{if } \ t-s\le l_1^2(t)\\ \begin{cases} (t-s)^{-\frac{b+1+\alpha}{2}} & \text{if } \ b<d\\ (t-s)^{-\frac{d+1+\alpha}{2}}\langle \ln(\frac{t-s}{l_1^2(t)})\rangle & \text{if } \ b=d \\ (t-s)^{-\frac{d+1+\alpha}{2}} l_1^{d-b}(t) & \text{if } \ b>d\end{cases} & \text{if } \ l_1^2(t)<t-s\le l_2^2(t)\\ \begin{cases} (t-s)^{-\frac{d+1+\alpha}{2}} l_2^{d-b}(t) & \text{if } \ b<d \\ (t-s)^{-\frac{d+1+\alpha}{2}}\langle\ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } \ b=d\\ (t-s)^{-\frac{d+1+\alpha}{2}} l_1^{d-b}(t) & \text{if } \ b>d\end{cases} & \text{if } \ t-s>l_2^2(t)\end{cases} ds\\
&\lesssim v(t) \begin{cases} \begin{cases} l_2^{1-b-\alpha}(t) & \text{if } \ b\le 1-\alpha\\ l_1^{1-b-\alpha}(t) & \text{if } \ b>1-\alpha\end{cases} & \text{if } \ t_*-t\le l_1^2(t)\\ \begin{cases}\begin{cases} l_2^{1-b-\alpha}(t) & \text{if } \ b\le1-\alpha\\ (t_*-t)^{\frac{1-b-\alpha}{2}} & \text{if } \ b>1-\alpha\end{cases} & \text{if } \ b<d\\ (t_*-t)^{\frac{1-d-\alpha}{2}}\langle\ln(\frac{t_*-t}{l_1^2(t)})\rangle & \text{if } \ b=d\\ (t_*-t)^{\frac{1-d-\alpha}{2}} l_1^{d-b}(t) & \text{if } \ b>d\end{cases} & \text{if } \ l_1^2(t)<t_*-t\le l_2^2(t)\\ (t_*-t)^{\frac{1-d-\alpha}{2}}\begin{cases} l_2^{d-b}(t) & \text{if } \ b<d\\ \langle\ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } \ b=d\\ l_1^{d-b}(t) & \text{if } \ b>d\end{cases} & \text{if } \ t_*-t\ge l_2^2(t)\end{cases}\\
&\lesssim v(t) \begin{cases} \begin{cases} l_2^{1-b-\alpha}(t) & \text{if } \ b\le 1-\alpha\\ l_1^{1-b-\alpha}(t) & \text{if } \ b>1-\alpha\end{cases} & \text{if } \ t_*-t\le l_1^2(t)\\ \begin{cases}\begin{cases} l_2^{1-b-\alpha}(t) & \text{if } \ b\le1-\alpha\\ l_1^{1-b-\alpha}(t) & \text{if } \ b>1-\alpha\end{cases} & \text{if } \ b<d\\ l_1^{1-d-\alpha}(t) & \text{if } \ b=d\\ l_1^{1-b-\alpha}(t) & \text{if } \ b>d\end{cases} & \text{if } \ l_1^2(t)<t_*-t\le l_2^2(t)\\ \begin{cases} l_2^{1-b-\alpha}(t) & \text{if } \ b<d\\ l_2^{1-d-\alpha}(t)\langle\ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } \ b=d\\ l_2^{1-d-\alpha}(t) l_1^{d-b}(t) & \text{if } \ b>d\end{cases} & \text{if } \ t_*-t\ge l_2^2(t)\end{cases}\\
&\sim v(t)\begin{cases} l_2^{1-b-\alpha}(t) & \text{if } \ b\le 1-\alpha\\ l_1^{1-b-\alpha}(t) & \text{if } \ b>1-\alpha\end{cases}
\end{aligned}
$$

where we used the following calculation in the second "$\lesssim$". For $l_1^2(t)<t_*-t\le l_2^2(t)$,

$$
\begin{aligned}
&l_2^{1-d-\alpha}(t)\begin{cases} l_2^{d-b}(t) & \text{if } \ b<d\\ \langle\ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } \ b=d\\ l_1^{d-b}(t) & \text{if } \ b>d\end{cases} + \int_{t-l_2^2(t)}^{t-(t_*-t)} \begin{cases} (t-s)^{-\frac{b+1+\alpha}{2}} & \text{if } \ b<d\\ (t-s)^{-\frac{d+1+\alpha}{2}}\langle\ln(\frac{t-s}{l_1^2(t)})\rangle & \text{if } \ b=d\\ (t-s)^{-\frac{d+1+\alpha}{2}} l_1^{d-b}(t) & \text{if } \ b>d\end{cases} ds\\
&\lesssim l_2^{1-d-\alpha}(t)\begin{cases} l_2^{d-b}(t) & \text{if } \ b<d\\ \langle\ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } \ b=d\\ l_1^{d-b}(t) & \text{if } \ b>d\end{cases} + \begin{cases}\begin{cases} l_2^{1-b-\alpha}(t) & \text{if } \ b\le1-\alpha\\ (t_*-t)^{\frac{1-b-\alpha}{2}} & \text{if } \ b>1-\alpha\end{cases} & \text{if } \ b<d\\ (t_*-t)^{\frac{1-d-\alpha}{2}}\langle\ln(\frac{t_*-t}{l_1^2(t)})\rangle & \text{if } \ b=d\\ (t_*-t)^{\frac{1-d-\alpha}{2}} l_1^{d-b}(t) & \text{if } \ b>d\end{cases}\\
&\sim \begin{cases}\begin{cases} l_2^{1-b-\alpha}(t) & \text{if } \ b\le1-\alpha\\ (t_*-t)^{\frac{1-b-\alpha}{2}} & \text{if } \ b>1-\alpha\end{cases} & \text{if } \ b<d\\ (t_*-t)^{\frac{1-d-\alpha}{2}}\langle\ln(\frac{t_*-t}{l_1^2(t)})\rangle & \text{if } \ b=d\\ (t_*-t)^{\frac{1-d-\alpha}{2}} l_1^{d-b}(t) & \text{if } \ b>d\end{cases}
\end{aligned}
$$

where for $b=d$, we used

$$\int_{t-l_2^2(t)}^{t-(t_*-t)}(t-s)^{-\frac{d+1+\alpha}{2}}\langle\ln(\frac{t-s}{l_1^2(t)})\rangle ds = l_1^{1-d-\alpha}(t)\int_{\frac{t_*-t}{l_1^2(t)}}^{\frac{l_2^2(t)}{l_1^2(t)}} z^{-\frac{d+1+\alpha}{2}}\langle\ln z\rangle dz \lesssim (t_*-t)^{\frac{1-d-\alpha}{2}}\langle\ln(\frac{t_*-t}{l_1^2(t)})\rangle.$$

For $t_*-t\le l_1^2(t)$,

$$\begin{cases} l_2^{1-b-\alpha}(t) & \text{if } \ b\le 1-\alpha \\ l_1^{1-b-\alpha}(t) & \text{if } \ b>1-\alpha. \end{cases}$$

For I_{13},

$$I_{13}\lesssim v(t)(t_*-t)^{-\frac{\alpha}{2}}\int_{t-(t_*-t)}^{t}\begin{cases}(t-s)^{-\frac12}l_1^{-b}(t) & & \text{if } \ t-s\le l_1^2(t)\\ \begin{cases}(t-s)^{-\frac12-\frac b2} & \text{if } \ b<d\\ (t-s)^{-\frac12-\frac b2}\langle\ln(\frac{t-s}{l_1^2(t)})\rangle & \text{if } \ b=d\\ (t-s)^{-\frac12-\frac d2}l_1^{d-b}(t) & \text{if } \ b>d\end{cases} & & \text{if } \ l_1^2(t)<t-s\le l_2^2(t)\\ \begin{cases}(t-s)^{-\frac12-\frac d2}l_2^{d-b}(t) & \text{if } \ b<d\\ (t-s)^{-\frac12-\frac d2}\langle\ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } \ b=d\\ (t-s)^{-\frac12-\frac d2}l_1^{d-b}(t) & \text{if } \ b>d\end{cases} & & \text{if } \ t-s>l_2^2(t)\end{cases} ds$$

$$+v(t)(t_*-t)^{-\frac{\alpha}{2}}\begin{cases}(t_*-t)^{\frac12}l_1^{-b}(t) & & \text{if } \ t_*-t\le l_1^2(t)\\ \begin{cases}(t_*-t)^{\frac12-\frac b2} & \text{if } \ b<d\\ (t_*-t)^{\frac12-\frac b2}\langle\ln(\frac{t_*-t}{l_1^2(t)})\rangle & \text{if } \ b=d\\ (t_*-t)^{\frac12-\frac d2}l_1^{d-b}(t) & \text{if } \ b>d\end{cases} & & \text{if } \ l_1^2(t)<t_*-t\le l_2^2(t)\\ \begin{cases}(t_*-t)^{\frac12-\frac d2}l_2^{d-b}(t) & \text{if } \ b<d\\ (t_*-t)^{\frac12-\frac d2}\langle\ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } \ b=d\\ (t_*-t)^{\frac12-\frac d2}l_1^{d-b}(t) & \text{if } \ b>d\end{cases} & & \text{if } \ t_*-t>l_2^2(t)\end{cases}$$

$$\lesssim v(t)\begin{cases} l_2^{1-\alpha-b}(t) & b\le 1-\alpha\\ l_1^{1-\alpha-b}(t) & b>1-\alpha\end{cases}$$

where we used (B.5) in the last "$\lesssim$".

For I_2,

$$|I_2|\lesssim \int_t^{t_*} v(s)(t_*-s)^{-\frac{d+1}{2}}\int_{\mathbb{R}^d} e^{-\beta(\frac{|x_*-y|}{\sqrt{t_*-s}})^{2-\delta}}|y-q|^{-b}\mathbf{1}_{\{l_1(s)\le|y-q|\le l_2(s)\}}dyds$$

$$\lesssim \int_t^{t_*} v(s)\begin{cases}(t_*-s)^{-\frac12}l_1^{-b}(s) & & \text{if } \ t_*-s\le l_1^2(s)\\ \begin{cases}(t_*-s)^{-\frac12-\frac b2} & \text{if } \ b<d\\ (t_*-s)^{-\frac{d+1}{2}}\langle\ln(\frac{t_*-s}{l_1^2(s)})\rangle & \text{if } \ b=d\\ (t_*-s)^{-\frac{d+1}{2}}l_1^{d-b}(s) & \text{if } \ b>d\end{cases} & & \text{if } \ l_1^2(s)<t_*-s\le l_2^2(s)\\ \begin{cases}(t_*-s)^{-\frac{d+1}{2}}l_2^{d-b}(s) & \text{if } \ b<d\\ (t_*-s)^{-\frac{d+1}{2}}\langle\ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } \ b=d\\ (t_*-s)^{-\frac{d+1}{2}}l_1^{d-b}(s) & \text{if } \ b>d\end{cases} & & \text{if } \ t_*-s>l_2^2(s)\end{cases} ds$$

by (B.2).

When $t_*-t\le (T-t)/2$, then for $s\in(t,t_*)$,

$$\frac{T-t}{2}\le T-t_*\le T-s\le 2(T-t_*)\le 2(T-t).$$

It follows that

$$|I_2| \lesssim v(t)\int_{t_*-(t_*-t)}^{t_*} \begin{cases} (t_*-s)^{-\frac{1}{2}} l_1^{-b}(t) & \text{if } \ t_*-s \le l_1^2(t) \\ \begin{cases} (t_*-s)^{-\frac{1}{2}-\frac{b}{2}} & \text{if } \ b<d \\ (t_*-s)^{-\frac{d+1}{2}} \langle \ln(\frac{t_*-s}{l_1^2(t)})\rangle & \text{if } \ b=d \\ (t_*-s)^{-\frac{d+1}{2}} l_1^{d-b}(t) & \text{if } \ b>d \end{cases} & \text{if } \ l_1^2(t) < t_*-s \le l_2^2(t) \\ \begin{cases} (t_*-s)^{-\frac{d+1}{2}} l_2^{d-b}(t) & \text{if } \ b<d \\ (t_*-s)^{-\frac{d+1}{2}} \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } \ b=d \\ (t_*-s)^{-\frac{d+1}{2}} l_1^{d-b}(t) & \text{if } \ b>d \end{cases} & \text{if } \ t_*-s > l_2^2(t) \end{cases} ds$$

$$\lesssim (t_*-t)^{\frac{\alpha}{2}} v(t) \begin{cases} l_2^{1-\alpha-b}(t) & b \le 1-\alpha \\ l_1^{1-\alpha-b}(t) & b > 1-\alpha \end{cases}$$

by (B.5).

When $t_*-t > (T-t)/2$,

$$|I_2| \lesssim (T-t_*)^{\frac{\alpha}{2}} v(t_*) \begin{cases} l_2^{1-\alpha-b}(t_*) & b \le 1-\alpha \\ l_1^{1-\alpha-b}(t_*) & b > 1-\alpha \end{cases} + \int_t^{t_*-(T-t_*)} v(s)(T-s)^{-\frac{d+1}{2}} \begin{cases} l_2^{d-b}(s) & \text{if } \ b<d \\ \langle \ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } \ b=d \, ds \\ l_1^{d-b}(s) & \text{if } \ b>d \end{cases}$$

since

$$\int_{t_*-(T-t_*)}^{t_*} v(s) \begin{cases} (t_*-s)^{-\frac{1}{2}} l_1^{-b}(s) & \text{if } \ t_*-s \le l_1^2(s) \\ \begin{cases} (t_*-s)^{-\frac{1}{2}-\frac{b}{2}} & \text{if } \ b<d \\ (t_*-s)^{-\frac{d+1}{2}} \langle \ln(\frac{t_*-s}{l_1^2(s)})\rangle & \text{if } \ b=d \\ (t_*-s)^{-\frac{d+1}{2}} l_1^{d-b}(s) & \text{if } \ b>d \end{cases} & \text{if } \ l_1^2(s) < t_*-s \le l_2^2(s) \\ \begin{cases} (t_*-s)^{-\frac{d+1}{2}} l_2^{d-b}(s) & \text{if } \ b<d \\ (t_*-s)^{-\frac{d+1}{2}} \langle \ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } \ b=d \\ (t_*-s)^{-\frac{d+1}{2}} l_1^{d-b}(s) & \text{if } \ b>d \end{cases} & \text{if } \ t_*-s > l_2^2(s) \end{cases} ds$$

$$\lesssim (T-t_*)^{\frac{\alpha}{2}} v(2t_*-T) \begin{cases} l_2^{1-\alpha-b}(2t_*-T) & b \le 1-\alpha \\ l_1^{1-\alpha-b}(2t_*-T) & b > 1-\alpha \end{cases}$$

$$\sim (T-t_*)^{\frac{\alpha}{2}} v(t_*) \begin{cases} l_2^{1-\alpha-b}(t_*) & b \le 1-\alpha \\ l_1^{1-\alpha-b}(t_*) & b > 1-\alpha \end{cases},$$

and

$$\int_t^{t_*-(T-t_*)} v(s) \begin{cases} (t_*-s)^{-\frac{1}{2}} l_1^{-b}(s) & \text{if } \ t_*-s \le l_1^2(s) \\ \begin{cases} (t_*-s)^{-\frac{1}{2}-\frac{b}{2}} & \text{if } \ b<d \\ (t_*-s)^{-\frac{d+1}{2}} \langle \ln(\frac{t_*-s}{l_1^2(s)})\rangle & \text{if } \ b=d \\ (t_*-s)^{-\frac{d+1}{2}} l_1^{d-b}(s) & \text{if } \ b>d \end{cases} & \text{if } \ l_1^2(s) < t_*-s \le l_2^2(s) \\ \begin{cases} (t_*-s)^{-\frac{d+1}{2}} l_2^{d-b}(s) & \text{if } \ b<d \\ (t_*-s)^{-\frac{d+1}{2}} \langle \ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } \ b=d \\ (t_*-s)^{-\frac{d+1}{2}} l_1^{d-b}(s) & \text{if } \ b>d \end{cases} & \text{if } \ t_*-s > l_2^2(s) \end{cases} ds$$

$$\lesssim \int_t^{t_*-(T-t_*)} v(s)(T-s)^{-\frac{d+1}{2}} \begin{cases} l_2^{d-b}(s) & \text{if } \ b<d \\ \langle \ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } \ b=d \, ds \\ l_1^{d-b}(s) & \text{if } \ b>d \end{cases}$$

since $\frac{T-s}{2} \le t_*-s \le T-s$ and $l_2^2(s) \lesssim T-s$.

□

B.3. **Convolution about** $v(t)|x-q|^{-b}\mathbf{1}_{\{|x-q|\geq (T-t)^{\frac{1}{2}}\}}$. For

$$|f(x,t)|\leq v(t)|x-q|^{-b}\mathbf{1}_{\{|x-q|\geq (T-t)^{\frac{1}{2}}\}}$$

consider

$$\psi(x,t)=\int_0^t\int_{\mathbb{R}^d}\Gamma(x,t,y,s)f(y,s)dyds.$$

Claim: for $b\geq 0$, $0<\alpha<1$, $0\leq t<t_*\leq T$,

$$|\psi(x,t)|\lesssim \int_0^t v(s)(T-s)^{-\frac{b}{2}}ds,$$

$$|\nabla\psi(x,t)|\lesssim \int_0^t v(s)(T-s)^{-\frac{b+1}{2}}ds,$$

$$|\psi(x,t)-\psi(x,T)|\lesssim (T-t)\int_0^{t-(T-t)} v(s)(T-s)^{-1-\frac{b}{2}}ds+v(t)(T-t)^{1-\frac{b}{2}}+\int_t^T v(s)(T-s)^{-\frac{b}{2}}ds, \tag{B.14}$$

$$\begin{aligned}&|\nabla\psi(x,t)-\nabla\psi(x,T)|\\ \lesssim\ & C(\alpha)\,(T-t)^{\frac{\alpha}{2}}\left[\int_0^{t-(T-t)} v(s)(T-s)^{-\frac{1+b+\alpha}{2}}ds+v(t)(T-t)^{\frac{1-b-\alpha}{2}}\right]+\int_t^T v(s)(T-s)^{-\frac{1+b}{2}}ds.\end{aligned} \tag{B.15}$$

$$\begin{aligned}&|\nabla_x\psi(x,t)-\nabla_x\psi(x_*,t_*)|\\ \lesssim\ & C(\alpha)\left(|x-x_*|+\sqrt{|t-t_*|}\right)^{\alpha}\left[\int_0^{t-(T-t)} v(s)(T-s)^{-\frac{1+b+\alpha}{2}}ds+v(t)(T-t)^{\frac{1-b-\alpha}{2}}\right]\\ &+\mathbf{1}_{\{t_*>\frac{T+t}{2}\}}\int_t^{t_*} v(s)(T-s)^{-\frac{1+b}{2}}ds.\end{aligned} \tag{B.16}$$

Proof of (B.14).

$$\begin{aligned}\psi(x,t)-\psi(x,T)=&\int_0^{t-(T-t)}\int_{\mathbb{R}^d}(\Gamma(x,t,y,s)-\Gamma(x,T,y,s))f(y,s)dyds\\ &+\int_{t-(T-t)}^t\int_{\mathbb{R}^d}(\Gamma(x,t,y,s)-\Gamma(x,T,y,s))f(y,s)dyds-\int_t^T\int_{\mathbb{R}^d}\Gamma(x,T,y,s)f(y,s)dyds:=I_1+I_2+I_3.\end{aligned}$$

By (7.5) and (B.3),

$$\begin{aligned}|I_1|&\lesssim (T-t)\int_0^{t-(T-t)}\int_{\mathbb{R}^d}\int_0^1|(\partial_t\Gamma)(x,\theta t+(1-\theta)T,y,s)||f(y,s)|d\theta dyds\\ &\lesssim (T-t)\int_0^{t-(T-t)} v(s)(T-s)^{-1-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-c\left(\frac{|x-y|}{\sqrt{T-s}}\right)^{2-\delta}}|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq (T-s)^{\frac{1}{2}}\}}dyds\\ &\lesssim (T-t)\int_0^{t-(T-t)} v(s)(T-s)^{-1-\frac{b}{2}}ds,\\ |I_2|&\lesssim \int_{t-(T-t)}^t v(s)(t-s)^{-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-c\left(\frac{|x-y|}{\sqrt{t-s}}\right)^{2-\delta}}|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq (T-s)^{\frac{1}{2}}\}}dyds\\ &\quad+\int_{t-(T-t)}^t v(s)(T-s)^{-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-c\left(\frac{|x-y|}{\sqrt{T-s}}\right)^{2-\delta}}|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq (T-s)^{\frac{1}{2}}\}}dyds\\ &\lesssim v(t)(T-t)^{1-\frac{b}{2}},\\ |I_3|&\lesssim \int_t^T\int_{\mathbb{R}^d}(T-s)^{-\frac{d}{2}}e^{-c\left(\frac{|x-y|}{\sqrt{T-s}}\right)^{2-\delta}}v(s)|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq (T-s)^{\frac{1}{2}}\}}dyds\lesssim\int_t^T v(s)(T-s)^{-\frac{b}{2}}ds.\end{aligned}$$

This concludes (B.14). ☐

Proof of (B.15).

$$
\begin{aligned}
&\partial_{x_i}\psi(x,t)-\partial_{x_i}\psi(x,T)\\
&=\int_0^t\int_{\mathbb{R}^d}(\partial_{x_i}\Gamma(x,t,y,s)-\partial_{x_i}\Gamma(x,T,y,s))f(y,s)dyds-\int_t^T\int_{\mathbb{R}^d}\partial_{x_i}\Gamma(x,T,y,s)f(y,s)dyds\\
&=: I_1+I_2.
\end{aligned}
$$

For I_1, by (7.7),

$$
\begin{aligned}
|I_1|\lesssim{}& C(\alpha)(T-t)^{\frac{\alpha}{2}}\int_0^t\int_{\mathbb{R}^d}(T-s)^{-\frac{\alpha}{2}}\left[(t-s)^{-\frac{d+1}{2}}e^{-c\left(\frac{|x-y|}{\sqrt{t-s}}\right)^{2-\delta}}+(T-s)^{-\frac{d+1}{2}}e^{-c\left(\frac{|x-y|}{\sqrt{T-s}}\right)^{2-\delta}}\right]\\
&*v(s)|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq(T-s)^{\frac12}\}}dyds\\
={}& C(\alpha)\,(T-t)^{\frac{\alpha}{2}}\left(\int_0^{t-(T-t)}+\int_{t-(T-t)}^t\cdots\right)=: C(\alpha)\,(T-t)^{\frac{\alpha}{2}}\,(I_{11}+I_{12}).
\end{aligned}
$$

For I_{11},

$$
\begin{aligned}
I_{11}&\lesssim\int_0^{t-(T-t)}\int_{\mathbb{R}^d}(t-s)^{-\frac{d+1+\alpha}{2}}\left[e^{-c\left(\frac{|x-y|}{\sqrt{t-s}}\right)^{2-\delta}}+e^{-c\left(\frac{|x_*-y|}{\sqrt{t-s}}\right)^{2-\delta}}\right]v(s)|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq(T-s)^{\frac12}\}}dyds\\
&\lesssim\int_0^{t-(T-t)}v(s)(T-s)^{-\frac{1+b+\alpha}{2}}ds
\end{aligned}
$$

by the same calculation in (B.8).

For I_{12}, since $T-t\leq T-s\leq 2(T-t)$,

$$
\begin{aligned}
I_{12}={}&\int_{t-(T-t)}^t\int_{\mathbb{R}^d}(T-s)^{-\frac{\alpha}{2}}\left[(t-s)^{-\frac{d+1}{2}}e^{-c\left(\frac{|x-y|}{\sqrt{t-s}}\right)^{2-\delta}}+(T-s)^{-\frac{d+1}{2}}e^{-c\left(\frac{|x_*-y|}{\sqrt{T-s}}\right)^{2-\delta}}\right]\\
&*v(s)|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq(T-s)^{\frac12}\}}dyds\\
\lesssim{}& v(t)(T-t)^{-\frac{\alpha}{2}}\int_{t-(T-t)}^t(t-s)^{-\frac{d+1}{2}}\int_{\mathbb{R}^d}e^{-c\left(\frac{|x-y|}{\sqrt{t-s}}\right)^{2-\delta}}|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq(T-t)^{\frac12}\}}dyds\\
&+v(t)(T-t)^{-\frac{d+1+\alpha}{2}}\int_{t-(T-t)}^t\int_{\mathbb{R}^d}e^{-c\left(\frac{|x_*-y|}{\sqrt{T-t}}\right)^{2-\delta}}|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq(T-t)^{\frac12}\}}dyds\\
\lesssim{}& v(t)(T-t)^{\frac{1-b-\alpha}{2}}
\end{aligned}
$$

where we used (B.3) in the last "$\lesssim$".

For I_2, by (B.3),

$$
|I_2|\lesssim\int_t^T v(s)(T-s)^{-\frac{d+1}{2}}\int_{\mathbb{R}^d}e^{-c\left(\frac{|x_*-y|}{\sqrt{T-s}}\right)^{2-\delta}}|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq(T-s)^{\frac12}\}}dyds\lesssim\int_t^T v(s)(T-s)^{-\frac{1+b}{2}}ds.
$$

□

Proof of (B.16).

$$
\begin{aligned}
&\partial_{x_i}\psi(x,t)-\partial_{x_i}\psi(x_*,t_*)\\
&=\int_0^t\int_{\mathbb{R}^d}(\partial_{x_i}\Gamma(x,t,y,s)-\partial_{x_i}\Gamma(x_*,t_*,y,s))f(y,s)dyds-\int_t^{t_*}\int_{\mathbb{R}^d}\partial_{x_i}\Gamma(x_*,t_*,y,s)f(y,s)dyds\\
&=: I_1+I_2.
\end{aligned}
$$

For I_1, by (7.7),

$$\begin{aligned}|I_1| &\lesssim C(\alpha)\left(|x-x_*|+\sqrt{|t-t_*|}\right)^{\alpha}\\&\quad\times\int_0^t\int_{\mathbb{R}^d}(t_*-s)^{-\frac{\alpha}{2}}\left[(t-s)^{-\frac{d+1}{2}}e^{-c\left(\frac{|x-y|}{\sqrt{t-s}}\right)^{2-\delta}}+(t_*-s)^{-\frac{d+1}{2}}e^{-c\left(\frac{|x_*-y|}{\sqrt{t_*-s}}\right)^{2-\delta}}\right]\\&\quad * v(s)|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq(T-s)^{\frac12}\}}dyds\\&= C(\alpha)\left(|x-x_*|+\sqrt{|t-t_*|}\right)^{\alpha}\left(\int_0^{t-(T-t)}+\int_{t-(T-t)}^{t-(t_*-t)}+\int_{t-(t_*-t)}^{t}\cdots\right)\\&=: C(\alpha)\left(|x-x_*|+\sqrt{|t-t_*|}\right)^{\alpha}(I_{11}+I_{12}+I_{13}).\end{aligned}$$

For I_{11},

$$\begin{aligned}I_{11} &\lesssim \int_0^{t-(T-t)}\int_{\mathbb{R}^d}(t-s)^{-\frac{d+1+\alpha}{2}}\left[e^{-c\left(\frac{|x-y|}{\sqrt{t-s}}\right)^{2-\delta}}+e^{-c\left(\frac{|x_*-y|}{\sqrt{t-s}}\right)^{2-\delta}}\right]v(s)|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq(T-s)^{\frac12}\}}dyds\\&\lesssim \int_0^{t-(T-t)}v(s)(T-s)^{-\frac{1+b+\alpha}{2}}ds\end{aligned}$$

by the same calculation in (B.8).

For I_{12},

$$\begin{aligned}I_{12} &\lesssim v(t)\int_{t-(T-t)}^{t-(t_*-t)}\int_{\mathbb{R}^d}(t-s)^{-\frac{d+1+\alpha}{2}}\left[e^{-c\left(\frac{|x-y|}{\sqrt{t-s}}\right)^{2-\delta}}+e^{-c\left(\frac{|x_*-y|}{\sqrt{t-s}}\right)^{2-\delta}}\right]|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq(T-t)^{\frac12}\}}dyds\\&\leq v(t)\int_{t-(T-t)}^{t}\int_{\mathbb{R}^d}(t-s)^{-\frac{d+1+\alpha}{2}}\left[e^{-c\left(\frac{|x-y|}{\sqrt{t-s}}\right)^{2-\delta}}+e^{-c\left(\frac{|x_*-y|}{\sqrt{t-s}}\right)^{2-\delta}}\right]|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq(T-t)^{\frac12}\}}dyds\\&\lesssim v(t)(T-t)^{\frac{1-b-\alpha}{2}}\end{aligned}$$

by the same calculation in (B.8) where we used $\alpha<1$.

For I_{13}, since $t_*-t\leq t_*-s\leq 2(t_*-t)$, $T-t\leq T-s\leq 2(T-t)$,

$$\begin{aligned}I_{13} &= \int_{t-(t_*-t)}^{t}\int_{\mathbb{R}^d}(t_*-s)^{-\frac{\alpha}{2}}\left[(t-s)^{-\frac{d+1}{2}}e^{-c\left(\frac{|x-y|}{\sqrt{t-s}}\right)^{2-\delta}}+(t_*-s)^{-\frac{d+1}{2}}e^{-c\left(\frac{|x_*-y|}{\sqrt{t_*-s}}\right)^{2-\delta}}\right]\\&\quad * v(s)|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq(T-s)^{\frac12}\}}dyds\\&\lesssim v(t)(t_*-t)^{-\frac{\alpha}{2}}\int_{t-(t_*-t)}^{t}(t-s)^{-\frac{d+1}{2}}\int_{\mathbb{R}^d}e^{-c\left(\frac{|x-y|}{\sqrt{t-s}}\right)^{2-\delta}}|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq(T-t)^{\frac12}\}}dyds\\&\quad + v(t)(t_*-t)^{-\frac{d+1+\alpha}{2}}\int_{t-(t_*-t)}^{t}\int_{\mathbb{R}^d}e^{-c\left(\frac{|x_*-y|}{\sqrt{t_*-t}}\right)^{2-\delta}}|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq(T-t)^{\frac12}\}}dyds\\&\lesssim v(t)(T-t)^{-\frac{b}{2}}(t_*-t)^{\frac{1-\alpha}{2}}\end{aligned}$$

where we used (B.3) in the last "$\lesssim$".

For I_2, by (B.3),

$$\begin{aligned}|I_2| &\lesssim \int_t^{t_*}v(s)(t_*-s)^{-\frac{d+1}{2}}\int_{\mathbb{R}^d}e^{-c\left(\frac{|x_*-y|}{\sqrt{t_*-s}}\right)^{2-\delta}}|y-q|^{-b}\mathbf{1}_{\{|y-q|\geq(T-s)^{\frac12}\}}dyds\\&\lesssim \int_t^{t_*}v(s)(t_*-s)^{-\frac12}(T-s)^{-\frac{b}{2}}ds\\&\lesssim v(t)(T-t)^{-\frac{b}{2}}(t_*-t)^{\frac12}\mathbf{1}_{\{t_*\leq\frac{T+t}{2}\}}+\int_t^{t_*}v(s)(T-s)^{-\frac{1+b}{2}}ds\mathbf{1}_{\{t_*>\frac{T+t}{2}\}}.\end{aligned}$$

Indeed, when $t_*-t\leq(T-t)/2$, then for $s\in(t,t_*)$,

$$\frac{T-t}{2}\leq T-t_*\leq T-s\leq 2(T-t_*)\leq 2(T-t).$$

It follows that

$$|I_2|\lesssim v(t)(T-t)^{-\frac{b}{2}}(t_*-t)^{\frac12}.$$

When $t_* - t > (T-t)/2$,

$$\int_{t_*-(T-t_*)}^{t_*} v(s)(t_*-s)^{-\frac{1}{2}}(T-s)^{-\frac{b}{2}}ds \lesssim v(2t_*-T)(T-t_*)^{\frac{1}{2}-\frac{b}{2}} \sim v(t_*)(T-t_*)^{\frac{1}{2}-\frac{b}{2}}.$$

$$\int_{t}^{t_*-(T-t_*)} v(s)(t_*-s)^{-\frac{1}{2}}(T-s)^{-\frac{b}{2}}ds \sim \int_{t}^{t_*-(T-t_*)} v(s)(T-s)^{-\frac{1+b}{2}}ds$$

since $\frac{T-s}{2} \le t_* - s \le T-s$.

