Mathematical Models and Methods in Applied Sciences

Existence of Multi-spikes in the Keller-Segel model with Logistic Growth

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The Keller–Segel model is a paradigm to describe the chemotactic mechanism, which plays a vital role on the physiological and pathological activities of uni-cellular and multi-cellular organisms. One of the most interesting variants is the coupled system with the intrinsic growth, which admits many complex non-trivial patterns. This paper is devoted to the construction of multi-spiky solutions to the Keller–Segel models with the logistic source in 2D. Assuming that the chemo-attractive rate is large, we employ the inner-outer gluing scheme to nonlocal cross-diffusion system and prove the existence of multiple boundary and interior spikes. The numerical simulations are presented to highlight our theoretical results.

Keywords: Chemotaxis Models; Logistic Growth; Spiky Solutions; Gluing Method.

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1. Introduction

In this paper, we shall investigate the existence of non-constant steady states to the following system with the no-flux boundary condition:

$$\begin{cases} u_t = \nabla \cdot (\nabla u - \chi u \nabla v) + \mu u(\bar{u} - u) & \text{in } \ \Omega \times (0, T_{\max}), \\ v_t = \Delta v - v + u & \text{in } \ \Omega \times (0, T_{\max}), \\ \dfrac{\partial u}{\partial \boldsymbol{\nu}} = \dfrac{\partial v}{\partial \boldsymbol{\nu}} = 0 & \text{on } \ \partial\Omega \times (0, T_{\max}), \\ u(x,0) = u_0(x) \geq, \not\equiv 0, v(x,0) = v_0(x) > 0, & \text{in } \ \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with piecewisely smooth boundary $\partial\Omega$, $\boldsymbol{\nu}$ denotes the unit outer normal and the initial condition $(u_0, v_0) \in L^\infty(\Omega) \times W^{2,\infty}(\Omega)$. Here $u(x,t)$ and $v(x,t)$ represent the cellular density and chemical concentration at space–time (x,t), respectively; while $\chi > 0$ measures the chemoattractive rate and $\mu > 0$, $\bar{u} > 0$ describe the strength of logistic growth and the carrying capacity of the habitat for cells, respectively.

Our goal is to construct the multi-spiky steady states of (1.1) in the asymptotically limit of $\chi \gg 1$. Before stating the existence results, we shall introduce the context of chemotaxis and the applications of Keller–Segel models.

1.1. *Chemotaxis*

Chemotaxis is the mechanism by which the cells or bacteria direct their movements in response to the chemical stimulus gradient in the environment. Such mechanisms can be observed in a variety of physiological and pathological processes, such as embryonic morphogenesis, wound healing, cell-cell interactions in the immune system, the proliferation of cancer cells, the migration of neurons, etc. The comprehensive discussion of its applications are shown in Refs. 9 and 23.

In terms of the type of chemical stimulus, the chemotactic movement is divided into the chemoattractants and the chemorepellents. In general, cell and bacteria are attracted then move towards a higher concentration of the beneficial chemicals such as food, and repelled by harmful ones such as poison then move away from unfavorable environments. The former process is named chemoattractants or positive chemotaxis and the latter one is called chemorepellents or negative chemotaxis. The chemoattractive movement is more interesting since the combination of this effect and random diffusion can promote the occurrence of Turing's instability, which stabilize the non-constant steady states.

1.2. *Keller–Segel Models*

One of the most interesting phenomenon in the chemotactic process is the self-organized aggregation in which the unicellular organisms aggregate and eventually form a stable spatial inhomogeneous pattern. To quantitatively analyze this phenomenon, Keller and Segel [14,15] proposed the following strongly coupled reaction diffusion system:

$$
\begin{cases}
u_t = \nabla \cdot (\overbrace{d_1 \nabla u}^{\text{random (flux)}} - \overbrace{\chi u \nabla v}^{\text{chemotactic (flux)}}) + \overbrace{f(u)}^{\text{Source}}, \, x \in \Omega, t > 0, \\
v_t = \overbrace{d_2 \Delta v}^{\text{chemical diffusion}} + \overbrace{g(u,v)}^{\text{chemical creation/consumption}}, \, x \in \Omega, t > 0,
\end{cases}
\tag{1.2}
$$

where d_1 and d_2 denote the cellular diffusion rate and the strength of chemical diffusion, respectively. In many cases, the solution to (1.2) can converge to the non-constant steady states as $t \to \infty$. These non-trivial stationary solutions with concentrated structures can be well-used to simulate the cellular aggregation.

Since system (1.2) has the relatively simple structure but admit rich dynamics, lots of researchers have studied it over the last few decades, see the reviews in Refs. 9, 10 and the recent ones in Refs. 1, 2. We would like to mention that there are two well-accepted approaches to qualitatively study (1.2).

The first one is to analyze the global well-posedness of the time-dependent system (1.2). The goal is to prove the global existence and uniformly boundedness of the solution to system (1.2), then further investigate their large time behaviors. Moreover, the structure of stationary solutions are studied to model the cellular aggregation phenomenon. It is well known that Keller and Segel [13] adopt this method to discuss the stability property of constant steady states, then established the necessary condition for the formation of non-trivial patterns. Schaaf [22] utilized this idea to investigate the small amplitude stationary solutions in higher dimensions via Crandall–Rabinowitz techniques. [4,21] The study of large amplitude stationary solutions was initiated by Lin, Ni and Takagi. [17,19,20]

The other method is to show that the time-dependent system admits finite or infinite time "blow-up" solutions in which the cellular density u collapses to the linear combination of several δ-functions plus some regular parts (See Refs. 3, 18, 11, 8 and 6). This phenomenon is so-called "chemotactic collapse" and there are plenty of references on it. It is known that in the absence of source, (1.2) admits the collapsing steady states in 2D. The seminal works can be traced to the research finished by Nanjundiah [18] and Childress and Percus.[3] Moreover, Senba and Suzuki [24,25] characterized the asymptotic behavior of "blow-up" solutions in terms of Green's functions. They showed that the chemical concentration converges to the finite sum of Neumann Greens' functions with the coefficients are equal to 8π and 4π when they are located at the interior of domain and the boundary, respectively. Del pino and Wei [6] further proved the existence of the profile of the cellular density u with finite sum of Dirac and locations of blow-up points are determined by the Neumann Green's function. There are also some results for the blow-up phenomenon in higher dimensions. For instance, Winkler [29] proved that when $g = u - v$ and $f = 0$, for any given initial cellular mass $m > 0$, (1.2) has finite time "blow-up" solutions. For other blow-up/prevention of blow-up mechanisms we refer to Refs. 1 and 2.

1.3. *Motivations and Main Results*

Unlike the classical Keller–Segel models, there does not exists the chemotactic collapsing phenomenon in 2D for system (1.2) with the logistic growth. In fact, Winkler [27] showed that the solution globally exists and is uniformly bounded. The author [28] further proved that (1.2) possesses a unique global-in-time classical solution when $f(u) = ku - \mu u^2$ and μ is sufficiently large in any dimensions. For other results of the global existence and uniformly boundedness, we refer the reader to Refs. 29, 30, 31 and 32.

Regarding the chemotactic rate χ as the parameter, Wang et al. [26,12] obtained the existence of non-constant steady states and study their stability via the bifurcation method. There are also some interesting numerical results shown in Ref. 12, which implies system (1.1) has rich complex nontrivial steady states such as boundary spikes, interior spikes, stripes. Our aim in this paper is to study and construct the multi-spikes in the asymptotically limit of $\chi \gg 1$.

Now, we are concerned with the following stationary problem of (1.1):

$$\begin{cases} \nabla \cdot (\nabla u - \chi u \nabla v) + \mu u(\bar{u} - u) = 0 & \text{in } \Omega, \\ \Delta v - v + u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{\nu}} = \dfrac{\partial v}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $f(u) = \mu u(\bar{u} - u)$. Before constructing the desired solutions to (1.3), it is necessary to introduce the results in the case of $\mu = 0$. In the absence of $f(u)$, we have from the first equation in (1.3) that $u = C_1 e^{\chi v}$, where $C_1 > 0$ is a constant. Upon substituting it into the second equation, one finds $\bar{v} := \chi v$ satisfies the following equation:

$$\Delta \bar{v} - \bar{v} + \tilde{\epsilon}^2 e^{\bar{v}} = 0, \tag{1.4}$$

where $\tilde{\epsilon}^2 := C_1 \chi$. Senba and Suzuki [24,25] showed that in the case of finite mass $\int_\Omega \tilde{\epsilon}^2 e^{\bar{v}} < +\infty$, for sufficiently small $\tilde{\epsilon}$, the blow-up family of solutions to (1.4) must satisfy

$$\bar{v} \to 8\pi \sum_{j=1}^{k} G(x, \xi_j) + 4\pi \sum_{j=k+1}^{m} G(x, \xi_j) \text{ uniformly on } \bar{\Omega} \backslash \{\xi_1, \cdots, \xi_m\},$$

where $G(x, y)$ is the Green's function, which is the solution of the following equation:

$$\begin{cases} \Delta G - G + \delta_y = 0, & x \in \Omega, \\ \frac{\partial G}{\partial \boldsymbol{\nu}} = 0, & x \in \partial\Omega. \end{cases}$$

The existence of such solutions was proved by Del pino and Wei in Ref. 6 via the localized energy method. Furthermore they showed that the profile of $\bar{v}$ is

$$\bar{v}_{\tilde{\epsilon}}(x) = \big(\Gamma_{\hat{\mu}, \tilde{\epsilon}, \xi}(x) + \hat{c} H(x, \xi)\big), \tag{1.5}$$

where

$$\Gamma_{\hat{\mu}, \tilde{\epsilon}, \xi}(x) := \log \frac{8 \hat{\mu}^2 \tilde{\epsilon}^4}{(\hat{\mu}^2 \tilde{\epsilon}^2 + |x - \xi|^2)^2}, \tag{1.6}$$

and $H \in C^{1,\alpha}$ is the regular part of Green's function $G(x,\xi)$, which satisfies

$$\begin{cases} -\Delta_x H + H = -\dfrac{4}{\hat{c}} \log \dfrac{1}{|x-\xi|} & \text{in } \Omega, \\ \dfrac{\partial H}{\partial \boldsymbol{\nu}} = \dfrac{4}{\hat{c}} \dfrac{(x-\xi)\cdot \boldsymbol{\nu}}{|x-\xi|^2} & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

In particular,

$$\Gamma(y) = \log \frac{8\hat{\mu}^2}{(\hat{\mu}^2 + |y|^2)^2}$$

is the solution of the following Liouville equation:

$$\Delta_y \Gamma + e^{\Gamma} = 0 \ \text{ in } \ \mathbb{R}^2. \tag{1.8}$$

If we further define $c_0 = \frac{1}{\chi\tilde{\epsilon}^2}$, we obtain the approximate solution of u is

$$u_0 = c_0 e^{\Gamma} = \frac{8\hat{\mu}^2 c_0}{(\hat{\mu}^2 + |y|^2)^2}.$$

Now, we consider the influence of the logistic growth term to our approximate solution. By imposing the following integral constraint:

$$\int_{\mathbb{R}^2} u_0(\bar{u} - u_0)dy = 0,$$

one can show that $c_0 \sim \frac{3}{8}\hat{\mu}^2\bar{u}$.

From the discussion shown as above, we already obtained the leading order term of our ansatz. However, we choose $\tilde{\epsilon}^2 = C_1\chi$, which depends on C_1. To reduce the ambiguity and construct multi-spikes, we define

$$\varepsilon^2 := \frac{1}{\chi}, \ u = \bar{c}_0 e^{\Gamma},$$

then rewrite (1.6) as the following form:

$$\Gamma_{\hat{\mu},\varepsilon,\xi}(x) := \log \frac{8\bar{\mu}^2\varepsilon^4}{(\bar{\mu}^2\varepsilon^2 + |x-\xi|^2)^2}, \tag{1.9}$$

where $\bar{\mu}$ can be determined later on. Now, the solution $\bar{v}_\varepsilon$ can be rewritten as

$$\bar{v}_\varepsilon(x) \sim \big(\Gamma_{\bar{\mu},\varepsilon,\xi}(x) + \hat{c}H(x,\xi)\big) - 2\log \bar{c}_0.$$

Here $\xi \in \bar{\Omega}$ represents the location of the single spike and $\hat{c}$, $\bar{\mu}$ depending on ξ are positive constants. Moreover, if $\xi \in \partial\Omega$, $\hat{c} = 4\pi$; otherwise, $\hat{c} = 8\pi$. To make the error be small, we can choose $\bar{\mu}$ as the following form:

$$\log 8\bar{\mu}^2 = \hat{c}H(\xi,\xi) - 2\log \bar{c}_0.$$

Then we have from $u = \bar{c}_0 e^{\Gamma}$ that

$$u(x) \sim \frac{8\bar{c}_0\bar{\mu}^2\varepsilon^4}{(\bar{\mu}^2\varepsilon^2 + |x-\xi|^2)^2}, \tag{1.10}$$

where $\bar{c}_0$ is independent of ε. We similarly utilize the mass constraint to obtain that $\bar{c}_0 \sim \frac{3}{8}\bar{\mu}^2\bar{u}$.

We choose the linear combination of (1.5) and (1.10) as the rough ansatz of the multi-spikes to (1.3). By employing the gluing method, we can prove the existence of this type of stationary solutions to (1.1) and our results can be summarized as follows:

Theorem 1.1. *Assume that k, l are non-negative integers with $k+l \geq 1$ and $\mu\bar{u} < \bar{C}$. Then for sufficiently large $\chi := \frac{1}{\varepsilon^2}$, there exists a solution (u_χ, v_χ) to (1.3) satisfying the following form:*

$$u_\chi(x) = \sum_{j=1}^{m} c_j U_j\left(\frac{x-\xi_j}{\varepsilon}\right) + o(1); \tag{1.11}$$

$$v_\chi(x) = \varepsilon^2 \sum_{j=1}^{m} c_j \left[\Gamma_{\mu_j,\varepsilon,\xi_j}(x) + \hat{c}_j H(x,\xi_j) - 2\log c_j\right] + o(1), \tag{1.12}$$

where H_j is defined as the solution of (1.7), U_j and $\Gamma_{\mu_j,\varepsilon,\xi_j}(x)$ are given by (2.3) and (1.9), respectively. Moreover, $\xi_j \in \Omega$ and $\hat{c}_j = 8\pi$ for $j \leq k$; $\xi_j \in \partial\Omega$ and $\hat{c}_j = 4\pi$ for $k < j \leq m$. In addition, the m -tuple $(\xi_1^\varepsilon, \cdots, \xi_m^\varepsilon)$ converges to a critical point of $\mathcal{J}_m$ as $\varepsilon \to 0$, where $\mathcal{J}_m$ is defined by

$$\mathcal{J}_m = \sum_{j=1}^{m} \hat{c}_j^2 H(x_j, x_j) + \sum_{j\neq l} \hat{c}_j \hat{c}_l G(x_j, x_l). \tag{1.13}$$

In particular, the critical points of $\mathcal{J}_m$ are assumed to be non-degenerate and $\bar{C} := \sum_{j=1}^{m} \hat{c}_j C_\Omega$, where C_Ω is the positive lower bound of Green's function $G(x,y)$; $c_j := \frac{3\mu_j^2\bar{u}}{8} + O\left(\frac{1}{\sqrt{\chi}}\right)$ and μ_j is determined by

$$\log 8\mu_j^2 = \hat{c}_j H(\xi_j,\xi_j) + \sum_{l\neq j} \hat{c}_l G(\xi_j,\xi_l) - 2\sum_{j=1}^{m} \log c_j.$$

Theorem 1.1 demonstrates that when the intrinsic growth rate is small, system (1.3) admits mutli-spikes for sufficiently large χ. It is worthwhile to mention that in contrast to the multi-spikes of classical Keller–Segel models,[6] the heights of each spike in (1.11) are $O(1)$ rather than $O(\chi)$. Our theoretical results are identical with the conclusion involving the global existence, which is the time-dependent (1.1) does not have any "blow-up" solution in 2D. [32]

We prove Theorem 1.1 by the inner-outer gluing method. In the absence of the logistic source, i.e. $f(u) = 0$, one works with a local semilinear elliptic equation (1.4). However with the source term $f(u) \neq 0$, we are forced to work with a nonlocal elliptic equation written as

$$S(u) := \Delta u + \chi\nabla\cdot(u\nabla(\Delta-1)^{-1}u) + f(u) = 0,$$

where $(\Delta-1)^{-1}$ denotes the inverse of $\Delta-1$ under Neumann boundary condition. Here we borrow the gluing ideas from Ref. 5 in which infinite-time blow-up solutions are constructed to the Keller-Segel system in the absence of $f(u)=0$. The gluing methods we used in this paper are quite flexible and may be useful in dealing with other cross-diffusion systems. As far as we know this seems to be the first existence of spiky solutions to Keller-Segel system with logistic terms. In a future work we shall study the stability of these solutions.