□

Appendix C. Derivation of the weighted topology for the outer problem

Proposition C.1. *For*

$$|f| \lesssim \sum_{j=1}^{N}\left(\varrho_1^{[j]} + \varrho_2^{[j]}\right) + \varrho_3,$$

suppose that

$$\begin{aligned} &0<\Theta<\beta<\frac{1}{2},\ 0<\alpha<1,\ \Theta+\frac{1}{2}-\beta-\frac{\alpha}{2}<0,\\ &0<\sigma_0<\beta,\ \beta-\sigma_0-\frac{\alpha}{2}<0,\ 1-\sigma_0-(1+\alpha)(1-\beta)<0,\\ &\Theta+2\sigma_0-\beta<0, \end{aligned} \tag{C.1}$$

then we have

$$|\mathcal{T}_2^{\text{out}}[f]| \lesssim |\ln T|\lambda_*^{\Theta+1}(0)R(0), \tag{C.2}$$

$$|\nabla_x \mathcal{T}_2^{\text{out}}[f]| \lesssim \lambda_*^{\Theta}(0), \tag{C.3}$$

$$|\mathcal{T}_2^{\text{out}}[f](x,t) - \mathcal{T}_2^{\text{out}}[f](x,T)| \lesssim |\ln(T-t)|\lambda_*^{\Theta+1}R, \tag{C.4}$$

$$|\nabla_x \mathcal{T}_2^{\text{out}}[f](x,t) - \nabla_x \mathcal{T}_2^{\text{out}}[f](x,T)| \lesssim C(\alpha)\lambda_*^{\Theta}, \tag{C.5}$$

for $0<t<t_*\le T$, $t_*-t<\frac{1}{4}(T-t)$,

$$|\nabla_x \mathcal{T}_2^{\text{out}}[f](x,t) - \nabla_x \mathcal{T}_2^{\text{out}}[f](x_*,t_*)| \lesssim C(\alpha)\left(|x-x_*| + \sqrt{|t-t_*|}\right)^{\alpha}\lambda_*^{\Theta}(t)(\lambda_* R)^{-\alpha}(t). \tag{C.6}$$

Proof. **Convolution estimate about $\varrho_1^{[j]}$.** For

$$|f| \lesssim \varrho_1^{[j]} := \lambda_*^{\Theta}(\lambda_* R)^{-1}\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}},$$

$$\begin{aligned} \mathcal{T}_2^{\text{out}}[f] \lesssim\ & \int_0^{t-(T-t)} \lambda_*^{\Theta}(s)(\lambda_* R)^{-1}(s)(T-s)^{-1}(\lambda_* R)^2(s)ds + \lambda_*^{\Theta}(\lambda_* R)^{-1}(\lambda_* R)^2|\ln(T-t)|\\ =\ & \int_0^{t-(T-t)} \lambda_*^{\Theta}(s)\lambda_* R(T-s)^{-1}ds + \lambda_*^{\Theta}\lambda_* R|\ln(T-t)|\\ \lesssim\ & \lambda_*^{\Theta}(0)(\lambda_* R)(0)|\ln T| \end{aligned} \tag{C.7}$$

provided

$$1+\Theta-\beta>0. \tag{C.8}$$

$$\begin{aligned} |\nabla_x \mathcal{T}_2^{\text{out}}[f]| \lesssim\ & \int_0^{t-(T-t)} \lambda_*^{\Theta}(s)(\lambda_* R)^{-1}(s)(T-s)^{-\frac{3}{2}}(\lambda_* R)^2(s)ds + \lambda_*^{\Theta}(\lambda_* R)^{-1}\lambda_* R\\ =\ & \int_0^{t-(T-t)} \lambda_*^{\Theta}(s)(\lambda_* R)(s)(T-s)^{-\frac{3}{2}}ds + \lambda_*^{\Theta} \lesssim \lambda_*^{\Theta}(0) \end{aligned} \tag{C.9}$$

provided

$$\beta<\frac{1}{2},\ \Theta+\frac{1}{2}-\beta>0. \tag{C.10}$$

$$
\begin{aligned}
&|\mathcal{T}_2^{\text{out}}[f](x,t)-\mathcal{T}_2^{\text{out}}[f](x,T)| \\
\lesssim\ &(T-t)\int_0^{t-(T-t)}\lambda_*^{\Theta}(s)(\lambda_*R)^{-1}(s)(T-s)^{-2}(\lambda_*R)^2(s)ds \\
&+\lambda_*^{\Theta}(\lambda_*R)^{-1}\int_{t-(T-t)}^{t}\begin{cases}1 & \text{if } \ t-s\le(\lambda_*R)^2(t)\\ (\lambda_*R)^2(t)(t-s)^{-1} & \text{if } \ t-s>(\lambda_*R)^2(t)\end{cases}ds \\
&+\lambda_*^{\Theta}(\lambda_*R)^{-1}(\lambda_*R)^2+\int_t^T(T-s)^{-1}\lambda_*^{\Theta}(s)(\lambda_*R)^{-1}(s)(\lambda_*R)^2(s)ds \\
\lesssim\ &(T-t)\int_0^{t-(T-t)}\lambda_*^{\Theta}(s)(\lambda_*R)(s)(T-s)^{-2}ds+\lambda_*^{\Theta}(\lambda_*R)|\ln(T-t)|+\int_t^T(T-s)^{-1}\lambda_*^{\Theta}(s)(\lambda_*R)(s)ds \\
\lesssim\ &\lambda_*^{\Theta}(\lambda_*R)|\ln(T-t)|
\end{aligned}
\tag{C.11}
$$

provided

$$0<\beta-\Theta<1. \tag{C.12}$$

$$
\begin{aligned}
&|\nabla\mathcal{T}_2^{\text{out}}[f](x,t)-\nabla\mathcal{T}_2^{\text{out}}[f](x,T)| \\
\lesssim\ &C(\alpha)\left[(T-t)^{\frac{\alpha}{2}}\int_0^{t-(T-t)}\lambda_*^{\Theta}(s)(\lambda_*R)^{-1}(s)(T-s)^{-\frac{3+\alpha}{2}}(\lambda_*R)^2(s)ds+\lambda_*^{\Theta}(\lambda_*R)^{-1}\lambda_*R\right] \\
&+\int_t^T\lambda_*^{\Theta}(s)(\lambda_*R)^{-1}(s)(T-s)^{-\frac{3}{2}}(\lambda_*R)^2(s)ds \\
=\ &C(\alpha)\left[(T-t)^{\frac{\alpha}{2}}\int_0^{t-(T-t)}\lambda_*^{\Theta}(s)(\lambda_*R)(s)(T-s)^{-\frac{3+\alpha}{2}}ds+\lambda_*^{\Theta}\right]+\int_t^T\lambda_*^{\Theta}(s)(\lambda_*R)(s)(T-s)^{-\frac{3}{2}}ds \\
\lesssim\ &C(\alpha)\lambda_*^{\Theta}
\end{aligned}
\tag{C.13}
$$

provided

$$0<\alpha<1,\ \beta<\frac{1}{2},\ 0<\Theta+\frac{1}{2}-\beta<\frac{\alpha}{2}. \tag{C.14}$$

For $0 < \alpha < 1$ and $0 < t < t_* \le T$,

$$
\begin{aligned}
&|\nabla \mathcal{T}_2^{\text{out}}[f](x,t) - \nabla \mathcal{T}_2^{\text{out}}[f](x_*,t_*)| \\
\lesssim\ & C(\alpha)\left(|x-x_*| + \sqrt{|t-t_*|}\right)^{\alpha} \left[\int_0^{t-(T-t)} \lambda_*^{\Theta}(s)(\lambda_* R)^{-1}(s)(T-s)^{-\frac{3+\alpha}{2}}(\lambda_* R)^2(s)ds \right.\\
&\left. + \lambda_*^{\Theta}(t)(\lambda_* R)^{-1}(t)(\lambda_* R)^{1-\alpha}(t)\right] \\
&+ \mathbf{1}_{\{t_* > \frac{T+t}{2}\}}\left[(T-t_*)^{\frac{\alpha}{2}}\lambda_*^{\Theta}(t_*)(\lambda_* R)^{-1}(t_*)(\lambda_* R)^{1-\alpha}(t_*)\right. \\
&\left. + \int_t^{t_*-(T-t_*)} \lambda_*^{\Theta}(s)(\lambda_* R)^{-1}(s)(T-s)^{-\frac{3}{2}}(\lambda_* R)^2(s)ds\right] \qquad \text{(C.15)}\\
=\ & C(\alpha)\left(|x-x_*| + \sqrt{|t-t_*|}\right)^{\alpha}\left[\int_0^{t-(T-t)} \lambda_*^{\Theta}(s)(\lambda_* R)(s)(T-s)^{-\frac{3+\alpha}{2}}ds + \lambda_*^{\Theta}(t)(\lambda_* R)^{-\alpha}(t)\right] \\
&+ \mathbf{1}_{\{t_* > \frac{T+t}{2}\}}\left[(T-t_*)^{\frac{\alpha}{2}}\lambda_*^{\Theta}(t_*)(\lambda_* R)^{-\alpha}(t_*) + \int_t^{t_*-(T-t_*)} \lambda_*^{\Theta}(s)(\lambda_* R)(s)(T-s)^{-\frac{3}{2}}ds\right] \\
\lesssim\ & C(\alpha)\left(|x-x_*| + \sqrt{|t-t_*|}\right)^{\alpha}\lambda_*^{\Theta}(t)(\lambda_* R)^{-\alpha}(t) \\
&+ \mathbf{1}_{\{t_* > \frac{T+t}{2}\}}\left[(T-t_*)^{\frac{\alpha}{2}}\lambda_*^{\Theta}(t_*)(\lambda_* R)^{-\alpha}(t_*) + \lambda_*^{\Theta}(t)(\lambda_* R)(t)(T-t)^{-\frac{1}{2}}\right] \\
\lesssim\ & C(\alpha)\left(|x-x_*| + \sqrt{|t-t_*|}\right)^{\alpha}\lambda_*^{\Theta}(t)(\lambda_* R)^{-\alpha}(t) + \mathbf{1}_{\{t_* > \frac{T+t}{2}\}}(T-t_*)^{\frac{\alpha}{2}}\lambda_*^{\Theta}(t_*)(\lambda_* R)^{-\alpha}(t_*)
\end{aligned}
$$

provided

$$\Theta + \frac{1}{2} - \beta < \frac{\alpha}{2},\ \beta < \frac{1}{2}. \qquad \text{(C.16)}$$

Convolution estimate about $\varrho_2^{[j]}$.

Recall $\varrho_2^{[j]} = T^{-\sigma_0}\frac{\lambda_*^{1-\sigma_0}}{|x-q^{[j]}|^2}\mathbf{1}_{\{\lambda_* R \le |x-q^{[j]}| \le d_q\}}$. Consider

$$|f| \le \lambda_*^{1-\sigma_0}|x-q^{[j]}|^{-2}\mathbf{1}_{\{\lambda_* R \le |x-q^{[j]}| \le (T-t)^{\frac{1}{2}}\}} + \lambda_*^{1-\sigma_0}|x-q^{[j]}|^{-2}\mathbf{1}_{\{(T-t)^{\frac{1}{2}} < |x-q^{[j]}| \le d_q\}}.$$

Then

$$
\begin{aligned}
\mathcal{T}_2^{\text{out}}[f] \lesssim\ & \int_0^{t-(T-t)} \lambda_*^{1-\sigma_0}(s)(T-s)^{-1}|\ln(T-s)|ds + \lambda_*^{1-\sigma_0}|\ln(T-t)|^2 + \int_0^t \lambda_*^{1-\sigma_0}(s)(T-s)^{-1}ds \qquad \text{(C.17)}\\
\lesssim\ & \lambda_*^{1-\sigma_0}(0)(\ln T)^2.
\end{aligned}
$$

$$
\begin{aligned}
|\nabla \mathcal{T}_2^{\text{out}}[f]| \lesssim\ & \int_0^{t-(T-t)} \lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3}{2}}|\ln(T-s)|ds + \lambda_*^{1-\sigma_0}(\lambda_* R)^{-1} + \int_0^t \lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3}{2}}ds \qquad \text{(C.18)}\\
\lesssim\ & \lambda_*^{1-\sigma_0}(0)(\lambda_* R)^{-1}(0)
\end{aligned}
$$

provided

$$\sigma_0 < \beta < \frac{1}{2}. \qquad \text{(C.19)}$$

$$\begin{aligned}
&|\mathcal{T}_2^{\rm out}[f](x,t)-\mathcal{T}_2^{\rm out}[f](x,T)| \\
\lesssim\ & (T-t)\int_0^{t-(T-t)} \lambda_*^{1-\sigma_0}(s)(T-s)^{-2}|\ln(T-s)|ds \\
&+\lambda_*^{1-\sigma_0}\int_{t-(T-t)}^{t}\begin{cases}(\lambda_*R)^{-2} & \text{if }\ t-s\le(\lambda_*R)^2(t)\\ (t-s)^{-1}\langle\ln(\frac{t-s}{(\lambda_*R)^2(t)})\rangle & \text{if }\ (\lambda_*R)^2(t)<t-s\le T-t\end{cases} ds \\
&+\lambda_*^{1-\sigma_0}|\ln(T-t)|+\int_t^T (T-s)^{-1}\lambda_*^{1-\sigma_0}(s)|\ln(T-s)|ds \\
&+(T-t)\int_0^{t-(T-t)}\lambda_*^{1-\sigma_0}(s)(T-s)^{-2}ds+\lambda_*^{1-\sigma_0}+\int_t^T\lambda_*^{1-\sigma_0}(s)(T-s)^{-1}ds \\
\lesssim\ & \lambda_*^{1-\sigma_0}\ln^2(T-t)
\end{aligned} \tag{C.20}$$

provided

$$0<\sigma_0<1,\ \beta<\frac{1}{2}. \tag{C.21}$$

where we used

$$\int_{t-(T-t)}^{t-(\lambda_*R)^2(t)}(t-s)^{-1}\langle\ln(\frac{t-s}{(\lambda_*R)^2(t)})\rangle ds=\int_1^{\frac{T-t}{(\lambda_*R)^2(t)}} z^{-1}\langle\ln z\rangle dz=O(\ln^2(T-t)).$$

For $0<\alpha<1$,

$$\begin{aligned}
&|\nabla\mathcal{T}_2^{\rm out}[f](x,t)-\nabla\mathcal{T}_2^{\rm out}[f](x,T)| \\
\lesssim\ & C(\alpha)\left[(T-t)^{\frac{\alpha}{2}}\int_0^{t-(T-t)}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3+\alpha}{2}}|\ln(T-s)|ds+\lambda_*^{1-\sigma_0}(\lambda_*R)^{-1}\right] \\
&+\int_t^T\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3}{2}}|\ln(T-s)|ds \\
&+C(\alpha)\,(T-t)^{\frac{\alpha}{2}}\left[\int_0^{t-(T-t)}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3+\alpha}{2}}ds+\lambda_*^{1-\sigma_0}(T-t)^{\frac{-1-\alpha}{2}}\right]+\int_t^T\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3}{2}}ds \\
\lesssim\ & C(\alpha)\left[(T-t)^{\frac{\alpha}{2}}\int_0^{t-(T-t)}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3+\alpha}{2}}|\ln(T-s)|ds+\lambda_*^{1-\sigma_0}(\lambda_*R)^{-1}\right]\lesssim C(\alpha)\lambda_*^{1-\sigma_0}(\lambda_*R)^{-1}
\end{aligned} \tag{C.22}$$

provided

$$\beta<\frac{1}{2},\ \sigma_0<\frac{1}{2},\ \beta-\sigma_0<\frac{\alpha}{2}. \tag{C.23}$$

where we used the following estimate in the last "$\lesssim$": If $1-\sigma_0-\frac{1+\alpha}{2}\le 0$, then

$$(T-t)^{\frac{\alpha}{2}}\int_0^{t-(T-t)}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3+\alpha}{2}}|\ln(T-s)|ds\lesssim\lambda_*^{1-\sigma_0}(T-t)^{-\frac{1}{2}-\epsilon}|\ln(T-t)|\ll\lambda_*^{1-\sigma_0}(\lambda_*R)^{-1}$$

where $\epsilon=0$ when $1-\sigma_0-\frac{1+\alpha}{2}<0$ and $0<\epsilon<\frac{1}{2}-\beta$ when $1-\sigma_0-\frac{1+\alpha}{2}=0$;

If $1-\sigma_0-\frac{1+\alpha}{2}>0$, then

$$(T-t)^{\frac{\alpha}{2}}\int_0^{t-(T-t)}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3+\alpha}{2}}|\ln(T-s)|ds\lesssim(T-t)^{\frac{\alpha}{2}}\lambda_*^{1-\sigma_0}(0)T^{-\frac{1+\alpha}{2}}|\ln T|\ll\lambda_*^{1-\sigma_0}(\lambda_*R)^{-1}$$

when $\beta<\frac{1}{2},\beta-\sigma_0<\frac{\alpha}{2}$.

For $0<\alpha<1$ and $0<t<t_*\le T$,

$$
\begin{aligned}
&|\nabla \mathcal{T}_2^{\text{out}}[f](x,t)-\nabla \mathcal{T}_2^{\text{out}}[f](x_*,t_*)|\\
\lesssim\ & C(\alpha)\left(|x-x_*|+\sqrt{|t-t_*|}\right)^{\alpha}\left[\int_0^{t-(T-t)}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3+\alpha}{2}}|\ln(T-s)|ds+\lambda_*^{1-\sigma_0}(t)(\lambda_*R)^{-1-\alpha}(t)\right]\\
&+\mathbf{1}_{\{t_*>\frac{T+t}{2}\}}\left[(T-t_*)^{\frac{\alpha}{2}}\lambda_*^{1-\sigma_0}(t_*)(\lambda_*R)^{-1-\alpha}(t_*)+\int_t^{t_*-(T-t_*)}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3}{2}}|\ln(T-s)|ds\right]\\
&+C(\alpha)\left(|x-x_*|+\sqrt{|t-t_*|}\right)^{\alpha}\left[\int_0^{t-(T-t)}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3+\alpha}{2}}ds+\lambda_*^{1-\sigma_0}(t)(T-t)^{\frac{-1-\alpha}{2}}\right]\\
&+\mathbf{1}_{\{t_*>\frac{T+t}{2}\}}\int_t^{t_*}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3}{2}}ds\\
\lesssim\ & C(\alpha)\left(|x-x_*|+\sqrt{|t-t_*|}\right)^{\alpha}\left[\int_0^{t-(T-t)}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3+\alpha}{2}}|\ln(T-s)|ds+\lambda_*^{1-\sigma_0}(t)(\lambda_*R)^{-1-\alpha}(t)\right]\\
&+\mathbf{1}_{\{t_*>\frac{T+t}{2}\}}\left[(T-t_*)^{\frac{\alpha}{2}}\lambda_*^{1-\sigma_0}(t_*)(\lambda_*R)^{-1-\alpha}(t_*)+\int_t^{t_*-(T-t_*)}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3}{2}}|\ln(T-s)|ds\right]\\
&+\mathbf{1}_{\{t_*>\frac{T+t}{2}\}}\int_t^{t_*}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3}{2}}ds\\
\lesssim\ & C(\alpha)\left(|x-x_*|+\sqrt{|t-t_*|}\right)^{\alpha}\lambda_*^{1-\sigma_0}(t)(\lambda_*R)^{-1-\alpha}(t)\\
&+\mathbf{1}_{\{t_*>\frac{T+t}{2}\}}\left[(T-t_*)^{\frac{\alpha}{2}}\lambda_*^{1-\sigma_0}(t_*)(\lambda_*R)^{-1-\alpha}(t_*)+\int_t^{t_*-(T-t_*)}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3}{2}}|\ln(T-s)|ds\right]\\
&+\mathbf{1}_{\{t_*>\frac{T+t}{2}\}}\int_t^{t_*}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3}{2}}ds
\end{aligned}
\tag{C.24}
$$

provided

$$
\beta<\frac{1}{2},\ 1-\sigma_0-(1+\alpha)(1-\beta)<0 \tag{C.25}
$$

where we used the following estimate in the last "$\lesssim$": If $1-\sigma_0-\frac{1+\alpha}{2}\le 0$,

$$
\int_0^{t-(T-t)}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3+\alpha}{2}}ds\lesssim\lambda_*^{1-\sigma_0}(t)(T-t)^{-\frac{1+\alpha}{2}-\epsilon}\ll\lambda_*^{1-\sigma_0}(t)(\lambda_*R)^{-1-\alpha}(t)
$$

where $\epsilon=0$ when $1-\sigma_0-\frac{1+\alpha}{2}<0$ and $0<\epsilon<(1+\alpha)(\frac{1}{2}-\beta)$ when $1-\sigma_0-\frac{1+\alpha}{2}=0$;

If $1-\sigma_0-\frac{1+\alpha}{2}>0$,

$$
\int_0^{t-(T-t)}\lambda_*^{1-\sigma_0}(s)(T-s)^{-\frac{3+\alpha}{2}}ds\lesssim\lambda_*^{1-\sigma_0}(0)T^{-\frac{1+\alpha}{2}}\ll\lambda_*^{1-\sigma_0}(t)(\lambda_*R)^{-1-\alpha}(t)
$$

when $\beta<\frac{1}{2},1-\sigma_0-(1+\alpha)(1-\beta)<0$.

Convolution estimate about ϱ_3.

Recall $\varrho_3=T^{-\sigma_0}$. Consider

$$
|f|\le 1=\mathbf{1}_{\{|x|\le\sqrt{T-t}\}}+\mathbf{1}_{\{|x|>\sqrt{T-t}\}}.
$$

Then

$$
|\mathcal{T}_d^{\text{out}}[f]|\lesssim T. \tag{C.26}
$$

$$
|\nabla\mathcal{T}_d^{\text{out}}[f]|\lesssim T^{\frac{1}{2}}. \tag{C.27}
$$

$$
|\mathcal{T}_d^{\text{out}}[f](x,t)-\mathcal{T}_d^{\text{out}}[f](x,T)|\lesssim(T-t)|\ln(T-t)|. \tag{C.28}
$$

For $0<\alpha<1$,

$$\begin{aligned}&|\nabla\mathcal{T}_d^{\text{out}}[f](x,t)-\nabla\mathcal{T}_d^{\text{out}}[f](x,T)|\\&\lesssim C(\alpha)\left[(T-t)^{\frac{\alpha}{2}}\int_0^{t-(T-t)}(T-s)^{-\frac{1+\alpha}{2}}ds+(T-t)^{\frac12}\right]\lesssim C(\alpha)T^{\frac{1-\alpha}{2}}(T-t)^{\frac{\alpha}{2}}.\end{aligned}\tag{C.29}$$

For $0<\alpha<1$ and $0<t<t_*\le T$,

$$\begin{aligned}&|\nabla\mathcal{T}_d^{\text{out}}[f](x,t)-\nabla\mathcal{T}_d^{\text{out}}[f](x_*,t_*)|\\&\lesssim C(\alpha)\left(|x-x_*|+\sqrt{|t-t_*|}\right)^{\alpha}\left[\int_0^{t-(T-t)}(T-s)^{-\frac{1+\alpha}{2}}ds+(T-t)^{\frac{1-\alpha}{2}}\right]\\&\quad+\mathbf{1}_{\{t_*>\frac{T+t}{2}\}}\left[(T-t_*)^{\frac12}+\int_t^{t_*}(T-s)^{-\frac12}ds\right]\\&\lesssim C(\alpha)\left(|x-x_*|+\sqrt{|t-t_*|}\right)^{\alpha}T^{\frac{1-\alpha}{2}}+\mathbf{1}_{\{t_*>\frac{T+t}{2}\}}(T-t)^{\frac12}.\end{aligned}\tag{C.30}$$

In sum, for

$$|f|\lesssim\sum_{j=1}^{N}\left(\varrho_1^{[j]}+\varrho_2^{[j]}\right)+\varrho_3,$$

combining (C.7), (C.17) and (C.26), we have

$$|\mathcal{T}_d^{\text{out}}[f]|\lesssim\lambda_*^{\Theta}(0)(\lambda_*R)(0)|\ln T|+T^{-\sigma_0}\lambda_*^{1-\sigma_0}(0)(\ln T)^2+T^{1-\sigma_0}\lesssim\lambda_*^{\Theta}(0)(\lambda_*R)(0)|\ln T|\tag{C.31}$$

where in the last "$\lesssim$", we require

$$\Theta+2\sigma_0-\beta<0,\ \sigma_0>0.\tag{C.32}$$

Combining (C.9), (C.18) and (C.27), we have

$$|\nabla_x\mathcal{T}_2^{\text{out}}[f]|\lesssim\lambda_*^{\Theta}(0)+T^{-\sigma_0}\lambda_*^{1-\sigma_0}(0)(\lambda_*R)^{-1}(0)+T^{\frac12-\sigma_0}\lesssim\lambda_*^{\Theta}(0)\tag{C.33}$$

where in the last "$\lesssim$" we used

$$\Theta+2\sigma_0-\beta<0,\ \sigma_0>0,\ \beta<\frac12.\tag{C.34}$$

Combining (C.11), (C.20) and (C.28), then

$$\begin{aligned}|\mathcal{T}_2^{\text{out}}[f](x,t)-\mathcal{T}_2^{\text{out}}[f](x,T)|&\lesssim\lambda_*^{\Theta}(\lambda_*R)|\ln(T-t)|+T^{-\sigma_0}\lambda_*^{1-\sigma_0}\ln^2(T-t)+T^{-\sigma_0}(T-t)|\ln(T-t)|\\&\lesssim\lambda_*^{\Theta}(\lambda_*R)|\ln(T-t)|\end{aligned}\tag{C.35}$$

where in the last "$\lesssim$" we used

$$\Theta+2\sigma_0-\beta<0,\ \sigma_0>0.\tag{C.36}$$

Combining (C.13), (C.22) and (C.29), then

$$\begin{aligned}&|\nabla\mathcal{T}_2^{\text{out}}[f](x,t)-\nabla\mathcal{T}_2^{\text{out}}[f](x,T)|\lesssim C(\alpha)\left[\lambda_*^{\Theta}+T^{-\sigma_0}\lambda_*^{1-\sigma_0}(\lambda_*R)^{-1}+T^{-\sigma_0}T^{\frac{1-\alpha}{2}}(T-t)^{\frac{\alpha}{2}}\right]\\&=C(\alpha)\lambda_*^{\Theta}\left[1+T^{-\sigma_0}\lambda_*^{1-\sigma_0-\Theta}(\lambda_*R)^{-1}+T^{-\sigma_0}T^{\frac{1-\alpha}{2}}(T-t)^{\frac{\alpha}{2}-\Theta}\right]\lesssim C(\alpha)\lambda_*^{\Theta}\end{aligned}\tag{C.37}$$

where in the last "$\lesssim$" we used

$$\Theta+2\sigma_0-\beta<0,\ \Theta-\frac{\alpha}{2}<0,\ \Theta+\sigma_0-\frac12<0.\tag{C.38}$$

Combining (C.15), (C.24), (C.30), then for $0<t<t_*\le T$, $t_*-t<\frac14(T-t)$,

$$\begin{aligned}&|\nabla\mathcal{T}_2^{\text{out}}[f](x,t)-\nabla\mathcal{T}_2^{\text{out}}[f](x_*,t_*)|\\&\lesssim C(\alpha)\left(|x-x_*|+\sqrt{|t-t_*|}\right)^{\alpha}\left[\lambda_*^{\Theta}(t)(\lambda_*R)^{-\alpha}(t)+T^{-\sigma_0}\lambda_*^{1-\sigma_0}(t)(\lambda_*R)^{-1-\alpha}(t)+T^{-\sigma_0}T^{\frac{1-\alpha}{2}}\right]\\&=C(\alpha)\left(|x-x_*|+\sqrt{|t-t_*|}\right)^{\alpha}\lambda_*^{\Theta}(t)(\lambda_*R)^{-\alpha}(t)\\&\quad\times\left[1+T^{-\sigma_0}\lambda_*^{1-\sigma_0-\Theta}(t)(\lambda_*R)^{-1}(t)+T^{-\sigma_0}T^{\frac{1-\alpha}{2}}\lambda_*^{-\Theta}(t)(\lambda_*R)^{\alpha}(t)\right]\\&\lesssim C(\alpha)\left(|x-x_*|+\sqrt{|t-t_*|}\right)^{\alpha}\lambda_*^{\Theta}(t)(\lambda_*R)^{-\alpha}(t)\end{aligned}\tag{C.39}$$

where in the last "$\lesssim$" we used

$$\Theta + 2\sigma_0 - \beta < 0,\ \Theta - \alpha(1-\beta) < 0,\ \Theta + \sigma_0 - \frac{1}{2} - \alpha(\frac{1}{2} - \beta) < 0. \tag{C.40}$$

Collecting (C.8), (C.10), (C.12), (C.14), (C.16), (C.19), (C.21), (C.23), (C.25), (C.32), (C.34), (C.36), (C.38) and (C.40), we conclude the restrictions (C.1) on the parameters.

□

Appendix D. Estimates of $\mathcal{G}$ and $\mathcal{H}_j$

D.1. **Estimates for terms involving $\Phi_{\rm out}$, $\Phi_{\rm in}^{[j]}$.** In this section, we first derive some estimates for $\Phi_{\rm out}$, $\Phi_{\rm in}$ that will be used frequently in the estimate of $\mathcal{G}$.

For $\Phi_{\rm out} \in B_{\rm out}$ and all $j = 1, 2, \ldots, N$,

$$\begin{aligned}&|\Phi_{\rm out}(x,t)| = \left|\Phi_{\rm out}(x,t) - \Phi_{\rm out}(x,T) + \Phi_{\rm out}(x,T) - \Phi_{\rm out}(q^{[j]},T)\right| \\ &\lesssim \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left[|\ln(T-t)|\lambda_*^{\Theta+1}(t)R(t) + (T-t)\|Z_*\|_{C^3(\mathbb{R}^2)} + |x-q^{[j]}|\left(\lambda_*^{\Theta}(0) + \|Z_*\|_{C^3(\mathbb{R}^2)}\right)\right].\end{aligned}$$

Thus

$$\begin{aligned}&|\Phi_{\rm out}| \\ &\lesssim \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\min\Big\{|\ln(T-t)|\lambda_*^{\Theta+1}(t)R(t) + (T-t)\|Z_*\|_{C^3(\mathbb{R}^2)} + \inf_{j=1,\ldots,N}|x-q^{[j]}|\left(\lambda_*^{\Theta}(0) + \|Z_*\|_{C^3(\mathbb{R}^2)}\right), \\ &|\ln T|\lambda_*^{\Theta+1}(0)R(0) + \|Z_*\|_{C^3(\mathbb{R}^2)}\Big\},\end{aligned} \tag{D.1}$$

which implies

$$\begin{aligned}&|\Phi_{\rm out}| \lesssim \sum_{j=1}^N \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha} \\ &\times\left[|\ln(T-t)|\lambda_*^{\Theta+1}(t)R(t) + (T-t)\|Z_*\|_{C^3(\mathbb{R}^2)} + |x-q^{[j]}|\left(\lambda_*^{\Theta}(0) + \|Z_*\|_{C^3(\mathbb{R}^2)}\right)\right] \\ &+ \mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(|\ln T|\lambda_*^{\Theta+1}(0)R(0) + \|Z_*\|_{C^3(\mathbb{R}^2)}\right) \\ &\lesssim \sum_{j=1}^N \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(|\ln(T-t)|\lambda_*^{\Theta+1}R + \lambda_j\rho_j\right) + \mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}.\end{aligned} \tag{D.2}$$

By (5.50), we have

$$|\nabla_x\Phi_{\rm out}| \le \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(\lambda_*^{\Theta}(0) + \|Z_*\|_{C^3(\mathbb{R}^2)}\right). \tag{D.3}$$

By (5.44) and (4.21), one has

$$\begin{aligned}&|\Phi - \Phi_{\rm out}| = \left|\sum_{j=1}^N\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}(y^{[j]},t) + \eta_{d_q}^{[j]}\Phi_0^{*[j]}(r_j,t)\right)\right| \\ &\lesssim \sum_{j=1}^N\left[\eta_R^{[j]}\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0}\langle\rho_j\rangle^{-l} + \eta_{d_q}^{[j]}\left(z_j\mathbf{1}_{\{z_j^2<t\}} + t|\ln T|^{-1}z_j^{-1}\mathbf{1}_{\{z_j^2\ge t\}}\right)\right] \\ &\lesssim \sum_{j=1}^N\left[\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\left(\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0}\langle\rho_j\rangle^{-l} + \lambda_*\langle\rho_j\rangle\right) + \mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\lambda_*\langle\rho_j\rangle\right].\end{aligned} \tag{D.4}$$

By (5.44), we get

$$\begin{aligned}&\left|\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}(y^{[j]},t)\right)\right| = \left|\eta_R^{[j]}\nabla_x\left(Q_{\gamma_j}\Phi_{\rm in}^{[j]}(y^{[j]},t)\right) + Q_{\gamma_j}\Phi_{\rm in}^{[j]}(y^{[j]},t)\nabla_x\eta_R^{[j]}\right| \\ &\lesssim \eta_R^{[j]}\lambda_j^{-1}\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0}\langle y^{[j]}\rangle^{-l-1} + (\lambda_jR)^{-1}\mathbf{1}_{\{\lambda_jR\le|x-\xi^{[j]}|\le 2\lambda_jR\}}\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0}\langle y^{[j]}\rangle^{-l} \\ &\lesssim \mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}\lambda_*^{\nu-\delta_0-1}\langle\rho_j\rangle^{-l-1}.\end{aligned} \tag{D.5}$$

By (4.21), we have

$$\begin{aligned}&\left|\nabla_x\left(\eta_{d_q}^{[j]}\Phi_0^{*[j]}(r_j,t)\right)\right|=\left|\eta_{d_q}^{[j]}\nabla_x\Phi_0^{*[j]}(r_j,t)+\Phi_0^{*[j]}(r_j,t)\nabla_x\eta_{d_q}^{[j]}\right|\\&\lesssim\eta_{d_q}^{[j]}+t^{\frac12}\mathbf{1}_{\{d_q\le|x-\xi^{[j]}|\le 2d_q\}}\lesssim\mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}.\end{aligned}\tag{D.6}$$

Combining (D.5) and (D.6), we have

$$|\nabla_x(\Phi-\Phi_{\rm out})|\lesssim\sum_{j=1}^N\left[\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\left(\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0-1}\langle\rho_j\rangle^{-l-1}+1\right)+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\right].\tag{D.7}$$

$$\begin{aligned}\Delta_x(\Phi-\Phi_{\rm out})&=\sum_{j=1}^N\Delta_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}(y^{[j]},t)+\eta_{d_q}^{[j]}\Phi_0^{*[j]}(r_j,t)\right)\\&=\sum_{j=1}^N\Big[\eta_R^{[j]}\Delta_x\left(Q_{\gamma_j}\Phi_{\rm in}^{[j]}(y^{[j]},t)\right)+2\nabla_x\eta_R^{[j]}\nabla_x\left(Q_{\gamma_j}\Phi_{\rm in}^{[j]}(y^{[j]},t)\right)+Q_{\gamma_j}\Phi_{\rm in}^{[j]}(y^{[j]},t)\Delta_x\eta_R^{[j]}\\&\quad+\eta_{d_q}^{[j]}\Delta_x\Phi_0^{*[j]}(r_j,t)+2\nabla_x\eta_{d_q}^{[j]}\nabla_x\Phi_0^{*[j]}(r_j,t)+\Phi_0^{*[j]}(r_j,t)\Delta_x\eta_{d_q}^{[j]}\Big].\end{aligned}$$

By (4.21) and (5.44), it holds that

$$\begin{aligned}&|\Delta_x(\Phi-\Phi_{\rm out})|\\&\lesssim\sum_{j=1}^N\Big[\eta_R^{[j]}\lambda_j^{-2}\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0}\langle y^{[j]}\rangle^{-l-2}+(\lambda_jR)^{-1}\mathbf{1}_{\{\lambda_jR\le|x-\xi^{[j]}|\le2\lambda_jR\}}\lambda_j^{-1}\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0}\langle y^{[j]}\rangle^{-l-1}\\&\quad+(\lambda_jR)^{-2}\mathbf{1}_{\{\lambda_jR\le|x-\xi^{[j]}|\le2\lambda_jR\}}\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0}\langle y^{[j]}\rangle^{-l}+\eta_{d_q}^{[j]}\lambda_j^{-1}\langle\rho_j\rangle^{-1}+t\mathbf{1}_{\{d_q\le|x-\xi^{[j]}|\le2d_q\}}\Big]\\&\lesssim\sum_{j=1}^N\left(\mathbf{1}_{\{|x-\xi^{[j]}|\le2\lambda_jR\}}\lambda_j^{-2}\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0}\langle\rho_j\rangle^{-l-2}+\lambda_j^{-1}\langle\rho_j\rangle^{-1}\mathbf{1}_{\{|x-\xi^{[j]}|\le2d_q\}}\right)\\&\lesssim\sum_{j=1}^N\left[\mathbf{1}_{\{|x-q^{[j]}|\le3\lambda_*R\}}\left(\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0-2}\langle\rho_j\rangle^{-l-2}+\lambda_*^{-1}\langle\rho_j\rangle^{-1}\right)+\lambda_*^{-1}\langle\rho_j\rangle^{-1}\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\right].\end{aligned}\tag{D.8}$$

Combining (D.2) and (D.4), we have

$$\begin{aligned}|\Phi|\lesssim&\sum_{j=1}^N\Big[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\mathbf{1}_{\{|x-q^{[j]}|\le3\lambda_*R\}}\left(\lambda_*^{\nu-\delta_0}\langle\rho_j\rangle^{-l}+\lambda_*\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{\Theta+1}R\right)\\&+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\left(\lambda_*\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{\Theta+1}R\right)\Big]\\&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}.\end{aligned}\tag{D.9}$$

Integrating (D.3), (D.7) and (D.13), we have

$$\begin{aligned}&|\nabla_x\Phi|\\&\lesssim\sum_{j=1}^N\Big\{\mathbf{1}_{\{|x-q^{[j]}|\le3\lambda_*R\}}\left[\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0-1}\langle\rho_j\rangle^{-l-1}+1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\right]\\&\quad+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left[1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\right]\Big\}\\&\quad+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\\&\lesssim\sum_{j=1}^N\Big[\mathbf{1}_{\{|x-q^{[j]}|\le3\lambda_*R\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\left(\lambda_*^{\nu-\delta_0-1}\langle\rho_j\rangle^{-l-1}+1\right)\\&\quad+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\Big]+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}.\end{aligned}\tag{D.10}$$

Recalling (5.1) yields

$$\left(\sum_{j=1}^{N}\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\cdot U_*=\sum_{j=1}^{N}\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\cdot\left(U_*-U^{[j]}\right), \tag{D.11}$$

which implies

$$\left|\left(\sum_{j=1}^{N}\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\cdot U_*\right|\lesssim\sum_{j=1}^{N}\eta_R^{[j]}\left|\Phi_{\rm in}^{[j]}\right|\lambda_*\lesssim\sum_{j=1}^{N}\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0+1}\langle\rho_j\rangle^{-l}. \tag{D.12}$$

By (D.12), (4.21) and (D.2), we obtain

$$\begin{aligned}
&|\Phi\cdot U_*|=\left|\sum_{j=1}^{N}\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\cdot\left(U_*-U^{[j]}\right)+\sum_{j=1}^{N}\eta_{d_q}^{[j]}\Phi_0^{*[j]}\cdot U_*+\Phi_{\rm out}\cdot U_*\right|\\
\lesssim&\sum_{j=1}^{N}\Bigg[\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\Big\{\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0+1}\langle\rho_j\rangle^{-l}+\lambda_*\langle\rho_j\rangle\\
&+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left[|\ln(T-t)|\lambda_*^{\Theta+1}(t)R(t)+(T-t)\|Z_*\|_{C^3(\mathbb{R}^2)}+|x-q^{[j]}|\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\right]\Big\}\\
&+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left[\lambda_*\langle\rho_j\rangle+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(|\ln(T-t)|\lambda_*^{\Theta+1}R+\lambda_j\rho_j\right)\right]\Bigg]\\
&+\mathbf{1}_{\{\cap_{j=1}^{N}\{|x-q^{[j]}|\ge 3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(|\ln T|\lambda_*^{\Theta+1}(0)R(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\\
\lesssim&\sum_{j=1}^{N}\Bigg[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\left(\lambda_*^{\nu-\delta_0+1}\langle\rho_j\rangle^{-l}+\lambda_*\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{\Theta+1}R\right)\\
&+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\left(|\ln(T-t)|\lambda_*^{\Theta+1}R+\lambda_j\langle\rho_j\rangle\right)\Bigg]+\mathbf{1}_{\{\cap_{j=1}^{N}\{|x-q^{[j]}|\ge 3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}.
\end{aligned} \tag{D.13}$$

By (5.44) and (D.5), we have

$$\begin{aligned}
&\left|\nabla_x\left[\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\cdot\left(U_*-U^{[j]}\right)\right]\right|=\left|\left(U_*-U^{[j]}\right)\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)+\left(\sum_{k\ne j}\nabla_xU^{[k]}\right)\cdot\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right|\\
\lesssim&\,\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\left(\lambda_*\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0-1}\langle\rho_j\rangle^{-l-1}+\lambda_*\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0}\langle\rho_j\rangle^{-l}\right)\\
\lesssim&\,\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0}\langle\rho_j\rangle^{-l-1}.
\end{aligned} \tag{D.14}$$

(4.21) and (D.6) imply

$$\begin{aligned}
&\left|\nabla_x\left(\eta_{d_q}^{[j]}\Phi_0^{*[j]}\cdot U_*\right)\right|=\left|U_*\cdot\nabla_x\left(\eta_{d_q}^{[j]}\Phi_0^{*[j]}\right)+\eta_{d_q}^{[j]}\Phi_0^{*[j]}\cdot\nabla_xU_*\right|\\
\lesssim&\,\mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\left[1+\left(z_j\mathbf{1}_{\{z_j^2<t\}}+t|\ln T|^{-1}z_j^{-1}\mathbf{1}_{\{z_j^2\ge t\}}\right)\sum_{m=1}^{N}\lambda_m^{-1}\langle\rho_m\rangle^{-2}\right]\\
\lesssim&\,\mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}.
\end{aligned} \tag{D.15}$$

By (D.2), (D.3) and (D.33), we have

$$
\begin{aligned}
&|\nabla_x(\Phi_{\text{out}}\cdot U_*)| = \left|\Phi_{\text{out}}\cdot\nabla_x U_* + U_*\cdot\nabla_x\Phi_{\text{out}}\right| \\
\lesssim & \left(\sum_{j=1}^N \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\lambda_*^{-1}\langle\rho_j\rangle^{-2} + \mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\geq 3d_q\}\}}\lambda_*\right) \\
&\times\left\{\sum_{j=1}^N \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}\left(|\ln(T-t)|\lambda_*^{\Theta+1}R+\lambda_j\rho_j\right) + \mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\geq 3d_q\}\}}\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}\right\} \qquad \text{(D.16)}\\
&+\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right) \\
\lesssim & \sum_{j=1}^N \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}\left(|\ln(T-t)|\lambda_*^{\Theta}R\langle\rho_j\rangle^{-2}+1\right) \\
&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\geq 3d_q\}\}}\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right).
\end{aligned}
$$

Combining (D.14), (D.15) and (D.16), we have

$$
\begin{aligned}
&|\nabla_x(\Phi\cdot U_*)| \\
\lesssim & \sum_{j=1}^N \Big[\mathbf{1}_{\{|x-q^{[j]}|\leq 3\lambda_* R\}}\left(1+\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\text{in}}^{[j]}\|_{\text{in},\nu-\delta_0,l}^{[j]}\right)\left(|\ln(T-t)|\lambda_*^{\Theta}R\langle\rho_j\rangle^{-2}+1\right) \qquad \text{(D.17)}\\
&+\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}\right)\Big] + \mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\geq 3d_q\}\}}\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right).
\end{aligned}
$$

D.2. **Estimate of $\nabla_x A$.** Recall $\nabla_x A$ given in (5.12). This subsection is devoted to the proof of the following Claim.

Claim: Suppose

$$
\begin{aligned}
&\Theta<\beta,\quad \Theta+\beta-1<0,\quad \beta<\frac{1}{2},\quad \Theta+\beta+2\delta_0-2\nu<0,\\
&3\beta<1+\Theta,\quad \Theta+\beta+4\delta_0-4\nu+1<0,\quad \Theta+\beta(3-l)+\delta_0-\nu-2<0.
\end{aligned}
\qquad \text{(D.18)}
$$

Then for $\epsilon>0$ sufficiently small, we have

$$
\begin{aligned}
\nabla_x A = & -U_*\cdot\nabla_x U_* - \Phi\cdot\nabla_x\Phi + O\Bigg(\sum_{j=1}^N\Bigg[\left(1+\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\text{in}}\|_{\text{in},\nu-\delta_0,l}^{[j]}\right)^4\mathbf{1}_{\{|x-q^{[j]}|\leq 3\lambda_* R\}}\\
&\times\left(\lambda_*^{\epsilon+1}\lambda_*^{\Theta}(\lambda_* R)^{-1}+\lambda_*\langle\rho_j\rangle\right)\\
&+\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}\right)^4\left(|\ln(T-t)|\lambda_*^{\Theta+1}R+\lambda_j\langle\rho_j\rangle\right)\Bigg] \qquad \text{(D.19)}\\
&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\geq 3d_q\}\}}\left(1+\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}\right)^4\Bigg).
\end{aligned}
$$

Proof of (D.19). Let us simplify (5.12) first

$$
|\Pi_{U_*^{\perp}}\Phi|^2 = |\Phi|^2+\left(|U_*|^2-2\right)(\Phi\cdot U_*)^2,\quad U_*\cdot\Pi_{U_*^{\perp}}\Phi=\left(1-|U_*|^2\right)(\Phi\cdot U_*),
$$

where

$$
\Phi\cdot U_* = \sum_{j=1}^N \eta_R^{[j]}Q_{\gamma_j}\Phi_{\text{in}}^{[j]}\cdot\left(U_*-U^{[j]}\right)+\sum_{j=1}^N\eta_{d_q}^{[j]}\Phi_0^{*[j]}\cdot U_*+\Phi_{\text{out}}\cdot U_*.
$$

Then

$$
\begin{aligned}
\nabla_x\left(|\Pi_{U_*^{\perp}}\Phi|^2\right) &= 2\Phi\cdot\nabla_x\Phi+2(\Phi\cdot U_*)^2U_*\cdot\nabla_x U_*+2\left(|U_*|^2-2\right)(\Phi\cdot U_*)\nabla_x(\Phi\cdot U_*),\\
\nabla_x\left(U_*\cdot\Pi_{U_*^{\perp}}\Phi\right) &= \left(1-|U_*|^2\right)\nabla_x(\Phi\cdot U_*)-2(\Phi\cdot U_*)U_*\cdot\nabla_x U_*.
\end{aligned}
$$

By (5.5), (4.2) and (5.3), we have

$$
(1+A)|U_*|^2+(U_*\cdot\Pi_{U_*^{\perp}}\Phi)=(1+A)|U_*|^2+(1-|U_*|^2)(\Phi\cdot U_*)=1+O\left(\lambda_*+|\Phi|^2\right),
$$

which implies

$$\left[(1+A)|U_*|^2 + (U_* \cdot \Pi_{U_*^\perp}\Phi)\right]^{-1} = \left[(1+A)|U_*|^2 + (1-|U_*|^2)(\Phi\cdot U_*)\right]^{-1} = 1 + O\left(\lambda_* + |\Phi|^2\right). \tag{D.20}$$

Thus we obtain

$$\begin{aligned}
&\nabla_x A = -\left(1 + O\left(\lambda_* + |\Phi|^2\right)\right)\\
&\quad \times \Big\{(1+A)^2 U_*\cdot\nabla_x U_* + \Phi\cdot\nabla_x\Phi + (\Phi\cdot U_*)^2\, U_*\cdot\nabla_x U_* + \left(|U_*|^2 - 2\right)(\Phi\cdot U_*)\,\nabla_x(\Phi\cdot U_*)\\
&\quad + (1+A)\left[\left(1-|U_*|^2\right)\nabla_x(\Phi\cdot U_*) - 2(\Phi\cdot U_*)\,U_*\cdot\nabla_x U_*\right]\Big\}\\
= &-\left(1 + O\left(\lambda_* + |\Phi|^2\right)\right)\Big\{\left[1 + A(2+A) - 2(1+A)(\Phi\cdot U_*) + (\Phi\cdot U_*)^2\right] U_*\cdot\nabla_x U_* + \Phi\cdot\nabla_x\Phi\\
&\quad + \left(|U_*|^2 - 2\right)(\Phi\cdot U_*)\,\nabla_x(\Phi\cdot U_*) + (1+A)\left(1-|U_*|^2\right)\nabla_x(\Phi\cdot U_*)\Big\}\\
= &-\Big\{\left[1 + A(2+A) - 2(1+A)(\Phi\cdot U_*) + (\Phi\cdot U_*)^2\right] U_*\cdot\nabla_x U_* + \Phi\cdot\nabla_x\Phi\\
&\quad + \left(|U_*|^2 - 2\right)(\Phi\cdot U_*)\,\nabla_x(\Phi\cdot U_*) + (1+A)\left(1-|U_*|^2\right)\nabla_x(\Phi\cdot U_*)\Big\}\\
&\quad + O\left(\lambda_* + |\Phi|^2\right)\Big\{\left[1 + A(2+A) - 2(1+A)(\Phi\cdot U_*) + (\Phi\cdot U_*)^2\right] U_*\cdot\nabla_x U_* + \Phi\cdot\nabla_x\Phi\\
&\quad + \left(|U_*|^2 - 2\right)(\Phi\cdot U_*)\,\nabla_x(\Phi\cdot U_*) + (1+A)\left(1-|U_*|^2\right)\nabla_x(\Phi\cdot U_*)\Big\}\\
= &-U_*\cdot\nabla_x U_* - \left(1 + O\left(\lambda_* + |\Phi|^2\right)\right)\left[-2(1+A)(\Phi\cdot U_*) + (\Phi\cdot U_*)^2\right] U_*\cdot\nabla_x U_*\\
&\quad + O\left(\lambda_* + |\Phi|^2\right) U_*\cdot\nabla_x U_* - \left(1 + O\left(\lambda_* + |\Phi|^2\right)\right)\Phi\cdot\nabla_x\Phi\\
&\quad - \left(1 + O\left(\lambda_* + |\Phi|^2\right)\right)\left(|U_*|^2 - 2\right)(\Phi\cdot U_*)\,\nabla_x(\Phi\cdot U_*)\\
&\quad - \left(1 + O\left(\lambda_* + |\Phi|^2\right)\right)(1+A)\left(1-|U_*|^2\right)\nabla_x(\Phi\cdot U_*)\\
= &-U_*\cdot\nabla_x U_* + \left(2\Phi\cdot U_* + O\left(\lambda_* + |\Phi|^2\right)\right) U_*\cdot\nabla_x U_* - \left(1 + O\left(\lambda_* + |\Phi|^2\right)\right)\Phi\cdot\nabla_x\Phi\\
&\quad - \left(1 + O\left(\lambda_* + |\Phi|^2\right)\right)\left(|U_*|^2 - 2\right)(\Phi\cdot U_*)\,\nabla_x(\Phi\cdot U_*)\\
&\quad - \left(1 + O\left(\lambda_* + |\Phi|^2\right)\right)(1+A)\left(1-|U_*|^2\right)\nabla_x(\Phi\cdot U_*)
\end{aligned} \tag{D.21}$$

where we have used $A(2+A) = O(\lambda_* + |\Phi|^2)$ by (5.5).

By (D.9) and (D.13), we have

$$
\begin{aligned}
&|\Phi\cdot U_*|+\lambda_*+|\Phi|^2\\
\lesssim&\sum_{j=1}^{N}\Bigg[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\left\{\lambda_*^{\nu-\delta_0+1}\langle\rho_j\rangle^{-l}+\lambda_*\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{\Theta+1}R\right\}\\
&+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\left(|\ln(T-t)|\lambda_*^{\Theta+1}R+\lambda_j\langle\rho_j\rangle\right)\Bigg]\\
&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\\
&+\sum_{j=1}^{N}\Bigg[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2\\
&\times\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\left(\lambda_*^{2\nu-2\delta_0}\langle\rho_j\rangle^{-2l}+\lambda_*^2\langle\rho_j\rangle^2+|\ln(T-t)|^2\lambda_*^{2\Theta+2}R^2\right)\\
&+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\left(\lambda_*^2\langle\rho_j\rangle^2+|\ln(T-t)|^2\lambda_*^{2\Theta+2}R^2\right)\Bigg]\\
&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\\
\lesssim&\sum_{j=1}^{N}\Bigg[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\left(\lambda_*^{2\nu-2\delta_0}\langle\rho_j\rangle^{-l}+\lambda_*\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{\Theta+1}R\right)\\
&+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\left(\lambda_*\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{\Theta+1}R\right)\Bigg]\\
&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2.
\end{aligned}
\tag{D.22}
$$

Notice $\lambda_*^{\epsilon+1}\lambda_*^{\Theta}(\lambda_*R)^{-1}=\lambda_*^{\epsilon+\Theta+\beta}$. Then using (D.37) and (D.22), we get

$$
\begin{aligned}
&\left|\left(2\Phi\cdot U_*+O\left(\lambda_*+|\Phi|^2\right)\right)U_*\cdot\nabla_x U_*\right|\\
\lesssim&\sum_{j=1}^{N}\Bigg\{\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\Big[\lambda_*^{2\nu-2\delta_0}\langle\rho_j\rangle^{-l-2}+\lambda_*\langle\rho_j\rangle^{-1}\\
&+|\ln(T-t)|\lambda_*^{\Theta+1}R\langle\rho_j\rangle^{-2}\Big]\\
&+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\left(\lambda_*\langle\rho_j\rangle^{-1}+|\ln(T-t)|\lambda_*^{\Theta+1}R\langle\rho_j\rangle^{-2}\right)\Bigg\}\\
&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\lambda_*^2\\
\lesssim&\sum_{j=1}^{N}\Bigg[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\lambda_*^{\epsilon+1}\lambda_*^{\Theta}(\lambda_*R)^{-1}\\
&+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\lambda_*\langle\rho_j\rangle^{-1}\Bigg]\\
&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\lambda_*^2
\end{aligned}
\tag{D.23}
$$

for some $\epsilon>0$ where in the last "$\lesssim$", we require

$$\Theta<\beta,\quad \Theta+\beta+2\delta_0-2\nu<0,\quad \Theta+\beta-1<0,\quad \beta<\frac{1}{2}. \tag{D.24}$$

By (D.9) and (D.10), it follows that

$$
\begin{aligned}
&\left|O\left(\lambda_*+|\Phi|^2\right)\Phi\cdot\nabla_x\Phi\right|\\
\lesssim&\left\{\sum_{j=1}^N\Big[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^3\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\right.\\
&\times\left(\lambda_*^{3\nu-3\delta_0}\langle\rho_j\rangle^{-3l}+\lambda_*^3\langle\rho_j\rangle^3+|\ln(T-t)|^3\lambda_*^{3\Theta+3}R^3\right)\\
&+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^3\left(\lambda_*^3\langle\rho_j\rangle^3+|\ln(T-t)|^3\lambda_*^{3\Theta+3}R^3\right)\Big]\\
&\left.+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^3\right\}\\
&\times\left\{\sum_{j=1}^N\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\Big[\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\left(\lambda_*^{\nu-\delta_0-1}\langle\rho_j\rangle^{-l-1}+1\right)\right.\\
&\left.+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\Big]+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right\}\\
\lesssim&\sum_{j=1}^N\Big[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^4\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\\
&\times\left(\lambda_*^{4\nu-4\delta_0-1}+\lambda_*^3\langle\rho_j\rangle^3+\lambda_*^{\nu-\delta_0+2}\langle\rho_j\rangle^{2-l}+|\ln(T-t)|^3\lambda_*^{\nu-\delta_0+3\Theta+2}R^3\right)\\
&+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^4\left(\lambda_*^3\langle\rho_j\rangle^3+|\ln(T-t)|^3\lambda_*^{3\Theta+3}R^3\right)\Big]\\
&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^4\\
\lesssim&\sum_{j=1}^N\Big[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^4\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\lambda_*^{\epsilon+1}\lambda_*^{\Theta}(\lambda_*R)^{-1}\\
&+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^4\left(\lambda_*^3\langle\rho_j\rangle^3+|\ln(T-t)|^3\lambda_*^{3\Theta+3}R^3\right)\Big]\\
&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^4
\end{aligned}
\tag{D.25}
$$

where for the last "$\lesssim$", we require

$$
\Theta+\beta+4\delta_0-4\nu+1<0,\quad \Theta+4\beta<3,\quad \Theta+\beta(3-l)+\delta_0-\nu-2<0. \tag{D.26}
$$

Combining (D.13) and (D.17), we have

$$
\begin{aligned}
&\left|\left(\Phi\cdot U_*\right)\nabla_x\left(\Phi\cdot U_*\right)\right|\\
\lesssim&\left\{\sum_{j=1}^N\left[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\left(\lambda_*^{\nu-\delta_0+1}\langle\rho_j\rangle^{-l}+\lambda_*\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{\Theta+1}R\right)\right.\right.\\
&\left.\left.+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\left(|\ln(T-t)|\lambda_*^{\Theta+1}R+\lambda_j\langle\rho_j\rangle\right)\right]+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right\}\\
&\times\left\{\sum_{j=1}^N\Big[\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\left(|\ln(T-t)|\lambda_*^{\Theta}R\langle\rho_j\rangle^{-2}+1\right)\right.\\
&\left.+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\Big]+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\right\}\\
\lesssim&\sum_{j=1}^N\left[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\right.\\
&\times\left(|\ln(T-t)|\lambda_*^{\nu-\delta_0+1+\Theta}R\langle\rho_j\rangle^{-l}+\lambda_*\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{\Theta+1}R\langle\rho_j\rangle^{-1}+|\ln(T-t)|^2\lambda_*^{2\Theta+1}R^2\right)\\
&\left.+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\left(|\ln(T-t)|\lambda_*^{\Theta+1}R+\lambda_j\langle\rho_j\rangle\right)\right]+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}^2.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
&|(\Phi\cdot U_*)\nabla_x(\Phi\cdot U_*)|\\
\lesssim&\sum_{j=1}^N\Bigg[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\left(\lambda_*^{\epsilon+1}\lambda_*^{\Theta}(\lambda_*R)^{-1}+\lambda_*\langle\rho_j\rangle\right)\\
&+\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\left(|\ln(T-t)|\lambda_*^{\Theta+1}R+\lambda_j\langle\rho_j\rangle\right)\Bigg]\\
&+\left(\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}
\end{aligned}
\tag{D.27}
$$

provided

$$
\Theta-\beta<0,\quad 3\beta<1+\Theta. \tag{D.28}
$$

By (4.2) and (D.17), one has

$$
\begin{aligned}
&\left|\left(1-|U_*|^2\right)\nabla_x(\Phi\cdot U_*)\right|\\
\lesssim&\sum_{j=1}^N\Bigg[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\Big(|\ln T|^{-1}|\ln(T-t)|^2\lambda_*\\
&+|\ln(T-t)|\lambda_*^{1+\Theta}R(t)\langle\rho_j\rangle^{-2}\Big)+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\lambda_*\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\Bigg]\\
&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|>3d_q\}\}}\lambda_*\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\\
\lesssim&\sum_{j=1}^N\Bigg[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_*R\}}\lambda_*^{\epsilon+1}\lambda_*^{\Theta}(\lambda_*R)^{-1}\\
&+\mathbf{1}_{\{3\lambda_*R<|x-q^{[j]}|<3d_q\}}\lambda_*\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\Bigg]+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|>3d_q\}\}}\lambda_*\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}
\end{aligned}
\tag{D.29}
$$

provided

$$
\Theta+\beta<1,\quad \beta<\frac12. \tag{D.30}
$$

Combining (D.23), (D.25), (D.27) and (D.29), we conclude the validity of (D.19) under the assumptions (D.18) for the parameters, and these are from (D.24), (D.26), (D.28) and (D.30).

□

D.3. **Estimate of $\mathcal{G}$.**

Lemma D.1. *Suppose $\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}\le C_{\rm in}$ for $j=1,2,\dots,N$, under the parameter assumptions*

$$
\begin{aligned}
&\Theta<\beta,\quad \Theta+\beta+\delta_0-\nu<0,\quad 3\beta<1+\Theta,\quad \delta_0<\beta<\frac12,\quad \beta(l+1)-1+\nu-\delta_0-\Theta>0,\\
&\Theta+2\beta-1<0,\quad \delta_0<\nu,\quad 2\beta+\delta_0-\nu<0,\quad \Theta+\beta+1+3\delta_0-3\nu<0,\\
&\Theta+\beta+4\delta_0-4\nu+1<0.
\end{aligned}
\tag{D.31}
$$

Then for $\epsilon>0$ sufficiently small,

$$
\|\mathcal{G}\|_{**}\lesssim T^{\epsilon}. \tag{D.32}
$$

where $\|\cdot\|_{**}$ is defined in (5.49).