The remaining part of this paper is organized as follows. In section 2, we introduce the idea for the choice of good ansatz. The section 3 is devoted to the linear theory of the inner and outer operators. In section 4, we utilize the inner-outer gluing method to prove the existence of the multi-spikes satisfying (1.11) and (1.12) rigorously. The construction of boundary spikes are discussed in Section 5. Finally, in Section 6, we perform the numerical simulations of the pattern formation within (1.3).

2. The Ansatz of the solution

In this section, we shall present the choice of our ansatz with correction terms and derive the error generated by this approximation. Noting that v can be rewritten as the form $v=-(\Delta-1)^{-1}u$, we have from the u-equation that

$$S(u)=\Delta u+\chi\nabla\cdot(u\nabla(\Delta-1)^{-1}u)+f(u)=0,$$

where $f(u):=\mu u(\bar{u}-u)$.

According to the argument in Subsection 1.3, one obtains for $\varepsilon\ll 1$, $\bar{v}$ has the following form:

$$\bar{v}(x)=\sum_{j=1}^{m}\left[\Gamma_{\mu_j,\varepsilon,\xi_j}(x)+\hat{c}_jH(x,\xi_j)-2\log c_j\right]+O(\varepsilon^{\alpha}), \tag{2.1}$$

where $\alpha\in(0,1)$ is a constant, $m>0$ denotes the number of spikes, $\hat{c}_j$ are defined in Theorem 1.1, $\Gamma_{\mu_j,\varepsilon,\xi_j}(x)$ is given by (1.9) and $H(x,\xi_j)$ satisfies the equation (1.7). On the other hand, in the region $|x-\xi_j|<\delta\varepsilon$ with $\delta>0$, we have the fact that $u=c_je^{\bar{v}}$, where c_j are constants determined by the mass constraints for each spike. It follows that

$$\begin{aligned}u=&\frac{c_j\varepsilon^4}{(\mu_j^2\varepsilon^2+|x-\xi_j|^2)^2}\exp\left\{\hat{c}_jH(x,\xi_j)+\sum_{l\neq j}\left(\log\frac{1}{(\mu_l^2\varepsilon^2+|x-\xi_l|^2)^2}+\hat{c}_lH(x,\xi_l)\right)\right.\\&\left.\qquad\qquad-2\sum_{j=1}^{m}\log c_j+O(\varepsilon^{\alpha})\right\},\end{aligned}$$

where ξ_j is the centre of the j-th spike. Moreover, μ_j is chosen to satisfy

$$\log 8\mu_j^2=\hat{c}_jH(\xi_j,\xi_j)+\sum_{l\neq j}\hat{c}_lG(\xi_l,\xi_j)-2\sum_{j=1}^{m}\log c_j. \tag{2.2}$$

With the help of (2.2), we have for any x satisfies $|x-\xi_j|<\delta\varepsilon$,

$$u(x)=\frac{8c_j\mu_j^2\varepsilon^4}{(\mu_j^2\varepsilon^2+|x-\xi_j|^2)^2}+o(1).$$

Now, we define

$$U_j(y)=\frac{8\mu_j^2}{(\mu_j^2+|y|^2)^2} \tag{2.3}$$

to obtain that u can be approximated by

$$u=\sum_{j=1}^m c_jU_j\Big(\frac{x-\xi_j}{\varepsilon}\Big)+o(1) \text{ in } \Omega. \tag{2.4}$$

From (2.1), we find $v:=-(\Delta-1)^{-1}u$ can be written as

$$v=\varepsilon^2\sum_{j=1}^m c_j\Big(\log\frac{8\mu_j^2\varepsilon^4}{(\mu_j^2\varepsilon^2+|x-\xi_j|^2)^2}+\hat{c}_jH(x,\xi_j)-2\log c_j\Big)+o(1). \tag{2.5}$$

The next step is to compute the error generated by the leading terms (2.4) and (2.5). We denote u_0 and $\bar{v}_0$ as $u_0:=\sum\limits_{j=1}^m c_jU_j\big(\frac{x-\xi_j}{\varepsilon}\big)$ and $\bar{v}_0=\sum\limits_{j=1}^m(\Gamma_{\mu_j,\varepsilon,\xi_j}+\hat{c}_jH_j-2\log c_j)$, then calculate

$$S(u_0)=\Delta_x u_0-\nabla_x u_0\cdot\nabla_x\bar{v}_0-u_0\Delta_x\bar{v}_0+f(u_0).$$

In the region $|x-\xi_j|<\varepsilon\delta$, we have the fact that

$$\begin{cases}\nabla_x u_0=\frac{-32c_j\varepsilon^4\mu_j^2(x-\xi_j)}{(\mu_j^2\varepsilon^2+|x-\xi_j|^2)^3}+\frac{1}{\varepsilon}\sum_{l\neq j}c_l\nabla U_l\big(\frac{x-\xi_l}{\varepsilon}\big),\\ \nabla_x\bar{v}_0=\frac{-4(x-\xi_j)}{\mu_j^2\varepsilon^2+|x-\xi_j|^2}+\nabla_x\tilde{H}_j^\varepsilon,\end{cases} \tag{2.6}$$

and

$$\begin{cases}\Delta_x u_0=c_j\mu_j^2\varepsilon^4\Big(\frac{128|x-\xi_j|^2-64\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2+|x-\xi_j|^2)^4}\Big)+\frac{1}{\varepsilon^2}\sum_{l\neq j}c_l\Delta U_l\big(\frac{x-\xi_l}{\varepsilon}\big),\\ \Delta_x\bar{v}_0=-\frac{1}{\varepsilon^2}U_j\big(\frac{x-\xi_j}{\varepsilon}\big)+\Delta_x\tilde{H}_j^\varepsilon,\end{cases} \tag{2.7}$$

where

$$\tilde{H}_j^\varepsilon(x)=\hat{c}_jH(x,\xi_j)+\sum_{l\neq j}\Big(\log\frac{8\mu_l^2}{(\mu_l^2\varepsilon^2+|x-\xi_l|^2)^2}+\hat{c}_lH(x,\xi_l)\Big). \tag{2.8}$$

Then we use (2.6) and (2.7) to compute the terms $\nabla_x u_0\cdot\nabla_x\bar{v}_0$ and $u_0\Delta_x\bar{v}_0$, which are

$$\begin{aligned}\nabla_x u_0\cdot\nabla_x\bar{v}_0=&\frac{128c_j\varepsilon^4\mu_j^2|x-\xi_j|^2}{(\mu_j^2\varepsilon^2+|x-\xi_j|^2)^4}-\frac{4}{\varepsilon}\Big(\sum_{l\neq j}c_l\nabla U_l\Big(\frac{x-\xi_l}{\varepsilon}\Big)\Big)\cdot\frac{(x-\xi_j)}{\mu_j^2\varepsilon^2+|x-\xi_j|^2}\\&+\frac{c_j}{\varepsilon}\nabla U_j\Big(\frac{x-\xi_j}{\varepsilon}\Big)\cdot\nabla_x\tilde{H}_j^\varepsilon+\frac{1}{\varepsilon}\sum_{l\neq j}c_l\nabla U_l\Big(\frac{x-\xi_l}{\varepsilon}\Big)\cdot\nabla_x\tilde{H}_j^\varepsilon,\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}u_0\Delta_x\bar{v}_0 =& -c_jU_j^2\Big(\frac{x-\xi_j}{\varepsilon}\Big)-\Big(\sum_{l\neq j}c_lU_l\Big(\frac{x-\xi_l}{\varepsilon}\Big)\Big)\cdot U_j\Big(\frac{x-\xi_j}{\varepsilon}\Big)\\ &+c_jU_j\Big(\frac{x-\xi_j}{\varepsilon}\Big)\Delta_x\tilde{H}_j^\varepsilon+\sum_{l\neq j}c_lU_l\Big(\frac{x-\xi_j}{\varepsilon}\Big)\Delta_x\tilde{H}_j^\varepsilon.\end{aligned}\tag{2.10}$$

By collecting (2.7), (2.9) and (2.10), we obtain

$$\begin{aligned}S(u_0)=&c_j\mu_j^2\varepsilon^4\left(\frac{128|x-\xi_j|^2-64\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2+|x-\xi_j|^2)^4}\right)-\left[\frac{128c_j\varepsilon^4\mu_j^2|x-\xi_j|^2}{(\mu_j^2\varepsilon^2+|x-\xi_j|^2)^4}-c_jU_j^2\left(\frac{x-\xi_j}{\varepsilon}\right)\right]\\ &+\frac{1}{\varepsilon^2}\sum_{l\neq j}c_j\Delta U_l\Big(\frac{x-\xi_l}{\varepsilon}\Big)-\Big[-\frac{4}{\varepsilon}\sum_{l\neq j}c_l\nabla U_l\Big(\frac{x-\xi_l}{\varepsilon}\Big)\cdot\frac{(x-\xi_j)}{\mu_j^2\varepsilon^2+|x-\xi_j|^2}\\ &+\frac{1}{\varepsilon}\sum_{l\neq j}c_l\nabla U_l\Big(\frac{x-\xi_l}{\varepsilon}\Big)\cdot\nabla_x\tilde{H}_j^\varepsilon-\Big(\sum_{j\neq l}c_lU_l\Big(\frac{x-\xi_l}{\varepsilon}\Big)\Big)\cdot U_j\Big(\frac{x-\xi_j}{\varepsilon}\Big)\\ &+\sum_{l\neq j}c_lU_l(\frac{x-\xi_l}{\varepsilon})\Delta_x\tilde{H}_j^\varepsilon\Big]-\overbrace{\Big[\frac{c_j}{\varepsilon}\nabla U_j\Big(\frac{x-\xi_j}{\varepsilon}\Big)\cdot\nabla_x\tilde{H}_j^\varepsilon+c_jU_j\Big(\frac{x-\xi_j}{\varepsilon}\Big)\Delta_x\tilde{H}_j^\varepsilon\Big]}^{I_{21}}\\ &+\mu c_jU_j\Big(\frac{x-\xi_j}{\varepsilon}\Big)\Big[\bar{u}-c_0U_j\Big(\frac{x-\xi_j}{\varepsilon}\Big)\Big]\\ &+\mu\sum_{l\neq j}c_lU_l\Big(\frac{x-\xi_j}{\varepsilon}\Big)\Big[\bar{u}-\Big(\sum_{l=1}U_l\Big(\frac{x-\xi_l}{\varepsilon}\Big)\Big)-U_j\Big(\frac{x-\xi_j}{\varepsilon}\Big)\Big].\end{aligned}$$

We can observe from $S(u_0)$ that I_{21} has the main contribution to the error and the other terms can be neglected. To estimate I_{21}, we need to define $H_1^j(x,\xi_j):=-|x-\xi_j|^2\log|x-\xi_j|$ and decompose $\tilde{H}_j^\varepsilon$ into $H_1^j(x,\xi_j)+\bar{H}_j^\varepsilon(x,\xi_j)$ since $\tilde{H}_j^\varepsilon$ is only a class of $C^{1,\alpha}$ functions. Noting that $\bar{H}_j^\varepsilon$ has the good regularity, it is available for us to expand $\nabla\bar{H}_j^\varepsilon$ at ξ_j up to the $O(\varepsilon^{1+\alpha})$ term.

Now, we have I_{21} can be rewritten as:

$$I_{21}=c_j\nabla_x\cdot(U_j\nabla\tilde{H}_j^\varepsilon)=\overbrace{c_j\nabla_x\cdot(U_j\nabla_xH_1^j)}^{I_{211}}+\overbrace{c_j\nabla_x\cdot(U_j\nabla_x\bar{H}_j^\varepsilon)}^{I_{212}},$$

where

$$\bar{H}_j^\varepsilon:=\hat{c}_jH_2(x,\xi_j)+\sum_{l\neq j}\Big(\log\frac{8\mu_l^2}{(\mu_l^2\varepsilon^2+|x-\xi_l|^2)^2}+\hat{c}_lH(x,\xi_l)\Big),\tag{2.11}$$

and $H_2\in C^{2,\alpha}$ satisfies

$$\begin{cases}-\Delta H_2^j+H_2^j=-\frac{4}{\hat{c}_j}+\frac{1}{\hat{c}_j}|x-\xi_j|^2\log|x-\xi_j|, & x\in\Omega,\\ \frac{\partial H_2^j}{\partial\boldsymbol{\nu}}=\frac{4}{\hat{c}_j}\frac{(x-\xi_j)\cdot\boldsymbol{\nu}}{|x-\xi_j|^2}+\frac{1}{\hat{c}_j}(2\log|x-\xi_j|+1)\cdot(x-\xi_j)\cdot\boldsymbol{\nu}, & x\in\partial\Omega.\end{cases}$$

By analyzing I_{211} and I_{212}, we can obtain the estimate of I_{21}. To this end, we compute ∇H_1^j and ΔH_2^j, which are

$$\nabla H_1^j=-(2\log|x-\xi_j|+1)\cdot(x-\xi_j)\text{ and }\Delta H_1^j=-4\log|x-\xi_j|-4.$$

Upon substituting ∇H_1^j and ∇H_2^j into I_{211}, we have

$$\begin{aligned}
I_{211} =& \frac{c_j}{\varepsilon}\nabla_y U_j \cdot \nabla_x H_1^j + c_j U_j \Delta_x H_1^j \\
=& -\frac{32\mu^2 c_j y}{(\mu^2+|y|^2)^3}\frac{1}{\varepsilon}\Big[-2(\log|x-\xi_j|+1)\cdot(x-\xi_j)\Big] \\
& + \frac{8\mu c_j}{(\mu^2+|y|^2)^2}(-4\log|x-\xi_j|-4) \\
=& \frac{64\mu^2 c_j|y|^2}{(\mu^2+|y|^2)^3}\Big(\log\frac{1}{\varepsilon}+\log|y|\Big) + \frac{64\mu^2 c_j|y|^2}{(\mu^2+|y|^2)^3} \\
& - \frac{32\mu^2 c_j}{(\mu^2+|y|^2)^2}\Big(\log\frac{1}{\varepsilon}+\log|y|\Big) - \frac{32\mu^2 c_j}{(\mu^2+|y|^2)^2} \\
=& O\Big(\log\frac{1}{\varepsilon}\Big).
\end{aligned} \tag{2.12}$$

To tackle the term I_{212}, we firstly expand $\nabla \bar{H}_j^\varepsilon$ at ξ_j to obtain

$$\nabla \bar{H}_j^\varepsilon(x) = \nabla \bar{H}_j^\varepsilon(\xi_j) + \nabla^2 \bar{H}_j^\varepsilon(\xi_j)(x-\xi_j) + O(|x-\xi_j|^{1+\alpha}), \tag{2.13}$$

where $\alpha\in(0,1)$. In this case, I_{212} can be written as

$$\begin{aligned}
I_{212} =& c_j\nabla_x\cdot[U_j(\nabla_x \bar{H}_j^\varepsilon(\xi_j) + \nabla^2 \bar{H}_j^\varepsilon(\xi_j)(x-\xi_j))] + O(\varepsilon^\alpha) \\
=& \frac{c_j}{\varepsilon}\nabla_y U_j\cdot\nabla_x \bar{H}_j^\varepsilon + c_j\nabla_x\cdot[U_j\nabla^2\bar{H}_j^\varepsilon(\xi_j)\varepsilon y] + O(\varepsilon^\alpha) = O\Big(\frac{1}{\varepsilon}\Big).
\end{aligned} \tag{2.14}$$

Combining (2.12) and (2.14), we obtain I_{21} is the $O\big(\frac{1}{\varepsilon}\big)$ term. It follows that the leading order term of $\varepsilon^2 S(u_0)$ is $\mathrm{O}(\varepsilon)$. Moreover, we find that except I_{21}, other terms in $S(u_0)$ are both $\mathrm{O}(\varepsilon^2)$. Therefore, we can use the $\mathrm{O}(\varepsilon^2)$ correction term in the ansatz of u and adjust ξ_j to eliminate the $O(\varepsilon)$ error.

In the gluing method, it is necessary to impose the orthogonality condition on the right hand side such that the solution ϕ has the fast decay property. In $S(u_0)$, we find that $\nabla_x\cdot(U_j\nabla_x\tilde{H}_j^\varepsilon)$ is the leading order term. Thus, we need to calculate its mass and first moment to check the effect on the orthogonality condition for each spike. First of all, we investigate the mass of $\nabla_x\cdot(U_j\nabla_x\tilde{H}_j^\varepsilon)$. By using the divergence theorem, one obtains

$$\begin{aligned}
\int_{B_{R_\varepsilon}}\nabla_x\cdot\big(U_j\nabla_x H_1^j\big)dy =& \int_{B_{R_\varepsilon}}\nabla_y\cdot(U_j(2\log|\varepsilon y|+1)y)dy \\
=& \int_{\partial B_{R_\varepsilon}} U_j(2\log|\varepsilon y|+1)y\cdot\boldsymbol{\nu}dS_y = O(\varepsilon^2).
\end{aligned} \tag{2.15}$$

For the term $\nabla\cdot(U_j\nabla\bar{H}_j^\varepsilon)$, we only need to analyze the mass of $\nabla\cdot(U_j\nabla\bar{H}_j^\varepsilon(\xi_j))$ and $\nabla\cdot(U_j\nabla^2\bar{H}_j^\varepsilon(\xi_j))$ since other terms are higher order and negligible. The integral of $\nabla\cdot(U_j\nabla\bar{H}_j^\varepsilon(\xi_j))$ is

$$\int_{B_{R_\varepsilon}}\nabla_x\cdot(U_j\nabla\bar{H}_j^\varepsilon(\xi_j))dy = \frac{1}{\varepsilon}\int_{\partial B_{R_\varepsilon}} U_j\nabla\bar{H}_j^\varepsilon(\xi_j)\cdot\boldsymbol{\nu}dS_y = O(\varepsilon^2), \tag{2.16}$$

where $R_\varepsilon := R/\varepsilon$ with constant $R>0$. To study $\nabla\cdot(U_j\nabla^2\bar{H}_j^\varepsilon(\xi_j))$, we let $A := \nabla^2\bar{H}_j^\varepsilon(\xi_j)$ to find

$$\int_{B_{R_\varepsilon}}\nabla_x\cdot(U_jAy)dy=\int_{\partial B_{R_\varepsilon}}U_jAy\cdot\boldsymbol{\nu}dS_y=O(\varepsilon^2). \tag{2.17}$$

Combining (2.15), (2.16) and (2.17), we prove that

$$\int_{B_{R_\varepsilon}}\nabla_x\cdot(U_j\nabla_x\tilde{H}_j^\varepsilon)dy=O(\varepsilon^2).$$

Now, we discuss the first moment of $\nabla_x\cdot(U_j\nabla_x\tilde{H}_j^\varepsilon)$. Similarly as above, for the term $\nabla_x\cdot\big(U_j\nabla_xH_1^j\big)$, we have from the integration by parts that for $i=1,2$,

$$\begin{aligned}\int_{B_{R_\varepsilon}}\nabla_x\cdot\big(U_j\nabla_xH_1^j\big)y_idy&=\int_{B_{R_\varepsilon}}\nabla_y\cdot(U_j(2\log|\varepsilon y|+1)\cdot y)y_idy\\&=\int_{\partial B_{R_\varepsilon}}U_jy_i(2\log|\varepsilon y|+1)y\cdot\boldsymbol{\nu}dS_y\\&\quad-\int_{B_{R_\varepsilon}}U_j(2\log|\varepsilon y|+1)y_idy=O(\varepsilon).\end{aligned}$$

and for $\nabla_x\cdot(U_jAy)$, we obtain

$$\int_{B_{R_\varepsilon}}\nabla_y\cdot(U_jAy)y_idy=\int_{\partial B_{R_\varepsilon}}U_jAy\cdot\boldsymbol{\nu}y_idS_y-\int_{B_{R_\varepsilon}}U_jAy\cdot e_idy=O(\varepsilon),$$

where $e_1=(1,0)$ and $e_2=(0,1)$. However, for $\nabla_x\cdot\big(U_j\nabla\bar{H}_j^\varepsilon(\xi_j)\big)$, we calculate to find

$$\begin{aligned}&\int_{B_{R_\varepsilon}}\nabla_x\cdot\big(U_j\nabla\bar{H}_j^\varepsilon(\xi_j)\big)y_idy=\frac{1}{\varepsilon}\int_{B_{R_\varepsilon}}\nabla_y\cdot(U_j\nabla\bar{H}_j^\varepsilon(\xi_j))y_idy\\=&\frac{1}{\varepsilon}\int_{\partial B_{R_\varepsilon}}y_iU_j\nabla\bar{H}_j^\varepsilon(\xi_j)\cdot\boldsymbol{\nu}dS_y-\frac{1}{\varepsilon}\int_{B_{R_\varepsilon}}U_j\nabla\bar{H}_j^\varepsilon(\xi_j)\cdot e_idy=O\Big(\frac{1}{\varepsilon}\Big),\end{aligned}$$

which implies for $i=1,2$,

$$\int_{B_{R_\varepsilon}}\nabla_x\cdot(U_j\nabla_x\tilde{H}_j^\varepsilon)y_idy=O\Big(\frac{1}{\varepsilon}\Big).$$

In summary, we obtain that $\nabla\bar{H}_j^\varepsilon$ plays the significant role on the error analysis. Moreover, it does not effect the mass orthogonality condition but effect the first moment condition. It is left to determine the leading order term of the spike height c_j. By decomposing c_j into $c_{j0}+c_{j1}$, we have from the mass constraint $\int_{\mathbb{R}^2}U_j(\bar{u}-c_{j0}U_j)\mathrm{d}y=0$ and (2.3) that $c_{j0}=\frac{3}{8}\mu_j^2\bar{u}$.

We next construct the $\mathrm{O}(\varepsilon^2)$ correction term so as to balance the logistic source. To be more precisely, we choose the ansatz of u and v as the following form:

$$u=\sum_{j=1}^mc_jU_j\Big(\frac{x-\xi_j}{\varepsilon}\Big)+\mu\varepsilon^2\sum_{j=1}^mc_j\varphi_j\Big(\frac{x-\xi_j}{\varepsilon}\Big)+\phi\Big(\frac{x}{\varepsilon}\Big), \tag{2.18}$$

and

$$\begin{aligned} v = \varepsilon^2 \sum_{j=1}^{m} c_j \Big(\log \frac{8\mu_j^2 \varepsilon^4}{(\mu_j^2 \varepsilon^2 + |x-\xi_j|^2)^2} + \hat{c}_j H_j(x,\xi_j) - 2\log c_j \Big) \\ + \mu \varepsilon^4 \sum_{j=1}^{m} c_j \Big[\psi_j \Big(\frac{x-\xi_j}{\varepsilon} \Big) + \mathcal{H}_j(x,\xi_j) \Big] + \varepsilon^2 \bar{\psi}\Big(\frac{x}{\varepsilon}\Big), \end{aligned} \tag{2.19}$$

where φ_j is the correction term of U_j, $\psi_j(y) = (-\Delta_y)^{-1}\varphi_j$ and $\mathcal{H}_j$ is the correction term of ψ_j, which satisfies

$$\Delta_x \Big[\psi_j \Big(\frac{x-\xi_j}{\varepsilon} \Big) + \mathcal{H}_j \Big] - \Big[\psi_j \Big(\frac{x-\xi_j}{\varepsilon} \Big) + \mathcal{H}_j \Big] + \varphi_j \Big(\frac{x-\xi_j}{\varepsilon} \Big) = 0, \ \forall x \in \Omega.$$

Here ϕ and $\bar{\psi}$ are $o(1)$ terms what we need to glue in Section 4. We further have φ_j satisfies the following equation:

$$\begin{cases} \nabla \cdot (U_j \nabla g_j) + \mu_j U_j (\bar{u} - c_{0j} U_j) = 0 & \text{in } \mathbb{R}^2, \\ g_j := \dfrac{\varphi_j}{U_j} - \psi_j, \quad -\Delta_y \psi_j = \varphi_j & \text{in } \mathbb{R}^2. \end{cases} \tag{2.20}$$

Since U_j is radial, we are able to obtain the explicit solution of (φ, ψ) under the assumption that φ and ψ are radial by solving the ODE system.

Without confusing the reader, we drop "j" and use c_0, U, g, φ and ψ to replace c_{0j}, U_j, g_j and φ_j, ψ_j. We firstly rewrite the g-equation in (2.20) as the following form:

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(Ur \frac{\mathrm{d}g}{\mathrm{d}r} \Big) + \mu r U (\bar{u} - c_0 U) = 0.$$

Since $c_0 = \frac{3}{8}\mu^2 \bar{u}$, we integrate the ODE to obtain

$$\frac{\mathrm{d}g}{\mathrm{d}r} = \mu \frac{\bar{u}}{2} r + \frac{\bar{u}\mu^3 r}{2(\mu^2 + r^2)}.$$

We further solve it to get g has the following form:

$$g(r) = \frac{\bar{u}}{4}\mu r^2 + \frac{\bar{u}\mu^3}{4} \ln(\mu^2 + r^2). \tag{2.21}$$

Noting that ψ satisfies the following equation:

$$-\Delta \psi = U\psi + Ug \ \text{ in } \ \mathbb{R}^2,$$

we use the variation-of-constant formula to find

$$\psi(r) = z_0(r) \int_r^a U(\rho) g(\rho) \tilde{z}_0(\rho) \rho \mathrm{d}\rho + \tilde{z}_0(r) \int_0^r U(\rho) g(\rho) z_0(\rho) \rho \mathrm{d}\rho, \tag{2.22}$$

where a is a large positive constant, z_0 is defined by $z_0 = \frac{\mu^2 - |y|^2}{\mu^2 + |y|^2}$ and $\tilde{z}$ is the other linear independent kernel. Upon substituting (2.21) into (2.22), we can obtain the closed form of ψ. Moreover, the solution $\varphi := Ug + U\psi$ can be expressed in terms of (2.21) and (2.22).

We similarly compute the error generated by the new ansatz (2.18) and (2.19), then establish the inner and outer equation of ϕ and $\bar{\psi}$. We define

$$u_1 := \sum_{j=1}^{m} c_j U_j\Big(\frac{x-\xi_j}{\varepsilon}\Big) + \mu\varepsilon^2 \sum_{j=1}^{m} c_j \varphi_j\Big(\frac{x-\xi_j}{\varepsilon}\Big),$$

and

$$\bar{v}_1 := \sum_{j=1}^{m} [\Gamma_{\mu_j,\varepsilon,\xi_j} + \hat{c}_j H_j + \log 8\mu_j^2 - 2\log c_j] + \mu\varepsilon^2 \sum_{j=1}^{m} \Big[\psi_j\Big(\frac{x-\xi_j}{\varepsilon}\Big) + \mathcal{H}_j(x,\xi_j)\Big],$$

and substitute $u = u_1 + \phi$ and $\bar{v} = \bar{v}_1 + \bar{\psi}$ into $S(u)$ to obtain

$$S(u_1+\phi) = \nabla_x \cdot \big(\nabla_x(u_1+\phi) - (u_1+\phi)\nabla_x(\bar{v}_1+\bar{\psi})\big) + \mu(u_1+\phi)(\bar{u} - (u_1+\phi)) = 0.$$

To analyze it, we let $y = \frac{x}{\varepsilon}$ and $\xi_j' = \frac{\xi}{\varepsilon}$ then calculate ∇u, $\nabla \bar{v}$ and Δu, $\Delta \bar{v}$ to obtain

$$\nabla_x u = \frac{1}{\varepsilon}\sum_{j=1}^{m} c_j \nabla U_j(y-\xi_j') + \mu\varepsilon \sum_{j=1}^{m} c_j \nabla\varphi_j(y-\xi_j') + \frac{1}{\varepsilon}\nabla_y \phi; \tag{2.23}$$

$$\begin{aligned}\nabla_x \bar{v} =& \frac{1}{\varepsilon}\sum_{j=1}^{m} \nabla_y \Gamma(y-\xi_j') + \sum_{j=1}^{m} \nabla_x H_j^{\varepsilon}(\varepsilon y, \xi) \\ &+ \mu\varepsilon \sum_{j=1}^{m} [\nabla_y \psi_j(y-\xi_j') + \mu\varepsilon^2 \nabla_x \mathcal{H}_j^{\varepsilon}(\varepsilon y,\xi)] + \frac{1}{\varepsilon}\nabla_y\bar{\psi},\end{aligned} \tag{2.24}$$

and

$$\Delta_x u = \frac{1}{\varepsilon^2}\sum_{j=1}^{m} c_j \Delta U_j(y-\xi_j') + \mu \sum_{j=1}^{m} c_j \Delta\varphi_j(y-\xi_j') + \frac{1}{\varepsilon^2}\Delta_y \phi; \tag{2.25}$$

$$\begin{aligned}\Delta_x \bar{v} =& -\frac{1}{\varepsilon^2}\sum_{j=1}^{m} U_j(y-\xi_j') + \sum_{j=1}^{m} \Delta H_j^{\varepsilon} + \mu \sum_{j=1}^{m} \Delta\psi_j(y-\xi_j') \\ &+ \mu\varepsilon^2 \sum_{j=1}^{m} \Delta \mathcal{H}_j^{\varepsilon}(\varepsilon y, \varepsilon) + \frac{1}{\varepsilon^2}\Delta_y \bar{\psi}.\end{aligned} \tag{2.26}$$

By using (2.23), (2.24), (2.25) and (2.26), we obtain $S(u) = \sum_{k=1}^{8} \tilde{I}_k$, where

$$\tilde{I}_1 = -\frac{1}{\varepsilon^2}\sum_{l=1}^{m}\sum_{j\neq l} c_l U_l(y-\xi_l') \cdot \Delta\Gamma_j(y-\xi_j), \tag{2.27}$$

$$
\begin{aligned}
\tilde{I}_2 = &-\frac{1}{\varepsilon^2}\sum_{l=1}^{m}\sum_{j\neq l} c_l \nabla U_l(y-\xi_l')\cdot\nabla\Big(\Gamma(y-\xi_j')+\varepsilon\hat{c}_j H_j(\varepsilon y,\xi_j)\Big)\\
&-\frac{1}{\varepsilon}\sum_{j=1}^{m} c_j\hat{c}_j\nabla U_j(y-\xi_j')\cdot\nabla H_j(\varepsilon y,\xi_j)\\
&+\sum_{l=1}^{m}\sum_{j=1}^{m} c_l\hat{c}_j U_l(y-\xi_l')\cdot\Delta H_j(\varepsilon y,\xi_j),
\end{aligned} \tag{2.28}
$$

$$
\begin{aligned}
\tilde{I}_3 = -\mu\sum_{l=1}\sum_{j\neq l}\Big(& c_l\nabla U_l(y-\xi_l')\cdot\nabla\psi_j(y-\xi_j')+c_l\nabla\varphi_l(y-\xi_l')\cdot\nabla\Gamma_j(y-\xi_j')\\
&+c_l U_l(y-\xi_l')\cdot\Delta\psi_j(y-\xi_j')+c_l\varphi_l(y-\xi_l')\cdot\Delta\Gamma_j(y-\xi_j')\Big),
\end{aligned} \tag{2.29}
$$

$$
\begin{aligned}
\tilde{I}_4 := &-\mu\sum_{j=1}^{m}\sum_{l=1}^{m}\Big(\varepsilon\nabla\mathcal{H}_l(\varepsilon y,\xi_l)\cdot c_j\nabla U_j(y-\xi_j')\\
&+\varepsilon c_l\hat{c}_j\nabla\varphi_l(y-\xi_l')\cdot\nabla H_j(\varepsilon y,\xi_j)+\varepsilon^2 c_l U_l(y-\xi_l')\cdot\Delta\mathcal{H}_j(\varepsilon y,\xi_j)\\
&+\varepsilon^2\hat{c}_j c_l\varphi_l(y-\xi_l')\cdot\Delta H_j(\varepsilon y,\xi_j)\Big)
\end{aligned} \tag{2.30}
$$

$$
\begin{aligned}
\tilde{I}_5 = &-\mu^2\sum_{j=1}^{m}\sum_{l=1}^{m} c_l\varepsilon^2\nabla\varphi_l(y-\xi_l')\cdot\big(\nabla\psi_j(y-\xi_j')+\varepsilon\nabla\mathcal{H}_j(\varepsilon y,\xi_j)\big)\\
&-\mu^2\sum_{j=1}^{m}\sum_{l=1}^{m} c_l\varepsilon^2\varphi_l(y-\xi_l')\cdot\big(\Delta\psi_j(y-\xi_j')+\varepsilon^2\Delta\mathcal{H}_j(\varepsilon y,\xi_j)\big)
\end{aligned} \tag{2.31}
$$

$$
\begin{aligned}
\tilde{I}_6 = &-\frac{1}{\varepsilon}\nabla\phi\cdot\sum_{j=1}\hat{c}_j\nabla H_j(\varepsilon y,\xi_j)-\phi\cdot\sum_{j=1}\hat{c}_j\Delta H_j(\varepsilon y,\xi_j)\\
&-\frac{1}{\varepsilon}\nabla\phi\cdot\sum_{j=1}\big(\mu\varepsilon\nabla\psi_j(y-\xi_j')+\mu\varepsilon^2\nabla\mathcal{H}_j(\varepsilon y,\xi_j)\big)\\
&-\phi\cdot\sum_{j=1}\big(\mu\Delta\psi_j(y-\xi_j')+\mu\varepsilon^2\Delta\mathcal{H}_j(\varepsilon y,\xi_j)\big)\\
&-\mu\Big(\sum_{j=1}\nabla\varphi_j(y-\xi_j')\Big)\cdot\nabla\bar{\psi}-\mu\Big(\sum\varphi_j(y-\xi_j')\Big)\cdot\Delta\bar{\psi}\\
&-\frac{1}{\varepsilon^2}(\phi\Delta\bar{\psi}+\nabla\phi\cdot\nabla\bar{\psi}),
\end{aligned} \tag{2.32}
$$

$$
\begin{aligned}
\tilde{I}_7 =&\mu\Big(\sum_{j=1} c_j U_j(y-\xi_j')+\mu\varepsilon^2\sum_{j=1} c_j\varphi_j(y-\xi_j')+\phi\Big)\\
&\times\Big(\bar{u}-\sum_{j=1} c_j U_j(y-\xi_j')-\mu\varepsilon^2\sum c_j\varphi_j(y-\xi_j')-\phi\Big),
\end{aligned} \tag{2.33}
$$

and

$$\begin{aligned}\tilde{I}_8 :=& \frac{1}{\varepsilon^2}\Delta\phi(y) - \frac{1}{\varepsilon^2}\Big(\sum_{l=1}^{m}\nabla U_l(y-\xi_l')\Big)\cdot\nabla\bar{\psi}(y) - \frac{1}{\varepsilon^2}\big(\sum_{l=1}^{m}U_l\big)\cdot\Delta\bar{\psi}(y)\\ &- \frac{1}{\varepsilon^2}\nabla\phi(y)\cdot\sum_{l=1}^{m}\nabla\Gamma(y-\xi_l') - \frac{1}{\varepsilon^2}\Big(\phi(y)\cdot\sum_{l=1}\Delta\Gamma(y-\xi_l')\Big).\end{aligned} \tag{2.34}$$