Proof. First, we prepare some useful formulas here.

$$
|\nabla_x U_*|\lesssim\sum_{j=1}^N\lambda_*^{-1}\langle\rho_j\rangle^{-2}\lesssim\sum_{j=1}^N\mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\lambda_*^{-1}\langle\rho_j\rangle^{-2}+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\lambda_*. \tag{D.33}
$$

$$
|\Delta_x U_*|\lesssim\sum_{j=1}^N\lambda_*^{-2}\langle\rho_j\rangle^{-4}\lesssim\sum_{j=1}^N\mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\lambda_*^{-2}\langle\rho_j\rangle^{-4}+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\lambda_*^{2}. \tag{D.34}
$$

$$|U_* \cdot \nabla_x U_*| = \left| \sum_{j=1}^{N} \sum_{k \neq j} \left(U^{[k]} - U_\infty \right) \cdot \nabla_x U^{[j]} \right| \lesssim \sum_{j=1}^{N} \sum_{k \neq j} \langle \rho_k \rangle^{-1} \lambda_j^{-1} \langle \rho_j \rangle^{-2}. \tag{D.35}$$

For fixed $j = 1, 2, \ldots, N$, we have

$$\begin{aligned} &\mathbf{1}_{\{|x-q^{[j]}|<3d_q\}} |U_* \cdot \nabla_x U_*| \lesssim \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}} \left[\lambda_j^{-1} \langle \rho_j \rangle^{-2} \sum_{k \neq j} \langle \rho_k \rangle^{-1} + \sum_{m \neq j} \sum_{k \neq m} \langle \rho_k \rangle^{-1} \lambda_m^{-1} \langle \rho_m \rangle^{-2} \right] \\ &\lesssim \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}} \left(\langle \rho_j \rangle^{-2} + \lambda_* \sum_{m \neq j} \sum_{k \neq m} \langle \rho_k \rangle^{-1} \right) \sim \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}} \left(\langle \rho_j \rangle^{-2} + \lambda_* \langle \rho_j \rangle^{-1} \right) \\ &\sim \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}} \langle \rho_j \rangle^{-2}, \\ &\quad \mathbf{1}_{\cap_{j=1}^N \{|x-q^{[j]}| \geq 3d_q\}} |U_* \cdot \nabla_x U_*| \lesssim \lambda_*^2, \end{aligned} \tag{D.36}$$

and thus

$$|U_* \cdot \nabla_x U_*| \lesssim \sum_{j=1}^{N} \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}} \langle \rho_j \rangle^{-2} + \mathbf{1}_{\{\cap_{j=1}^N |x-q^{[j]}| \geq 3d_q\}} \lambda_*^2. \tag{D.37}$$

Notice

$$\left| \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \left(U^{[j]} - U_* \right) \right| \lesssim \sum_{j=1}^{N} \lambda_j^{-2} \langle \rho_j \rangle^{-4} \sum_{k \neq j} \langle \rho_k \rangle^{-1}.$$

$$\begin{aligned} &\mathbf{1}_{\{|x-q^{[m]}|<3d_q\}} \sum_{j=1}^{N} \lambda_j^{-2} \langle \rho_j \rangle^{-4} \sum_{k \neq j} \langle \rho_k \rangle^{-1} \lesssim \mathbf{1}_{\{|x-q^{[m]}|<3d_q\}} \left(\lambda_*^{-1} \langle \rho_m \rangle^{-4} + \sum_{j \neq m} \lambda_*^2 \sum_{k \neq j} \langle \rho_k \rangle^{-1} \right) \\ &\lesssim \mathbf{1}_{\{|x-q^{[m]}|<3d_q\}} \left(\lambda_*^{-1} \langle \rho_m \rangle^{-4} + \lambda_*^2 \langle \rho_m \rangle^{-1} \right), \\ &\qquad \mathbf{1}_{\{\cap_{m=1}^N \{|x-q^{[m]}| \geq 3d_q\}\}} \sum_{j=1}^{N} \lambda_j^{-2} \langle \rho_j \rangle^{-4} \sum_{k \neq j} \langle \rho_k \rangle^{-1} \lesssim \lambda_*^3. \end{aligned}$$

Thus we have

$$\begin{aligned} &\left| \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \left(U^{[j]} - U_* \right) \right| \lesssim \sum_{j=1}^{N} \lambda_j^{-2} \langle \rho_j \rangle^{-4} \sum_{k \neq j} \langle \rho_k \rangle^{-1} \\ &\lesssim \sum_{j=1}^{N} \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}} \left(\lambda_*^{-1} \langle \rho_j \rangle^{-4} + \lambda_*^2 \langle \rho_j \rangle^{-1} \right) + \mathbf{1}_{\{\cap_{j=1}^N \{|x-q^{[j]}| \geq 3d_q\}\}} \lambda_*^3. \end{aligned} \tag{D.38}$$

Notice

$$\begin{aligned} &|\nabla_x U_*|^2 + U_* \cdot \Delta_x U_* \\ &= \left| \sum_{j=1}^{N} \nabla_x U^{[j]} \right|^2 - U_* \cdot \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 U^{[j]} \\ &= \left| \sum_{j=1}^{N} \nabla_x U^{[j]} \right|^2 - \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 - \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \left(U_* - U^{[j]} \right) \cdot U^{[j]} \\ &= \sum_{j=1}^{N} \sum_{k \neq j} \nabla_x U^{[j]} \cdot \nabla_x U^{[k]} - \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \left(U_* - U^{[j]} \right) \cdot U^{[j]}. \end{aligned}$$

Then

$$\left| |\nabla_x U_*|^2 + U_* \cdot \Delta_x U_* \right| \lesssim \sum_{j=1}^{N} \sum_{k \neq j} \lambda_j^{-1} \lambda_k^{-1} \langle \rho_j \rangle^{-2} \langle \rho_k \rangle^{-2} + \sum_{j=1}^{N} \sum_{k \neq j} \lambda_j^{-2} \langle \rho_j \rangle^{-4} \langle \rho_k \rangle^{-1}.$$

More explicitly, one has

$$
\begin{aligned}
&\mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\left||\nabla_x U_*|^2+U_*\cdot\Delta_x U_*\right|\\
\lesssim\ &\mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\Big(\sum_{k\neq j}\lambda_j^{-1}\lambda_k^{-1}\langle\rho_j\rangle^{-2}\langle\rho_k\rangle^{-2}+\sum_{m\neq j}\sum_{k\neq m}\lambda_m^{-1}\lambda_k^{-1}\langle\rho_m\rangle^{-2}\langle\rho_k\rangle^{-2}\\
&+\sum_{k\neq j}\lambda_j^{-2}\langle\rho_j\rangle^{-4}\langle\rho_k\rangle^{-1}+\sum_{m\neq j}\sum_{k\neq m}\lambda_m^{-2}\langle\rho_m\rangle^{-4}\langle\rho_k\rangle^{-1}\Big)\\
\lesssim\ &\mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\left(\langle\rho_j\rangle^{-2}+\sum_{m\neq j}\sum_{k\neq m}\langle\rho_k\rangle^{-2}+\lambda_*^{-1}\langle\rho_j\rangle^{-4}+\sum_{m\neq j}\sum_{k\neq m}\lambda_*^{2}\langle\rho_k\rangle^{-1}\right)\\
\sim\ &\mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\left(\langle\rho_j\rangle^{-2}+\lambda_*^{-1}\langle\rho_j\rangle^{-4}\right),\\
&\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\geq 3d_q\}\}}\left||\nabla_x U_*|^2+U_*\cdot\Delta_x U_*\right|\lesssim\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\geq 3d_q\}\}}\lambda_*^2,
\end{aligned}
$$

and thus

$$
\left||\nabla_x U_*|^2+U_*\cdot\Delta_x U_*\right|\lesssim\sum_{j=1}^N\mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\left(\langle\rho_j\rangle^{-2}+\lambda_*^{-1}\langle\rho_j\rangle^{-4}\right)+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\geq 3d_q\}\}}\lambda_*^2. \tag{D.39}
$$

$$
\begin{aligned}
|U_*\wedge\Delta_x U_*|=&\left|U_*\wedge\sum_{j=1}^N\left|\nabla_x U^{[j]}\right|^2U^{[j]}\right|\lesssim\sum_{j=1}^N\lambda_j^{-2}\langle\rho_j\rangle^{-4}\sum_{k\neq j}\langle\rho_j\rangle^{-1}\\
\lesssim&\sum_{j=1}^N\mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\left(\lambda_*^{-1}\langle\rho_j\rangle^{-4}+\lambda_*^2\langle\rho_j\rangle^{-1}\right)+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\geq 3d_q\}\}}\lambda_*^3
\end{aligned}
\tag{D.40}
$$

where we have used (D.38) in the last "$\lesssim$".

• By (D.1) and (D.3), we have

$$
\begin{aligned}
&\left|\left(1-\eta_R^{[j]}\right)\left(a-bU^{[j]}\wedge\right)\left[|\nabla_x U^{[j]}|^2\Phi_{\rm out}-2\nabla_x\left(U^{[j]}\cdot\Phi_{\rm out}\right)\nabla_x U^{[j]}\right]\right|\\
\lesssim\ &\mathbf{1}_{\{|x-q^{[j]}|\geq\lambda_*R/2\}}\left(\lambda_j^{-2}\langle\rho_j\rangle^{-4}|\Phi_{\rm out}|+|\nabla_x\Phi_{\rm out}|\,\lambda_j^{-1}\langle\rho_j\rangle^{-2}\right)\\
\lesssim\ &\mathbf{1}_{\{|x-q^{[j]}|\geq\lambda_*R/2\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\\
&\times\Big\{\lambda_j^{-2}\langle\rho_j\rangle^{-4}\left[|\ln(T-t)|\lambda_*^{\Theta+1}R(t)+(T-t)\|Z_*\|_{C^3(\mathbb{R}^2)}+\lambda_j\rho_j\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\right]\\
&+\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\lambda_j^{-1}\langle\rho_j\rangle^{-2}\Big\}\\
\lesssim\ &\mathbf{1}_{\{|x-q^{[j]}|\geq\lambda_*R/2\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\\
&\times\Big\{\langle\rho_j\rangle^{-2}\left[|\ln(T-t)|\lambda_*^{\Theta-1}R^{-1}+(T-t)\lambda_j^{-2}R^{-2}\|Z_*\|_{C^3(\mathbb{R}^2)}+\lambda_j^{-1}R^{-1}\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\right]\\
&+\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\lambda_j^{-1}\langle\rho_j\rangle^{-2}\Big\}\\
\lesssim\ &T^{\epsilon}\left(\varrho_2^{[j]}+\varrho_3\right).
\end{aligned}
\tag{D.41}
$$

•
$$
\begin{aligned}
&\Big|\left(1-\eta_R^{[j]}\right)\Big\{-\partial_t(\eta_{d_q}^{[j]}\Phi_0^{*[j]})+\left(a-bU^{[j]}\wedge\right)\Big[\Delta_x(\eta_{d_q}^{[j]}\Phi_0^{*[j]})+|\nabla_x U^{[j]}|^2\eta_{d_q}^{[j]}\Phi_0^{*[j]}\\
&-2\nabla_x\left(U^{[j]}\cdot\eta_{d_q}^{[j]}\Phi_0^{*[j]}\right)\nabla_x U^{[j]}\Big]-\partial_t U^{[j]}\Big\}\Big|\\
=&\Big|\left(1-\eta_R^{[j]}\right)\Big\{-\eta_{d_q}^{[j]}\partial_t\Phi_0^{*[j]}-\Phi_0^{*[j]}\partial_t\eta_{d_q}^{[j]}+\left(a-bU^{[j]}\wedge\right)\Big[\eta_{d_q}^{[j]}\Delta_x\Phi_0^{*[j]}+2\nabla_x\eta_{d_q}^{[j]}\nabla_x\Phi_0^{*[j]}+\Phi_0^{*[j]}\Delta_x\eta_{d_q}^{[j]}\\
&+|\nabla_x U^{[j]}|^2\eta_{d_q}^{[j]}\Phi_0^{*[j]}-2\left(U^{[j]}\cdot\Phi_0^{*[j]}\right)\nabla_x\eta_{d_q}^{[j]}\nabla_x U^{[j]}-2\eta_{d_q}^{[j]}\nabla_x\left(U^{[j]}\cdot\Phi_0^{*[j]}\right)\nabla_x U^{[j]}\Big]-\partial_t U^{[j]}\Big\}\Big|\\
=&\Big|\left(1-\eta_R^{[j]}\right)\eta_{d_q}^{[j]}\Big\{-\partial_t\Phi_0^{*[j]}+\left(a-bU^{[j]}\wedge\right)\Big[\Delta_x\Phi_0^{*[j]}+|\nabla_x U^{[j]}|^2\Phi_0^{*[j]}-2\nabla_x\left(U^{[j]}\cdot\Phi_0^{*[j]}\right)\nabla_x U^{[j]}\Big]\\
&-\partial_t U^{[j]}\Big\}-(1-\eta_{d_q}^{[j]})\partial_t U^{[j]}+\left(1-\eta_R^{[j]}\right)\Big\{-\Phi_0^{*[j]}\partial_t\eta_{d_q}^{[j]}+\left(a-bU^{[j]}\wedge\right)\Big[2\nabla_x\eta_{d_q}^{[j]}\nabla_x\Phi_0^{*[j]}+\Phi_0^{*[j]}\Delta_x\eta_{d_q}^{[j]}\\
&-2\left(U^{[j]}\cdot\Phi_0^{*[j]}\right)\nabla_x\eta_{d_q}^{[j]}\nabla_x U^{[j]}\Big]\Big\}\Big|\\
\lesssim&\ \mathbf{1}_{\{\lambda_j R\le|x-\xi^{[j]}|\le 2d_q\}}\left(\lambda_*^{-1}\langle\rho_j\rangle^{-2}+|\dot\lambda_*|\langle\rho_j\rangle^{-1}+|\dot\xi^{[j]}|\right)\\
&+\mathbf{1}_{\{|x-\xi^{[j]}|\ge d_q\}}\left[\left(\lambda_j^{-1}|\dot\lambda_j|+|\dot\gamma_j|\right)\langle\rho_j\rangle^{-1}+\lambda_j^{-1}|\dot\xi^{[j]}|\langle\rho_j\rangle^{-2}\right]+\mathbf{1}_{\{d_q\le|x-\xi^{[j]}|\le 2d_q\}}\\
\lesssim&\ \left(\mathbf{1}_{\{\lambda_j R/2\le|x-q^{[j]}|\le\lambda_j R\}}+\mathbf{1}_{\{\lambda_j R<|x-q^{[j]}|\le 3d_q\}}\right)\lambda_*|x-q^{[j]}|^{-2}+\mathbf{1}_{\{\lambda_j R/2\le|x-q^{[j]}|\le 3d_q\}}\left(|\dot\lambda_*|R^{-1}+|\dot\xi^{[j]}|\right)\\
&+\mathbf{1}_{\{|x-\xi^{[j]}|\ge d_q\}}\left[\left(|\dot\lambda_j|+\lambda_j|\dot\gamma_j|\right)+\lambda_j|\dot\xi^{[j]}|\right]+\mathbf{1}_{\{d_q\le|x-\xi^{[j]}|\le 2d_q\}}\\
\lesssim&\ T^\epsilon\left(\varrho_1^{[j]}+\varrho_2^{[j]}+\varrho_3\right)
\end{aligned}
$$

provided

$$\Theta<\beta \tag{D.42}$$

where we have used (4.47), (4.8) and (4.21) in the first "$\lesssim$".

• By (4.43) and (4.45), we have

$$\left|\eta_R^{[j]}\left(e^{i\theta_j}\tilde M_1^{[j]}+e^{-i\theta_j}M_{-1}^{[j]}\right)_{\mathcal{C}_j^{-1}}\right|\lesssim\eta_R^{[j]}|\dot\xi^{[j]}|\lesssim T^\epsilon\varrho_1.$$

• Since $|\dot\gamma_j|\lesssim(T-t)^{-1}$, $|\dot\xi^{[j]}|\lesssim O(R_0^{-2})$, one has

$$\left|\eta_R^{[j]}Q_{\gamma_j}\left[\left(\lambda_j^{-1}\dot\lambda_j y^{[j]}+\lambda_j^{-1}\dot\xi^{[j]}\right)\cdot\nabla_{y^{[j]}}\Phi_{\rm in}^{[j]}-\dot\gamma_j J\Phi_{\rm in}^{[j]}\right]\right|\lesssim\eta_R^{[j]}(T-t)^{-1}\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0}\lesssim T^\epsilon\varrho_1^{[j]}$$

for $\epsilon>0$ sufficiently small provided

$$\Theta+\delta_0+\beta-\nu<0. \tag{D.43}$$

•
$$
\begin{aligned}
&\left|Q_{\gamma_j}\left\{-\Phi_{\rm in}^{[j]}\partial_t\eta_R^{[j]}+\left(a-bW^{[j]}\wedge\right)\left[\Phi_{\rm in}^{[j]}\Delta_x\eta_R^{[j]}+2\nabla_x\eta_R^{[j]}\nabla_x\Phi_{\rm in}^{[j]}+W^{[j]}\cdot\Phi_{\rm in}^{[j]}\left(-2\nabla_x\eta_R^{[j]}\nabla_x W^{[j]}\right)\right]\right\}\right|\\
=&\left|-\Phi_{\rm in}^{[j]}\partial_t\eta_R^{[j]}+\left(a-bW^{[j]}\wedge\right)\left[\Phi_{\rm in}^{[j]}\Delta_x\eta_R^{[j]}+2\nabla_x\eta_R^{[j]}\nabla_x\Phi_{\rm in}^{[j]}+W^{[j]}\cdot\Phi_{\rm in}^{[j]}\left(-2\nabla_x\eta_R^{[j]}\nabla_x W^{[j]}\right)\right]\right|\\
=&\Big|\Phi_{\rm in}^{[j]}\left(\nabla\eta\right)\left(\frac{x-\xi^{[j]}}{\lambda_j R}\right)\cdot\left(\frac{\dot\xi^{[j]}}{\lambda_j R}+\frac{x-\xi^{[j]}}{\lambda_j R}\frac{(\lambda_j R)'}{\lambda_j R}\right)\\
&+\left(a-bW^{[j]}\wedge\right)\Big[\Phi_{\rm in}^{[j]}(\lambda_j R)^{-2}\left(\Delta\eta\right)\left(\frac{x-\xi^{[j]}}{\lambda_j R}\right)+2(\lambda_j R)^{-1}\left(\nabla\eta\right)\left(\frac{x-\xi^{[j]}}{\lambda_j R}\right)\lambda_j^{-1}\nabla_{y^{[j]}}\Phi_{\rm in}^{[j]}\\
&-2\left(W^{[j]}\cdot\Phi_{\rm in}^{[j]}\right)(\lambda_j R)^{-1}\left(\nabla\eta\right)\left(\frac{x-\xi^{[j]}}{\lambda_j R}\right)\lambda_j^{-1}\nabla_{y^{[j]}}W^{[j]}\Big]\Big|\\
\lesssim&\ \|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\mathbf{1}_{\{\lambda_j R\le|x-\xi^{[j]}|\le 2\lambda_j R\}}\Big[(T-t)^{-1}\lambda_*^{\nu-\delta_0}\langle y^{[j]}\rangle^{-l}+(\lambda_j R)^{-2}\lambda_*^{\nu-\delta_0}\langle y^{[j]}\rangle^{-l}\\
&+(\lambda_j R)^{-1}\lambda_j^{-1}\lambda_*^{\nu-\delta_0}\langle y^{[j]}\rangle^{-l-1}+(\lambda_j R)^{-1}\lambda_j^{-1}\langle y^{[j]}\rangle^{-2}\lambda_*^{\nu-\delta_0}\langle y^{[j]}\rangle^{-l}\Big]\\
\sim&\ \|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}\mathbf{1}_{\{\lambda_j R\le|x-\xi^{[j]}|\le 2\lambda_j R\}}(\lambda_j R)^{-2}\lambda_*^{\nu-\delta_0}R^{-l}\lesssim T^\epsilon\varrho_1
\end{aligned} \tag{D.44}
$$

provided

$$\delta_0 < \beta < \frac{1}{2}, \quad \nu - \delta_0 + \beta l - (1-\beta) > \Theta. \tag{D.45}$$

•

$$\begin{aligned} & \left(U_* - U^{[j]}\right) \wedge \Big\{ \Delta_x \left(\eta_{d_q}^{[j]} \Phi_0^{*[j]}\right) + \eta_R^{[j]} Q_{\gamma_j} \Delta_x \Phi_{\text{in}}^{[j]} + Q_{\gamma_j} \left(\Phi_{\text{in}}^{[j]} \Delta_x \eta_R^{[j]} + 2\nabla_x \eta_R^{[j]} \nabla_x \Phi_{\text{in}}^{[j]}\right) \\ & - 2\nabla_x \left(U^{[j]} \cdot \Phi_{\text{out}}\right) \nabla_x U^{[j]} - 2\nabla_x \left[U^{[j]} \cdot \left(\eta_R^{[j]} Q_{\gamma_j} \Phi_{\text{in}}^{[j]} + \eta_{d_q}^{[j]} \Phi_0^{*[j]}\right)\right] \nabla_x U^{[j]} \Big\} \\ = \; & \left(U_* - U^{[j]}\right) \wedge \Big\{ \Delta_x \left(\eta_{d_q}^{[j]} \Phi_0^{*[j]}\right) - 2\nabla_x \left(U^{[j]} \cdot \eta_{d_q}^{[j]} \Phi_0^{*[j]}\right) \nabla_x U^{[j]} \\ & + Q_{\gamma_j} \left(\Phi_{\text{in}}^{[j]} \Delta_x \eta_R^{[j]} + 2\nabla_x \eta_R^{[j]} \nabla_x \Phi_{\text{in}}^{[j]}\right) - 2\left(W^{[j]} \cdot \Phi_{\text{in}}^{[j]}\right) \nabla_x \eta_R^{[j]} \nabla_x U^{[j]} \\ & + \eta_R^{[j]} Q_{\gamma_j} \Delta_x \Phi_{\text{in}}^{[j]} - 2\eta_R^{[j]} \nabla_x \left(W^{[j]} \cdot \Phi_{\text{in}}^{[j]}\right) \nabla_x U^{[j]} - 2\nabla_x \left(U^{[j]} \cdot \Phi_{\text{out}}\right) \nabla_x U^{[j]} \Big\}, \end{aligned}$$

where by (4.21), it follows that

$$\begin{aligned} & \left| \left(U_* - U^{[j]}\right) \wedge \left[\Delta_x \left(\eta_{d_q}^{[j]} \Phi_0^{*[j]}\right) - 2\nabla_x \left(U^{[j]} \cdot \eta_{d_q}^{[j]} \Phi_0^{*[j]}\right) \nabla_x U^{[j]}\right] \right| \\ = & \Big| \left(U_* - U^{[j]}\right) \wedge \Big[\Phi_0^{*[j]} \Delta_x \eta_{d_q}^{[j]} + 2\nabla_x \eta_{d_q}^{[j]} \nabla_x \Phi_0^{*[j]} + \eta_{d_q}^{[j]} \Delta_x \Phi_0^{*[j]} \\ & - 2\left(U^{[j]} \cdot \Phi_0^{*[j]}\right) \nabla_x \eta_{d_q}^{[j]} \nabla_x U^{[j]} - 2\eta_{d_q}^{[j]} \left(U^{[j]} \cdot \nabla_x \Phi_0^{*[j]}\right) \nabla_x U^{[j]} - 2\eta_{d_q}^{[j]} \left(\Phi_0^{*[j]} \cdot \nabla_x U^{[j]}\right) \nabla_x U^{[j]}\Big]\Big| \\ \lesssim & \lambda_* \left(1 + \lambda_j^{-1} \langle \rho_j \rangle^{-1} + \lambda_*^{-1} \langle \rho_j \rangle^{-2} + \lambda_j \langle \rho_j \rangle \lambda_*^{-2} \langle \rho_j \rangle^{-4}\right) \mathbf{1}_{\{|x-q_j| \le 3d_q\}} \lesssim T^\epsilon \varrho_3. \end{aligned}$$

•

$$\begin{aligned} & \left| \left(U_* - U^{[j]}\right) \wedge \left[Q_{\gamma_j} \left(\Phi_{\text{in}}^{[j]} \Delta_x \eta_R^{[j]} + 2\nabla_x \eta_R^{[j]} \nabla_x \Phi_{\text{in}}^{[j]}\right) - 2\left(W^{[j]} \cdot \Phi_{\text{in}}^{[j]}\right) \nabla_x \eta_R^{[j]} \nabla_x U^{[j]}\right] \right| \\ \lesssim & \lambda_* \left| \Phi_{\text{in}}^{[j]} \Delta_x \eta_R^{[j]} + 2\nabla_x \eta_R^{[j]} \nabla_x \Phi_{\text{in}}^{[j]} - 2\left(W^{[j]} \cdot \Phi_{\text{in}}^{[j]}\right) \nabla_x \eta_R^{[j]} \nabla_x W^{[j]} \right| \lesssim T^\epsilon \varrho_1 \end{aligned}$$

by the similar estimate in (D.44).

•

$$\begin{aligned} & \left| \left(U_* - U^{[j]}\right) \wedge \eta_R^{[j]} \left[Q_{\gamma_j} \Delta_x \Phi_{\text{in}}^{[j]} - 2\nabla_x \left(W^{[j]} \cdot \Phi_{\text{in}}^{[j]}\right) \nabla_x U^{[j]}\right] \right| \\ = & \lambda_j^{-2} \left| \left(U_* - U^{[j]}\right) \wedge \eta_R^{[j]} \left[Q_{\gamma_j} \Delta_{y^{[j]}} \Phi_{\text{in}}^{[j]} - 2\nabla_{y^{[j]}} \left(W^{[j]} \cdot \Phi_{\text{in}}^{[j]}\right) \nabla_{y^{[j]}} U^{[j]}\right] \right| \\ = & \lambda_j^{-2} \left| \left(U_* - U^{[j]}\right) \wedge \eta_R^{[j]} \left[Q_{\gamma_j} \Delta_{y^{[j]}} \Phi_{\text{in}}^{[j]} - 2\left(W^{[j]} \cdot \nabla_{y^{[j]}} \Phi_{\text{in}}^{[j]} + \Phi_{\text{in}}^{[j]} \cdot \nabla_{y^{[j]}} W^{[j]}\right) \nabla_{y^{[j]}} U^{[j]}\right] \right| \\ \lesssim & \mathbf{1}_{\{|x-\xi^{[j]}| \le 2\lambda_j R\}} \lambda_*^{-1} \|\Phi_{\text{in}}\|_{\text{in},\nu-\delta_0,l}^{[j]} \lambda_*^{\nu-\delta_0} \langle y^{[j]} \rangle^{-l-2} \lesssim T^\epsilon \varrho_1 \end{aligned}$$

provided

$$\Theta + \beta + \delta_0 - \nu < 0. \tag{D.46}$$

•

$$\begin{aligned} & \left(U_* - U^{[j]}\right) \wedge \left[\nabla_x \left(U^{[j]} \cdot \Phi_{\text{out}}\right) \nabla_x U^{[j]}\right] = \left(U_* - U^{[j]}\right) \wedge \left[\left(U^{[j]} \cdot \nabla_x \Phi_{\text{out}} + \Phi_{\text{out}} \cdot \nabla_x U^{[j]}\right) \nabla_x U^{[j]}\right] \\ \lesssim & \sum_{k \ne j} \langle \rho_k \rangle^{-1} \left(|\nabla_x \Phi_{\text{out}}| + |\Phi_{\text{out}}| \lambda_j^{-2} \langle \rho_j \rangle^{-4}\right), \end{aligned}$$

where

$$\sum_{k \ne j} \langle \rho_k \rangle^{-1} |\nabla_x \Phi_{\text{out}}| \lesssim \sum_{k \ne j} \langle \rho_k \rangle^{-1} \|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha} \left(\lambda_*^\Theta(0) + \|Z_*\|_{C^3(\mathbb{R}^2)}\right) \lesssim T^\epsilon \varrho_3.$$

By (D.1), we have

$$
\begin{aligned}
&\eta_R^{[j]}\sum_{k\neq j}\langle\rho_k\rangle^{-1}|\Phi_{\rm out}|\lambda_j^{-2}\langle\rho_j\rangle^{-4}\lesssim\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\eta_R^{[j]}\sum_{k\neq j}\lambda_k\lambda_j^{-2}\langle\rho_j\rangle^{-4}\\
&\times\left[|\ln(T-t)|\lambda_*^{\Theta+1}(t)R(t)+(T-t)\|Z_*\|_{C^3(\mathbb{R}^2)}+\inf_{m=1,\dots,N}|x-q^{[m]}|\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\right]\\
\lesssim{}&\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\eta_R^{[j]}\sum_{k\neq j}\lambda_k\lambda_j^{-2}\langle\rho_j\rangle^{-4}\\
&\times\left[|\ln(T-t)|\lambda_*^{\Theta+1}(t)R(t)+(T-t)\|Z_*\|_{C^3(\mathbb{R}^2)}+\left(\lambda_jR+|\xi^{[j]}-q^{[j]}|\right)\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\right]\\
\lesssim{}&\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\eta_R^{[j]}\lambda_*^{-1}\langle\rho_j\rangle^{-4}\lambda_*R\left(|\ln T|\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\\
\lesssim{}&T^{\epsilon}\varrho_1^{[j]}
\end{aligned}
$$

provided

$$2\beta+\Theta-1<0. \tag{D.47}$$

For all $k\neq j$,

$$
\begin{aligned}
&\eta_R^{[k]}\left(1-\eta_R^{[j]}\right)\langle\rho_k\rangle^{-1}|\Phi_{\rm out}|\lambda_j^{-2}\langle\rho_j\rangle^{-4}\\
\lesssim{}&\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\eta_R^{[k]}\left(1-\eta_R^{[j]}\right)\langle\rho_k\rangle^{-1}\lambda_j^{2}\left(|\ln T|\lambda_*^{\Theta+1}(0)R(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\\
\lesssim{}&T^{\epsilon}\varrho_1^{[k]}.
\end{aligned}
$$

$$
\begin{aligned}
&\left(1-\eta_R^{[k]}\right)\left(1-\eta_R^{[j]}\right)\langle\rho_k\rangle^{-1}|\Phi_{\rm out}|\lambda_j^{-2}\langle\rho_j\rangle^{-4}\\
\lesssim{}&\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(1-\eta_R^{[k]}\right)\left(1-\eta_R^{[j]}\right)\lambda_*^3|x-q^{[k]}|^{-1}|x-q^{[j]}|^{-4}\left(|\ln T|\lambda_*^{\Theta+1}(0)R(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\\
\lesssim{}&T^{\epsilon}\left(\varrho_2^{[j]}+\varrho_3\right)
\end{aligned}
$$

since

$$
\begin{aligned}
&\mathbf{1}_{\{|x-q^{[k]}|\le d_q\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(1-\eta_R^{[k]}\right)\left(1-\eta_R^{[j]}\right)\lambda_*^3|x-q^{[k]}|^{-1}|x-q^{[j]}|^{-4}\left(|\ln T|\lambda_*^{\Theta+1}(0)R(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\\
\lesssim{}&\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\mathbf{1}_{\{|x-q^{[k]}|\le d_q\}}\left(1-\eta_R^{[k]}\right)\left(1-\eta_R^{[j]}\right)\lambda_*^3(\lambda_kR)^{-1}\left(|\ln T|\lambda_*^{\Theta+1}(0)R(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\\
\lesssim{}&T^{\epsilon}\varrho_3,\\
&\mathbf{1}_{\{|x-q^{[j]}|\le d_q\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(1-\eta_R^{[k]}\right)\left(1-\eta_R^{[j]}\right)\lambda_*^3|x-q^{[k]}|^{-1}|x-q^{[j]}|^{-4}\left(|\ln T|\lambda_*^{\Theta+1}(0)R(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\\
\lesssim{}&\mathbf{1}_{\{|x-q^{[j]}|\le d_q\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(1-\eta_R^{[k]}\right)\left(1-\eta_R^{[j]}\right)\lambda_*R^{-2}|x-q^{[j]}|^{-2}\left(|\ln T|\lambda_*^{\Theta+1}(0)R(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\\
\lesssim{}&T^{\epsilon}\varrho_2^{[j]},
\end{aligned}
$$

$$
\begin{aligned}
&\left(1-\mathbf{1}_{\{|x-q^{[k]}|\le d_q\}}-\mathbf{1}_{\{|x-q^{[j]}|\le d_q\}}\right)\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(1-\eta_R^{[k]}\right)\left(1-\eta_R^{[j]}\right)\\
&\times\lambda_*^3|x-q^{[k]}|^{-1}|x-q^{[j]}|^{-4}\left(|\ln T|\lambda_*^{\Theta+1}(0)R(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\lesssim T^{\epsilon}\varrho_3.
\end{aligned}
$$

• For

$$(a-bU_*\wedge)\left\{-2\sum_{j=1}^N\nabla_x\left[\Phi\cdot\left(U_*-U^{[j]}\right)\right]\nabla_xU^{[j]}\right\},$$

it suffices to estimate

$$\nabla_x\left[\Phi\cdot\left(U_*-U^{[j]}\right)\right]\nabla_xU^{[j]}=\left[\left(U_*-U^{[j]}\right)\cdot\nabla_x\Phi+\Phi\cdot\nabla_x\left(U_*-U^{[j]}\right)\right]\nabla_xU^{[j]}.$$

Then for any fixed j,

$$
\begin{aligned}
&\left|\nabla_x\left[\Phi\cdot\left(U_*-U^{[j]}\right)\right]\nabla_xU^{[j]}\right|\\
\lesssim{}&\left(|\nabla_x\Phi|\sum_{k\neq j}\langle\rho_k\rangle^{-1}+|\Phi|\sum_{k\neq j}\lambda_*^{-1}\langle\rho_k\rangle^{-2}\right)\lambda_*^{-1}\langle\rho_j\rangle^{-2}\\
\lesssim{}&|\nabla_x\Phi|\langle\rho_j\rangle^{-1}\sum_{k\neq j}\left(\min\{\langle\rho_j\rangle,\langle\rho_k\rangle\}\right)^{-1}+|\Phi|\sum_{k\neq j}\left(\min\{\langle\rho_j\rangle,\langle\rho_k\rangle\}\right)^{-2}
\end{aligned}
$$

since

$$\langle\rho_j\rangle\langle\rho_k\rangle \gtrsim \lambda_*^{-1}\min\{\langle\rho_j\rangle,\langle\rho_k\rangle\} \quad \text{for} \quad j\neq k. \tag{D.48}$$

Proof of (D.48). For $|x-\xi^{[j]}|\leq\frac{|\xi^{[j]}-\xi^{[k]}|}{2}$, then $\langle\rho_k\rangle\sim\lambda_*^{-1}$, which implies

$$\langle\rho_j\rangle\langle\rho_k\rangle\sim\lambda_*^{-1}\langle\rho_j\rangle\sim\lambda_*^{-1}\min\{\langle\rho_j\rangle,\langle\rho_k\rangle\}.$$

For $|x-\xi^{[k]}|\leq\frac{|\xi^{[j]}-\xi^{[k]}|}{2}$, similarly, we have

$$\langle\rho_j\rangle\langle\rho_k\rangle\sim\lambda_*^{-1}\min\{\langle\rho_j\rangle,\langle\rho_k\rangle\}.$$