In particular, $\tilde{I}_7$ can be decomposed into the following form:

$$\begin{aligned}\tilde{I}_7 =&\mu\sum_{j=1}^{m}c_{1j}U_j\Big(\bar{u}-2c_0\sum_{j=1}^{m}U_j-\sum_{j=1}^{m}c_{1j}U_j\Big)-\mu c_0\sum_{j\neq l}c_{1j}U_jU_l\\ &-\mu\Big(\mu\varepsilon^2\sum_{j=1}c_j\varphi_j+\phi-\bar{u}+2\sum_{j=1}c_jU_j\Big)\Big(\mu\varepsilon^2\sum_{j=1}c_j\varphi_j+\phi\Big),\end{aligned} \tag{2.35}$$

where $c_{j1} := c_j - c_{j0}$ with $c_{j0} = \frac{3\mu_j^2\bar{u}}{8}$. We collect (2.27)–(2.33) and define $-\tilde{I}_8$ shown in (2.34) as $L[\phi]$ to formulate the equation of ϕ as

$$\begin{cases} L[\phi] = -\Delta\phi + \nabla\cdot\big(W\nabla\psi\big) + \nabla\cdot\big(\phi\nabla V\big) = \varepsilon^2\sum\limits_{k=1}^{7}\tilde{I}_k(\phi,\mathbf{p}) & \text{in } \ \Omega_\varepsilon,\\ \dfrac{\partial\phi}{\partial\boldsymbol{\nu}} = 0 & \text{on } \ \partial\Omega_\varepsilon. \end{cases} \tag{2.36}$$

where

$$W(y) = \sum_{j=1}^{m}\frac{8\mu_j^2}{(\mu_j^2+|y-\xi_j'|^2)^2} \quad , \quad V(y) = \sum_{j=1}^{m}\log\frac{8\mu_j^2}{(\mu_j^2+|y-\xi_j'|^2)^2}, \tag{2.37}$$

and

$$\mathbf{p} = (c_1,\dots,c_m,\xi_1,\cdots,\xi_m).$$

We can construct the multi-spikes with the form of (1.11) and (1.12) to system (1.3) by proving the existence of $(\phi,\mathbf{p})$ to (2.36). It is necessary to point out that the behavior of the spiky solutions is distinct in the inner and outer regions. Thus, ϕ should be decomposed into the sum of inner and outer solutions. To show their existence, we need to formulate the inner and outer operators then analyze the properties. In Section 3, the inner and outer linear theories will be established.

3. Linear Theory

The key ingredient in the gluing method is the formation of the linear theory. Noting that the structure of (2.36) is similar as that in Ref. 5. Therefore, we can use the same idea to investigate the inner and outer linear theory, respectively.

3.1. *Inner Linear Theory*

In the inner region, we shall study the situation of each spike. To be more precisely, we consider the region $|x-\xi_j|<\delta$, where $\delta>0$ is a constant. By defining ξ_j as the center of the j-th spike and using the scaling: $y=\frac{x-\xi_j}{\varepsilon}$, we obtain that the region becomes $|y|<\frac{\delta}{\varepsilon}$. For sufficiently small ε, one has the region is approximated by the whole space $\mathbb{R}^2$. Therefore, the inner operator can be defined by

$$L_j[\phi]:=-\Delta_y\phi+\nabla\cdot(U_j\nabla_y\psi)+\nabla\cdot(\phi\nabla_y\ln U_j), \tag{3.1}$$

where $\psi=(-\Delta)^{-1}\phi$. We denote $-h$ as the error then utilize (3.1) to formulate the following inner problem:

$$L_j[\phi]=h \ \text{ in } \ \mathbb{R}^2. \tag{3.2}$$

For the simplification of calculation, we assume $\mu=1$ and write (2.3) as $U=\frac{8}{(1+|y|^2)^2}$. We would like to mention that the constant μ can not influence the structure of kernels to the operator (3.1). Moreover, the inner norm is given by $\|\cdot\|_\nu$, which is

$$\|h\|_\nu:=\sup_{y\in\mathbb{R}^2}|h|(1+|y|)^\nu,$$

where $\nu>0$ is a constant. First of all, we assume that the location ξ_j is in the interior of Ω. With the help of Fourier projection, one can obtain the following Lemma:

Lemma 3.1. *Assume that h satisfies*

$$\int_{\mathbb{R}^2}h(y)\mathrm{d}y=0,\quad \int_{\mathbb{R}^2}h(y)y_j\mathrm{d}y=0 \ \ \textit{for} \ \ j=1,2, \tag{3.3}$$

then for any $\|h\|_{4+\sigma}<\infty$ with $\sigma\in(0,1)$, there exists the solution $\phi=\mathcal{T}_{in}^j[h]$ to (3.2) *such that*

$$\|\phi\|_{2+\sigma}\le C\|h\|_{4+\sigma}, \tag{3.4}$$

where $\mathcal{T}_{in}^j[h]$ is a continuous linear operator from the Banach space $\mathcal{C}^*$ of all functions h in L^∞ for which $\|h\|_{4+\sigma}<\infty$ into L^∞.

Proof. Thanks to the results obtained in Ref. 5, we have (3.2) can be rewritten as the following equation:

$$\begin{cases}\nabla\cdot(U\nabla g)=h & \text{in } \ \mathbb{R}^2,\\ g=\dfrac{\phi}{U}-\psi, \quad -\Delta\psi=\phi & \text{in } \ \mathbb{R}^2.\end{cases} \tag{3.5}$$

Define $\tilde{g}:=Ug$, then we have from the Fourier expansion that

$$h=\sum_{k=0}^{\infty}\frac{1}{k!}h_k(r)e^{ik\theta},\qquad \psi=\sum_{k=0}^{\infty}\frac{1}{k!}\psi_k(r)e^{ik\theta}, \tag{3.6}$$

and

$$\tilde{g} = \sum_{k=1}^{\infty} \frac{1}{k!}\tilde{g}_k(r)e^{ik\theta}, \quad g = \sum_{k=0}^{\infty} \frac{1}{k!}g_k(r)e^{ik\theta}, \tag{3.7}$$

where $h_k(r)$, $\psi_k(r)$ and $g_k(r)$, $\tilde{g}_k(r)$ are both radial. Upon substituting (3.6) and (3.7) into (3.5), we obtain for $k \geq 0$,

$$\begin{cases} \dfrac{\mathrm{d}^2\tilde{g}_k}{\mathrm{d}r^2} + \dfrac{1}{r}\dfrac{\mathrm{d}\tilde{g}_k}{\mathrm{d}r} - \dfrac{k^2}{r^2}\tilde{g}_k + \dfrac{4r}{(1+r^2)}\dfrac{\mathrm{d}\tilde{g}_k}{\mathrm{d}r} + \tilde{g}_k U - h_k = 0, \\ \dfrac{\mathrm{d}^2\psi_k}{\mathrm{d}r^2} + \dfrac{1}{r}\dfrac{\mathrm{d}\psi_k}{\mathrm{d}r} - \dfrac{k^2}{r^2}\psi_k + \dfrac{8}{(1+r^2)^2}\psi_k + \tilde{g}_k = 0. \end{cases} \tag{3.8}$$

We observe that (3.8) are ODE systems and we can solve them to find solutions. First of all, we investigate the homogeneous problems of the second equations in (3.8), which are

$$\frac{\mathrm{d}^2\psi_k}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\psi_k}{\mathrm{d}r} - \frac{k^2}{r^2}\psi_k + \frac{8}{(1+r^2)^2}\psi_k = 0. \tag{3.9}$$

We have the fact that there exists the regular fundamental solutions to (3.9) when $k = 0, 1$. Hence, in these cases, we need to impose the orthogonality condition so as to rule out the regular kernels and guarantee the fast decay properties of solutions. When $k \geq 2$, we find (3.9) does not admit any regular kernel. It follows that we do not need to consider the orthogonality conditions when $k \geq 2$ and can easily construct the barrier to show the decay property of ψ_k. We next analyze the delicate case $k = 1$.

For the $\tilde{g}$-equation in (3.8), we would like to construct the barrier so as to give the decay estimate. To this end, we denote the operator $\mathcal{L}$ as

$$\mathcal{L}[w] := -\frac{\mathrm{d}^2}{\mathrm{d}r^2}w - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}w + \frac{1}{r^2}w - Uw - \frac{4r}{1+r^2}\frac{\mathrm{d}}{\mathrm{d}r}w, \tag{3.10}$$

and the barrier w is given by

$$w = \frac{C\|h\|_{4+\sigma}}{(1+r)^{2+\sigma}}, \tag{3.11}$$

where $\sigma > 0$ is a small constant and $C > 0$ will be determined later on. We combine (3.11) and (3.10) to get $\hat{w} := w - \tilde{g}_1$ satisfies

$$\begin{aligned} \mathcal{L}[\hat{w}] = \frac{C\|h\|_{4+\sigma}}{(1+r)^{4+\sigma}}\Bigg[& -(2+\sigma)(3+\sigma) + \frac{1+r}{r}(2+\sigma) + \frac{(1+r)^2}{r^2} \\ & - U(1+r)^2 + 4(2+\sigma)\frac{r(1+r)}{1+r^2}\Bigg] + h_1. \end{aligned} \tag{3.12}$$

We choose the constant R large enough such that $\mathcal{L}[\hat{w}] > 0$ for $r > R$. With the fixed $R > 0$, we further set a large constant $C > 0$ to obtain

$$\frac{C\|h\|_{4+\sigma}}{(1+R)^{2+\sigma}} - \max_{y \in \bar{B}_R(0)} \tilde{g}_1 > 0, \tag{3.13}$$

where $\tilde{g}_1$ is bounded in $B_R(0)$. By using the maximum principle, we have from (3.12) and (3.13) that

$$\tilde{g}_1 \leq \frac{C\|h\|_{4+\sigma}}{(1+r)^{2+\sigma}} \quad \text{for} \quad r > R. \tag{3.14}$$

In addition, for $y \in B_R(0)$, we have from the boundedness of $\tilde{g}_1$ that

$$\tilde{g}_1(r) < \max_{y \in \bar{B}_R(0)} \tilde{g}_1 < \frac{C\|h\|_{4+\sigma}}{(1+R)^{2+\sigma}} < \frac{C\|h\|_{4+\sigma}}{(1+r)^{2+\sigma}}. \tag{3.15}$$

Combining (3.14) and (3.15), one can show that w is a sup-solution of $\tilde{g}_1$ and

$$\tilde{g}_1(r) < \frac{C\|h\|_{4+\sigma}}{(1+r)^{2+\sigma}} \text{ in } \mathbb{R}^2.$$

Proceeding similarly with $-\tilde{g}_1$, we get $-\tilde{g}$ have the same estimate, which implies

$$\|\tilde{g}_1\|_{L^\infty} \leq \frac{C\|h\|_{4+\sigma}}{(1+r)^{2+\sigma}} \quad \text{in} \quad \mathbb{R}^2. \tag{3.16}$$

For the second equation in (3.8), we have to check the first moment orthogonality condition in (3.3) so as to obtain the good estimate. By testing y_i, $i = 1, 2$ against the g-equation in (3.5), one has

$$\int_{\mathbb{R}^2} \nabla \cdot (U \nabla g) y_i \mathrm{d}y = \int_{\mathbb{R}^2} h y_i \mathrm{d}y = 0.$$

The left hand side can be written as

$$\int_{\mathbb{R}^2} \nabla \cdot (U \nabla g) y_i \mathrm{d}y = -\int_{\mathbb{R}^2} U \nabla g \cdot e_i \mathrm{d}y = \int_{\mathbb{R}^2} g U \nabla \ln U \cdot e_i \mathrm{d}y, \tag{3.17}$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. For $i = 1$, we further calculate to get

$$\begin{aligned} \int_{\mathbb{R}^2} g U \nabla \ln U \cdot e_1 dy &= \int_0^{2\pi} \int_0^\infty \sum_{j=1}^{\infty} \frac{1}{k!} g_k(r) e^{ik\theta} U(r) z_1(r) r \cos\theta \mathrm{d}r \mathrm{d}\theta \\ &= \pi \int_0^\infty g_1(r) U(r) z_1(r) r \mathrm{d}r. \end{aligned} \tag{3.18}$$

In light of the right hand side in (3.17), we find from (3.18) that

$$\int_0^\infty g_1(r) U(r) z_1(r) r \mathrm{d}r = 0.$$

Then, we can use the variation of parameters formula to choose the solution of the second equation in (3.8) as

$$\psi_1(r) = z_2(r) \int_0^r z_1 U g_1 \rho \mathrm{d}\rho + z_1(r) \int_r^\infty z_2 U g_1 \rho \mathrm{d}\rho, \tag{3.19}$$

where the kernels z_1 and z_2 are given by

$$z_1 = \frac{\mathrm{d}}{\mathrm{d}r} \ln U, \quad z_2 = \frac{r^4 + 4r^2 \log r - 1}{r(r^2+1)}.$$

Now, we have from (3.16) and (3.19) that

$$|\psi_1| \leq C\|h\|_{4+\sigma}(1+r)^{1-\sigma}. \tag{3.20}$$

We next discuss the case $k = 0$. Similarly as above, there exists one of the fundamental solutions corresponding to the second equation in (3.8) is regular. Thus, we have to check the orthogonality conditions. The system satisfied by (g_0, ψ_0) reads

$$\begin{cases} \nabla \cdot (U\nabla g_0) = h_0, \\ \dfrac{\mathrm{d}^2\psi_0}{\mathrm{d}r^2} + \dfrac{1}{r}\dfrac{\mathrm{d}\psi_0}{\mathrm{d}r} + \dfrac{8}{(1+r^2)^2}\psi_0 + \tilde{g}_0 = 0. \end{cases} \tag{3.21}$$

Focusing on the mass orthogonality condition in (3.3), we have

$$0 = \int_{\mathbb{R}^2} h\mathrm{d}y = \int_0^{2\pi}\int_0^\infty \sum_{k=0}^\infty h_k(r)e^{ik\theta}r\mathrm{d}r\mathrm{d}\theta = \int_{\mathbb{R}^2} h_0(r)r\mathrm{d}r.$$

Then we choose the solution of the first equation in (3.21) as

$$g_0(r) = \int_r^\infty \frac{1}{\rho U(\rho)}\int_0^\rho h_0(s)s\mathrm{d}s\mathrm{d}\rho.$$

It follows from the assumption $\|h_0\|_{4+\sigma} < \infty$ that

$$|g_0(r)| \leq C\|h_0\|_{4+\sigma}r^{2-\sigma}. \tag{3.22}$$

Since $\tilde{g}_0 = Ug$, one further finds from (3.22) that

$$|\tilde{g}_0(r)| \leq \frac{C\|h\|_{4+\sigma}}{(1+r)^{2+\sigma}}. \tag{3.23}$$

For the ψ-equation in the case that $k = 0$, the kernels z_{01} and z_{02} are given by

$$z_{01} = \frac{r^2-1}{r^2+1}, \qquad z_{02} = \frac{(r^2-1)\log r - 2}{r^2+1},$$

It is similar to check the orthogonality condition in (3.3) to obtain that

$$\int_0^\infty z_{01}(\rho)\tilde{g}_0(\rho)\rho\mathrm{d}\rho = 0.$$

Then by using the variation of parameters formula, we have

$$\psi_0(r) = z_{02}(\rho)\int_0^r z_{01}(\rho)\tilde{g}_0(\rho)\rho\mathrm{d}\rho + z_{01}(\rho)\int_\rho^\infty z_{02}(\rho)\tilde{g}_0(\rho)\rho\mathrm{d}\rho. \tag{3.24}$$

Upon substituting (3.23) into (3.24), one can show the following estimate

$$|\psi_0(r)| \leq C\|h_0\|_{4+\sigma}(1+\log r). \tag{3.25}$$

Finally, we investigate the case that $k \geq 2$. Since there does not exist any regular kernel for the second equation in (3.8), we do not require to impose any orthogonality condition. Similarly as above, we construct the barrier to estimate $\tilde{g}_k$ and ψ_k via the maximum principle. For the first equation in (3.8), one has

$$|\tilde{g}_k| \le C\|h\|_{4+\sigma}\frac{1}{(1+r)^{2+\sigma}} \quad \text{in} \quad \mathbb{R}^2,$$

where $C > 0$ is a large constant. For the ψ-equation, we denote the operator $\mathcal{L}_1$ as

$$\mathcal{L}_1[w] = -\frac{\mathrm{d}^2}{\mathrm{d}r}w - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}w + \frac{k^2}{r^2}w - \frac{8}{(1+r^2)^2}w,$$

and the barrier function as

$$w_k = C\|h\|_{4+\sigma}(1+r)^{1-\sigma}.$$

By straightforward calculation, one has $\hat{w}_k := w_k - \psi_k$ satisfy

$$\begin{aligned}\mathcal{L}_1[\hat{w}_k] = \frac{C\|h\|_{4+\sigma}}{(1+r)^{1+\sigma}}\Bigg[&-(1-\sigma)(-\sigma) - (1-\sigma)\frac{1+r}{r} \\ &+ k^2\frac{(1+r)^2}{r^2} - \frac{8(1+r)^2}{(1+r^2)^2}\Bigg] - \tilde{g}_k.\end{aligned} \tag{3.26}$$

By similarly taking $R > 0$ large enough, we have $\mathcal{L}_1[\hat{w}_k] > 0$ for any $r > R$. With the fixed R, we choose a large constant $C > 0$ such that

$$\frac{C\|h\|_{4+\sigma}}{(1+R)^{1+\sigma}} - \max_{y\in\bar{B}_R(0)}|\psi_k| \ge 0. \tag{3.27}$$

Invoking (3.26) and (3.27), one can obtain from the maximum principle that

$$\psi_k \le C\|h\|_{4+\sigma}(1+r)^{1-\sigma} \text{ in } \mathbb{R}^2.$$

We similarly apply the maximum principle to $-\psi_k$ and get

$$|\psi_k| \le C\|h\|_{4+\sigma}(1+r)^{1-\sigma} \text{ in } \mathbb{R}^2. \tag{3.28}$$

In summary, we have for $k \ge 0$, $\|\tilde{g}_k(r)\|_{2+\sigma} < \infty$ since h satisfies $\|h\|_{4+\sigma} < \infty$ and ψ_k satisfy (3.20), (3.25) and (3.28) when $k = 0$, $k = 1$ and $k \ge 2$. Since there exists the representation formula of ϕ, one gets (3.5) admits the solution satisfying

$$\phi = Ug + U\psi = \sum_{k=0}^{\infty}\frac{1}{k!}(g_k + \psi_k)e^{ik\theta}.$$

Thus, we use the relationship between ϕ and ψ to find

$$\begin{aligned}|\phi| \le |Ug| + |U\psi| &\le \sum_{k=0}^{\infty}\frac{1}{k!}|\tilde{g}_k| + \sum_{k=0}^{\infty}\frac{1}{k!}|U\psi_k| \\ &\le \sum_{k=0}^{\infty}\frac{1}{k!}\left[C\frac{\|h\|_{4+\sigma}}{(1+r)^{2+\sigma}} + \frac{C\|h\|_{4+\sigma}}{(1+r)^{3+\sigma}}\right] \le \frac{C\|h\|_{4+\sigma}}{(1+r)^{2+\sigma}},\end{aligned}$$

which shows (3.4) and completes the proof of Lemma 3.1. □

Lemma 3.1 is devoted to the formation of inner linear theory for the interior spikes. It is necessary to consider the case that ξ_j is located at the boundary. In this situation, we can regard the domain of the ϕ-equation as the half space $\mathbb{R}^2_+$ rather than $\mathbb{R}^2$, where $\mathbb{R}^2_+ := \{(y_1, y_2)|y_2 > 0\}$. To tackle this issue, the natural idea is to extend ϕ evenly so as to straightforward use Lemma 3.1. Based on this idea, we define some bridge function ϑ then transform our problem into the new one holds in the whole space $\mathbb{R}^2$. By defining the norm $\|\cdot\|_{\nu,H}$ as

$$\|h\|_{\nu,H} = \sup_{y\in\mathbb{R}^2_+} |h|(1+|y|)^\nu,$$

we establish the linear theory of the boundary point, which can be summarized as the following Lemma:

Lemma 3.2. *Given any function h and $\beta(x)$ satisfying*

$$\int_{\mathbb{R}^2_+} h dy - \int_{\partial\mathbb{R}^2_+} \beta dS_y = 0, \quad \int_{\mathbb{R}^2_+} h y_1 dy - \int_{\partial\mathbb{R}^2_+} \beta y_1 dS_y = 0, \tag{3.29}$$

and $\|h\|_{4+\sigma,H} < \infty$ with $\sigma \in (0,1)$, we have the problem

$$\begin{cases} L[\phi] = h & \text{in } \mathbb{R}^2_+, \\ U\dfrac{\partial g}{\partial \boldsymbol{\nu}} = \beta(x) & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{3.30}$$

admits a solution satisfying the following estimate:

$$\|\phi\|_{2+\sigma,H} \leq C\|h\|_{4+\sigma,H} \tag{3.31}$$

where $C > 0$ is a constant and $g = \frac{\phi}{U} - \psi$. Moreover, ϕ satisfies $\phi = \mathcal{T}_H[h]$, where $\mathcal{T}_H[h]$ is defined by a linear operator.

Proof. For any given β defined on $\partial\mathbb{R}^2_+$, there exists a function pair (ϕ_0, ψ_0) such that

$$\frac{\partial \phi_0}{\partial \boldsymbol{\nu}} - U\frac{\partial \psi_0}{\partial \boldsymbol{\nu}} = \beta \quad \partial\mathbb{R}^2_+,$$

where $\|\phi_0\|_{2+\sigma,H} \leq C\|h\|_{4+\sigma,H}$. Then, we define $\vartheta := \frac{\phi_0}{U} - \psi_0$ and find ϑ satisfies

$$\int_{\mathbb{R}^2_+} U\nabla\vartheta \cdot e_1 dy = 0, \tag{3.32}$$

where $\boldsymbol{e}_1 = (1, 0)$. Now, the problem (3.30) is transformed into the following form:

$$\begin{cases} \nabla\cdot(U\nabla\tilde{g}) = h - \nabla\cdot(U\nabla\vartheta) & \text{in } \mathbb{R}^2_+ \\ U\dfrac{\partial \tilde{g}}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{3.33}$$

where $\tilde{g} := g - \vartheta$. By defining the the solution of (3.33) as $(\phi_\vartheta, \psi_\vartheta)$ and

$$\tilde{g}_0 := \begin{cases} \tilde{g}(y_1, y_2) & \text{for } y_2 \geq 0; \\ \tilde{g}(y_1, -y_2) & \text{for } y_2 < 0, \end{cases}$$

we have the equation in (3.33) is evenly extended into the whole space, which is

$$\nabla\cdot(U\nabla\tilde{g}_0)=\tilde{h}\ \text{ in }\ \mathbb{R}^2,$$

where

$$\tilde{h}(x,y)=\begin{cases}h(y_1,y_2)-\nabla(U\nabla\vartheta)(y_1,y_2) & \text{for }\ y_2\geq 0,\\ h(y_1,-y_2)-\nabla(U\nabla\vartheta)(y_1,-y_2) & \text{for }\ y_2<0.\end{cases}$$

It is easy to check that $\|\tilde{h}\|_{4+\sigma}<\infty$ due to $\|h\|_{4+\sigma,H}<\infty$. The key step is the verification of the orthogonality condition. To finish it , we first obtain from the property of even function that

$$\int_{\mathbb{R}^2_-} h(y_1,-y_2)-\nabla(U\nabla\vartheta)(y_1,-y_2)dy=\int_{\mathbb{R}^2_+} h(y_1,y_2)-\nabla(U\nabla\vartheta)(y_1,y_2)dy,$$

and

$$\int_{\mathbb{R}^2_-}\big[h(y_1,-y_2)-\nabla(U\nabla\vartheta)(x_1,-x_2)\big]y_1dy=\int_{\mathbb{R}^2_+}\big[h(y_1,y_2)-\nabla(U\nabla\vartheta)(y_1,y_2)\big]y_1dy.$$

Then, by using condition (3.29), we have from the divergence Theorem that

$$\begin{aligned}\int_{\mathbb{R}^2}\tilde{h}dy&=2\int_{\mathbb{R}^2_+}h-\nabla(U\nabla\vartheta)dy=2\int_{\mathbb{R}^2_+}hdy-2\int_{\partial\mathbb{R}^2_+}(U\nabla\vartheta)\cdot\boldsymbol{\nu}dS_y\\&=2\int_{\mathbb{R}^2_+}hdy-2\int_{\partial\mathbb{R}^2_+}U\frac{\partial\vartheta}{\partial\boldsymbol{\nu}}dS_y=2\int_{\mathbb{R}^2_+}hdy-2\int_{\partial\mathbb{R}^2_+}\beta dS_y=0,\end{aligned}$$

which implies the mass condition in (3.3). For the first moment condition, the integration by parts and (3.32) tell us

$$\begin{aligned}\int_{\mathbb{R}^2_+}\tilde{h}y_1dy&=2\int_{\mathbb{R}^2_+}[h-\nabla(U\nabla\vartheta)]y_1dy\\&=2\int_{\mathbb{R}^2_+}hy_1dy-2\int_{\partial\mathbb{R}^2_+}y_1U\nabla\vartheta\cdot\boldsymbol{\nu}dS_y+2\int_{\mathbb{R}^2_+}U\nabla\vartheta\cdot e_1dy\\&=2\int_{\mathbb{R}^2_+}hy_1dy-2\int_{\partial\mathbb{R}^2_+}\beta y_1dS_y=0.\end{aligned}\tag{3.34}$$

Since $\tilde{h}$ is even with respect to y_1, we can easily obtain from (3.34) that $\int_{\mathbb{R}^2}\tilde{h}y_1dy=0$, which completes the verification of orthogonality condition (3.3). Therefore, we can utilize the results shown in Lemma 3.1 to find there exists the solution $(\tilde{\phi},\tilde{\psi})$ to the following system:

$$\begin{cases}-\Delta\tilde{\psi}=U\tilde{\psi}+U\tilde{g}, & \text{in }\ \mathbb{R}^2,\\ -\Delta\tilde{\psi}=\tilde{\phi} & \text{in }\ \mathbb{R}^2.\end{cases}$$

In particular, $\tilde{\phi}$ satisfies the following estimate :

$$|\tilde{\phi}|\leq C\frac{\|h\|_{4+\sigma,H}}{(1+r)^{2+\sigma}}.\tag{3.35}$$

Since $\tilde{\phi}$ is even, it can be defined as the even extension of ϕ. By using (3.35), we further show that ϕ is the solution of (3.30) and satisfies

$$\|\phi\|_{2+\sigma,H} = \|\tilde{\phi} + \phi_0\|_{2+\sigma,H} \leq \|\tilde{\phi}\|_{2+\sigma,H} + \|\phi_0\|_{2+\sigma,H} \leq C\|h\|_{4+\sigma,H},$$

which completes the proof of (3.31) and this Lemma. □

After establishing the inner linear theories for the interior spikes and boundary spikes, we focus on the outer problem and discuss a priori estimates and the existence of the outer solution.

3.2. *Outer Linear Theory*

Due to the appearance of logistic source $f(u)$, we have the structure of outer operator is not the same as that in Ref. 5. By analyzing the first equation of (2.36) in the outer region, one finds the operator becomes

$$\bar{L}[\phi] = -\Delta\phi + \nabla\cdot(W\nabla\bar{\psi}) + \nabla\cdot(\phi\nabla\bar{V}), \tag{3.36}$$

where

$$W = \sum_{j=1}^{m} \frac{8\mu_j^2}{(\mu_j^2 + |y-\xi'|^2)^2} \text{ and } \bar{V}(y) = \sum_{j=1}^{m}\Big(\log\frac{8\mu_j^2\varepsilon^4}{(\mu_j^2+|y-\xi_j'|^2)^2} + \hat{c}_j H^\varepsilon(x,\xi_j)\Big).$$

We substitute $\Delta\bar{V} = \varepsilon^2\bar{V} - W$ into (3.36) and expand it to obtain

$$\begin{aligned}\bar{L}[\phi] =& -\Delta\phi + \nabla\phi\cdot\nabla\bar{V} + \varepsilon^2\bar{V}\phi\\ &+ \nabla W\cdot\nabla\bar{\psi} + W\Delta\bar{\psi} - W\phi := I_{31} + I_{32}.\end{aligned}$$

Due to the decay property of W, we have I_{32} can be neglected in the outer region and I_{31} is the important term. On the other hand, the leading term in the logistic source is $\varepsilon^2\mu\bar{u}\phi$. Combining this term with I_{31}, we define the outer operator as

$$L^o[\phi] = -\Delta\phi + \nabla\phi\cdot\nabla\bar{V} - \varepsilon^2(\mu\bar{u} - \bar{V})\phi \ \text{ in } \ \Omega_\varepsilon,$$

where $\varepsilon y = x$ and $\varepsilon\xi_j' = \xi_j$. Moreover, the outer norm $\|\cdot\|_{\nu,o}$, $\nu > 0$ is given by

$$\|h\|_{\nu,o} := \sup_{y\in\Omega_\varepsilon} \frac{|h|}{\sum\limits_{j=1}^{m}(1+|y-\xi_j'|)^{-\nu}}.$$

We shall construct the barrier to give a priori estimate of ϕ then prove its existence. Our results are summarized as the following lemma:

Lemma 3.3. *Assume that* $\|h\|_{b+2,o} < \infty$ *and* $\mu\bar{u} < \bar{C}$, *where* $\bar{C} := \sum\limits_{j=1}^{m}\hat{c}_j C_\Omega$ *and* C_Ω *is the positive lower bound of Green's function, then the problem*

$$\begin{cases} L^o[\phi] = h & \text{in } \ \Omega_\varepsilon,\\ \dfrac{\partial\phi}{\partial\boldsymbol{\nu}} = 0 & \text{on } \ \partial\Omega_\varepsilon \end{cases} \tag{3.37}$$

admits the solution $\phi = \mathcal{T}_o(h)$ satisfying

$$\|\phi\|_{b,o} \le C\|h\|_{b+2,o}, \tag{3.38}$$

where $C > 0$, $b > 0$ are constants and $\mathcal{T}_o(h)$ is a continuous linear mapping.