For $|x-\xi^{[j]}|>\frac{|\xi^{[j]}-\xi^{[k]}|}{2}$ and $|x-\xi^{[k]}|>\frac{|\xi^{[j]}-\xi^{[k]}|}{2}$,

$$\langle\rho_j\rangle\langle\rho_k\rangle\sim\lambda_*^{-2}|x-\xi^{[j]}||x-\xi^{[k]}|\gtrsim\lambda_*^{-1}\min\{\langle\rho_j\rangle,\langle\rho_k\rangle\}.$$

□

For $k\neq j$, by (D.1) and (D.4), we have

$$|\Phi|\left(\min\{\langle\rho_j\rangle,\langle\rho_k\rangle\}\right)^{-2}\lesssim T^{\epsilon}\varrho_3.$$

By (D.3) and (4.21), we have

$$\begin{aligned}
&|\nabla_x\Phi|\langle\rho_j\rangle^{-1}\left(\min\{\langle\rho_j\rangle,\langle\rho_k\rangle\}\right)^{-1}\\
\lesssim&\ \langle\rho_j\rangle^{-1}\left(\min\{\langle\rho_j\rangle,\langle\rho_k\rangle\}\right)^{-1}\Big[\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\\
&+\sum_{m=1}^{N}\left(\mathbf{1}_{\{|x-\xi^{[m]}|\leq2\lambda_mR\}}\lambda_m^{-1}\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[m]}\lambda_*^{\nu-\delta_0}\langle\rho_m\rangle^{-l-1}+\mathbf{1}_{\{|x-\xi^{[m]}|\leq2d_q\}}\right)\Big]\\
\lesssim&\ T^{\epsilon}\left(\sum_{m=1}^{N}\varrho_1^{[m]}+\varrho_3\right)
\end{aligned}$$

provided

$$\Theta+\beta+\delta_0-\nu<0. \tag{D.49}$$

- $$(a-bU_*\wedge)\left\{-2\sum_{j=1}^{N}\nabla_x\left[U^{[j]}\cdot\sum_{k\neq j}\left(\eta_R^{[k]}Q_{\gamma_k}\Phi_{\rm in}^{[k]}+\eta_{d_q}^{[k]}\Phi_0^{*[k]}\right)\right]\nabla_xU^{[j]}\right\}.$$

For $k\neq j$, we get

$$\begin{aligned}
&\nabla_x\left[U^{[j]}\cdot\left(\eta_R^{[k]}Q_{\gamma_k}\Phi_{\rm in}^{[k]}\right)\right]\nabla_xU^{[j]}\\
=&\left[\eta_R^{[k]}\left(W^{[j]}\cdot\nabla_x\Phi_{\rm in}^{[k]}+\Phi_{\rm in}^{[k]}\cdot\nabla_xW^{[j]}\right)+\left(W^{[j]}\cdot\Phi_{\rm in}^{[k]}\right)\nabla_x\eta_R^{[k]}\right]\nabla_xU^{[j]}\\
=&\left[\eta_R^{[k]}\left(W^{[j]}\cdot\lambda_k^{-1}\nabla_{y^{[k]}}\Phi_{\rm in}^{[k]}+\Phi_{\rm in}^{[k]}\cdot\lambda_j^{-1}\nabla_{y^{[j]}}W^{[j]}\right)+\left(W^{[j]}\cdot\Phi_{\rm in}^{[k]}\right)\nabla_x\eta_R^{[k]}\right]\lambda_j^{-1}\nabla_{y^{[j]}}U^{[j]}\\
\lesssim&\ \|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[k]}\lambda_*^{\nu-\delta_0}\Big[\mathbf{1}_{\{|x-\xi^{[k]}|\leq2\lambda_kR\}}\left(\lambda_k^{-1}\langle\rho_k\rangle^{-l-1}+\langle\rho_k\rangle^{-l}\lambda_j^{-1}\langle\rho_j\rangle^{-2}\right)\\
&+\langle\rho_k\rangle^{-l}(\lambda_kR)^{-1}\mathbf{1}_{\{\lambda_kR\leq|x-\xi^{[k]}|\leq2\lambda_kR\}}\Big]\lambda_j^{-1}\langle\rho_j\rangle^{-2}\\
\sim&\ \mathbf{1}_{\{|x-\xi^{[k]}|\leq2\lambda_kR\}}\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[k]}\lambda_*^{\nu-\delta_0}\left(\lambda_k^{-1}\langle\rho_k\rangle^{-l-1}+\langle\rho_k\rangle^{-l}\lambda_j\right)\lambda_j\\
\lesssim&\ \mathbf{1}_{\{|x-\xi^{[k]}|\leq2\lambda_kR\}}\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[k]}\lambda_*^{\nu-\delta_0}\left(\langle\rho_k\rangle^{-l-1}+\langle\rho_k\rangle^{-l}\lambda_*^2\right)\\
\lesssim&\ T^{\epsilon}\varrho_1^{[k]}
\end{aligned}$$

when

$$\delta_0<\nu; \tag{D.50}$$

and by (4.21),

$$\begin{aligned}
&\left|\nabla_x\left(U^{[j]}\cdot\eta_{d_q}^{[k]}\Phi_0^{*[k]}\right)\nabla_xU^{[j]}\right|=\left|\left[\left(U^{[j]}\cdot\Phi_0^{*[k]}\right)\nabla_x\eta_{d_q}^{[k]}+\eta_{d_q}^{[k]}\left(U^{[j]}\cdot\nabla_x\Phi_0^{*[k]}+\Phi_0^{*[k]}\cdot\nabla_xU^{[j]}\right)\right]\nabla_xU^{[j]}\right|\\
\lesssim&\ \lambda_j\mathbf{1}_{\{|x-\xi^{[k]}|\leq2d_q\}}\lesssim T^{\epsilon}\varrho_3.
\end{aligned}$$

•
$$\sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \left(a - bU^{[j]}\wedge\right) \sum_{k\neq j} \left(\eta_R^{[k]} Q_{\gamma_k} \Phi_{\rm in}^{[k]} + \eta_{d_q}^{[k]} \Phi_0^{*[k]}\right).$$

For $k \neq j$,
$$\left| |\nabla_x U^{[j]}|^2 \eta_R^{[k]} Q_{\gamma_k} \Phi_{\rm in}^{[k]} \right| \lesssim \lambda_j^2 \eta_R^{[k]} \|\Phi_{\rm in}\|_{{\rm in},\nu-\delta,l}^{[k]} \lambda_*^{\nu-\delta_0} \langle \rho_k \rangle^{-l} \lesssim T^{\epsilon} \varrho_1^{[k]}$$
and by (4.21),
$$|\nabla_x U^{[j]}|^2 |\eta_{d_q}^{[k]} \Phi_0^{*[k]}| \lesssim \lambda_j^2 \lesssim T^{\epsilon} \varrho_3.$$

• By (D.1), (D.4) and (D.48),
$$\left| a\Phi \sum_{j\neq k} \nabla_x U^{[j]} \cdot \nabla_x U^{[k]} \right| \lesssim |\Phi| \sum_{j\neq k} \lambda_j^{-1} \langle \rho_j \rangle^{-2} \lambda_k^{-1} \langle \rho_k \rangle^{-2} \lesssim |\Phi| \sum_{j\neq k} \left(\min\{\langle \rho_j \rangle, \langle \rho_k \rangle\}\right)^{-2} \lesssim T^{\epsilon} \varrho_3.$$

• By (5.5), (4.8) and (D.22),
$$|[(\Phi\cdot U_*) - A]\partial_t U_*| \lesssim \left(|\Phi\cdot U_*| + \lambda_* + |\Phi|^2\right) \sum_{j=1}^{N} \left[\left(\lambda_j^{-1}|\dot\lambda_j| + |\dot\gamma_j|\right) \langle \rho_j \rangle^{-1} + \lambda_j^{-1} |\dot\xi^{[j]}| \langle \rho_j \rangle^{-2} \right]$$
$$\begin{aligned}
&\lesssim \Bigg\{ \sum_{j=1}^{N} \Bigg[\left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha} + \|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2 \\
&\quad \times \mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}} \left(\lambda_*^{2\nu-2\delta_0} \langle \rho_j \rangle^{-l} + \lambda_* \langle \rho_j \rangle + |\ln(T-t)| \lambda_*^{\Theta+1} R\right) \\
&\quad + \mathbf{1}_{\{3\lambda_* R < |x-q^{[j]}| < 3d_q\}} \left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2 \left(\lambda_* \langle \rho_j \rangle + |\ln(T-t)| \lambda_*^{\Theta+1} R\right) \Bigg] \\
&\quad + \mathbf{1}_{\{\cap_{j=1}^{N}\{|x-q^{[j]}|\ge 3d_q\}\}} \left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2 \Bigg\} \sum_{j=1}^{N} \left(\lambda_j^{-1}|\dot\lambda_j| + |\dot\gamma_j| + \lambda_j^{-1} |\dot\xi^{[j]}|\right) \langle \rho_j \rangle^{-1} \\
&\lesssim T^{\epsilon} \left(\sum_{j=1}^{N} \varrho_1^{[j]} + \varrho_3 \right)
\end{aligned}$$
provided
$$\Theta + \beta + 2\delta_0 - 2\nu < 0. \tag{D.51}$$

•
$$\begin{aligned}
&\sum_{j=1}^{2} \eta_R^{[j]} \left(U^{[j]} - U_*\right) \Bigg\{ -2a \left(\nabla_x W^{[j]} \cdot \nabla_x \Phi_{\rm in}^{[j]}\right) + a |\nabla_x U^{[j]}|^2 \left(U^{[j]} \cdot \Phi_{\rm out}\right) \\
&+ \Big\{ -\partial_t(\Phi_0^{*[j]}) + \left(a - bU^{[j]}\wedge\right) \left[\Delta_x \Phi_0^{*[j]} + |\nabla_x U^{[j]}|^2 \Phi_0^{*[j]} - 2\nabla_x \left(U^{[j]} \cdot \Phi_0^{*[j]}\right) \nabla_x U^{[j]}\right] \\
&- \partial_t U^{[j]} \Big\} \cdot U^{[j]} \Bigg\}.
\end{aligned}$$

For above terms, we first estimate
$$\begin{aligned}
&\left| \eta_R^{[j]} \left(U^{[j]} - U_*\right) \left(\nabla_x W^{[j]} \cdot \nabla_x \Phi_{\rm in}^{[j]}\right) \right| \\
&= \left| \lambda_j^{-2} \eta_R^{[j]} \left(U^{[j]} - U_*\right) \left(\nabla_{y^{[j]}} W^{[j]} \cdot \nabla_{y^{[j]}} \Phi_{\rm in}^{[j]}\right) \right| \\
&\lesssim \lambda_j^{-2} \mathbf{1}_{\{|x-\xi^{[j]}|\le 2\lambda_j R\}} \lambda_* \langle \rho_j \rangle^{-2} \|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]} \lambda_*^{\nu-\delta_0} \langle \rho_j \rangle^{-l-1} \\
&\sim \mathbf{1}_{\{|x-\xi^{[j]}|\le 2\lambda_j R\}} \|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]} \lambda_*^{\nu-\delta_0-1} \langle \rho_j \rangle^{-l-3} \\
&\lesssim T^{\epsilon} \varrho_1^{[j]}
\end{aligned}$$
when
$$\Theta + \delta_0 + \beta - \nu < 0. \tag{D.52}$$

Secondly, by (D.2),

$$\begin{aligned}&\left|\eta_R^{[j]}\left(U^{[j]}-U_*\right)|\nabla_x U^{[j]}|^2\left(U^{[j]}\cdot\Phi_{\rm out}\right)\right|\\ \lesssim\ & \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\eta_R^{[j]}\lambda_*^{-1}\langle\rho_j\rangle^{-4}\lambda_* R\left(|\ln T|\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\lesssim T^{\epsilon}\varrho_1^{[j]}\end{aligned}$$

provided

$$\Theta+2\beta-1<0. \tag{D.53}$$

Thirdly, by (4.40), we obtain

$$\begin{aligned}&\Bigg|\eta_R^{[j]}\left(U^{[j]}-U_*\right)\Bigg\{\Big\{-\partial_t(\Phi_0^{*[j]})+\left(a-bU^{[j]}\wedge\right)\left[\Delta_x\Phi_0^{*[j]}+|\nabla_x U^{[j]}|^2\Phi_0^{*[j]}-2\nabla_x\left(U^{[j]}\cdot\Phi_0^{*[j]}\right)\nabla_x U^{[j]}\right]\\ &-\partial_t U^{[j]}\Big\}\cdot U^{[j]}\Bigg\}\Bigg|\lesssim\eta_R^{[j]}\lambda_*\left(|\dot\xi^{[j]}|\langle\rho_j\rangle^{-1}+|\lambda_j|^{-1}\langle\rho_j\rangle^{-2}\right)\lesssim T^{\epsilon}\varrho_3.\end{aligned}$$

- By (D.20), we have

$$\begin{aligned}&(\Phi\wedge U_*)\left[(1+A)|U_*|^2+\left(U_*\cdot\Pi_{U_*^{\perp}}\Phi\right)\right]^{-1}\left[\Phi+(1+A-\Phi\cdot U_*)U_*\right]\cdot\Delta_x(\Phi-\Phi_{\rm out})\\ &-\left(AU_*+\Pi_{U_*^{\perp}}\Phi\right)\wedge\Delta_x(\Phi-\Phi_{\rm out})\\ =\ &(\Phi\wedge U_*)\left(1+O\left(\lambda_*+|\Phi|^2\right)\right)\left[\Phi+(1+A-\Phi\cdot U_*)U_*\right]\cdot\Delta_x(\Phi-\Phi_{\rm out})\\ &-\left[AU_*+\Phi-(\Phi\cdot U_*)U_*\right]\wedge\Delta_x(\Phi-\Phi_{\rm out})\\ =\ &(\Phi\wedge U_*)\left[U_*\cdot\Delta_x(\Phi-\Phi_{\rm out})\right]-\Phi\wedge\Delta_x(\Phi-\Phi_{\rm out})\\ &+(\Phi\wedge U_*)\left[\Phi+(A-\Phi\cdot U_*)U_*\right]\cdot\Delta_x(\Phi-\Phi_{\rm out})\\ &+(\Phi\wedge U_*)O\left(\lambda_*+|\Phi|^2\right)\left[\Phi+(1+A-\Phi\cdot U_*)U_*\right]\cdot\Delta_x(\Phi-\Phi_{\rm out})\\ &-\left[AU_*-(\Phi\cdot U_*)U_*\right]\wedge\Delta_x(\Phi-\Phi_{\rm out}).\end{aligned}$$

For above terms, we estimate by (5.5)

$$\begin{aligned}&\Big|(\Phi\wedge U_*)\left[\Phi+(A-\Phi\cdot U_*)U_*\right]\cdot\Delta_x(\Phi-\Phi_{\rm out})\\ &+(\Phi\wedge U_*)O\left(\lambda_*+|\Phi|^2\right)\left[\Phi+(1+A-\Phi\cdot U_*)U_*\right]\cdot\Delta_x(\Phi-\Phi_{\rm out})\\ &-\left[AU_*-(\Phi\cdot U_*)U_*\right]\wedge\Delta_x(\Phi-\Phi_{\rm out})\Big|\\ \lesssim\ &\left(\lambda_*+|\Phi|^2+|\Phi\cdot U_*|\right)|\Delta_x(\Phi-\Phi_{\rm out})|.\end{aligned}$$

Using (D.22) and (D.8), we have

$$\begin{aligned}&\left(\lambda_*+|\Phi|^2+|\Phi\cdot U_*|\right)|\Delta_x(\Phi-\Phi_{\rm out})|\\ \lesssim\ &\Bigg\{\sum_{j=1}^N\Big[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\left(\lambda_*^{2\nu-2\delta_0}\langle\rho_j\rangle^{-l}+\lambda_*\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{\Theta+1}R\right)\\ &+\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\left(\lambda_*\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{\Theta+1}R\right)\Big]\\ &+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\Bigg\}\\ &\times\sum_{j=1}^N\left[\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\left(\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0-2}\langle\rho_j\rangle^{-l-2}+\lambda_*^{-1}\langle\rho_j\rangle^{-1}\right)+\lambda_*^{-1}\langle\rho_j\rangle^{-1}\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\right]\\ \sim\ &\sum_{j=1}^N\Big[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^3\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\Big(\lambda_*^{2\nu-2\delta_0+1}\langle\rho_j\rangle^{1-l}+\lambda_*^2\langle\rho_j\rangle^2\\ &+|\ln(T-t)|\lambda_*^{\Theta+2}R\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{2\nu-2\delta_0+\Theta+1}R\langle\rho_j\rangle^{-l}+|\ln(T-t)|\lambda_*^{\Theta+2}R\langle\rho_j\rangle\\ &+|\ln(T-t)|^2\lambda_*^{2\Theta+2}R^2\Big)+\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\Big]\lesssim T^{\epsilon}\varrho_3.\end{aligned}$$

We need more refined estimates for the other part. Recalling (5.1), we have

$$\begin{aligned}
&(\Phi \wedge U_*)\left[U_* \cdot \Delta_x\left(\Phi-\Phi_{\text{out}}\right)\right]-\Phi \wedge \Delta_x\left(\Phi-\Phi_{\text{out}}\right) \\
= &-\Phi \wedge\left\{\Delta_x\left(\Phi-\Phi_{\text{out}}\right)-\left[U_* \cdot \Delta_x\left(\Phi-\Phi_{\text{out}}\right)\right] U_*\right\} \\
= &-\left(\sum_{j=1}^N \eta_{d_q}^{[j]} \Phi_0^{*[j]}+\Phi_{\text{out}}\right) \wedge\left\{\Delta_x\left(\Phi-\Phi_{\text{out}}\right)-\left[U_* \cdot \Delta_x\left(\Phi-\Phi_{\text{out}}\right)\right] U_*\right\} \\
&-\sum_{j=1}^N \eta_R^{[j]} Q_{\gamma_j} \Phi_{\text{in}}^{[j]} \wedge\left\{\left[U^{[j]} \cdot \Delta_x\left(\Phi-\Phi_{\text{out}}\right)\right] U^{[j]}-\left[U_* \cdot \Delta_x\left(\Phi-\Phi_{\text{out}}\right)\right] U_*\right\} \\
&-\sum_{j=1}^N \eta_R^{[j]} Q_{\gamma_j} \Phi_{\text{in}}^{[j]} \wedge\left\{\Delta_x\left(\Phi-\Phi_{\text{out}}\right)-\left[U^{[j]} \cdot \Delta_x\left(\Phi-\Phi_{\text{out}}\right)\right] U^{[j]}\right\}.
\end{aligned}$$

By (4.21) and (D.2), one has

$$\begin{aligned}
&\left|\sum_{j=1}^N \eta_{d_q}^{[j]} \Phi_0^{*[j]}+\Phi_{\text{out}}\right| \\
\lesssim &\sum_{j=1}^N \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\left[\lambda_j\langle\rho_j\rangle+\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}\left(|\ln(T-t)|\lambda_*^{\Theta+1}R+\lambda_j\rho_j\right)\right]+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\geq 3d_q\}\}}\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}.
\end{aligned} \tag{D.54}$$

Then using (D.8), we get

$$\begin{aligned}
&\left|\left(\sum_{j=1}^N \eta_{d_q}^{[j]} \Phi_0^{*[j]}+\Phi_{\text{out}}\right) \wedge\left\{\Delta_x\left(\Phi-\Phi_{\text{out}}\right)-\left[U_* \cdot \Delta_x\left(\Phi-\Phi_{\text{out}}\right)\right] U_*\right\}\right| \\
\lesssim &\left\{\sum_{j=1}^N \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\left[\lambda_j\langle\rho_j\rangle+\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}\left(|\ln(T-t)|\lambda_*^{\Theta+1}R+\lambda_j\rho_j\right)\right]+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\geq 3d_q\}\}}\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}\right\} \\
&\times \sum_{j=1}^N\left[\mathbf{1}_{\{|x-q^{[j]}|\leq 3\lambda_* R\}}\left(\|\Phi_{\text{in}}^{[j]}\|_{\text{in},\nu-\delta_0,l}^{[j]}\lambda_*^{\nu-\delta_0-2}\langle\rho_j\rangle^{-l-2}+\lambda_*^{-1}\langle\rho_j\rangle^{-1}\right)+\lambda_*^{-1}\langle\rho_j\rangle^{-1}\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\right] \\
\lesssim &\sum_{j=1}^N\left[\mathbf{1}_{\{|x-q^{[j]}|\leq 3\lambda_* R\}}\left(1+\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\text{in}}^{[j]}\|_{\text{in},\nu-\delta_0,l}^{[j]}\right)^2\right. \\
&\times\left(\lambda_*^{\nu-\delta_0-1}\langle\rho_j\rangle^{-l-1}+1+|\ln(T-t)|\lambda_*^{\nu-\delta_0+\Theta-1}R\langle\rho_j\rangle^{-l-2}+|\ln(T-t)|\lambda_*^{\Theta}R\langle\rho_j\rangle^{-1}\right) \\
&\left.+\left(1+\|\Phi_{\text{out}}\|_{\sharp,\Theta,\alpha}\right)\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\right] \lesssim T^{\epsilon}\left(\sum_{j=1}^N \varrho_1^{[j]}+\varrho_3\right)
\end{aligned}$$

provided

$$\Theta+\beta+\delta_0-\nu<0, \quad 2\beta+\delta_0-\nu<0. \tag{D.55}$$

By (D.8) and (5.44), we have

$$\begin{aligned}
&\left|\eta_R^{[j]} Q_{\gamma_j} \Phi_{\rm in}^{[j]} \wedge \left\{\left[U^{[j]} \cdot \Delta_x\left(\Phi-\Phi_{\rm out}\right)\right] U^{[j]}-\left[U_* \cdot \Delta_x\left(\Phi-\Phi_{\rm out}\right)\right] U_*\right\}\right| \\
= &\left|\eta_R^{[j]} Q_{\gamma_j} \Phi_{\rm in}^{[j]} \wedge \left\{\left[\left(U^{[j]}-U_*\right) \cdot \Delta_x\left(\Phi-\Phi_{\rm out}\right)\right] U^{[j]}+\left[U_* \cdot \Delta_x\left(\Phi-\Phi_{\rm out}\right)\right]\left(U^{[j]}-U_*\right)\right\}\right| \\
\lesssim &\ \eta_R^{[j]} \lambda_*\left|\Phi_{\rm in}^{[j]}\right|\left|\Delta_x\left(\Phi-\Phi_{\rm out}\right)\right| \\
\lesssim &\ \mathbf{1}_{\{|x-q^{[j]}| \leq 3 \lambda_* R\}} \lambda_*\|\Phi_{\rm in}^{[j]}\|_{{\rm in}, \nu-\delta_0, l}^{[j]} \lambda_*^{\nu-\delta_0}\langle\rho_j\rangle^{-l} \\
&\times \sum_{k=1}^N\left[\mathbf{1}_{\{|x-q^{[k]}| \leq 3 \lambda_* R\}}\left(\|\Phi_{\rm in}^{[k]}\|_{{\rm in}, \nu-\delta_0, l}^{[k]} \lambda_*^{\nu-\delta_0-2}\langle\rho_k\rangle^{-l-2}+\lambda_*^{-1}\langle\rho_k\rangle^{-1}\right)+\lambda_*^{-1}\langle\rho_k\rangle^{-1} \mathbf{1}_{\{3 \lambda_* R<|x-q^{[k]}|<3 d_q\}}\right] \\
\lesssim &\ \mathbf{1}_{\{|x-q^{[j]}| \leq 3 \lambda_* R\}}\left(1+\|\Phi_{\rm in}^{[j]}\|_{{\rm in}, \nu-\delta_0, l}^{[j]}\right)^2\left(\lambda_*^{2 \nu-2 \delta_0-1}\langle\rho_j\rangle^{-2 l-2}+\lambda_*^{\nu-\delta_0}\langle\rho_j\rangle^{-l-1}\right) \lesssim T^\epsilon \varrho_1^{[j]}
\end{aligned}$$

provided

$$\Theta+\beta+2 \delta_0-2 \nu<0. \tag{D.56}$$

Notice

$$\eta_R^{[j]} Q_{\gamma_j} \Phi_{\rm in}^{[j]} \wedge\left\{\Delta_x\left(\Phi-\Phi_{\rm out}\right)-\left[U^{[j]} \cdot \Delta_x\left(\Phi-\Phi_{\rm out}\right)\right] U^{[j]}\right\}$$

is a vector parallel with $U^{[j]}$. That is,

$$\eta_R^{[j]} Q_{\gamma_j} \Phi_{\rm in}^{[j]} \wedge\left\{\Delta_x\left(\Phi-\Phi_{\rm out}\right)-\left[U^{[j]} \cdot \Delta_x\left(\Phi-\Phi_{\rm out}\right)\right] U^{[j]}\right\}=\eta_R^{[j]} f_j(x, t) U^{[j]}$$

where

$$|f_j(x, t)|=\left|Q_{\gamma_j} \Phi_{\rm in}^{[j]} \wedge\left\{\Delta_x\left(\Phi-\Phi_{\rm out}\right)-\left[U^{[j]} \cdot \Delta_x\left(\Phi-\Phi_{\rm out}\right)\right] U^{[j]}\right\}\right| .$$

By U_*-operation, we only need to estimate the following term.

$$\left|\eta_R^{[j]} f_j(x, t)\left(U^{[j]}-U_*\right)\right| \lesssim \eta_R^{[j]} \lambda_*\left|\Phi_{\rm in}^{[j]}\right|\left|\Delta_x\left(\Phi-\Phi_{\rm out}\right)\right| \lesssim T^\epsilon \varrho_1^{[j]}$$

by the same calculations as the above terms under the assumptions (D.56) on the parameters.

Under the parameter assumptions (D.18), by (D.19) and (D.33), we have

$$\begin{aligned}
&\left|\left(\nabla_x A+U_* \cdot \nabla_x U_*+\Phi \cdot \nabla_x \Phi\right) \nabla_x U_*\right| \\
\lesssim &\left\{\sum_{j=1}^N\left[\left(1+\|\Phi_{\rm out}\|_{\sharp, \Theta, \alpha}+\|\Phi_{\rm in}\|_{{\rm in}, \nu-\delta_0, l}^{[j]}\right)^4 \mathbf{1}_{\{|x-q^{[j]}| \leq 3 \lambda_* R\}}\left(\lambda_*^{\epsilon+1} \lambda_*^{\Theta}(\lambda_* R)^{-1}+\lambda_*\langle\rho_j\rangle\right)\right.\right. \\
&\left.+\mathbf{1}_{\{3 \lambda_* R<|x-q^{[j]}|<3 d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp, \Theta, \alpha}\right)^4\left(|\ln (T-t)| \lambda_*^{\Theta+1} R+\lambda_j\langle\rho_j\rangle\right)\right] \\
&\left.+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}| \geq 3 d_q\}\}}\left(1+\|\Phi_{\rm out}\|_{\sharp, \Theta, \alpha}\right)^4\right\}\left(\sum_{j=1}^N \mathbf{1}_{\{|x-q^{[j]}|<3 d_q\}} \lambda_*^{-1}\langle\rho_j\rangle^{-2}+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}| \geq 3 d_q\}\}} \lambda_*\right) \\
\lesssim &\ T^\epsilon\left(\sum_{j=1}^N \varrho_1^{[j]}+\varrho_3\right) .
\end{aligned} \tag{D.57}$$

- $$\left(A-\Phi \cdot U_*\right) \Delta_x U_*=-\left(A-\Phi \cdot U_*\right) \sum_{j=1}^N|\nabla_x U^{[j]}|^2 U^{[j]} .$$

By U_*-operation and (5.5), it suffices to estimate

$$\left|\left(A-\Phi \cdot U_*\right) \sum_{j=1}^N|\nabla_x U^{[j]}|^2\left(U^{[j]}-U_*\right)\right| \lesssim\left(\lambda_*+|\Phi|^2+|\Phi \cdot U_*|\right) \sum_{j=1}^N \lambda_j^{-2}\langle\rho_j\rangle^{-4} \sum_{k \neq j}\langle\rho_k\rangle^{-1}$$

which will be dealt with uniformly in (D.58) later.

• By (D.20), one has

$$\begin{aligned}
&\left|(\Phi\wedge U_*)\left[(1+A)|U_*|^2+\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\right]^{-1}(1+A-\Phi\cdot U_*)\left(2\nabla_x\Phi\cdot\nabla_xU_*\right)-(\Phi\wedge U_*)\left(2\nabla_x\Phi\cdot\nabla_xU_*\right)\right|\\
=&\left|(\Phi\wedge U_*)\left(1+O\left(\lambda_*+|\Phi|^2+|\Phi\cdot U_*|\right)\right)^2\left(2\nabla_x\Phi\cdot\nabla_xU_*\right)-(\Phi\wedge U_*)\left(2\nabla_x\Phi\cdot\nabla_xU_*\right)\right|\\
\lesssim&\,|\Phi|\left(\lambda_*+|\Phi|^2+|\Phi\cdot U_*|\right)|\nabla_x\Phi\cdot\nabla_xU_*|\\
\lesssim&\left(\lambda_*+|\Phi|^2+|\Phi\cdot U_*|\right)|\Phi|\,|\nabla_xU_*|^2+\left(\lambda_*+|\Phi|^2+|\Phi\cdot U_*|\right)|\Phi|\,|\nabla_x\Phi|^2
\end{aligned}$$

which will be controlled by (D.58) and (D.60) later.

•

$$\begin{aligned}
&\Bigg|\Bigg[|\nabla_xA|^2|U_*|^2+2(1+A)\nabla_xA\cdot(U_*\cdot\nabla_xU_*)+A(2+A)\,|\nabla_xU_*|^2\\
&+2\sum_{k=1}^{2}\Big\{\left[(\partial_{x_k}A)\,U_*\cdot\partial_{x_k}\Phi+A\partial_{x_k}U_*\cdot\partial_{x_k}\Phi\right]-\partial_{x_k}(U_*\cdot\Phi)\left[|U_*|^2\partial_{x_k}A+(1+A)U_*\cdot\partial_{x_k}U_*\right]\\
&-(U_*\cdot\Phi)\left[(\partial_{x_k}A)\,U_*\cdot\partial_{x_k}U_*+(1+A)\,|\partial_{x_k}U_*|^2\right]\Big\}\\
&+\sum_{k=1}^{2}|\partial_{x_k}\Phi-U_*\partial_{x_k}(\Phi\cdot U_*)-(\Phi\cdot U_*)\partial_{x_k}U_*|^2\Bigg]\Pi_{U_*^\perp}\Phi\Bigg|\\
&+\Bigg|-b\Bigg[-2^{-1}(\Phi\wedge U_*)\left[(1+A)|U_*|^2+\left(U_*\cdot\Pi_{U_*^\perp}\Phi\right)\right]^{-1}\Big\{2\,(1+A-\Phi\cdot U_*)\,(\Phi\cdot\Delta_xU_*)\\
&+2(|U_*|^2-2)|\nabla_x(\Phi\cdot U_*)|^2+2|\nabla_x\Phi|^2+8[(\Phi\cdot U_*)-(1+A)](U_*\cdot\nabla_xU_*)\cdot\nabla_x(\Phi\cdot U_*)\\
&+2|U_*|^2|\nabla_xA|^2+4\left[-2(\Phi\cdot U_*)U_*\cdot\nabla_xU_*+(1-|U_*|^2)\nabla_x(\Phi\cdot U_*)\right]\cdot\nabla_xA\\
&+8(1+A)\,(U_*\cdot\nabla_xU_*)\cdot\nabla_xA+2\left[(\Phi\cdot U_*)-(1+A)\right]^2\left(|\nabla_xU_*|^2+U_*\cdot\Delta_xU_*\right)\Big\}\\
&-(\Pi_{U_*^\perp}\Phi+AU_*)\wedge\left[2\nabla_x(\Phi\cdot U_*)\,\nabla_xU_*\right]+[A-(\Phi\cdot U_*)]\Phi\wedge\Delta_xU_*\\
&+\Pi_{U_*^\perp}\Phi\wedge(2\nabla_xA\nabla_xU_*)+\left[(\Phi\cdot U_*)^2-2A(\Phi\cdot U_*)-2(\Phi\cdot U_*)\right]U_*\wedge\Delta_xU_*\\
&+(1+A)U_*\wedge\left[A\Delta_xU_*+2\left(\nabla_xA+U_*\cdot\nabla_xU_*+\Phi\cdot\nabla_x\Phi\right)\nabla_xU_*+\Delta_xU_*-2\left(U_*\cdot\nabla_xU_*\right)\nabla_xU_*\right]\Bigg]\\
&+2bAU_*\wedge\left[(\Phi\cdot\nabla_x\Phi)\,\nabla_xU_*\right]\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \Big(|\nabla_x A|^2 + |\nabla_x A|\,|U_*\cdot\nabla_x U_*| + \left(\lambda_* + |\Phi|^2\right)|\nabla_x U_*|^2\\
&\quad + |\nabla_x A||\nabla_x\Phi| + (\lambda_* + |\Phi|^2)|\nabla_x U_*||\nabla_x\Phi| + |\nabla_x A|\,|\nabla_x\left(U_*\cdot\Phi\right)| + |U_*\cdot\nabla_x U_*|\,|\nabla_x\left(U_*\cdot\Phi\right)|\\
&\quad + |\nabla_x A||U_*\cdot\nabla_x U_*|\,|U_*\cdot\Phi| + |\nabla_x U_*|^2\,|U_*\cdot\Phi|\\
&\quad + |\nabla_x\Phi|^2 + |\nabla_x\left(\Phi\cdot U_*\right)|^2 + |\Phi\cdot U_*|^2|\nabla_x U_*|^2\Big)|\Phi|\\
&\quad + |\Phi|\Big(\left|\Phi\cdot\sum_{j=1}^N|\nabla_x U^{[j]}|^2U^{[j]}\right|\\
&\quad + |\nabla_x\left(\Phi\cdot U_*\right)|^2 + |\nabla_x\Phi|^2 + |U_*\cdot\nabla_x U_*||\nabla_x\left(\Phi\cdot U_*\right)|\\
&\quad + |\nabla_x A|^2 + |\Phi\cdot U_*||U_*\cdot\nabla_x U_*||\nabla_x A| + \lambda_*|\nabla_x\left(\Phi\cdot U_*\right)||\nabla_x A|\\
&\quad + |U_*\cdot\nabla_x U_*||\nabla_x A| + \left||\nabla_x U_*|^2 + U_*\cdot\Delta_x U_*\right|\Big)\\
&\quad + \left(\lambda_* + |\Phi|\right)|\nabla_x\left(\Phi\cdot U_*\right)|\,|\nabla_x U_*| + \left(\lambda_* + |\Phi|^2 + |\Phi\cdot U_*|\right)|\Phi\wedge\Delta_x U_*|\\
&\quad + |\Phi|\,|\nabla_x A\nabla_x U_*| + |\Phi\cdot U_*|\,|U_*\wedge\Delta_x U_*|\\
&\quad + \left(\lambda_* + |\Phi|^2\right)|U_*\wedge\Delta_x U_*| + |U_*\wedge[\Delta_x U_* - 2\left(U_*\cdot\nabla_x U_*\right)\nabla_x U_*]|\\
&\quad + |\left(\nabla_x A + U_*\cdot\nabla_x U_* + \Phi\cdot\nabla_x\Phi\right)\nabla_x U_*|\\
&\quad + \left(\lambda_* + |\Phi|^2\right)|\Phi||\nabla_x\Phi||\nabla_x U_*|\\
&\lesssim |\nabla_x A|^2|\Phi| + \left(\lambda_* + |\Phi|^2 + |U_*\cdot\Phi|\right)|\Phi|\left(|\nabla_x U_*|^2 + |\Delta_x U_*|\right)\\
&\quad + |U_*\cdot\nabla_x U_*|^2\,|\Phi| + |\nabla_x\Phi|^2|\Phi| + \left|\Phi\cdot\sum_{j=1}^N|\nabla_x U^{[j]}|^2U^{[j]}\right||\Phi| + \left||\nabla_x U_*|^2 + U_*\cdot\Delta_x U_*\right||\Phi|\\
&\quad + \left(\lambda_* + |\Phi|\right)|\nabla_x\left(\Phi\cdot U_*\right)|\,|\nabla_x U_*| + |\Phi|\,|\nabla_x A\nabla_x U_*| + \left(\lambda_* + |\Phi|^2 + |U_*\cdot\Phi|\right)|U_*\wedge\Delta_x U_*|\\
&\quad + |U_*\wedge[\Delta_x U_* - 2\left(U_*\cdot\nabla_x U_*\right)\nabla_x U_*]| + |\left(\nabla_x A + U_*\cdot\nabla_x U_* + \Phi\cdot\nabla_x\Phi\right)\nabla_x U_*|\\
&\lesssim \left(\lambda_* + |\Phi|^2 + |U_*\cdot\Phi|\right)\Big[|\Phi|\left(|\nabla_x U_*|^2 + |\Delta_x U_*|\right) + |U_*\wedge\Delta_x U_*|\Big]\\
&\quad + |U_*\cdot\nabla_x U_*|^2\,|\Phi| + |\nabla_x\Phi|^2|\Phi| + \left|\Phi\cdot\sum_{j=1}^N|\nabla_x U^{[j]}|^2U^{[j]}\right||\Phi| + \left||\nabla_x U_*|^2 + U_*\cdot\Delta_x U_*\right||\Phi|\\
&\quad + \left(\lambda_* + |\Phi|\right)|\nabla_x\left(\Phi\cdot U_*\right)|\,|\nabla_x U_*| + |\Phi|\,|\left(U_*\cdot\nabla_x U_*\right)\nabla_x U_*| + |\Phi|\,|\left(\Phi\cdot\nabla_x\Phi\right)\nabla_x U_*|\\
&\quad + |U_*\wedge[\Delta_x U_* - 2\left(U_*\cdot\nabla_x U_*\right)\nabla_x U_*]| + O(T^\epsilon)\left(\sum_{j=1}^N\varrho_1^{[j]} + \varrho_3\right),
\end{aligned}
$$

where in the last "$\lesssim$", we require (D.18) and then by (D.19),

$$
\begin{aligned}
|\nabla_x A|^2|\Phi| &\lesssim |U_*\cdot\nabla_x U_*|^2\,|\Phi| + |\Phi|^3|\nabla_x\Phi|^2 + T^\epsilon\varrho_3,\\
|\Phi|\,|\nabla_x A\nabla_x U_*| &\lesssim |\Phi|\,|\left(\nabla_x A + U_*\cdot\nabla_x U_* + \Phi\cdot\nabla_x\Phi\right)\nabla_x U_*|\\
&\quad + |\Phi|\,|\left(U_*\cdot\nabla_x U_*\right)\nabla_x U_*| + |\Phi|\,|\left(\Phi\cdot\nabla_x\Phi\right)\nabla_x U_*|,
\end{aligned}
$$

and $|\left(\nabla_x A + U_*\cdot\nabla_x U_* + \Phi\cdot\nabla_x\Phi\right)\nabla_x U_*|$ has been controlled by (D.57).

- Combining (D.33), (D.34), (D.40), (D.22) and (D.9), we then obtain

$$
\begin{aligned}
&\left(\lambda_* + |\Phi|^2 + |U_*\cdot\Phi|\right)\left[|\Phi|\left(|\nabla_x U_*|^2 + |\Delta_x U_*|\right) + |U_*\wedge\Delta_x U_*|\right]\\
\lesssim\ &\left(\lambda_* + |\Phi|^2 + |U_*\cdot\Phi|\right)\left(|\Phi|\sum_{j=1}^{N}\lambda_*^{-2}\langle\rho_j\rangle^{-4} + \sum_{j=1}^{N}\lambda_j^{-2}\langle\rho_j\rangle^{-4}\sum_{k\neq j}\langle\rho_j\rangle^{-1}\right)\\
\lesssim\ &\left(\lambda_* + |\Phi|^2 + |U_*\cdot\Phi|\right)\\
&\times\left[\sum_{j=1}^{N}\mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\left(|\Phi|\lambda_*^{-2}\langle\rho_j\rangle^{-4} + \lambda_*^{-1}\langle\rho_j\rangle^{-4} + \lambda_*^{2}\langle\rho_j\rangle^{-1}\right) + \mathbf{1}_{\{\cap_{j=1}^{N}\{|x-q^{[j]}|\ge 3d_q\}\}}\left(|\Phi|\lambda_*^2 + \lambda_*^3\right)\right]\\
\lesssim\ &\Bigg\{\sum_{j=1}^{N}\Big[\left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha} + \|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\left(\lambda_*^{2\nu-2\delta_0}\langle\rho_j\rangle^{-l} + \lambda_*\langle\rho_j\rangle + |\ln(T-t)|\lambda_*^{\Theta+1}R\right)\\
&+ \mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\left(\lambda_*\langle\rho_j\rangle + |\ln(T-t)|\lambda_*^{\Theta+1}R\right)\Big]\\
&+ \mathbf{1}_{\{\cap_{j=1}^{N}\{|x-q^{[j]}|\ge 3d_q\}\}}\left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\Bigg\}\\
&\times\Bigg\{\sum_{j=1}^{N}\Big[\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha} + \|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\\
&\times\left(\lambda_*^{\nu-\delta_0-2}\langle\rho_j\rangle^{-l-4} + \lambda_*^{-1}\langle\rho_j\rangle^{-3} + |\ln(T-t)|\lambda_*^{\Theta-1}R\langle\rho_j\rangle^{-4}\right)\\
&+ \mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\lambda_*^{-1}\langle\rho_j\rangle^{-3}\Big] + \mathbf{1}_{\{\cap_{j=1}^{N}\{|x-q^{[j]}|\ge 3d_q\}\}}\left(\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\lambda_*^2 + \lambda_*^3\right)\Bigg\}\\
\lesssim\ &\sum_{j=1}^{N}\Big[\left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha} + \|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^3\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\\
&\times\left(\lambda_*^{3\nu-3\delta_0-2} + |\ln(T-t)|\lambda_*^{\nu-\delta_0+\Theta-1}R + |\ln(T-t)|^2\lambda_*^{2\Theta}R^2\right)\\
&+ \mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^3\langle\rho_j\rangle^{-2}\Big]\\
&+ \mathbf{1}_{\{\cap_{j=1}^{N}\{|x-q^{[j]}|\ge 3d_q\}\}}\lambda_*^2\left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^3 \lesssim T^{\epsilon}\left(\sum_{j=1}^{N}\varrho_1^{[j]} + \varrho_3\right)
\end{aligned}
\tag{D.58}
$$

provided

$$
\Theta + \beta + 1 + 3\delta_0 - 3\nu < 0, \quad 2\beta + \delta_0 - \nu < 0, \quad 3\beta < 1 + \Theta. \tag{D.59}
$$

- By (D.37), we get

$$
|U_*\cdot\nabla_x U_*|^2\,|\Phi| \lesssim T^{\epsilon}\varrho_3.
$$

-

$$
\begin{aligned}
&|\nabla_x \Phi|^2 |\Phi| \\
\lesssim &\left\{ \sum_{j=1}^{N} \left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha} + \|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2 \left[\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}} \left(\lambda_*^{2\nu-2\delta_0-2}\langle\rho_j\rangle^{-2l-2}+1\right)\right.\right. \\
&\left.\left. + \mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}} \left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2 \right] + \mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}} \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}^2 \right\} \\
&\times \left\{ \sum_{j=1}^{N} \left[\left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha} + \|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right) \mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}} \left(\lambda_*^{\nu-\delta_0}\langle\rho_j\rangle^{-l} + \lambda_*\langle\rho_j\rangle + |\ln(T-t)|\lambda_*^{\Theta+1}R\right)\right.\right. \\
&\left. + \mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}} \left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right) \left(\lambda_*\langle\rho_j\rangle + |\ln(T-t)|\lambda_*^{\Theta+1}R\right) \right] \\
&\left. + \mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}} \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha} \right\} \\
\lesssim &\sum_{j=1}^{N} \left[\left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha} + \|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^3 \mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}} \left(\lambda_*\langle\rho_j\rangle + \lambda_*^{3\nu-3\delta_0-2} + |\ln(T-t)|\lambda_*^{2\nu-2\delta_0+\Theta-1}R\right)\right. \\
&\left. + \mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}} \left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^3 \left(\lambda_*\langle\rho_j\rangle + |\ln(T-t)|\lambda_*^{\Theta+1}R\right) \right] + \mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}} \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}^3 \\
\lesssim &T^{\epsilon} \left(\sum_{j=1}^{N} \varrho_1^{[j]} + \varrho_3\right)
\end{aligned}
\tag{D.60}
$$

provided

$$
\Theta + \beta + 1 + 3\delta_0 - 3\nu < 0, \quad \beta + \delta_0 - \nu < 0. \tag{D.61}
$$

- By (D.38), we have

$$
\begin{aligned}
&\left|\Phi \cdot \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 U^{[j]}\right| |\Phi| \le \left|\sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \Phi \cdot \left(U^{[j]} - U_*\right)\right| |\Phi| + \left|\sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \Phi \cdot U_*\right| |\Phi| \\
\lesssim &|\Phi|^2 \sum_{j=1}^{N} \lambda_j^{-2}\langle\rho_j\rangle^{-4} \sum_{k\ne j}\langle\rho_k\rangle^{-1} + |\Phi\cdot U_*|\,|\Phi| \sum_{j=1}^{N} \lambda_*^{-2}\langle\rho_j\rangle^{-4} \lesssim T^{\epsilon}\left(\sum_{j=1}^{N} \varrho_1^{[j]} + \varrho_3\right)
\end{aligned}
$$

where the last step is derived by the same way as (D.58) under the parameter assumption (D.59).

• By (D.39) and (D.9), we get

$$
\begin{aligned}
&\left||\nabla_x U_*|^2 + U_*\cdot\Delta_x U_*\right||\Phi| \\
\lesssim &\left(\sum_{j=1}^N \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\left(\langle\rho_j\rangle^{-2}+\lambda_*^{-1}\langle\rho_j\rangle^{-4}\right)+\lambda_*^2\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\right) \\
&\times\Bigg\{\sum_{j=1}^N\Bigg[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\left(\lambda_*^{\nu-\delta_0}\langle\rho_j\rangle^{-l}+\lambda_*\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{\Theta+1}R\right) \\
&+\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\left(\lambda_*\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{\Theta+1}R\right)\Bigg] \\
&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\Bigg\} \\
\lesssim &\sum_{j=1}^N\Bigg[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\left(\lambda_*^{\nu-\delta_0-1}+|\ln(T-t)|\lambda_*^{\Theta}R\right) \\
&+\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right) \\
&\times\left(\lambda_*\langle\rho_j\rangle^{-1}+|\ln(T-t)|\lambda_*^{\Theta+1}R\langle\rho_j\rangle^{-2}+\langle\rho_j\rangle^{-3}+|\ln(T-t)|\lambda_*^{\Theta}R\langle\rho_j\rangle^{-4}\right)\Bigg] \\
&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\lambda_*^2\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\lesssim T^{\epsilon}\left(\sum_{j=1}^N\varrho_1^{[j]}+\varrho_3\right)
\end{aligned}
\tag{D.62}
$$

provided

$$
\Theta+\beta+\delta_0-\nu<0. \tag{D.63}
$$

• Combining (D.9), (D.17) and (D.33), one has

$$
\begin{aligned}
&(\lambda_*+|\Phi|)\left|\nabla_x(\Phi\cdot U_*)\right||\nabla_x U_*| \\
\lesssim &\Bigg\{\sum_{j=1}^N\Bigg[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\left(\lambda_*^{\nu-\delta_0}\langle\rho_j\rangle^{-l}+\lambda_*\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{\Theta+1}R\right) \\
&+\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\left(\lambda_*\langle\rho_j\rangle+|\ln(T-t)|\lambda_*^{\Theta+1}R\right)\Bigg] \\
&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\left(\lambda_*+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\Bigg\} \\
&\times\Bigg\{\sum_{j=1}^N\Bigg[\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\left(|\ln(T-t)|\lambda_*^{\Theta}R\langle\rho_j\rangle^{-2}+1\right) \\
&+\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\Bigg]+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\Bigg\} \\
&\times\left(\sum_{j=1}^N\mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\lambda_*^{-1}\langle\rho_j\rangle^{-2}+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\lambda_*\right) \\
\lesssim &\sum_{j=1}^N\Bigg[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\left(|\ln(T-t)|\lambda_*^{\nu-\delta_0+\Theta-1}R+|\ln(T-t)|^2\lambda_*^{2\Theta}R^2\right) \\
&+\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\left(\langle\rho_j\rangle^{-1}+|\ln(T-t)|\lambda_*^{\Theta}R\langle\rho_j\rangle^{-2}\right)\Bigg] \\
&+\mathbf{1}_{\{\cap_{j=1}^N\{|x-q^{[j]}|\ge 3d_q\}\}}\lambda_*\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\lesssim T^{\epsilon}\left(\sum_{j=1}^N\varrho_1^{[j]}+\varrho_3\right)
\end{aligned}
$$

provided

$$
2\beta+\delta_0-\nu<0,\quad 3\beta<1+\Theta. \tag{D.64}
$$

• By (D.33) and (D.37), we have

$$
\begin{aligned}
&|\Phi|\left|\left(U_* \cdot \nabla_x U_*\right) \nabla_x U_*\right| \\
\lesssim\ & |\Phi|\left(\sum_{j=1}^N \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\langle\rho_j\rangle^{-2}+\mathbf{1}_{\{\cap_{j=1}^N |x-q^{[j]}|\geq 3d_q\}}\lambda_*^2\right) \\
&\times\left(\sum_{j=1}^N \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\lambda_*^{-1}\langle\rho_j\rangle^{-2}+\mathbf{1}_{\{\cap_{j=1}^N \{|x-q^{[j]}|\geq 3d_q\}\}}\lambda_*\right) \\
=\ & |\Phi|\left(\sum_{j=1}^N \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\lambda_*^{-1}\langle\rho_j\rangle^{-4}+\mathbf{1}_{\{\cap_{j=1}^N \{|x-q^{[j]}|\geq 3d_q\}\}}\lambda_*^3\right) \lesssim T^\epsilon\left(\sum_{j=1}^N \varrho_1^{[j]}+\varrho_3\right)
\end{aligned}
$$

which is obtained by the same calculation as in (D.62) under the parameter assumption (D.63).

• By (D.9), (D.10) and (D.33), we get

$$
\begin{aligned}
&|\Phi|\left|\left(\Phi \cdot \nabla_x \Phi\right) \nabla_x U_*\right| \\
\lesssim\ & \Bigg\{\sum_{j=1}^N \Bigg[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2 \mathbf{1}_{\{|x-q^{[j]}|\leq 3\lambda_* R\}} \\
&\times\left(\lambda_*^{2\nu-2\delta_0}\langle\rho_j\rangle^{-2l}+\lambda_*^2\langle\rho_j\rangle^2+|\ln(T-t)|^2\lambda_*^{2\Theta+2}R^2\right) \\
&+\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2\left(\lambda_*^2\langle\rho_j\rangle^2+|\ln(T-t)|^2\lambda_*^{2\Theta+2}R^2\right)\Bigg] \\
&+\mathbf{1}_{\{\cap_{j=1}^N \{|x-q^{[j]}|\geq 3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}^2\Bigg\} \\
&\times\Bigg\{\sum_{j=1}^N\Bigg[\mathbf{1}_{\{|x-q^{[j]}|\leq 3\lambda_* R\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)\left(\lambda_*^{\nu-\delta_0-1}\langle\rho_j\rangle^{-l-1}+1\right) \\
&+\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\Bigg]+\mathbf{1}_{\{\cap_{j=1}^N \{|x-q^{[j]}|\geq 3d_q\}\}}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\Bigg\} \\
&\times\left(\sum_{j=1}^N \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}}\lambda_*^{-1}\langle\rho_j\rangle^{-2}+\mathbf{1}_{\{\cap_{j=1}^N \{|x-q^{[j]}|\geq 3d_q\}\}}\lambda_*\right) \\
\lesssim\ & \sum_{j=1}^N\Bigg[\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^3 \mathbf{1}_{\{|x-q^{[j]}|\leq 3\lambda_* R\}}\left(\lambda_*^{3\nu-3\delta_0-2}+|\ln(T-t)|^2\lambda_*^{2\Theta+\nu-\delta_0}R^2\right) \\
&+\mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}}\left(1+\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^3\lambda_*\Bigg]+\mathbf{1}_{\{\cap_{j=1}^N \{|x-q^{[j]}|\geq 3d_q\}\}}\lambda_*\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}^3 \\
\lesssim\ & T^\epsilon\left(\sum_{j=1}^N \varrho_1^{[j]}+\varrho_3\right)
\end{aligned}
$$

provided

$$
\Theta+\beta+1+3\delta_0-3\nu<0, \quad 3\beta<\Theta+1+\nu-\delta_0. \tag{D.65}
$$

• By (D.83), we have

$$
\left|U_* \wedge\left[\Delta_x U_*-2\left(U_* \cdot \nabla_x U_*\right) \nabla_x U_*\right]\right| \lesssim T^\epsilon\left[\sum_{j=1}^N\left(\varrho_1^{[j]}+\varrho_2^{[j]}\right)+\varrho_3\right].
$$

• Consider

$$
\begin{aligned}
&2\left(a-bU_*\wedge\right)\Bigg[\left(\nabla_x U_*\cdot\nabla_x\Phi\right)\Phi-\left(\Phi\cdot\nabla_x\Phi\right)\nabla_x U_*\\
&-\sum_{j=1}^{N}\left\{\left[\nabla_x U_*\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)-\left[\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\nabla_x U_*\right\}\\
&+\sum_{j=1}^{N}\left\{\left[\nabla_x U_*\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)-\left[\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\nabla_x U_*\right\}\\
&-\sum_{j=1}^{N}\left\{\left[\nabla_x U^{[j]}\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)-\left[\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\nabla_x U^{[j]}\right\}\Bigg]\\
&+2\left(a-bU_*\wedge\right)\sum_{j=1}^{N}\left\{\left[\nabla_x U^{[j]}\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)-\left[\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\nabla_x U^{[j]}\right\}\\
&-\sum_{j=1}^{N}2\left(a-bU^{[j]}\wedge\right)\left\{\left[\nabla_x U^{[j]}\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)-\left[\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\nabla_x U^{[j]}\right\}.
\end{aligned}
$$

We estimate by (D.5) that

$$
\begin{aligned}
&\left|\left[\nabla_x U_*\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)-\left[\nabla_x U^{[j]}\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right|\\
\lesssim\ &\lambda_*\left|\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right|\left|\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right|\\
\lesssim\ &\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\left(\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2\lambda_*^{2\nu-2\delta_0}\langle\rho_j\rangle^{-2l-1}\lesssim T^{\epsilon}\varrho_1^{[j]}.
\end{aligned}
$$

By (D.5),

$$
\begin{aligned}
&\left(U_*-U^{[j]}\right)\wedge\left[\nabla_x U^{[j]}\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\\
\lesssim\ &\mathbf{1}_{\{|x-q^{[j]}|\le 3\lambda_* R\}}\left(\|\Phi_{\rm in}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2\lambda_*^{2\nu-2\delta_0-1}\langle\rho_j\rangle^{-2l-3}\lesssim T^{\epsilon}\varrho_1^{[j]}
\end{aligned}
$$

provided

$$
\Theta+\beta+2\delta_0-2\nu<0. \tag{D.66}
$$

By the property of cut-off function,

$$
\sum_{j=1}^{N}\left[\nabla_x U_*\cdot\nabla_x\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\left(\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)=\left[\nabla_x U_*\cdot\nabla_x\left(\sum_{j=1}^{N}\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right)\right]\left(\sum_{j=1}^{N}\eta_R^{[j]}Q_{\gamma_j}\Phi_{\rm in}^{[j]}\right).
$$

Then by (D.33) and (D.10), it follows that

$$
\begin{aligned}
&\left|(\nabla_x U_* \cdot \nabla_x \Phi)\, \Phi - \left[\nabla_x U_* \cdot \nabla_x \left(\sum_{j=1}^{N} \eta_R^{[j]} Q_{\gamma_j} \Phi_{\rm in}^{[j]}\right)\right] \left(\sum_{j=1}^{N} \eta_R^{[j]} Q_{\gamma_j} \Phi_{\rm in}^{[j]}\right)\right| \\
= &\left|(\nabla_x U_* \cdot \nabla_x \Phi) \left(\Phi - \sum_{j=1}^{N} \eta_R^{[j]} Q_{\gamma_j} \Phi_{\rm in}^{[j]}\right) + \left[\nabla_x U_* \cdot \nabla_x \left(\Phi - \sum_{j=1}^{N} \eta_R^{[j]} Q_{\gamma_j} \Phi_{\rm in}^{[j]}\right)\right] \left(\sum_{j=1}^{N} \eta_R^{[j]} Q_{\gamma_j} \Phi_{\rm in}^{[j]}\right)\right| \\
\lesssim &\left(\sum_{j=1}^{N} \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}} \lambda_*^{-1} \langle \rho_j\rangle^{-2} + \mathbf{1}_{\{\cap_{j=1}^N \{|x-q^{[j]}|\geq 3d_q\}\}} \lambda_*\right) \\
&\times \Bigg[\Bigg\{\sum_{j=1}^{N} \Big[\mathbf{1}_{\{|x-q^{[j]}|\leq 3\lambda_* R\}} \left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha} + \|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right) \left(\lambda_*^{\nu-\delta_0-1} \langle\rho_j\rangle^{-l-1} + 1\right) \\
&+ \mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}} \left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\Big] + \mathbf{1}_{\{\cap_{j=1}^N \{|x-q^{[j]}|\geq 3d_q\}\}} \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\Bigg\} \\
&\times \Bigg\{\sum_{j=1}^{N} \Big[\left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right) \mathbf{1}_{\{|x-q^{[j]}|\leq 3\lambda_* R\}} \left(\lambda_* \langle\rho_j\rangle + |\ln(T-t)| \lambda_*^{\Theta+1} R\right) \\
&+ \mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}} \left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right) \left(\lambda_* \langle\rho_j\rangle + |\ln(T-t)| \lambda_*^{\Theta+1} R\right)\Big] \\
&+ \mathbf{1}_{\{\cap_{j=1}^N \{|x-q^{[j]}|\geq 3d_q\}\}} \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\Bigg\} \\
&+ \Bigg\{\sum_{j=1}^{N} \Big[\mathbf{1}_{\{|x-q^{[j]}|\leq 3\lambda_* R\}} \left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right) \\
&+ \mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}} \left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)\Big] + \mathbf{1}_{\{\cap_{j=1}^N \{|x-q^{[j]}|\geq 3d_q\}\}} \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\Bigg\} \\
&\times \left(\sum_{j=1}^{N} \mathbf{1}_{\{|x-q^{[j]}|\leq 3\lambda_* R\}} \|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]} \lambda_*^{\nu-\delta_0} \langle\rho_j\rangle^{-l}\right)\Bigg] \\
\lesssim &\left(\sum_{j=1}^{N} \mathbf{1}_{\{|x-q^{[j]}|<3d_q\}} \lambda_*^{-1} \langle \rho_j\rangle^{-2} + \mathbf{1}_{\{\cap_{j=1}^N \{|x-q^{[j]}|\geq 3d_q\}\}} \lambda_*\right) \\
&\times \Bigg\{\sum_{j=1}^{N} \Big[\left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha} + \|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l}^{[j]}\right)^2 \mathbf{1}_{\{|x-q^{[j]}|\leq 3\lambda_* R\}} \left(\lambda_* \langle\rho_j\rangle + |\ln(T-t)| \lambda_*^{\nu-\delta_0+\Theta} R\right) \\
&+ \mathbf{1}_{\{3\lambda_* R<|x-q^{[j]}|<3d_q\}} \left(1 + \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\right)^2 \left(\lambda_* \langle\rho_j\rangle + |\ln(T-t)| \lambda_*^{\Theta+1} R\right)\Big] + \mathbf{1}_{\{\cap_{j=1}^N \{|x-q^{[j]}|\geq 3d_q\}\}} \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}^2\Bigg\} \\
\lesssim &\, T^{\epsilon} \left(\sum_{j=1}^{N} \varrho_1^{[j]} + \varrho_3\right)
\end{aligned}
$$

provided

$$2\beta + \delta_0 - \nu < 0. \tag{D.67}$$

The other terms in this collection can be dealt in the same way.

- Before proceeding, we take a closer look at $-\Delta_x U_*$ and $\nabla_x(|U_*|^2)\nabla_x U_*$.

In the single bubble case $N = 1$, then $-\Delta_x U_*$ can be neglected by the U_*-operation and $\nabla_x(|U_*|^2)\nabla_x U_*$ vanishes automatically. But for case of multiple bubbles, the phenomenon is different, and interactions appear here.