Proof. We define the barrier w as

$$w = \sum_{j=1}^{m} w_j(x) + w_0 + \bar{w}_1 = \sum_{j=1}^{m} \frac{C_1}{(\mu_j^2 + |y-\xi_j'|)^b} + C_2 w_0 + C_3 \bar{w}_1, \tag{3.39}$$

where C_1, C_2 and C_3 are positive constants and functions w_0, $\bar{w}_1$ will be explained later on. For w_j, we have the fact that

$$-\Delta w_j = -\frac{C_1 b(b+1)}{(\mu_j^2 + |y-\xi_j'|)^{b+2}} + \frac{C_1 b}{(|y-\xi_j'|)(\mu_j^2 + |y-\xi_j'|)^{b+1}}, \tag{3.40}$$

and

$$\nabla w_j = -\frac{C_1 b(y-\xi_j')}{(\mu_j^2 + |y-\xi_j'|)^{b+2}}. \tag{3.41}$$

Before using the maximum principle, we need to check the boundary condition of w_j. If $\xi_j' \in \Omega_\varepsilon$, we have from (3.40) and (3.41) that w_j satisfies

$$\frac{\partial w_j}{\partial \boldsymbol{\nu}} = O(\varepsilon^{b+1}) \quad \text{on} \quad \partial\Omega_\varepsilon. \tag{3.42}$$

If $\xi_j' \in \partial\Omega_\varepsilon$, we write $\partial\Omega_\varepsilon$ near ξ_j' as the graph $(y_1, y_2) = (y_1, \rho(y_1))$ with $\rho(0) = 0$ and $\rho'(0) = 0$, then find

$$\begin{aligned}\frac{\partial w_j}{\partial \boldsymbol{\nu}} =& -\frac{b}{|y-\xi_j'|(\mu_j^2 + |y-\xi_j'|)^{1+b}} \cdot \frac{(y_2 - \xi_{j2}') - (y_1 - \xi_{j1}')\rho'(y_1)}{\sqrt{1+|\rho'(y)|^2}} \\ =& O(\varepsilon^{b+1}).\end{aligned} \tag{3.43}$$

Combining (3.42) and (3.43), we obtain that

$$\frac{\partial w_j}{\partial \boldsymbol{\nu}} = O(\varepsilon^{b+1}) \quad \text{on} \quad \partial\Omega_\varepsilon. \tag{3.44}$$

Let w_0 be the unique solution of

$$\begin{cases} -\Delta w_0 + \nabla\bar{V}\cdot\nabla w_0 + \varepsilon^2\Big(\sum G(y,\xi_j') - \mu\bar{u}\Big)w_0 = 0 & \text{in} \quad \Omega_\varepsilon, \\ \frac{\partial w_0}{\partial \boldsymbol{\nu}} = -\sum\limits_{j=1}^{m} \frac{\partial w_j}{\partial \boldsymbol{\nu}} & \text{on} \quad \partial\Omega_\varepsilon, \end{cases} \tag{3.45}$$

then we have $|w_0| \le C\varepsilon^b$ for some constant $C > 0$. Moreover, letting $\bar{w}_1 = \varepsilon^b$, we obtain from

$$\nabla\bar{V} = \sum_{j=1}\Big(-\frac{4(y-\xi_j')}{\mu_j^2 + |y-\xi_j'|^2} + \varepsilon\hat{c}_j\nabla H_j^\varepsilon\Big)$$

that for $y \in \Omega_\varepsilon$,

$$
\begin{aligned}
L^o[w] =& \sum_{j=1}^m \Big(-\frac{C_1 b(b+1)}{(\mu_j^2 + |y-\xi_j'|)^{b+2}} + \frac{C_1 b}{(|y-\xi_j'|)(\mu_j^2+|y-\xi_j'|)^{b+1}} \\
&+ \frac{4C_1 b|y-\xi_j'|^2}{(\mu_j^2+|y-\xi_j'|)^{b+2}(\mu_j^2+|y-\xi_j'|^2)} \Big) - \sum_{j=1}^m \frac{C_1 b(y-\xi_j')}{(1+|y-\xi_j'|)^{b+2}} \varepsilon \hat{c}_j \nabla H_j^\varepsilon \\
&+ C_1\varepsilon^2 \Big(\sum_{j=1} G(y,\xi_j') - \mu\bar{u} \Big) \sum_{j=1} \frac{1}{(1+|y-\xi_j'|)^b} \\
&+ C_3\varepsilon^2 \Big(\sum_{j=1} G(y,\xi_j') - \mu\bar{u} \Big) w_1 \qquad (3.46) \\
\geq & C_1 b(4-b) \sum_{j=1}^m \frac{1}{(\mu_j^2+|y-\xi_j'|)^{b+2}} - \sum_{j=1}^m \frac{C_1 b(y-\xi_j')}{(1+|y-\xi_j'|)^{b+2}} \varepsilon \hat{c}_j \nabla H_j^\varepsilon \\
&+ C_1\varepsilon^2 \Big(\sum_{j=1} G(y,\xi_j') - \mu\bar{u} \Big) \sum_{j=1} \frac{1}{(1+|y-\xi_j'|)^b} \\
&+ C_3\varepsilon^2 \Big(\sum_{j=1} G(y,\xi_j') - \mu\bar{u} \Big) w_1 := I_{33} + I_{34}.
\end{aligned}
$$

When $0 < b < 4$, we take $R > 0$ small enough such that

$$
I_{33} \geq \sum_{j=1}^m \frac{C_4}{(1+|y-\xi_j'|)^{b+2}} \quad \text{in} \quad |y-\xi'| < \frac{R}{\varepsilon}, \qquad (3.47)
$$

where $C_4 > 0$ is a constant. For $|y-\xi'| > \frac{R}{\varepsilon}$, one sets C_3 in I_{34} be large enough and obtains

$$
C_3\varepsilon^2 \Big(\sum_{j=1} G(y,\xi_j') - \mu\bar{u} \Big) w_1 \geq \sum_{j=1}^m \frac{C_4}{(1+|y-\xi_j'|)^{b+2}}. \qquad (3.48)
$$

We collect (3.47) and (3.48) to get from (3.46) that

$$
L^o[w] \geq \frac{C_4}{(1+|y-\xi_j'|)^{b+2}} \quad \text{in} \quad \Omega_\varepsilon.
$$

On the other hand, we combine (3.44) and (3.45) to show $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega_\varepsilon$. Now, by using the maximum principle, one has

$$
|\phi| < C_5 \|h\|_{\nu,o} w,
$$

where w is defined by (3.39) and $C_5 > 0$ is a constant. It follows that (3.38) holds due to the boundedness of w. The existence of ϕ can be obtained thanks to Fredholm alternative Theorem, which finishes the proof of this Lemma. □

Lemma 3.3 implies that when the effect of intrinsic growth is small, the outer problem (3.37) admits the decay solution ϕ if h has the decay property. Combining Lemma 3.1, Lemma 3.2 and Lemma 3.3, we established the linear theories for the inner and outer solutions. Our next goal is to glue them together then construct multi-spikes.

4. The Inner-outer Gluing System

In this section, we shall employ the so-called inner-outer gluing scheme to construct multiple interior spikes which satisfies stationary problem (1.3). Moreover, the boundary spikes can be similarly constructed and will be discussed in Section 5.

Before formulating the inner-outer gluing system, we need to give some notations and definitions. $L_j[\phi]$ are defined as the inner operators for each spike, which satisfy

$$L_j[\phi] = -\Delta\phi + \nabla\cdot[U_j(y-\xi_j')\nabla\bar{\psi}] + \nabla\cdot[\phi\nabla\Gamma_j(y-\xi_j')], \quad j=1,2,\cdots,m. \tag{4.1}$$

The inner and outer norms $\|\cdot\|_{\nu,j}$ and $\|\cdot\|_{\nu,o}$ are given by

$$\|h\|_{\nu,j} := \sup_{y\in\mathbb{R}^2} |h(y)|(1+|y-\xi_j'|)^\nu \text{ and } \|h\|_{\nu,o} := \sup_{y\in\Omega_\varepsilon} \frac{|h|}{\sum_{j=1}^m (1+|y-\xi_j'|)^{-\nu}}. \tag{4.2}$$

In addition, we define $\delta' := \inf_{l\neq j}|\xi_j-\xi_l|$ and the cut-off functions as $\eta_j := \eta(|y-\xi_j'|)$, where

$$\eta(r) = \begin{cases} 1, \ r<\delta/\varepsilon, \\ 0, \ r>2\delta/\varepsilon, \end{cases} \tag{4.3}$$

and $\delta>0$ is a fixed number. After presenting the important notations (4.1), (4.2) and (4.3), we further decompose ϕ and $\bar{\psi}$ into the following forms:

$$\begin{aligned} &\phi = \Big(\sum_{j=1}^m \varepsilon^{\gamma_1}\phi_j(y)\eta_j(y) + \varepsilon^{\gamma_2}\phi^o\Big), \qquad \bar{\psi}_j = (-\Delta)^{-1}\phi_j, \\ &\psi^o = \varepsilon^{\gamma_2}\Big((-\Delta+\varepsilon^2)^{-1}\phi^o\Big) \ \text{ and } \ \bar{\psi}_j' = (-\Delta+\varepsilon^2)^{-1}(\phi_j\eta_j), \end{aligned} \tag{4.4}$$

where $\gamma_1, \gamma_2 > 0$ will be determined later on. In light of the linear property of $L[\phi]$, the equation in (2.36) can be rewritten as

$$\begin{aligned} L[\phi] =& \varepsilon^{\gamma_1}\sum_{j=1}^m L[\phi_j\eta_j(y)] + \varepsilon^{\gamma_2}L[\phi^o] \\ =& \varepsilon^2\sum_{k=1}^5 \tilde{I}_k(\mathbf{p}) + \varepsilon^2\tilde{I}_6(\phi,\mathbf{p}) + \varepsilon^2\tilde{I}_7(\phi,\mathbf{p}), \end{aligned} \tag{4.5}$$

where $\tilde{I}_1$–$\tilde{I}_7$ are given by (2.27)–(2.33). Then (4.5) becomes

$$\begin{aligned} &\varepsilon^{\gamma_1}\sum_{j=1}^m L_j[\phi_j\eta_j] + \varepsilon^{\gamma_2}L^o[\phi^o] \\ &= \varepsilon^2\tilde{h}(\phi,\mathbf{p}) - \varepsilon^{\gamma_2}\nabla\cdot(W\nabla\bar{\psi}^o) + \varepsilon^{\gamma_2}W\phi^o - \varepsilon^{2+\gamma_2}\mu\bar{u}\phi^o \\ &+ \varepsilon^{1+\gamma_2}\sum_{j=1}^m \nabla_y\cdot\Big(\phi^o\nabla_x H_j(\varepsilon y,\xi_j)\Big) \\ &- \varepsilon^{\gamma_1}\sum_{j=1}^m\sum_{l\neq j}\big[\nabla\cdot(U_l\nabla\bar{\psi}_j') + \nabla\cdot(\phi_j\nabla\Gamma_l)\eta_j\big] - \varepsilon^{\gamma_1}\sum_{j=1}^m\sum_{l\neq j}\phi_j\nabla\eta_j\cdot\nabla\Gamma_l, \end{aligned} \tag{4.6}$$

where $\tilde{h}(\phi,\mathbf{p}) := \sum_{k=1}^{7} \tilde{I}_k(\phi_1,\cdots,\phi_j,\phi^o,\mathbf{p})$. Now, we define

$$\begin{aligned} F(\phi,\mathbf{p}) :=& \varepsilon^2\tilde{h}(\phi,\mathbf{p}) - \varepsilon^{\gamma_2}\nabla\cdot(W\nabla\bar{\psi}^o) + \varepsilon^{\gamma_2}W\phi^o - \varepsilon^{2+\gamma_2}\mu\bar{u}\phi^o \\ &+ \varepsilon^{1+\gamma_2}\sum_{j=1}^{m}\nabla_y\cdot\Big(\phi^o\nabla_x H_j(\varepsilon y,\xi_j)\Big) \\ &+ \varepsilon^{1+\gamma_1}\Big(\sum_{j=1}^{m}\phi_j\nabla\eta_j\Big)\cdot\Big(\sum_{j=1}^{m}\hat{c}_j\nabla H_j(\varepsilon y,\xi_j)\Big) \\ &- \varepsilon^{\gamma_1}\sum_{j=1}^{m}\sum_{l\neq j}\big[\nabla\cdot(U_l\nabla\bar{\psi}_j') + \nabla\cdot(\phi_j\nabla\Gamma_l)\eta_j\big]. \end{aligned} \tag{4.7}$$

On the other hand, we have from straightforward calculations that

$$\begin{aligned} L_j[\phi_j\eta_j] &= -\eta_j\Delta\phi_j - 2\nabla\phi_j\cdot\nabla\eta_j - \phi_j\Delta\eta_j + \nabla\cdot(U_j(y-\xi_j')\nabla\bar{\psi}_j') \\ &\quad + \nabla\cdot(\phi_j\nabla\Gamma(y-\xi_j'))\eta_j + \phi_j\nabla\eta_j\cdot\nabla\Gamma(y-\xi_j') \\ &= \eta_j L_j[\phi_j] - 2\nabla\phi_j\cdot\nabla\eta_j - \phi_j\Delta\eta_j + \phi_j\nabla\eta_j\cdot\nabla\Gamma(y-\xi_j') \\ &\quad + \nabla\cdot(U_j\nabla\bar{\psi}_j') - \nabla\cdot(U_j\nabla\bar{\psi}_j)\eta_j, \end{aligned} \tag{4.8}$$

and we further denote $F_j(\phi,\mathbf{p})$ and $J(\phi,\mathbf{p})$ by using (4.7) as

$$F_j(\phi,\mathbf{p}) = \Big(F(\phi,\mathbf{p}) - \varepsilon^{\gamma_1}\nabla\cdot(U_j\nabla\bar{\psi}_j') + \varepsilon^{\gamma_1}\nabla\cdot(U_j\nabla\bar{\psi}_j)\eta_j\Big) \tag{4.9}$$

and

$$\begin{aligned} J(\phi,\mathbf{p}) :=& F(\phi,\mathbf{p})\big(1-\sum_{j=1}^{m}\eta_j^2\big) + \varepsilon^{1+\gamma_1}\Big(\sum_{j=1}^{m}\phi_j\nabla\eta_j\Big)\cdot\Big(\sum_{j=1}^{m}\hat{c}_j\nabla H(\varepsilon y,\xi_j)\Big) \\ &+ \varepsilon^{\gamma_1}\sum_{j=1}^{m}\big(2\nabla\phi_j\cdot\nabla\eta_j + \phi_j\Delta\eta_j - \phi_j\nabla\eta_j\cdot\nabla\Gamma_j(y-\xi_j')\big) \\ &- \varepsilon^{\gamma_1}\sum_{j=1}^{m}\sum_{l\neq j}\phi_j\nabla\eta_j\cdot\nabla\Gamma_l. \end{aligned} \tag{4.10}$$

By collecting (4.6), (4.8), (4.9) and (4.10), we formulate the system satisfied by ϕ_j and ϕ^o as

$$\begin{cases} L_j[\phi_j] = \varepsilon^{-\gamma_1}F_j(\phi,\mathbf{p})\eta_j & \text{in } \mathbb{R}^2 \quad \text{for } j=1,\cdots,m, \\ L^o[\phi^o] = \varepsilon^{-\gamma_2}J(\phi,\mathbf{p}) & \text{in } \Omega_\varepsilon. \end{cases} \tag{4.11}$$

It is necessary to consider the orthogonality condition so as to use Lemma 3.1. To this end, we let compactly supported functions W_{0j}, $j=1,\cdots,m$ be radial with respect to ξ_j' and satisfy

$$\int_{\mathbb{R}^2} W_{0j}(y-\xi_j)dy = 1,$$

and compactly supported radial functions W_{ij}, $i = 1, 2$ satisfy

$$\int_{\mathbb{R}^2} W_{ij}\big(|y - \xi_j'|\big)\big(y - \xi_j'\big)_i dy = 1,$$

then modify (4.11) to obtain the following problem:

$$\begin{cases} L_j[\phi_j] = \varepsilon^{-\gamma_1} F_j(\phi, \mathbf{p})\eta_j - \sum\limits_{i=0,1,2} m_{ij}[\varepsilon^{-\gamma_1} F_j(\phi, \mathbf{p})\eta_j] W_{ij} \ \text{ for } \ j = 1, \cdots, m, \\ \\ L^o[\phi^o] = \varepsilon^{-\gamma_2} J(\phi, \mathbf{p}), \end{cases} \tag{4.12}$$

where $m_{ij}[h]$ are defined by

$$m_{0j}[h] = \int_{\mathbb{R}^2} h(y)\mathrm{d}y \ \text{ and } \ m_{ij}[h] = \int_{\mathbb{R}^2} h(y)(y - \xi_j')_i \mathrm{d}y \tag{4.13}$$

for $j = 1, \cdots, m$ and $i = 1, 2$.