Recall (3.12). Claim:

$$\begin{aligned}
&\Delta_x U_* = -\sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 U^{[j]} \\
&= \sum_{j=1}^{N}\sum_{k\neq j} \Big\{ 16\eta_R^{[j]} \lambda_j^{-2}\rho_j^2(\rho_j^2+1)^{-3}\lambda_k |q^{[j]}-q^{[k]}|^{-2} \left[q_1^{[j]} - q_1^{[k]} + i\left(q_2^{[j]}-q_2^{[k]}\right)\right] e^{i(\gamma_k-\gamma_j)} e^{-i\theta_j} \\
&\quad - 16\eta_R^{[j]} \lambda_j^{-2}(\rho_j^2+1)^{-3}\lambda_k |q^{[j]}-q^{[k]}|^{-2} \left[q_1^{[j]} - q_1^{[k]} - i\left(q_2^{[j]}-q_2^{[k]}\right)\right] e^{-i(\gamma_k-\gamma_j)} e^{i\theta_j} \Big\}_{\mathcal{C}_j^{-1}} \\
&\quad + O(T^\epsilon)\left[\sum_{j=1}^{N} \left(\varrho_1^{[j]} + \varrho_2^{[j]}\right) + \varrho_3 \right] - \Xi_1(x,t) U_*
\end{aligned} \tag{D.68}$$

for some scalar function $\Xi_1(x,t)$ when

$$\Theta + 2\beta - 1 < 0. \tag{D.69}$$

Under the assumption (D.69), then

$$\begin{aligned}
&\nabla_x(|U_*|^2)\nabla_x U_* = 2\left(U_*\cdot \nabla_x U_*\right)\nabla_x U_* \\
&= \sum_{j=1}^{N}\sum_{m\neq j} \Big\{ 16\eta_R^{[j]} \lambda_j^{-2}\rho_j^2(\rho_j^2+1)^{-3}\lambda_m |q^{[j]}-q^{[m]}|^{-2} \left[q_1^{[j]} - q_1^{[m]} + i\left(q_2^{[j]}-q_2^{[m]}\right)\right] e^{i(\gamma_m-\gamma_j)} e^{-i\theta_j} \\
&\quad - 16\eta_R^{[j]} \lambda_j^{-2}(\rho_j^2+1)^{-3}\lambda_m |q^{[j]}-q^{[m]}|^{-2} \left[q_1^{[j]} - q_1^{[m]} - i\left(q_2^{[j]}-q_2^{[m]}\right)\right] e^{-i(\gamma_m-\gamma_j)} e^{i\theta_j} \Big\}_{\mathcal{C}_j^{-1}} \\
&\quad + O(T^\epsilon)\left[\sum_{j=1}^{N} \left(\varrho_1^{[j]} + \varrho_2^{[j]}\right) + \varrho_3 \right].
\end{aligned} \tag{D.70}$$

Proof of (D.68).

$$\sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 U^{[j]} = \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \left(U^{[j]} - U_* + U_*\right) = \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \left(U^{[j]} - U_*\right) + \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 U_*,$$

where

$$\left| \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \left(U^{[j]} - U_*\right) \right| \lesssim \sum_{j=1}^{N} \lambda_j^{-2} \langle y^{[j]}\rangle^{-4} \sum_{k\neq j} \langle y^{[k]}\rangle^{-1}.$$

Then

$$\begin{aligned}
&\left(1-\sum_{j=1}^{N}\eta_R^{[j]}\right)\left| \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \left(U^{[j]} - U_*\right) \right| \lesssim \left(1-\sum_{j=1}^{N}\eta_R^{[j]}\right)\left(\sum_{j=1}^{N} \lambda_j^{-2} \langle y^{[j]}\rangle^{-4} \sum_{k\neq j} \langle y^{[k]}\rangle^{-1} \right) \\
&\lesssim \left(1-\sum_{j=1}^{N}\eta_R^{[j]}\right)\left(\sum_{j=1}^{N}\sum_{k\neq j} \lambda_*^3 |x-q^{[j]}|^{-4} |x-q^{[k]}|^{-1} \right) \\
&\lesssim \left(1-\sum_{j=1}^{N}\eta_R^{[j]}\right)\left(\sum_{j=1}^{N} \lambda_*^3 (\lambda_* R)^{-2} |x-q^{[j]}|^{-2} \right) = \lambda_* R^{-2} \left(1-\sum_{j=1}^{N}\eta_R^{[j]}\right)\left(\sum_{j=1}^{N} |x-q^{[j]}|^{-2} \right) \\
&\lesssim T^\epsilon \left(\sum_{j=1}^{N} \varrho_2^{[j]} + \varrho_3 \right)
\end{aligned}$$

where we have used

$$|x-q^{[j]}|^{-1}|x-q^{[k]}|^{-1} \lesssim \max\{|x-q^{[j]}|^{-1}, |x-q^{[k]}|^{-1}\} \ \text{ for } \ j\neq k. \tag{D.71}$$

For any fixed $j = 1, \dots, N$,

$$\eta_R^{[j]} \left| \sum_{j=1}^{N} |\nabla_x U^{[j]}|^2 \left(U^{[j]} - U_*\right) \right| \lesssim \eta_R^{[j]} \left(\lambda_*^{-1} \langle y^{[j]} \rangle^{-4} + \sum_{m \neq j} \lambda_m^{-2} \langle y^{[m]} \rangle^{-4} \sum_{k \neq m} \langle y^{[k]} \rangle^{-1} \right)$$
$$\lesssim \eta_R^{[j]} \left(\lambda_*^{-1} \langle y^{[j]} \rangle^{-4} + \lambda_*^2 (\langle y^{[j]} \rangle^{-1} + \lambda_*) \right) \lesssim \eta_R^{[j]} \lambda_*^{-1} \langle y^{[j]} \rangle^{-4}.$$

This estimate is too rough that can not be controlled by the outer topology. More sophisticated analysis will be applied. Indeed,

$$|\nabla_x U^{[j]}|^2 \left(U^{[j]} - U_*\right) = -2\lambda_j^{-2} |\nabla_{y^{[j]}} U^{[j]}|^2 \sum_{k \neq j} (|y^{[k]}|^2 + 1)^{-1} Q_{\gamma_k} \left[y_1^{[k]}, y_2^{[k]}, -1\right]^T$$
$$= -16\lambda_j^{-2} (\rho_j^2 + 1)^{-2} \sum_{k \neq j} (\rho_k^2 + 1)^{-1} \left[e^{i\gamma_k} \left(y_1^{[k]} + i y_2^{[k]} \right), -1 \right]^T.$$

Then

$$\eta_R^{[j]} \sum_{m \neq j} |\nabla_x U^{[m]}|^2 \left(U^{[m]} - U_*\right) = \eta_R^{[j]} \sum_{m \neq j} O(\lambda_m^2) \left(\langle \rho_j \rangle^{-1} + \sum_{k \neq m, j} \lambda_k \right) = \eta_R^{[j]} O(\lambda_*^2) \langle \rho_j \rangle^{-1} \lesssim T^{\epsilon} \varrho_1^{[j]}.$$

$$\eta_R^{[j]} |\nabla_x U^{[j]}|^2 \left(U^{[j]} - U_*\right) = -16 \eta_R^{[j]} \lambda_j^{-2} (\rho_j^2 + 1)^{-2} \sum_{k \neq j} (\rho_k^2 + 1)^{-1} \left[e^{i\gamma_k} \left(y_1^{[k]} + i y_2^{[k]} \right), -1 \right]^T.$$

and for $k \neq j$,

$$\rho_k^2 = \lambda_k^{-2} |x - \xi^{[k]}|^2 = \lambda_k^{-2} \left[|x - \xi^{[j]}|^2 + 2(x - \xi^{[j]}) \cdot (\xi^{[j]} - \xi^{[k]}) + |\xi^{[j]} - \xi^{[k]}|^2 \right]$$
$$= \lambda_k^{-2} |\xi^{[j]} - \xi^{[k]}|^2 \left\{ 1 + |\xi^{[j]} - \xi^{[k]}|^{-2} \left[|x - \xi^{[j]}|^2 + 2(x - \xi^{[j]}) \cdot (\xi^{[j]} - \xi^{[k]}) \right] \right\}.$$

Then

$$\left(\rho_k^2 + 1\right)^{-1} = \lambda_k^2 |\xi^{[j]} - \xi^{[k]}|^{-2} \left\{ 1 + |\xi^{[j]} - \xi^{[k]}|^{-2} \left[\lambda_k^2 + |x - \xi^{[j]}|^2 + 2(x - \xi^{[j]}) \cdot (\xi^{[j]} - \xi^{[k]}) \right] \right\}^{-1}.$$

Specially,

$$\eta_R^{[j]} \left(\rho_k^2 + 1\right)^{-1} = \eta_R^{[j]} \lambda_k^2 |\xi^{[j]} - \xi^{[k]}|^{-2} \left(1 + O\left(\lambda_k^2 + \lambda_j R \right) \right), \tag{D.72}$$

which implies

$$\eta_R^{[j]} |\nabla_x U^{[j]}|^2 \left(U^{[j]} - U_*\right)$$
$$= -16 \eta_R^{[j]} \lambda_j^{-2} (\rho_j^2 + 1)^{-2} \sum_{k \neq j} \lambda_k^2 |\xi^{[j]} - \xi^{[k]}|^{-2} \left(1 + O\left(\lambda_k^2 + \lambda_j R \right) \right) \left[e^{i\gamma_k} \left(y_1^{[k]} + i y_2^{[k]} \right), -1 \right]^T. \tag{D.73}$$

Notice

$$y_1^{[k]} + i y_2^{[k]} = \left[\lambda_j y_1^{[j]} + \xi_1^{[j]} - \xi_1^{[k]} + i \left(\lambda_j y_2^{[j]} + \xi_2^{[j]} - \xi_2^{[k]} \right) \right] \lambda_k^{-1}$$
$$= \lambda_j \lambda_k^{-1} \rho_j e^{i\theta_j} + \lambda_k^{-1} \left[\xi_1^{[j]} - \xi_1^{[k]} + i \left(\xi_2^{[j]} - \xi_2^{[k]} \right) \right].$$

Then by (3.14), we have

$$\left(\Pi_{U^{[j]\perp}} \left[e^{i\gamma_k} \left(y_1^{[k]} + i y_2^{[k]} \right), -1 \right]^T \right)_{\mathcal{C}_j} = \left(1 - \frac{2}{\rho_j^2 + 1} \mathrm{Re} \right) \left[\left(y_1^{[k]} + i y_2^{[k]} \right) e^{i(-\theta_j + \gamma_k - \gamma_j)} \right] + \frac{2\rho_j}{\rho_j^2 + 1}$$
$$= \lambda_j \lambda_k^{-1} \rho_j \left(1 - \frac{2}{\rho_j^2 + 1} \mathrm{Re} \right) \left[e^{i(\gamma_k - \gamma_j)} \right] + \frac{2\rho_j}{\rho_j^2 + 1} \tag{D.74}$$
$$+ \lambda_k^{-1} \left(1 - \frac{2}{\rho_j^2 + 1} \mathrm{Re} \right) \left\{ \left[\xi_1^{[j]} - \xi_1^{[k]} + i \left(\xi_2^{[j]} - \xi_2^{[k]} \right) \right] e^{i(-\theta_j + \gamma_k - \gamma_j)} \right\},$$

$$\left[e^{i\gamma_k} \left(y_1^{[k]} + i y_2^{[k]} \right), -1 \right]^T \cdot U^{[j]} = \frac{2\rho_j}{\rho_j^2 + 1} \mathrm{Re} \left[\left(y_1^{[k]} + i y_2^{[k]} \right) e^{i(-\theta_j + \gamma_k - \gamma_j)} \right] - \frac{\rho_j^2 - 1}{\rho_j^2 + 1}$$
$$= \lambda_j \lambda_k^{-1} \frac{2\rho_j^2}{\rho_j^2 + 1} \mathrm{Re} \left[e^{i(\gamma_k - \gamma_j)} \right] - \frac{\rho_j^2 - 1}{\rho_j^2 + 1} + \lambda_k^{-1} \frac{2\rho_j}{\rho_j^2 + 1} \mathrm{Re} \left\{ \left[\xi_1^{[j]} - \xi_1^{[k]} + i \left(\xi_2^{[j]} - \xi_2^{[k]} \right) \right] e^{i(-\theta_j + \gamma_k - \gamma_j)} \right\}.$$

Then

$$\begin{aligned}
&\eta_R^{[j]}\left\{\Pi_{U^{[j]\perp}}\left[|\nabla_x U^{[j]}|^2\left(U^{[j]}-U_*\right)\right]\right\}_{\mathcal{C}_j}\\
=&-16\eta_R^{[j]}\lambda_j^{-2}(\rho_j^2+1)^{-2}\sum_{k\neq j}\lambda_k^2|\xi^{[j]}-\xi^{[k]}|^{-2}\left(1+O\left(\lambda_k^2+\lambda_j R\right)\right)\\
&\times\left[\lambda_j\lambda_k^{-1}\rho_j\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[e^{i(\gamma_k-\gamma_j)}\right]+\frac{2\rho_j}{\rho_j^2+1}\right.\\
&\left.+\lambda_k^{-1}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\left[\xi_1^{[j]}-\xi_1^{[k]}+i\left(\xi_2^{[j]}-\xi_2^{[k]}\right)\right]e^{i(-\theta_j+\gamma_k-\gamma_j)}\right\}\right]\\
=&-16\eta_R^{[j]}\lambda_j^{-2}(\rho_j^2+1)^{-2}\sum_{k\neq j}\lambda_k^2|\xi^{[j]}-\xi^{[k]}|^{-2}\left(1+O\left(\lambda_k^2+\lambda_j R\right)\right)\\
&\times\left\{\lambda_j\lambda_k^{-1}\rho_j\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[e^{i(\gamma_k-\gamma_j)}\right]+\frac{2\rho_j}{\rho_j^2+1}\right\}\\
&-16\eta_R^{[j]}\lambda_j^{-2}(\rho_j^2+1)^{-2}\sum_{k\neq j}\lambda_k|\xi^{[j]}-\xi^{[k]}|^{-2}\left(1+O\left(\lambda_k^2+\lambda_j R\right)\right)\\
&\times\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\left[\xi_1^{[j]}-\xi_1^{[k]}+i\left(\xi_2^{[j]}-\xi_2^{[k]}\right)\right]e^{i(-\theta_j+\gamma_k-\gamma_j)}\right\},
\end{aligned}\tag{D.75}$$

where

$$\begin{aligned}
&\Big|16\eta_R^{[j]}\lambda_j^{-2}(\rho_j^2+1)^{-2}\sum_{k\neq j}\lambda_k^2|\xi^{[j]}-\xi^{[k]}|^{-2}\left(1+O\left(\lambda_k^2+\lambda_j R\right)\right)\\
&\times\left\{\lambda_j\lambda_k^{-1}\rho_j\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[e^{i(\gamma_k-\gamma_j)}\right]+\frac{2\rho_j}{\rho_j^2+1}\right\}\Big|\lesssim\eta_R^{[j]}\langle\rho_j\rangle^{-3}\lesssim T^{\epsilon}\varrho_1^{[j]},\\
&\Big|16\eta_R^{[j]}\lambda_j^{-2}(\rho_j^2+1)^{-2}\sum_{k\neq j}\lambda_k|\xi^{[j]}-\xi^{[k]}|^{-2}\left(O\left(\lambda_k^2+\lambda_j R\right)\right)\\
&\times\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\left[\xi_1^{[j]}-\xi_1^{[k]}+i\left(\xi_2^{[j]}-\xi_2^{[k]}\right)\right]e^{i(-\theta_j+\gamma_k-\gamma_j)}\right\}\Big|\lesssim T^{\epsilon}\varrho_1^{[j]},
\end{aligned}$$

provided

$$\Theta+2\beta-1<0.\tag{D.76}$$

$$\begin{aligned}
&\Big|-16\eta_R^{[j]}\lambda_j^{-2}(\rho_j^2+1)^{-2}\sum_{k\neq j}\lambda_k|\xi^{[j]}-\xi^{[k]}|^{-2}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\left[\xi_1^{[j]}-\xi_1^{[k]}+i\left(\xi_2^{[j]}-\xi_2^{[k]}\right)\right]e^{i(-\theta_j+\gamma_k-\gamma_j)}\right\}\\
&+16\eta_R^{[j]}\lambda_j^{-2}(\rho_j^2+1)^{-2}\sum_{k\neq j}\lambda_k|q^{[j]}-q^{[k]}|^{-2}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\left[q_1^{[j]}-q_1^{[k]}+i\left(q_2^{[j]}-q_2^{[k]}\right)\right]e^{i(-\theta_j+\gamma_k-\gamma_j)}\right\}\Big|\\
&\lesssim\eta_R^{[j]}|\ln T|^{-1}\ln^2(T-t)\langle\rho_j\rangle^{-4}\lesssim T^{\epsilon}\varrho_1^{[j]}.
\end{aligned}\tag{D.77}$$

$$
\begin{aligned}
& -16\eta_R^{[j]}\lambda_j^{-2}(\rho_j^2+1)^{-2}\sum_{k\neq j}\lambda_k|q^{[j]}-q^{[k]}|^{-2}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\left[q_1^{[j]}-q_1^{[k]}+i\left(q_2^{[j]}-q_2^{[k]}\right)\right]e^{i(-\theta_j+\gamma_k-\gamma_j)}\right\} \\
= & -16\eta_R^{[j]}\lambda_j^{-2}(\rho_j^2+1)^{-2}\sum_{k\neq j}\lambda_k|q^{[j]}-q^{[k]}|^{-2}\left[q_1^{[j]}-q_1^{[k]}+i\left(q_2^{[j]}-q_2^{[k]}\right)\right]e^{i(\gamma_k-\gamma_j)}e^{-i\theta_j} \\
& +16\eta_R^{[j]}\lambda_j^{-2}(\rho_j^2+1)^{-3}\sum_{k\neq j}\lambda_k|q^{[j]}-q^{[k]}|^{-2} \\
& \times\left\{\left[q_1^{[j]}-q_1^{[k]}+i\left(q_2^{[j]}-q_2^{[k]}\right)\right]e^{i(\gamma_k-\gamma_j)}e^{-i\theta_j}+\left[q_1^{[j]}-q_1^{[k]}-i\left(q_2^{[j]}-q_2^{[k]}\right)\right]e^{-i(\gamma_k-\gamma_j)}e^{i\theta_j}\right\} \\
= & -16\eta_R^{[j]}\lambda_j^{-2}\rho_j^2(\rho_j^2+1)^{-3}\sum_{k\neq j}\lambda_k|q^{[j]}-q^{[k]}|^{-2}\left[q_1^{[j]}-q_1^{[k]}+i\left(q_2^{[j]}-q_2^{[k]}\right)\right]e^{i(\gamma_k-\gamma_j)}e^{-i\theta_j} \\
& +16\eta_R^{[j]}\lambda_j^{-2}(\rho_j^2+1)^{-3}\sum_{k\neq j}\lambda_k|q^{[j]}-q^{[k]}|^{-2}\left[q_1^{[j]}-q_1^{[k]}-i\left(q_2^{[j]}-q_2^{[k]}\right)\right]e^{-i(\gamma_k-\gamma_j)}e^{i\theta_j}
\end{aligned}
\tag{D.78}
$$

which will be put into mode 1 and mode -1, respectively.

For the projection in $U^{[j]}$, we calculate

$$
\begin{aligned}
& \eta_R^{[j]}|\nabla_x U^{[j]}|^2\left(U^{[j]}-U_*\right)\cdot U^{[j]} \\
= & -16\eta_R^{[j]}\lambda_j^{-2}(\rho_j^2+1)^{-2}\sum_{k\neq j}\lambda_k^2|\xi^{[j]}-\xi^{[k]}|^{-2}\left(1+O\left(\lambda_k^2+\lambda_jR\right)\right) \\
& \times\left[\lambda_j\lambda_k^{-1}\frac{2\rho_j^2}{\rho_j^2+1}\mathrm{Re}\left[e^{i(\gamma_k-\gamma_j)}\right]-\frac{\rho_j^2-1}{\rho_j^2+1}+\lambda_k^{-1}\frac{2\rho_j}{\rho_j^2+1}\mathrm{Re}\left\{\left[\xi_1^{[j]}-\xi_1^{[k]}+i\left(\xi_2^{[j]}-\xi_2^{[k]}\right)\right]e^{i(-\theta_j+\gamma_k-\gamma_j)}\right\}\right].
\end{aligned}
\tag{D.79}
$$

Thus by U_*-operation,

$$
\left|\eta_R^{[j]}|\nabla_x U^{[j]}|^2\left[\left(U^{[j]}-U_*\right)\cdot U^{[j]}\right]\left(U^{[j]}-U_*\right)\right|\lesssim\eta_R^{[j]}\langle\rho_j\rangle^{-4}\lesssim T^\epsilon\varrho_1^{[j]}.
$$

In sum, we conclude the validity of (D.68). □

Proof of (D.70).

$$
\begin{aligned}
2\left(U_*\cdot\nabla_x U_*\right)\nabla_x U_* &= 2\left(\sum_{k=1}^N U_*\cdot\nabla_x U^{[k]}\right)\sum_{m=1}^N\nabla_x U^{[m]} \\
&= 2\left[\sum_{k=1}^N\sum_{n\neq k}\left(U^{[n]}-U_\infty\right)\cdot\nabla_x U^{[k]}\right]\sum_{m=1}^N\nabla_x U^{[m]}.
\end{aligned}
$$

By (D.71), we have

$$
\begin{aligned}
& \left(1-\sum_{j=1}^N\eta_R^{[j]}\right)\left|\left[\sum_{k=1}^N\sum_{n\neq k}\left(U^{[n]}-U_\infty\right)\cdot\nabla_x U^{[k]}\right]\sum_{m=1}^N\nabla_x U^{[m]}\right| \\
\lesssim & \left(1-\sum_{j=1}^N\eta_R^{[j]}\right)\sum_{k=1}^N\sum_{n\neq k}\sum_{m=1}^N\frac{\lambda_n}{|x-q^{[n]}|}\frac{\lambda_k}{|x-q^{[k]}|^2}\frac{\lambda_m}{|x-q^{[m]}|^2} \\
\lesssim & \left(1-\sum_{j=1}^N\eta_R^{[j]}\right)\sum_{k=1}^N\sum_{n\neq k}\sum_{m=1}^N\lambda_n\max\left\{|x-q^{[k]}|^{-1},|x-q^{[n]}|^{-1}\right\}\frac{\lambda_k}{|x-q^{[k]}|}\frac{\lambda_m}{|x-q^{[m]}|^2} \\
\lesssim & \left(1-\sum_{j=1}^N\eta_R^{[j]}\right)\sum_{m=1}^N R^{-2}\frac{\lambda_m}{|x-q^{[m]}|^2}\lesssim T^\epsilon\left(\sum_{j=1}^N\varrho_2^{[j]}+\varrho_3\right).
\end{aligned}
$$

For any fixed $j = 1, 2, \ldots, N$,

$$\left|\eta_R^{[j]} \left[\sum_{k=1}^{N} \sum_{n\neq k} \left(U^{[n]} - U_\infty\right) \cdot \nabla_x U^{[k]}\right] \sum_{m\neq j} \nabla_x U^{[m]}\right| \lesssim \eta_R^{[j]} \lesssim T^\epsilon \varrho_1^{[j]},$$

$$\left|\eta_R^{[j]} \left[\sum_{k\neq j} \sum_{n\neq k} \left(U^{[n]} - U_\infty\right) \cdot \nabla_x U^{[k]}\right] \nabla_x U^{[j]}\right| \lesssim \eta_R^{[j]} \lesssim T^\epsilon \varrho_1^{[j]}.$$

Thus we only need to focus on

$$2\eta_R^{[j]} \left[\sum_{m\neq j} \left(U^{[m]} - U_\infty\right) \cdot \nabla_x U^{[j]}\right] \nabla_x U^{[j]}.$$

For $m \neq j$,

$$\left[\left(U^{[m]} - U_\infty\right) \cdot \nabla_x U^{[j]}\right] \nabla_x U^{[j]} = \begin{bmatrix} \sum_{k=1}^{2} \left[\left(U^{[m]} - U_\infty\right) \cdot \partial_{x_k} U^{[j]}\right] \partial_{x_k} (U^{[j]})_1 \\ \sum_{k=1}^{2} \left[\left(U^{[m]} - U_\infty\right) \cdot \partial_{x_k} U^{[j]}\right] \partial_{x_k} (U^{[j]})_2 \\ \sum_{k=1}^{2} \left[\left(U^{[m]} - U_\infty\right) \cdot \partial_{x_k} U^{[j]}\right] \partial_{x_k} (U^{[j]})_3 \end{bmatrix}$$

$$= \lambda_j^{-2} \sum_{k=1}^{2} \left[\left(U^{[m]} - U_\infty\right) \cdot \partial_{y_k^{[j]}} U^{[j]}\right] \partial_{y_k^{[j]}} U^{[j]}.$$

Notice

$$\begin{aligned} \partial_{y_1^{[j]}} &= \frac{\partial \rho_j}{\partial y_1^{[j]}} \partial_{\rho_j} + \frac{\partial \theta_j}{\partial y_1^{[j]}} \partial_{\theta_j} = \cos\theta_j \partial_{\rho_j} - \frac{\sin\theta_j}{\rho_j} \partial_{\theta_j}, \\ \partial_{y_2^{[j]}} &= \frac{\partial \rho_j}{\partial y_2^{[j]}} \partial_{\rho_j} + \frac{\partial \theta_j}{\partial y_2^{[j]}} \partial_{\theta_j} = \sin\theta_j \partial_{\rho_j} + \frac{\cos\theta_j}{\rho_j} \partial_{\theta_j}. \end{aligned} \tag{D.80}$$

Then

$$\begin{aligned} \partial_{y_1^{[j]}} U^{[j]} =&\ \cos\theta_j \partial_{\rho_j} U^{[j]} - \frac{\sin\theta_j}{\rho_j} \partial_{\theta_j} U^{[j]} = \cos\theta_j w_{\rho_j} Q_{\gamma_j} E_1^{[j]} - \frac{\sin\theta_j}{\rho_j} \sin w(\rho_j) Q_{\gamma_j} E_2^{[j]} \\ =&\ -2(\rho_j^2+1)^{-1} \left(\cos\theta_j Q_{\gamma_j} E_1^{[j]} + \sin\theta_j Q_{\gamma_j} E_2^{[j]}\right), \\ \partial_{y_2^{[j]}} U^{[j]} =&\ \sin\theta_j \partial_{\rho_j} U^{[j]} + \frac{\cos\theta_j}{\rho_j} \partial_{\theta_j} U^{[j]} = \sin\theta_j w_{\rho_j} Q_{\gamma_j} E_1^{[j]} + \frac{\cos\theta_j}{\rho_j} \sin w(\rho_j) Q_{\gamma_j} E_2^{[j]} \\ =&\ -2(\rho_j^2+1)^{-1} \left(\sin\theta_j Q_{\gamma_j} E_1^{[j]} - \cos\theta_j Q_{\gamma_j} E_2^{[j]}\right). \end{aligned} \tag{D.81}$$

Thus

$$\begin{aligned} &\lambda_j^{-2} \sum_{k=1}^{2} \left[\left(U^{[m]} - U_\infty\right) \cdot \partial_{y_k^{[j]}} U^{[j]}\right] \partial_{y_k^{[j]}} U^{[j]} \\ =&\ 4\lambda_j^{-2} (\rho_j^2+1)^{-2} \Big\{ \left[\left(U^{[m]} - U_\infty\right) \cdot \left(\cos\theta_j Q_{\gamma_j} E_1^{[j]} + \sin\theta_j Q_{\gamma_j} E_2^{[j]}\right)\right] \left(\cos\theta_j Q_{\gamma_j} E_1^{[j]} + \sin\theta_j Q_{\gamma_j} E_2^{[j]}\right) \\ &+ \left[\left(U^{[m]} - U_\infty\right) \cdot \left(\sin\theta_j Q_{\gamma_j} E_1^{[j]} - \cos\theta_j Q_{\gamma_j} E_2^{[j]}\right)\right] \left(\sin\theta_j Q_{\gamma_j} E_1^{[j]} - \cos\theta_j Q_{\gamma_j} E_2^{[j]}\right) \Big\} \\ =&\ 4\lambda_j^{-2} (\rho_j^2+1)^{-2} \left\{ \left[\left(U^{[m]} - U_\infty\right) \cdot Q_{\gamma_j} E_1^{[j]}\right] Q_{\gamma_j} E_1^{[j]} + \left[\left(U^{[m]} - U_\infty\right) \cdot Q_{\gamma_j} E_2^{[j]}\right] Q_{\gamma_j} E_2^{[j]} \right\}. \end{aligned}$$

Notice

$$U^{[m]} - U_\infty = 2(|y^{[m]}|^2+1)^{-1} Q_{\gamma_m} \left[y_1^{[m]}, y_2^{[m]}, -1\right]^T = 2(\rho_m^2+1)^{-1} \left[e^{i\gamma_m} \left(y_1^{[m]} + i y_2^{[m]}\right), -1\right]^T.$$

Then it follows that

$$
\begin{aligned}
&\left(\lambda_j^{-2}\sum_{k=1}^{2}\left[\left(U^{[m]}-U_\infty\right)\cdot\partial_{y_k^{[j]}}U^{[j]}\right]\partial_{y_k^{[j]}}U^{[j]}\right)_{\mathcal{C}_j}\\
&=4\lambda_j^{-2}(\rho_j^2+1)^{-2}\left[\left(U^{[m]}-U_\infty\right)\cdot Q_{\gamma_j}E_1^{[j]}+i\left(U^{[m]}-U_\infty\right)\cdot Q_{\gamma_j}E_2^{[j]}\right]\\
&=8\lambda_j^{-2}(\rho_j^2+1)^{-2}(\rho_m^2+1)^{-1}\left(\Pi_{U^{[j]\perp}}\left[e^{i\gamma_m}\left(y_1^{[m]}+iy_2^{[m]}\right),-1\right]^T\right)_{\mathcal{C}_j}\\
&=8\lambda_j^{-2}(\rho_j^2+1)^{-2}(\rho_m^2+1)^{-1}\Bigg[\lambda_j\lambda_m^{-1}\rho_j\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[e^{i(\gamma_m-\gamma_j)}\right]+\frac{2\rho_j}{\rho_j^2+1}\\
&\quad+\lambda_m^{-1}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\left[\xi_1^{[j]}-\xi_1^{[m]}+i\left(\xi_2^{[j]}-\xi_2^{[m]}\right)\right]e^{i(-\theta_j+\gamma_m-\gamma_j)}\right\}\Bigg]
\end{aligned}
$$

where we have used (D.74) in the last "=".

We estimate

$$
\left|2\eta_R^{[j]}8\lambda_j^{-2}(\rho_j^2+1)^{-2}(\rho_m^2+1)^{-1}\left\{\lambda_j\lambda_m^{-1}\rho_j\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left[e^{i(\gamma_m-\gamma_j)}\right]+\frac{2\rho_j}{\rho_j^2+1}\right\}\right|\lesssim\eta_R^{[j]}\langle\rho_j\rangle^{-3}\lesssim T^\epsilon\varrho_1^{[j]}.
$$

By (D.72), one has

$$
\begin{aligned}
&2\eta_R^{[j]}8\lambda_j^{-2}(\rho_j^2+1)^{-2}(\rho_m^2+1)^{-1}\lambda_m^{-1}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\left[\xi_1^{[j]}-\xi_1^{[m]}+i\left(\xi_2^{[j]}-\xi_2^{[m]}\right)\right]e^{i(-\theta_j+\gamma_m-\gamma_j)}\right\}\\
&=16\eta_R^{[j]}\lambda_j^{-2}\lambda_m(\rho_j^2+1)^{-2}|\xi^{[j]}-\xi^{[m]}|^{-2}\left(1+O\left(\lambda_m^2+\lambda_jR\right)\right)\\
&\quad\times\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\left[\xi_1^{[j]}-\xi_1^{[m]}+i\left(\xi_2^{[j]}-\xi_2^{[m]}\right)\right]e^{i(-\theta_j+\gamma_m-\gamma_j)}\right\}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\Bigg|16\eta_R^{[j]}\lambda_j^{-2}\lambda_m(\rho_j^2+1)^{-2}|\xi^{[j]}-\xi^{[m]}|^{-2}O\left(\lambda_m^2+\lambda_jR\right)\\
&\times\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\left[\xi_1^{[j]}-\xi_1^{[m]}+i\left(\xi_2^{[j]}-\xi_2^{[m]}\right)\right]e^{i(-\theta_j+\gamma_m-\gamma_j)}\right\}\Bigg|\lesssim\eta_R^{[j]}R\langle\rho_j\rangle^{-4}\lesssim T^\epsilon\varrho_1^{[j]}
\end{aligned}
$$

provided

$$
\Theta+2\beta-1<0. \tag{D.82}
$$

$$
\begin{aligned}
&16\eta_R^{[j]}\lambda_j^{-2}\lambda_m(\rho_j^2+1)^{-2}|\xi^{[j]}-\xi^{[m]}|^{-2}\left(1-\frac{2}{\rho_j^2+1}\mathrm{Re}\right)\left\{\left[\xi_1^{[j]}-\xi_1^{[m]}+i\left(\xi_2^{[j]}-\xi_2^{[m]}\right)\right]e^{i(-\theta_j+\gamma_m-\gamma_j)}\right\}\\
&=16\eta_R^{[j]}\lambda_j^{-2}\rho_j^2(\rho_j^2+1)^{-3}\lambda_m|q^{[j]}-q^{[m]}|^{-2}\left[q_1^{[j]}-q_1^{[m]}+i\left(q_2^{[j]}-q_2^{[m]}\right)\right]e^{i(\gamma_m-\gamma_j)}e^{-i\theta_j}\\
&\quad-16\eta_R^{[j]}\lambda_j^{-2}(\rho_j^2+1)^{-3}\lambda_m|q^{[j]}-q^{[m]}|^{-2}\left[q_1^{[j]}-q_1^{[m]}-i\left(q_2^{[j]}-q_2^{[m]}\right)\right]e^{-i(\gamma_m-\gamma_j)}e^{i\theta_j}+O(T^\epsilon)\varrho_1^{[j]}.
\end{aligned}
$$

by the same estimate as in (D.77) and (D.78), which will be assigned into mode 1 and mode −1.

□

Under the parameter assumption (D.69), combining (D.68) and (D.70), we have

$$
\Delta_xU_*-2\left(U_*\cdot\nabla_xU_*\right)\nabla_xU_*=O(T^\epsilon)\left[\sum_{j=1}^{N}\left(\varrho_1^{[j]}+\varrho_2^{[j]}\right)+\varrho_3\right]-\Xi_1(x,t)U_*. \tag{D.83}
$$

Combining (D.18), (D.42), (D.43), (D.45), (D.46), (D.47), (D.49), (D.50), (D.51), (D.52), (D.53), (D.55), (D.56), (D.59), (D.61), (D.63), (D.64), (D.65), (D.66), (D.67) and (D.69), we get the parameter requirement (D.31).

□

D.4. **Estimate of $\mathcal{H}^{[j]}$.** For $|x-\xi^{[j]}|\le 2\lambda_j R$, by (5.50), we have

$$\begin{aligned}
&\left|D_x\Phi_{\rm out}(x,t)-D_x\Phi_{\rm out}(q^{[j]},T)\right|\\
&=\left|D_x\Phi_{\rm out}(x,t)-D_x\Phi_{\rm out}(q^{[j]},t)+D_x\Phi_{\rm out}(q^{[j]},t)-D_x\Phi_{\rm out}(q^{[j]},T)\right|\\
&\lesssim \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\Big\{C(\alpha)\left[\lambda_*^{\Theta}(t)(\lambda_*(t)R(t))^{-\alpha}+\|Z_*\|_{C^3(\mathbb{R}^2)}\right]|x-q^{[j]}|^{\alpha}\\
&\quad +C(\alpha)\left[\lambda_*^{\Theta}(t)+(T-t)^{\frac{\alpha}{2}}\|Z_*\|_{C^3(\mathbb{R}^2)}\right]\Big\}\\
&\lesssim \|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}C(\alpha)\left[\lambda_*^{\Theta}(t)+(T-t)^{\frac{\alpha}{2}}\|Z_*\|_{C^3(\mathbb{R}^2)}\right].
\end{aligned}\tag{D.84}$$

By (5.29) and (D.84), when T is sufficiently small, we have

$$\left[\partial_{x_1}(\Phi_{\rm out})_1+\partial_{x_2}(\Phi_{\rm out})_2+i\left(\partial_{x_1}(\Phi_{\rm out})_2-\partial_{x_2}(\Phi_{\rm out})_1\right)\right](q^{[j]},T)\ne 0.$$

By (D.84), we have

$$\begin{aligned}
&\Big|\lambda_j^{-1}\rho_j w_{\rho_j}^2(\rho_j)e^{-i\gamma_j}\Big\{\left[\partial_{x_1}(\Phi_{\rm out})_1+\partial_{x_2}(\Phi_{\rm out})_2+i\left(\partial_{x_1}(\Phi_{\rm out})_2-\partial_{x_2}(\Phi_{\rm out})_1\right)\right](x,t)\\
&-\left[\partial_{x_1}(\Phi_{\rm out})_1+\partial_{x_2}(\Phi_{\rm out})_2+i\left(\partial_{x_1}(\Phi_{\rm out})_2-\partial_{x_2}(\Phi_{\rm out})_1\right)\right](q^{[j]},T)\Big\}\Big|\\
&+\left|\lambda_j^{-1}w_{\rho_j}(\rho_j)\cos w(\rho_j)\left\{\left[-\partial_{x_1}(\Phi_{\rm out})_3+i\partial_{x_2}(\Phi_{\rm out})_3\right](x,t)-\left[-\partial_{x_1}(\Phi_{\rm out})_3+i\partial_{x_2}(\Phi_{\rm out})_3\right](q^{[j]},T)\right\}\right|\\
&+\Big|\lambda_j^{-1}\rho_j w_{\rho_j}^2(\rho_j)e^{i\gamma_j}\Big\{\left[\partial_{x_1}(\Phi_{\rm out})_1-\partial_{x_2}(\Phi_{\rm out})_2-i\left(\partial_{x_1}(\Phi_{\rm out})_2+\partial_{x_2}(\Phi_{\rm out})_1\right)\right](x,t)\\
&-\left[\partial_{x_1}(\Phi_{\rm out})_1-\partial_{x_2}(\Phi_{\rm out})_2-i\left(\partial_{x_1}(\Phi_{\rm out})_2+\partial_{x_2}(\Phi_{\rm out})_1\right)\right](q^{[j]},T)\Big\}\Big|\\
\lesssim\ &\lambda_j^{-1}\langle\rho_j\rangle^{-2}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}C(\alpha)\left[\lambda_*^{\Theta}(t)+(T-t)^{\frac{\alpha}{2}}\|Z_*\|_{C^3(\mathbb{R}^2)}\right].
\end{aligned}\tag{D.85}$$

Recall $\mathcal{H}^{[j]}$ defined in (5.20). By (D.5),

$$|\mathcal{H}^{[j]}_{\rm in}|\lesssim \|\Phi^{[j]}_{\rm in}\|^2_{{\rm in},\nu-\delta_0,l}\lambda_*^{2\nu-2\delta_0}\langle\rho_j\rangle^{-2l-3}\lesssim \lambda_*^{\nu+\epsilon}\langle\rho_j\rangle^{-2-l}\tag{D.86}$$

provided

$$2\delta_0<\nu.\tag{D.87}$$

Denote

$$\left(\mathcal{H}^{[j]}_1\right)_{\mathbb{C}_j,k}(\rho_j,t)=(2\pi)^{-1}\int_0^{2\pi}\left(\mathcal{H}^{[j]}_1\right)_{\mathbb{C}_j}(\rho_je^{i\theta_j},t)e^{-ik\theta_j}d\theta_j,\quad \left(\mathcal{H}^{[j]}_1\right)_{\mathbb{C}_j,\top}=\sum_{|k|\ge 2}e^{ik\theta_j}\left(\mathcal{H}^{[j]}_1\right)_{\mathbb{C}_j,k}.$$

By (4.41), (3.20) and (D.85), we get

$$\begin{aligned}
&\left|\left(\mathcal{H}^{[j]}_1\right)_{\mathbb{C}_j,0}\right|\lesssim \lambda_*^2\Big\{\lambda_*^{-1}\langle\rho_j\rangle^{-3}+|\dot\lambda_*|\langle\rho_j\rangle^{-1}+\lambda_j^{-1}\langle\rho_j\rangle^{-3}\left|\nabla_x\Phi_{\rm out}(q^{[j]},T)\right|\\
&\quad+\lambda_j^{-1}\langle\rho_j\rangle^{-2}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}C(\alpha)\left[\lambda_*^{\Theta}+(T-t)^{\frac{\alpha}{2}}\|Z_*\|_{C^3(\mathbb{R}^2)}\right]\Big\}\\
&\lesssim \lambda_*\langle\rho_j\rangle^{-3}+\lambda_*\langle\rho_j\rangle^{-3}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\\
&\quad+\lambda_*\langle\rho_j\rangle^{-2}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}C(\alpha)\left[\lambda_*^{\Theta}+(T-t)^{\frac{\alpha}{2}}\|Z_*\|_{C^3(\mathbb{R}^2)}\right]\\
&\lesssim \lambda_*^{\nu+\epsilon}\langle\rho_j\rangle^{-2-l}
\end{aligned}\tag{D.88}$$

for $|\rho_j|\le 3R$ and $\epsilon>0$ sufficiently small provided

$$0<\nu<1,\quad 0<l<1,\quad \nu-\Theta+\beta l-1<0,\quad \nu-\frac{\alpha}{2}+\beta l-1<0.\tag{D.89}$$

By (4.43), (3.20) and (D.85),

$$\begin{aligned}\left|\left(\mathcal{H}_1^{[j]}\right)_{\mathbb{C}_j,1}\right| &\lesssim \lambda_*^2\Big\{|\dot{\xi}^{[j]}|\lambda_*^{-1}\langle\rho_j\rangle^{-2}+\lambda_j^{-1}\langle\rho_j\rangle^{-2}\left|\nabla_x\Phi_{\rm out}(q^{[j]},T)\right| \\ &\quad+\lambda_j^{-1}\langle\rho_j\rangle^{-2}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}C(\alpha)\left[\lambda_*^{\Theta}(t)+(T-t)^{\frac{\alpha}{2}}\|Z_*\|_{C^3(\mathbb{R}^2)}\right]\Big\} \\ &\lesssim |\dot{\xi}^{[j]}|\lambda_*\langle\rho_j\rangle^{-2}+\lambda_*\langle\rho_j\rangle^{-2}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right) \\ &\quad+\lambda_*\langle\rho_j\rangle^{-2}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}C(\alpha)\left[\lambda_*^{\Theta}+(T-t)^{\frac{\alpha}{2}}\|Z_*\|_{C^3(\mathbb{R}^2)}\right] \\ &\lesssim \lambda_*^{\nu+\epsilon}\langle\rho_j\rangle^{-2-l}\end{aligned} \tag{D.90}$$

provided

$$\nu+\beta l-1<0. \tag{D.91}$$

By (D.85),

$$\left|\left(\mathcal{H}_1^{[j]}\right)_{\mathbb{C}_j,-1}\right| \lesssim \lambda_*\langle\rho_j\rangle^{-2}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}C(\alpha)\left[\lambda_*^{\Theta}+(T-t)^{\frac{\alpha}{2}}\|Z_*\|_{C^3(\mathbb{R}^2)}\right] \tag{D.92}$$

By (5.40) and Proposition 9.1, $\mathcal{T}_{\rm in}[(\mathcal{H}^{[j]})_{\mathbb{C}_j,-1}]\in\mathbf{B}_{\rm in}$ provided

$$\nu+\beta l-\delta_0-1<0. \tag{D.93}$$

By (3.20) and (D.85), we have

$$\begin{aligned}\left|\left(\mathcal{H}_1^{[j]}\right)_{\mathbb{C}_j,\top}\right| &\lesssim \lambda_*^2\Big\{\lambda_j^{-1}\langle\rho_j\rangle^{-3}\left|\nabla_x\Phi_{\rm out}(q^{[j]},T)\right| \\ &\quad+\lambda_j^{-1}\langle\rho_j\rangle^{-2}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}C(\alpha)\left[\lambda_*^{\Theta}(t)+(T-t)^{\frac{\alpha}{2}}\|Z_*\|_{C^3(\mathbb{R}^2)}\right]\Big\} \\ &\lesssim \lambda_*\langle\rho_j\rangle^{-3}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\left(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right) \\ &\quad+\lambda_*\langle\rho_j\rangle^{-2}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}C(\alpha)\left[\lambda_*^{\Theta}+(T-t)^{\frac{\alpha}{2}}\|Z_*\|_{C^3(\mathbb{R}^2)}\right] \\ &\lesssim \lambda_*^{\nu+\epsilon}\langle\rho_j\rangle^{-2-l}\end{aligned} \tag{D.94}$$

provided the same parameter assumption (D.89) holds.

In summary, collecting (D.87), (D.89), (D.91) and (D.93), the parameter restrictions for bounding $\mathcal{H}^{[j]}$ are given by

$$2\delta_0<\nu<1,\quad 0<l<1,\quad \nu+\beta l-1<0. \tag{D.95}$$

Hölder estimate for $\mathcal{H}^{[j]}$. Recall that $\mathcal{H}^{[j]}=\mathcal{H}_1^{[j]}+\mathcal{H}_{\rm in}^{[j]}$ where $\mathcal{H}_1^{[j]}$ and $\mathcal{H}_{\rm in}^{[j]}$ given in (5.21) and (5.22), respectively. For the C^2-regularity of the inner solutions, the Hölder continuity in $(y^{[j]},\tau_j)$ of $\mathcal{H}_j$ is needed (see Proposition 9.1).

Notice for $\Omega\subset\mathbb{R}^d\times\mathbb{R}$,

$$[fg]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}(\Omega)}\le[f]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}(\Omega)}\|g\|_{L^\infty(\Omega)}+[g]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}(\Omega)}\|f\|_{L^\infty(\Omega)}.$$

For brevity, we denote $Q_y:=Q^-((y,\tau_j),\frac{|y|}{2})$ defined in (5.41). From (5.44), we have

$$\begin{aligned}&\left[\left[\nabla_{y^{[j]}}W^{[j]}\cdot\nabla_{y^{[j]}}\left(\eta_R^{[j]}\Phi_{\rm in}^{[j]}\right)\right]\Phi_{\rm in}^{[j]}\right]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}(Q_y)} \\ \lesssim\ &\|\nabla_{y^{[j]}}W^{[j]}\|_{L^\infty(Q_y)}\|\nabla_{y^{[j]}}\left(\eta_R^{[j]}\Phi_{\rm in}^{[j]}\right)\|_{L^\infty(Q_y)}\left[\Phi_{\rm in}^{[j]}\right]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}(Q_y)} \\ &+\|\Phi_{\rm in}^{[j]}\|_{L^\infty(Q_y)}\|\nabla_{y^{[j]}}\left(\eta_R^{[j]}\Phi_{\rm in}^{[j]}\right)\|_{L^\infty(Q_y)}\left[\nabla_{y^{[j]}}W^{[j]}\right]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}(Q_y)} \\ &+\|\nabla_{y^{[j]}}W^{[j]}\|_{L^\infty(Q_y)}\|\Phi_{\rm in}^{[j]}\|_{L^\infty(Q_y)}\left[\nabla_{y^{[j]}}\left(\eta_R^{[j]}\Phi_{\rm in}^{[j]}\right)\right]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}(Q_y)} \\ \lesssim\ &\lambda_*^{2(\nu-\delta_0)}\langle y^{[j]}\rangle^{-3-2l-\varsigma_{\rm in}}\left(\|\Phi_{\rm in}^{[j]}\|_{{\rm in},\nu-\delta_0,l,\varsigma_{\rm in}}\right)^2.\end{aligned} \tag{D.96}$$

- Hölder continuity of the coupling terms from the outer problem

$$\lambda_j^2Q_{-\gamma_j}\Big\{\left(a-bU^{[j]}\wedge\right)\left[|\nabla_xU^{[j]}|^2\Pi_{U^{[j]\perp}}\Phi_{\rm out}-2\nabla_x\left(U^{[j]}\cdot\Phi_{\rm out}\right)\nabla_xU^{[j]}\right]\Big\}.$$

For p_j given in Proposition 6.1,

$$\begin{aligned}\left[\frac{p_j}{|p_j|}\right]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}} &\lesssim \|p_j\|_\infty \left[\frac{1}{|p_j|}\right]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}} + \left\||p_j|^{-1}\right\|_\infty [p_j]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}}\\
&\lesssim \lambda_*(\tau_j) \sup_{s\in\left(\max\{\tau_j-\frac{y^{[j]}}{2},\tau_0\},\tau_j\right)} \frac{\left|\lambda_j^{-1}(t(\tau_j))-\lambda_j^{-1}(t(s))\right|}{|\tau_j-s|^{\varsigma_H/2}} + \lambda_*^{-1}(\tau_j)[p_j]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}}\\
&\lesssim \lambda_*(\tau_j)\ln^2\tau_j \sup_{s\in\left(\max\{\tau_j-\frac{y^{[j]}}{2},\tau_0\},\tau_j\right)} \frac{|\tau_j-s|}{|\tau_j-s|^{\varsigma_H/2}} + \lambda_*^{-1}(\tau_j)[p_j]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}}\\
&\lesssim \lambda_*(\tau_j)\tau_j^{1-\frac{\varsigma_H}{2}}\ln^2\tau_j + \lambda_*^{-1}(\tau_j)[p_j]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}}\end{aligned}$$

since $\tau_j \sim \frac{\ln^4(T-t)}{T-t}$. Also, we have

$$\begin{aligned}[p_j]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}} &= [p_j]_{C^{\varsigma_H/2}_{\tau_j}}\\
&\lesssim \lambda_*^2(t(\tau_j))\|\dot p_j(t)\|_\infty \sup_{s\in\left(\max\{\tau_j-\frac{y^{[j]}}{2},\tau_0\},\tau_j\right)} \frac{|\tau_j-s|}{|\tau_j-s|^{\varsigma_H/2}}\\
&\lesssim \lambda_*^2(t(\tau_j))\tau_j^{1-\frac{\varsigma_H}{2}}\|\dot p_j(t)\|_\infty,\end{aligned}$$

and thus

$$\begin{aligned}\left[\frac{p_j}{|p_j|}\right]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}} &= \left[\frac{p_j}{|p_j|}\right]_{C^{\varsigma_H/2}_{\tau_j}}\\
&\lesssim \lambda_*(t(\tau_j))\tau_j^{1-\frac{\varsigma_H}{2}}\ln^2\tau_j\\
&\lesssim \tau_j^{-\frac{\varsigma_H}{2}}\ln^4\tau_j\\
&\lesssim \lambda_*^{\frac{\varsigma_H}{2}}(t(\tau_j))\ln^{4-\varsigma_H}\tau_j.\end{aligned} \tag{D.97}$$

From (D.97), we then have

$$\begin{aligned}&\left[\lambda_j^2 Q_{-\gamma_j}\left\{\left(a-bU^{[j]}\wedge\right)\left[|\nabla_x U^{[j]}|^2\Pi_{U^{[j]\perp}}\Phi_{\rm out} - 2\nabla_x\left(U^{[j]}\cdot\Phi_{\rm out}\right)\nabla_x U^{[j]}\right]\right\}\right]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}}\\
\lesssim\ &\left\|\frac{p_j}{|p_j|}\right\|_\infty \left\||\nabla_{y^{[j]}}U^{[j]}|^2\right\|_{L^\infty}[\Phi_{\rm out}]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}} + \left\||\nabla_{y^{[j]}}U^{[j]}|^2\right\|_{L^\infty}\|\Phi_{\rm out}\|_{L^\infty}\left[\frac{p_j}{|p_j|}\right]_{C^{\varsigma_H/2}_{\tau_j}}\\
&+\left\|\frac{p_j}{|p_j|}\right\|_\infty \|\Phi_{\rm out}\|_{L^\infty}\left[|\nabla_{y^{[j]}}U^{[j]}|^2\right]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}} + \lambda_*(t(\tau_j))\|U^{[j]}\nabla_{y^{[j]}}U^{[j]}\|_{L^\infty}\|\nabla_x\Phi_{\rm out}\|_{L^\infty}\left[\frac{p_j}{|p_j|}\right]_{C^{\varsigma_H/2}_{\tau_j}}\\
&+\lambda_*(t(\tau_j))\|U^{[j]}\nabla_{y^{[j]}}U^{[j]}\|_{L^\infty}\left\|\frac{p_j}{|p_j|}\right\|_\infty[\nabla_x\Phi_{\rm out}]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}}\\
&+\lambda_*(t(\tau_j))\|\nabla_x\Phi_{\rm out}\|_{L^\infty}\left\|\frac{p_j}{|p_j|}\right\|_\infty[U^{[j]}\nabla_{y^{[j]}}U^{[j]}]_{C^{\varsigma_H,\frac{\varsigma_H}{2}}}\\
\lesssim\ &\lambda_*^{\frac{\alpha}{2}+1}(t)\left[\lambda_*^{\Theta-\alpha}(t)R^{-\alpha}(t)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right]\langle y^{[j]}\rangle^{-3}\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\\
&+\lambda_*^{\frac{\varsigma_H}{2}}(t)\langle y^{[j]}\rangle^{-4}\left[\lambda_*^{\Theta+1}(t)R(t)|\ln(T-t)|+(T-t)\|Z_*\|_{C^3(\mathbb{R}^2)}+\lambda_*(t)\rho_j(\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)})\right]\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\\
&+\lambda_*^{1+\frac{\varsigma_H}{2}}(t)\langle y^{[j]}\rangle^{-2}\left[\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right]\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\\
&+\lambda_*^{1+\frac{\alpha}{2}}(t)\langle y^{[j]}\rangle^{-2}\left[\lambda_*^{\Theta}(t)(\lambda_*(t)R(t))^{-\alpha}+\|Z_*\|_{C^3(\mathbb{R}^2)}\right]\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\\
&+\lambda_*(t)\langle y^{[j]}\rangle^{-2-\varsigma_H}\left[\lambda_*^{\Theta}(0)+\|Z_*\|_{C^3(\mathbb{R}^2)}\right]\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}\\
\lesssim\ &\lambda_*^{\nu}(t(\tau_j))\langle y^{[j]}\rangle^{-2-l-\varsigma_H}\left(\|\Phi_{\rm out}\|_{\sharp,\Theta,\alpha}+\|Z_*\|_{C^3(\mathbb{R}^2)}\right)\end{aligned}$$

provided

$$\begin{cases} \alpha > \varsigma_H, \\ 1+\Theta-\frac{\alpha}{2}+\alpha\beta>0, \\ \Theta+1-\beta+\frac{\varsigma_H}{2}>\nu, \\ 1-\nu-\beta l>0. \end{cases} \tag{D.98}$$

Therefore, under the restriction (D.98), the coupling terms from the outer problem are in the desired weighted Hölder space.

- Estimate of

$$\begin{aligned} &\Pi_{U^{[j]\perp}}\mathcal{S}^{[j]} \\ &= \Pi_{U^{[j]\perp}}\Big(-\partial_t(\Phi_0^{*[j]}) + \left(a-bU^{[j]}\wedge\right)\left[\Delta_x\Phi_0^{*[j]} + |\nabla_x U^{[j]}|^2\Phi_0^{*[j]} - 2\nabla_x\left(U^{[j]}\cdot\Phi_0^{*[j]}\right)\nabla_x U^{[j]}\right] - \partial_t U^{[j]}\Big). \end{aligned}$$

Collecting the estimates in subsection 4.2.2, we have

$$\begin{aligned} &\mathcal{S}^{[j]} \\ = &-\partial_t(\Phi_0^{*[j]}) + (a-bU_\infty\wedge)\,\Delta_x\Phi_0^{*[j]} - \partial_t U^{[j]} \\ &- b\left(U^{[j]}-U_\infty\right)\wedge\Delta_x\Phi_0^{*[j]} + a|\nabla_x U^{[j]}|^2\Phi_0^{*[j]} + b|\nabla_x U^{[j]}|^2\Phi_0^{*[j]}\wedge U^{[j]} \\ &+ \left(a-bU^{[j]}\wedge\right)\left[-2\nabla_x\left(U^{[j]}\cdot\Phi_0^{*[j]}\right)\nabla_x U^{[j]}\right] \\ = &\Big[\Big\{\frac{\dot\xi^{[j]}\cdot y^{[j]}\rho_j^2}{(\rho_j^2+1)^{\frac{3}{2}}}\partial_{z_j}\Phi_0^{[j]} + \big[\frac{2\lambda_j^{-1}\dot\xi^{[j]}\cdot y^{[j]}}{(\rho_j^2+1)^2} + \frac{i\lambda_j^{-1}(\dot\xi_2^{[j]}y_1^{[j]}-\dot\xi_1^{[j]}y_2^{[j]})}{\rho_j^2+1}\big]\Phi_0^{[j]}\Big\}e^{i\theta_j},0\Big]^T \\ &+ \Big[\big[\frac{-\dot\lambda_j\rho_j^2}{(\rho_j^2+1)^{\frac{3}{2}}}\partial_{z_j}\Phi_0^{[j]} + \frac{2\lambda_j^{-1}\dot\lambda_j\rho_j^2}{(\rho_j^2+1)^2}\Phi_0^{[j]}\big]e^{i\theta_j},0\Big]^T \\ &+ \Big[(a-ib)\Big[\frac{-\rho_j^2}{(\rho_j^2+1)^2}\partial_{z_jz_j}\Phi_0^{[j]} + \frac{5\lambda_j^{-1}\rho_j^2}{(\rho_j^2+1)^{\frac{5}{2}}}\partial_{z_j}\Phi_0^{[j]} + \frac{\lambda_j^{-2}(3-5\rho_j^2)}{(\rho_j^2+1)^3}\Phi_0^{[j]}\Big]e^{i\theta_j},0\Big]^T \\ &+ \Big[\Big\{\frac{-2i\dot\gamma_j\rho_j}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^{\frac32}} + \frac{2\lambda_j^{-1}\dot\lambda_j\rho_j[2\rho_j+(\rho_j^2+1)^{\frac12}]}{[\rho_j+(\rho_j^2+1)^{\frac12}](\rho_j^2+1)^2}\Big\}e^{i(\theta_j+\gamma_j)}, \frac{4\lambda_j^{-1}\dot\lambda_j\rho_j^2}{(\rho_j^2+1)^2}\Big]^T + \mathcal{E}_1^{[j]} \\ &+ \frac{-2b}{\rho_j^2+1}\Big[(\Delta_x\Phi_0^{*[j]})_2, -(\Delta_x\Phi_0^{*[j]})_1, \rho_j\mathrm{Re}\Big[\Big((\Delta_x\Phi_0^{*[j]})_2 - i(\Delta_x\Phi_0^{*[j]})_1\Big)e^{-i(\theta_j+\gamma_j)}\Big]\Big]^T + \Big[\frac{8a\lambda_j^{-2}\rho_j^2}{(\rho_j^2+1)^3}\Phi_0^{[j]}e^{i\theta_j},0\Big]^T \\ &+ \frac{8b\lambda_j^{-2}}{(\rho_j^2+1)^3}\Big[(\rho_j^2-1)(\Phi_0^{*[j]})_2, -(\rho_j^2-1)(\Phi_0^{*[j]})_1, -2\rho_j\mathrm{Re}\Big[\Big((\Phi_0^{*[j]})_2 - i(\Phi_0^{*[j]})_1\Big)e^{-i(\theta_j+\gamma_j)}\Big]\Big]^T \\ &+ \Big[\frac{8\lambda_j^{-2}(3\rho_j^2-\rho_j^4)}{(\rho_j^2+1)^4}\mathrm{Re}(\Phi_0^{[j]}e^{-i\gamma_j}) + \frac{8\lambda_j^{-1}\rho_j^4}{(\rho_j^2+1)^{\frac72}}\mathrm{Re}(\partial_{z_j}\Phi_0^{[j]}e^{-i\gamma_j})\Big]\Big(aQ_{\gamma_j}E_1^{[j]} - bQ_{\gamma_j}E_2^{[j]}\Big), \end{aligned}$$

where the expressions for ζ_j, $\Phi_0^{[j]}$, $\partial_{z_j}\Phi_0^{[j]}$, $\partial_{z_jz_j}\Phi_0^{[j]}$ can be found in (4.14), (4.15), (4.16), (4.17), respectively. In order to get a desired Hölder property in $y^{[j]}$, we consider a slight variant of the above expression for $\mathcal{S}^{[j]}$, denoted by $\tilde{\mathcal{S}}^{[j]}$, for which the term $\Phi_0^{[j]}$ is replaced by a "regularized" version

$$\tilde\Phi_0^{[j]} = -\lambda_j\rho_j\int_0^t \frac{\dot p_j(s)}{t-s}K_0(\zeta_j)ds.$$

Here another choice is by the standard Duhamel's form. The difference $\Phi_0^{[j]} - \tilde\Phi_0^{[j]}$ actually leaves a smaller error. Indeed, the difference is given by

$$(\lambda_j\rho_j - z_j)\int_0^t \frac{\dot p_j(s)}{t-s}K_0(\zeta_j)ds = -\frac{\lambda_j}{\rho_j+\sqrt{\rho_j^2+1}}\int_0^t\frac{\dot p_j(s)}{t-s}K_0(\zeta_j)ds,$$

and we see there is extra λ_j which carries smallness. So in the non-local reduced problem, the errors produced by the difference is of smaller order. Now since $\Pi_{U^{[j]\perp}}\tilde{\mathcal{S}}^{[j]}$ is regular in the spatial variable $y^{[j]}$, we only need to consider its Hölder property in τ_j. A typical model to consider is the Hölder in τ_j for the non-local term

$$\int_0^{t-\lambda_j^2(t)} \frac{\dot{p}_j(s)}{t-s} ds,$$

which is from the non-local expression for $\tilde{\Phi}_0^{[j]}$. Recall

$$\frac{d\tau_j}{dt} = \lambda_j^{-2}(t).$$

Then we have

$$\begin{aligned}
&\left[\int_0^{t-\lambda_j^2(t)} \frac{\dot{p}_j(s)}{t-s} ds\right]_{C_{\tau_j}^{\varsigma_H/2}} \\
= &\left[\int_0^{1-\frac{\lambda_j^2(t)}{t}} \frac{\dot{p}_j(tz)}{1-z} dz\right]_{C_{\tau_j}^{\varsigma_H/2}} \\
= &\sup_{s\in\left(\max\{\tau_j-\frac{y^{[j]}}{2},\tau_0\},\tau_j\right)} \frac{\left|\int_0^{1-\frac{\lambda_j^2(t(\tau_j))}{t(\tau_j)}} \frac{\dot{p}_j(t(\tau_j)z)}{1-z} dz - \int_0^{1-\frac{\lambda_j^2(t(s))}{t(s)}} \frac{\dot{p}_j(t(s)z)}{1-z} dz\right|}{|\tau_j - s|^{\varsigma_H/2}}
\end{aligned}$$

where

$$\begin{aligned}
&\left|\int_0^{1-\frac{\lambda_j^2(t(\tau_j))}{t(\tau_j)}} \frac{\dot{p}_j(t(\tau_j)z)}{1-z} dz - \int_0^{1-\frac{\lambda_j^2(t(s))}{t(s)}} \frac{\dot{p}_j(t(s)z)}{1-z} dz\right| \\
= &\left|\int_{1-\frac{\lambda_j^2(t(s))}{t(s)}}^{1-\frac{\lambda_j^2(t(\tau_j))}{t(\tau_j)}} \frac{\dot{p}_j(t(\tau_j)z)}{1-z} dz + \int_0^{1-\frac{\lambda_j^2(t(s))}{t(s)}} \frac{\dot{p}_j(t(\tau_j)z) - \dot{p}_j(t(s)z)}{1-z} dz\right| \\
\lesssim &\|\dot{p}_j\|_\infty \left|\ln \frac{\lambda_j^2(t(s))t(\tau_j)}{t(s)\lambda_j^2(t(\tau_j))}\right| + [\dot{p}_j]_{\frac{\alpha}{2},m,\varpi} T^\epsilon [t(\tau_j) - t(s)]^{\alpha/2} \int_0^{1-\frac{\lambda_j^2(t(s))}{t(s)}} \frac{z^{\alpha/2}}{1-z} dz \\
\lesssim &\|\dot{p}_j\|_\infty \left|\ln \frac{\lambda_j^2(t(s))t(\tau_j)}{t(s)\lambda_j^2(t(\tau_j))}\right| + [\dot{p}_j]_{\frac{\alpha}{2},m,\varpi} T^\epsilon [t(\tau_j) - t(s)]^{\alpha/2}.
\end{aligned}$$

Here

$$\int \frac{z^{\alpha/2}}{1-z} dz = \frac{2z^{\frac{\alpha}{2}+1} {}_2F_1(1, \frac{\alpha}{2}+1; \frac{\alpha}{2}+2; z)}{\alpha+2} \sim O(1) \quad \text{for} \quad z<1.$$

For the first term we have

$$\begin{aligned}
&\sup_{s\in\left(\max\{\tau_j-\frac{y^{[j]}}{2},\tau_0\},\tau_j\right)} \frac{\|\dot{p}_j\|_\infty \left|\ln \frac{\lambda_j^2(t(s))t(\tau_j)}{t(s)\lambda_j^2(t(\tau_j))}\right|}{|\tau_j - s|^{\varsigma_H/2}} \\
\lesssim &\sup_{s\in\left(\max\{\tau_j-\frac{y^{[j]}}{2},\tau_0\},\tau_j\right)} \frac{\|\dot{p}_j\|_\infty \left|\ln \frac{\frac{1}{s}(T-\frac{1}{\tau_j})}{\frac{1}{\tau_j}(T-\frac{1}{s})}\right|}{|\tau_j - s|^{\varsigma_H/2}} \\
= &\sup_{s\in\left(\max\{\tau_j-\frac{y^{[j]}}{2},\tau_0\},\tau_j\right)} \frac{\|\dot{p}_j\|_\infty \left|\ln \frac{T\tau_j - 1}{Ts-1}\right|}{|\tau_j - s|^{\varsigma_H/2}} \\
\lesssim &\|\dot{p}_j\|_\infty \sup_{s\in\left(\max\{\tau_j-\frac{y^{[j]}}{2},\tau_0\},\tau_j\right)} \frac{|\tau_j - s|^{1-\varsigma_H/2}}{s(1-\theta)+\tau_j\theta} \\
\lesssim &\|\dot{p}_j\|_\infty \tau_j^{-\varsigma_H/2},
\end{aligned}$$

where we have used $|y^{[j]}|^2 \ll \tau_j$ by our choice of R. One has roughly

$$|t(\tau_j) - t(s)|^{\alpha/2} \approx \frac{1}{s} - \frac{1}{\tau_j}$$

so for the second term

$$\sup_{s\in\left(\max\{\tau_j - \frac{y^{[j]}}{2}, \tau_0\}, \tau_j\right)} \frac{[\dot{p}_j]_{\frac{\alpha}{2},m,\varpi} T^\epsilon [t(\tau_j) - t(s)]^{\alpha/2}}{|\tau_j - s|^{\varsigma_H/2}} \lesssim [\dot{p}_j]_{\frac{\alpha}{2},m,\varpi} T^\epsilon \tau_j^{-1+\frac{\alpha-\varsigma_H}{2}}$$

where we have used

$$\alpha > \varsigma_H.$$

Collecting above estimates, we have

$$\left[\int_0^{t-\lambda_j^2(t)} \frac{\dot{p}_j(s)}{t-s} ds\right]_{C_{\tau_j}^{\varsigma_H/2}} \lesssim \left(\|\dot{p}_j\|_\infty + [\dot{p}_j]_{\frac{\alpha}{2},m,\varpi}\right) \tau_j^{-\varsigma_H/2}.$$

Using this together with the Hölder property of $\dot{\xi}^{[j]}$ in τ_j, we conclude that $\Pi_{U^{[j]\perp}} \tilde{\mathcal{S}}^{[j]}$ is in the desired weighted Hölder space similarly as before.

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