Now, according to Lemma 3.1 and Lemma 3.3, we have there exists the solution $(\phi_1, \cdots, \phi_m, \phi^o)$ to (4.12) provided $\mathbf{p}$ satisfying

$$m_{ij}[\varepsilon^{-\gamma_1} F_j(\phi, \mathbf{p})\eta_j] = 0 \ \text{ for } \ i = 0, 1, 2 \ \text{ and } \ j = 1, \cdots, m. \tag{4.14}$$

Moreover, we have ϕ_j, $j = 1, \cdots, m$, ϕ_0 and $\mathbf{p}$ are given as

$$\begin{aligned} \phi_j &= \mathcal{A}_j(\phi_1, \cdots, \phi_m, \phi^o, \mathbf{p}_0 + \mathbf{p}_1), \\ \phi^o &= \mathcal{A}_o(\phi_1, \cdots, \phi_m, \phi^o, \mathbf{p}_0 + \mathbf{p}_1), \\ \mathbf{p} &= \mathcal{A}_p(\phi_1, \cdots, \phi_m, \phi^o, \mathbf{p}_0 + \mathbf{p}_1), \end{aligned} \tag{4.15}$$

where $\mathcal{A}_j$, $\mathcal{A}_0$ and $\mathcal{A}_p$ are linear operators and

$$\mathbf{p} = \mathbf{p}_0 + \mathbf{p}_1 \ \text{ with } \ \mathbf{p}_0 = (c_{10}, \cdots, c_{m0}, \xi_{10}, \cdots, \xi_{m0}), \tag{4.16}$$

where $(\xi_{10}, \cdots, \xi_{m0})$ is a critical point of $\mathcal{J}_m$. We utilize (4.15) and (4.16) to rewrite the solutions and operators as the following vector forms:

$$\vec{\phi} = (\phi_1, \cdots, \phi_m, \phi^o, \mathbf{p}_0 + \mathbf{p}_1) \tag{4.17}$$

and

$$\mathcal{A}(\vec{\phi}) = \big(\mathcal{A}_1(\vec{\phi}), \cdots, \mathcal{A}_m(\vec{\phi}), \mathcal{A}_o(\vec{\phi}), \mathcal{A}_p(\vec{\phi})\big). \tag{4.18}$$

Before using the fixed point Theorem to show the existence of the solution (4.17) with the operator form (4.18), we need to give the functional spaces and norms satisfied by $\vec{\phi}$. We define the following spaces:

$$\begin{aligned} X_j = \Big\{\phi \in & L^\infty(\mathbb{R}^2) : \nabla\phi \in L^\infty(\mathbb{R}^2);\ \|\phi\|_{2+\sigma,j} < \infty, \\ & \int_{\mathbb{R}^2} \phi(y)dy = 0 \ \text{ and } \ \int_{\mathbb{R}^2} \phi(y)\big(y - \xi_j'\big)_i dy = 0, \ \ i = 1, 2\Big\}, \end{aligned}$$

$$X_o = \Big\{\phi \in L^\infty(\Omega_\varepsilon) :\ \nabla\phi \in L^\infty(\mathbb{R}^2),\ \|\phi\|_{b,o} < \infty,\ \ \frac{\partial\phi}{\partial\boldsymbol{\nu}} = 0\Big\},$$

and

$$X_p = \Big\{(c_1, \cdots, c_m, \xi_1, \cdots, \xi_m) \in \mathbb{R}^m \times (\mathbb{R}^2)^m : \|\mathbf{p}\|_p = \sup_j |c_j| + \sup_j |\xi_j| < \infty\Big\},$$

then collect them to define X as

$$X := \prod_j X_j \times X_o \times X_p \tag{4.19}$$

with the norm as

$$\|\vec{\phi}\|_X = \sum_{j=1}^m \|\phi_j\|_{2+\sigma,j} + \|\phi^o\|_{b,o} + \|\mathbf{p}\|_p. \tag{4.20}$$

With the definitions of (4.19) and (4.20), we claim that for $\|\phi\|_X < 1$,

$$\|\mathcal{A}(\phi)\|_X < 1. \tag{4.21}$$

Focusing on the inner problem, we find from Lemma 3.1 that if

$$\varepsilon^{-\gamma_1}\|F_j(\phi, \mathbf{p})\|_{4+\sigma,j} < \infty,$$

then

$$\|\mathcal{A}_j(\phi)\|_{2+\sigma,j} \le C\|\varepsilon^{-\gamma_1}F_j(\phi, \mathbf{p})\|_{4+\sigma,j},$$

where $C > 0$ is a constant. Thus, it suffices to show $C\varepsilon^{-\gamma_1}\|F(\phi, \mathbf{p})\|_{4+\sigma,j} < 1$ so as to prove our claim for the inner operator, which is

$$\|\mathcal{A}_j(\vec{\phi})\|_{2+\sigma,j} < 1. \tag{4.22}$$

To state our analysis in a user-friendly way, we expand $\varepsilon^{-\gamma_1}F_j(\phi, \mathbf{P})$ to obtain that

$$\begin{aligned}
\varepsilon^{-\gamma_1}F_j(\phi, \mathbf{p}) =& \varepsilon^{2-\gamma_1}\sum_{j=1}^5 \tilde{I}_k \\
&+ \Big(\varepsilon^{2-\gamma_1}\tilde{I}_7 - \varepsilon^{2+\gamma_2-\gamma_1}\mu\bar{u}\phi^o - \varepsilon^{\gamma_2-\gamma_1}\nabla\cdot(W\nabla\bar{\psi}^o) + \varepsilon^{\gamma_2-\gamma_1}W\phi^o\Big) \\
&+ \Big[\varepsilon^{2-\gamma_1}\tilde{I}_6 + \varepsilon^{1+\gamma_2-\gamma_1}\nabla_y\cdot\Big(\phi^o\nabla_x H(\varepsilon y, \xi_j)\Big) \\
&- \Big(\sum_{j=1}^m\sum_{l\neq j}[\nabla\cdot(U_l\nabla\bar{\psi}_j') + \nabla\cdot(\phi_j\nabla\Gamma_l)\eta_j]\Big) \\
&+ \varepsilon\Big(\sum_{j=1}^m \phi_j\nabla\eta_j\Big)\cdot\Big(\sum_{j=1}^m \hat{c}_j\nabla H_j(\varepsilon y, \xi_j)\Big)\Big] \\
&- \nabla\cdot\Big(U_j(y-\xi_j')\nabla(\bar{\psi}_j - \bar{\psi}_j')\Big) := \tilde{II}_1 + \tilde{II}_2 - \tilde{II}_3 - \tilde{II}_4.
\end{aligned} \tag{4.23}$$

We will estimate $\tilde{II}_1$–$\tilde{II}_4$ term by term. For $\tilde{II}_1$, it is easy to verify that $\|\tilde{II}_1\|_{4+\sigma,j} = o(1)$. We only discuss $\nabla_y U \cdot \nabla_y \Gamma$ since others can be studied in the same way. In fact, for $y \in B_{2\delta/\varepsilon}(\xi_j')$, we have

$$\begin{aligned}
&\varepsilon^{-\gamma_1}\Big|(1+|y-\xi_j'|)^{4+\sigma}\nabla U_l(y-\xi_l')\cdot\nabla\Gamma_j(y-\xi_j')\Big| \\
\le& \varepsilon^{-\gamma_1}C\Big|(1+|y-\xi_j'|)^{4+\sigma}\frac{1}{(1+|y-\xi_l'|)^5}\cdot\frac{1}{1+|y-\xi_j'|}\Big| \le C\varepsilon^{2-\gamma_1-\sigma},
\end{aligned}$$

which implies

$$\|\nabla U_l(y-\xi_l')\cdot\nabla\Gamma_j(y-\xi_j')\|_{4+\sigma,j}=o(1),\tag{4.24}$$

since γ_1 is chosen to be less than one and $\sigma\in(0,1)$. For II_2, we calculate to find

$$|II_2|\le\varepsilon^{2-\gamma_1}|\tilde{I}_7-\varepsilon^{\gamma_2}\mu\bar{u}\phi^o|+\varepsilon^{\gamma_2-\gamma_1}\Big(|\nabla\cdot(W\nabla\bar{\psi}^o)|+|W\phi^o|\Big)$$

where $W(y)$ is given in (2.37) and $\tilde{I}_7$ is defined by (2.33). Thr main error in II_2 is generated by outer solutions and we only discuss $\varepsilon^{\gamma_2-\gamma_1}W\phi^o$ since the other terms can be analyzed in the same way. Assuming $\gamma_2>\gamma_1$, we find

$$\varepsilon^{\gamma_2-\gamma_1}|W\phi^o|\le\varepsilon^{\gamma_2-\gamma_1}C\sum_{j=1}^m\frac{1}{(1+|y-\xi_j'|^2)^2}\frac{\|\phi^o\|_{b,o}}{\sum\limits_{j=1}^m(1+|y-\xi_j'|)^{-b}}.\tag{4.25}$$

By choosing $b>2$, we can get from (4.25) that when $\varepsilon\ll1$,

$$\|\varepsilon^{\gamma_2-\gamma_1}W\phi^o\|_{4+\sigma,j}\le\frac{1}{16}.$$

Thus, we utilize the decay properties of U_j, W and φ_j to show that

$$\|II_2\|_{4+\sigma,j}=o(1).\tag{4.26}$$

For $\tilde{II}_4$, since

$$\bar{\psi}_j=C+O\Big(\frac{1}{|y|}\Big),\qquad\bar{\psi}_j'=O(\varepsilon^\sigma)\log|y|+C+O\Big(\frac{1}{|y|}\Big)\ \text{ as }\ |y|\to\infty,$$

one similarly obtains $\tilde{II}_4=o(1)$. We next focus on the leading term $\tilde{II}_3$ in (4.23), which is

$$\begin{aligned}\tilde{II}_3=&\Bigg[\varepsilon\sum_{j=1}^m\sum_{l=1}^m(\hat{c}_l\nabla\phi_j\cdot\nabla H_l(\varepsilon y,\xi_l))\eta_j+\sum_{j=1}^m\sum_{l\neq j}\nabla\cdot(\phi_j\nabla\Gamma_l(y-\xi_l'))\eta_j\\&+\varepsilon^2\sum_{j=1}^m\sum_{l=1}^m(\hat{c}_l\phi_j\Delta H_l(\varepsilon y,\xi_l))\eta_j\Bigg]\\&+\Bigg[\sum_{j=1}^m\sum_{l\neq j}\nabla\cdot(U_l\nabla\bar{\psi}_j')+\Big[\varepsilon^{1-\gamma_1}\nabla\phi\cdot\sum\Big(\mu\varepsilon\nabla\psi_j+\mu\varepsilon^2\nabla\mathcal{H}_j\Big)\Big]\Bigg]\\&+\varepsilon^{2-\gamma_1}\Bigg[\phi\cdot\sum_{j=1}\Big(\mu\Delta\psi_j+\mu\varepsilon^2\Delta_x\mathcal{H}_j\Big)+\mu\sum_{j=1}\nabla\varphi_j\cdot\nabla\bar{\psi}+\mu\sum\varphi_j\Delta\bar{\psi}\Bigg]\\&+\varepsilon^{-\gamma_1}(\phi\Delta\bar{\psi}+\nabla\phi\cdot\nabla\bar{\psi}):=III_A+III_B+III_C+III_D.\end{aligned}\tag{4.27}$$

By substituting (4.4) into (4.27), we find III_B and III_C can be easily estimated and satisfy $|III_B|=o(1)$ and $|III_C|=o(1)$. Moreover, we obtain from $\gamma_1>1-\sigma$ $\gamma_1<\gamma_2$ that $|III_D|=o(1)$. The worse term in (4.29) is III_A. In fact, we calculate to obtain

$$
\begin{aligned}
III_A =&\varepsilon \sum_{j=1}^{m} \Big[\nabla\phi_j \cdot \Big(\hat{c}_j \nabla H_j(\varepsilon y, \xi_j) + \sum_{l\neq j} \Big(\frac{1}{\varepsilon} \nabla\Gamma(y-\xi_l') + \hat{c}_l \nabla H_l \Big) \Big) \eta_j \Big] \\
&+ \varepsilon^2 \sum_{j=1}^{m} \Big[\phi_j \Big(\hat{c}_j \Delta H_j(\varepsilon y, \xi_j) + \sum_{l\neq j} \Big(\frac{1}{\varepsilon^2} \Delta\Gamma(y-\xi_l') + \hat{c}_l \Delta H_l \Big) \Big) \eta_j \Big] \\
=&\varepsilon \sum_{j=1} \nabla\phi_j \cdot \nabla_x \tilde{H}_j^\varepsilon \eta_j + \varepsilon^2 \sum_{j=1} \phi_j \Delta_x \tilde{H}_j^\varepsilon \eta_j = \varepsilon^2 \sum_{j=1} \nabla_x \cdot \big(\phi_j \nabla_x \tilde{H}_j^\varepsilon \big) \eta_j, \quad (4.28)
\end{aligned}
$$

where $\tilde{H}_j^\varepsilon$ is given by (2.8). We shall estimate (4.28) near each centre of spikes i.e. in $|y-\xi'| < \frac{2\delta}{\varepsilon}$. Firstly, we expand $\nabla \tilde{H}_j^\varepsilon$ as

$$
\nabla \tilde{H}_j^\varepsilon(x) = \nabla \tilde{H}_j^\varepsilon(\xi_j) + O(|x-\xi_j|^\alpha) = \nabla \tilde{H}_j^\varepsilon(\xi_j) + \varepsilon^\alpha O(|y-\xi_j'|^\alpha), \quad (4.29)
$$

where $\alpha \in (0,1)$. By substituting the second term of (4.29) into (4.28), we can show that the order satisfies $O(\varepsilon^{1+\alpha}|y|^{-(3+\sigma-\alpha)})$. To get the desired estimate, we choose δ small enough such that the coefficient is much less than one. Similarly, we consider the leading term in (4.29), then obtain

$$
\varepsilon |\nabla\phi_j \cdot \nabla \tilde{H}_j^\varepsilon(\xi_j)| \leq \frac{C\delta}{|1-y|^{4+\sigma}}.
$$

By taking $\delta > 0$ small enough, one has $\varepsilon \|\nabla\phi_j \cdot \nabla \tilde{H}_j^\varepsilon(\xi_j)\|_{4+\sigma,j} < \frac{1}{4}$. On the other hand, we let δ be small to find $\varepsilon^2 \|\phi_j \Delta \tilde{H}_j^\varepsilon\|_{4+\sigma,j} < \frac{1}{4}$. In summary, we can choose $\delta < \bar{C}_1$ with $\bar{C}_1$ being some $O(1)$ constant to obtain the desired estimate of III_A. By collecting (4.24), (4.26) and (4.29), one completes the proof of the claim (4.22).

Next, we proceed to prove that ϕ^o satisfies

$$
\|\mathcal{A}_o(\vec{\phi})\|_{b,o} < 1. \quad (4.30)
$$

Thanks to Lemma 3.3, one similarly has

$$
\|\mathcal{A}_o(\vec{\phi})\|_{b,o} \leq C \|J(\phi, \mathbf{p})\|_{b+2,o}, \quad (4.31)
$$

where $C > 0$. Then we are going to prove $C\|J(\phi, \boldsymbol{p})\|_{b+2,o} < 1$. Noting that the error terms involving with the inner solutions ϕ_j are the leading ones, we have for $\nabla\phi_j \cdot \nabla\eta_j$,

$$
\varepsilon^{\gamma_1-\gamma_2} |\nabla\phi_j \cdot \nabla\eta_j| \leq C\varepsilon^{\gamma_1-\gamma_2} \frac{1}{(1+|y-\xi_j'|)^{4+\sigma}} \leq C \frac{\varepsilon^{\gamma_2-\gamma_1}}{\delta^{2\gamma_2-2\gamma_1}} \frac{1}{(1+|y-\xi_j'|)^{4+\sigma+2\gamma_1-2\gamma_2}}.
$$

By choosing $\delta > \bar{C}_2\sqrt{\varepsilon}$, $2\gamma_1 - 2\gamma_2 = -\frac{\sigma}{2}$ and $b = 2 + \frac{\sigma}{2}$, one finds

$$
\varepsilon^{\gamma_1-\gamma_2} \|\nabla\phi_j \cdot \nabla\eta_j\|_{2+b,o} \leq \sigma^*,
$$

where $\sigma^* < 1$ is a small constant. Proceeding the other terms involving with the inner solutions in the similar way, one can prove $C\|J(\phi, \mathbf{p})\|_{b+2,o} < 1$, which implies the desired estimate (4.30) thanks to (4.31).

In summary, we choose $\sigma \in (0,1)$, $\delta \in (\bar{C}_2\sqrt{\varepsilon}, \bar{C}_1)$, $\gamma_1 < 1$, $\gamma_2 = \gamma_1 + \frac{\sigma}{4} < 1$, $b = 2 + \frac{\sigma}{2}$, then obtained (4.22) and (4.30). To prove our claim, it is only left to study $\|\mathcal{A}_p(\vec{\phi})\|_p$, which will be discussed later on. Now, we complete the proof of our claim (4.21).

The next step is to verify that $\mathcal{A}$ is a contraction mapping. In fact, it can be proved that there exist constants $\alpha_1, \alpha_2 \in (0,1)$ such that

$$\begin{cases} \|\mathcal{A}_j[\vec{\phi}_1] - \mathcal{A}_j[\vec{\phi}_2]\|_{2+\sigma,j} \le \bar{\alpha}_1 \|\vec{\phi}_1 - \vec{\phi}_2\|_X, \\ \|\mathcal{A}_o[\vec{\phi}_1] - \mathcal{A}_o[\vec{\phi}_2]\|_{b,o} \le \bar{\alpha}_2 \|\vec{\phi}_1 - \vec{\phi}_2\|_X \end{cases} \tag{4.32}$$

for any $\vec{\phi}_1$, $\vec{\phi}_2 \in X$ with $\|\vec{\phi}_1\|_X$, $\|\vec{\phi}_2\|_X \le 1$.

Now, we focus on the properties of operator $\mathcal{A}_p$. Note that $\boldsymbol{p}_0$ is defined as $(c_{10}, \cdots, c_{m0}, \xi_{10}, \xi_{20}, \cdots, \xi_{m0})$, where $c_{j0} = 3\mu_j^2 \bar{u}/8$ and $(\xi_{10}, \cdots, \xi_{m0})$ are the critical points of $\mathcal{J}_m$ given by (1.13), then we need to adjust $\mathbf{p}_1$ in order to guarantee the orthogonality condition shown in (4.14), i. e. $m_{i,j}[h] = 0$, where $m_{i,j}$ are given by (4.13). We claim that $\mathbf{p}_1$ is $o(1)$, which immediately implies that $\|\mathcal{A}_p\|_p$ is a contraction mapping and satisfies $\|\mathcal{A}_p\|_p < 1$.

While checking the mass condition, we find from the terms defined in (4.9) that the leading one is $\int_{\Omega_\varepsilon} f(U)\eta_j dy$. Then we calculate to obtain from (2.35) that

$$\int_{\Omega_\varepsilon} f(U)\eta_j dy = \int_{B_{2\delta/\varepsilon}(\xi_j')} (c_0 + c_{1j})U_j[\bar{u} - (c_0 + c_{1j})U_j]dy + O(\varepsilon^2),$$

where

$$\int_{B_{2\delta/\varepsilon}(\xi_j')} c_0 U_j[\bar{u} - c_0 U_j]dy = 0.$$

Now, we have

$$\int_{\Omega_\varepsilon} f(U)\eta_j dy = O(1)c_{1j} + O(\varepsilon^2). \tag{4.33}$$

Focusing on the term in the divergence form operator, we employ the divergence Theorem to estimate. One observes that the main remainder term is $\nabla_x \cdot (u\nabla H)$. To analyze it, we note that for $j = 1, \cdots, m$, H_j can be decomposed into $H_{j1} + H_{j2}$, where $H_{j1} := -|x - \xi_j|^2 \log|x - \xi_j|$. Then, we have the following equality:

$$\begin{aligned} &\frac{1}{\varepsilon}\sum_{l=1}^m \int_{\Omega_\varepsilon} \nabla_y \cdot (U_l \nabla H_l)\eta_j dy \\ =&\frac{1}{\varepsilon}\sum_{l=1}^m \int_{\partial B_{2\delta/\varepsilon}(\xi_j')} U_l \nabla H_l \cdot \nu \eta_j \mathrm{d}s - \frac{1}{\varepsilon}\sum_{l=1}^m \int_{B_{2\delta/\varepsilon}(\xi_j')} U_l \nabla H_l \cdot \nabla \eta_j \mathrm{d}y. \end{aligned} \tag{4.34}$$

Due to the decay property of cut-off functions, we find the first term in (4.34) vanishes. For the second term, we have the following expansion:

$$\nabla H_l(\varepsilon y, \xi_l) = \nabla H_{l1}(\varepsilon y, \xi_l) + \nabla H_{l2}(\xi_l, \xi_l) + \nabla^2 H_{l2}(\xi_l, \xi_l)\varepsilon(y - \xi_l') + O(\varepsilon^{1+\alpha}), \tag{4.35}$$

where $\alpha \in (0,1)$ is a constant. By substituting (4.35) into (4.34), one obtains for $l = 1, \cdots, m$,

$$\Big|\frac{1}{\varepsilon}\sum_{l=1}^{m}\int_{\Omega_\varepsilon} \nabla_y \cdot \Big(U_l(y-\xi_l')\nabla H_l(\varepsilon y, \xi_l)\Big)\eta_j dy\Big| = O(\varepsilon^2).$$

Similarly, we analyze other terms in the divergence form operator to get the errors involving with $\nabla\cdot(\varphi_l \nabla H_l)$ and $\nabla\cdot(\phi_l\nabla H_l)$ are $O(\varepsilon^2)$ and $O(\varepsilon^{1+\sigma'})$, where $\sigma' \in (0,1)$ is a constant. In particular, thanks to decomposition (4.4) and the fact $\gamma_1 := 1-\tilde{\delta}$ with $\tilde{\delta}$ being small, we have from the integration by parts that the error term related to $\nabla_x \cdot (\phi\nabla\bar{\psi})$ is $O(\varepsilon^{2+\sigma'})$. where $b' > 0$ is a small constant. Proceeding the other terms in the divergence form operator with the same argument, we finally obtain from (4.33) that $c_{j1} = o(\varepsilon)$.

We next study the first moment orthogonality condition shown in (4.14). In fact, the leading term is $\sum\limits_{l=1}\sum\limits_{j=1} \nabla_x \cdot (U_j \nabla_x(\Gamma_l + H_l)) - \nabla_x \cdot (U_j \nabla \Gamma_j)$. To estimate it, we use (2.13) to obtain for $i = 1, 2$,

$$\begin{aligned}
&\sum_{l=1}^{m}\sum_{j=1}^{m}\varepsilon\int_{\Omega_\varepsilon} \nabla_y \cdot \Big(U_j(y-\xi_j')\nabla H_l(\varepsilon y,\xi_l)\Big)\big(y-\xi_j'\big)_i \eta_j(y)\mathrm{d}y\\
&+\sum_{l=1}\sum_{j\neq l}\int_{\Omega_\varepsilon} \nabla_y\cdot(U_j(y-\xi_j)\nabla_y\Gamma_l(y-\xi_l))(y-\xi_j')_i\eta_j\mathrm{d}y\\
=&\Bigg(\sum_{l=1}^{m}\sum_{j=1}^{m}\varepsilon\int_{\Omega_\varepsilon} U_j\nabla H_l\cdot e_i\eta_j(y)\mathrm{d}y + \sum_{l=1}^{m}\sum_{j\neq l}^{m}\int_{\Omega_\varepsilon} U_j\nabla\Gamma_l\cdot e_i\eta_j(y)\mathrm{d}y\Bigg)\\
&+\Bigg(\sum_{l=1}^{m}\sum_{j=1}^{m}\varepsilon\int_{\Omega_\varepsilon} U_j\nabla H_l\cdot(y-\xi_j')_i\nabla\eta_j\mathrm{d}y + \sum_{l=1}^{m}\sum_{j\neq l}^{m}\int_{\Omega_\varepsilon} U_j\nabla\Gamma_l\cdot(y-\xi_j')_i\nabla\eta_j\mathrm{d}y\Bigg)\\
:=&I\bar{I}I_A + I\bar{I}I_B.
\end{aligned} \tag{4.36}$$

The next step is to discuss $I\bar{I}I_A$ and $I\bar{I}I_B$ given in (4.36). For $I\bar{I}I_B$, we have from the decay property of U_j and Γ_l that $|III_B| = O(\varepsilon^4)$. For $I\bar{I}I_A$, we have the fact that (2.8) can be decomposed as $H_{j1} + \bar{H}_j$, where $H_{j1} = -|x-\xi_j|^2 \log|x-\xi_j|$ and $\bar{H}_j$ is given by (2.11). Then one can only consider $\int_{B_{2\delta/\varepsilon}(\xi_j')} U_j \nabla \bar{H}_j \cdot e_i \eta_j dy$ with the help of the cut-off function. Noting that $\partial_{x_i}\bar{H}_j$ has the following expansion:

$$\nabla\bar{H}_j(\varepsilon y,\xi_j)\cdot e_i = \partial_{x_i}\bar{H}_j(\xi_j,\xi_j) + \varepsilon\nabla(\partial_{x_i}\bar{H}_j)(\xi_j,\xi_j)\cdot(y-\xi_j') + o(\varepsilon|y-\xi_j'|),$$

we further obtain

$$I\bar{I}I_A = \varepsilon\sum_{l=1}^{m}\partial_{x_i}\bar{H}_j(\xi_j,\xi_j) + O(\varepsilon^2). \tag{4.37}$$

Since $(\xi_{10}, \cdots, \xi_{m0})$ is a m-tuple critical point of $\mathcal{J}_m$ defined by (1.13), one has ξ_{j0} is the critical point of $\bar{H}_j$. On the other hand, we rewrite $\partial_{x_i}\bar{H}_j$ in the x-variable and expand it at ξ_{j0} to get

$$\partial_{x_i}\bar{H}_j(\xi_j,\xi_j) = \partial_{x_i}\bar{H}_j(\xi_{j0},\xi_{j0}) + \partial^2_{x_i}\bar{H}_j(\xi_{j0},\xi_{j0})\xi_{j1}^{(i)} + O(|\xi_{j1}|^2).$$

Thanks to the non-degeneracy of $\mathcal{J}_m$, $I\bar{I}I_A$ becomes

$$I\bar{I}I_A = \varepsilon\xi_{j1}^{(i)}\partial_{x_i}^2\bar{H}_j(\xi_{j0},\xi_{j0}) + O(\varepsilon^2) + O(|\xi_{j1}|^2) \quad \text{for} \quad i = 1, 2.$$

To determine ξ_{j1}, we need to analyze the other terms from the divergence form operator. Focusing on the error $\sum_{j=1}\sum_{l=1}\nabla_x\cdot(\varphi_j\nabla_x(\Gamma_l + H_l)) - \nabla_x\cdot(\varphi_j\nabla\Gamma_j)$, one has from the expansion of $\partial_{x_i}\bar{H}_j$ that its order is $o(\varepsilon^2)$. For terms $\nabla\cdot(\phi\nabla H)$ and $\nabla\cdot(\phi\nabla\bar{\psi})$, we similarly use decomposition (4.4) and the integration by parts to obtain that their first moments are $\mathrm{O}(\varepsilon^{1+\bar{\alpha}})$, where $\bar{\alpha} < 1$ but $\bar{\alpha} \approx 1$. Thus, we find from (4.37) that

$$\xi_{j1}^{(i)} = O(\varepsilon^{\bar{\alpha}}), \quad i = 1, 2.$$

This completes the proof of our claim that $\mathbf{p}_1 = o(1)$. Hence, when $\|\phi\|_X < 1$, $\|\mathcal{A}(\phi)\|_X < 1$ and $\mathcal{A}_p(\vec{\phi})$ is a contraction mapping.

Now, we define set $\mathcal{B}$ as

$$\mathcal{B} = \{\phi \in X : \|\phi\|_X < 1\}.$$

In light of (4.32), one finds

$$\mathcal{A}(\mathcal{B}) \subset \mathcal{B} \quad \text{and} \quad \|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_X \leq \frac{2}{3}\|\phi_1 - \phi_2\|_X \quad, \quad \forall\phi_1,\, \phi_2 \in \mathcal{B},$$

which implies that there exist the solution such that $\phi = \mathcal{A}(\phi)$.

Now, we have established the existence of multi-interior spikes rigorously. We next consider the case of boundary spikes.

5. Construction of Boundary Spike

This section is devoted to the existence of multi-boundary spikes. Since the centres of spikes are located at the boundary of Ω, it is necessary to use the transformation to straighten the boundary and study the inner problem in the half space $\mathbb{R}^2$. To be more precisely, we define the graph $\rho(x_1)$ as $\{(x_1, x_2) = (x_1, \rho(x_1)\}$ with $\rho(0) = \rho'(0) = 0$, then for $j = 1, \cdots, m$, transform (y_1, y_2) into

$$z_{1,j} = y_1 - \xi'_{j,(1)}, \quad z_{2,j} = y_2 - \xi'_{j,(2)} - \frac{1}{\varepsilon}\rho\big(\varepsilon\big(y_1 - \xi'_{j,(1)}\big)\big), \tag{5.1}$$

where $y_1 = x_1/\varepsilon$ and $y_2 = x_2/\varepsilon$. For convenience, we denote operator $\bar{P}_{\rho,\xi'_j}$ such that for any function w,

$$\bar{P}_{\rho,\xi'_j}w(y_1, y_2) = w(z_{1,j}, z_{2,j}). \tag{5.2}$$

In this transformation, the Laplace operator and Neumann boundary operator become

$$\begin{cases} \Delta_y w = \Delta_{z,j}w + (\rho'(\varepsilon z_{1,j}))^2\partial_{z_{2,j}z_{2,j}}w - 2\rho'(\varepsilon z_{1,j})\partial_{z_{1,j}z_{2,j}}w - \varepsilon\rho''(\varepsilon z_{1,j})\partial_{z_{2,j}}w, \\ \sqrt{1 + (\rho'(\varepsilon z_{1,j}))^2}\dfrac{\partial w}{\partial\boldsymbol{\nu}} = \rho'(\varepsilon z_{1,j})\partial_{z_{1,j}}w - [1 + (\rho'(\varepsilon z_{1,j}))^2]\partial_{z_{2,j}}w. \end{cases}$$

Without confusing readers, we use z_1 and z_2 to replace $z_{1,j}$ and $z_{2,j}$ but understand the z variables depend on ξ_j'. Then, it follows that

$$\begin{aligned}\Delta_y w =&\Delta_z w + (\rho''(0))^2\varepsilon^2 z_1^2 \partial_{z_2z_2} w\\ &- 2\rho''(0)\varepsilon z_1 \partial_{z_1z_2} w - \varepsilon\rho''(0)\partial_{z_2} w + O(\varepsilon^2)\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\nabla_y w_1 \cdot \nabla_y w_2 = \nabla_z w_1 \cdot \nabla_z w_2 &+ \frac{\partial w_1}{\partial z_2}\cdot\frac{\partial w_2}{\partial z_2}(\rho''(0))^2\varepsilon^2 z_1^2\\ &- \Big(\frac{\partial w_1}{\partial z_1}\cdot\frac{\partial w_2}{\partial z_2} + \frac{\partial w_1}{\partial z_2}\cdot\frac{\partial w_2}{\partial z_1}\Big)\rho''(0)\varepsilon z_1 + O(\varepsilon^2),\end{aligned} \tag{5.4}$$

where ρ and ρ' can be expanded as $\rho(\varepsilon z_1) = \frac{1}{2}\rho''(0)\varepsilon^2 z_1^2 + O(\varepsilon^2)$ and $\rho'(\varepsilon z_1) = \rho''(0)\varepsilon z_1 + O(\varepsilon^3)$. We can find there exist many extra terms except $\Delta_z w$ and $\nabla_z w_1 \cdot \nabla_z w_2$ in (5.3) and (5.4), respectively. However, they are both $o(1)$ and actually it is easy to establish the good estimate for these higher order terms, which will be explained later on.

For the multi-boundary spikes, we still use (2.18) and (2.19) as the ansatz of u and v, respectively. Similarly as shown in Section 4, we shall formulate the inner-outer gluing system. To this end, we first consider the effect of transformation (5.1) on the inner problem. Before that, we need to give some notations and definitions. The cut-off function η_H is defined by

$$\eta_{H,j}(z) = 1 \text{ for } z \in \bar{\mathbb{R}}^2_+ \cap \bar{B}_{\delta/\varepsilon}(0) \quad\text{and}\quad \eta_{H,j}(y) = 0 \text{ for } z \in \mathbb{R}^2_+ \cap B^c_{2\delta/\varepsilon}(0), \tag{5.5}$$

and thanks to (5.3) and (5.4), new error function $N_{\rho,j}$ is given as

$$\begin{aligned}N_{\rho,j} =&(\rho'(\varepsilon z_1))^2\Big[\frac{\partial^2(\phi_{H,j}\eta_{H,j})}{\partial z_2^2} - \Big(\frac{\partial U_j}{\partial z_2}\frac{\partial\bar\psi_{H,j}}{\partial z_2} + \frac{\partial^2\bar\psi_{H,j}}{\partial z_2^2}U_j\\ &+ \frac{\partial(\phi_{H,j}\eta_{H,j})}{\partial z_2}\frac{\partial\Gamma_j}{\partial z_2} + \frac{\partial^2\Gamma_j}{\partial z_2^2}(\phi_{H,j}\eta_{H,j})\Big)\Big]\\ &- \rho'(\varepsilon z_1)\Big[\frac{\partial^2(\phi_{H,j}\eta_{H,j})}{\partial z_1\partial z_2} - \Big(\frac{\partial U_j}{\partial z_1}\frac{\partial\bar\psi_{H,j}}{\partial z_2} + \frac{\partial U_j}{\partial z_2}\frac{\partial\bar\psi_{H,j}}{\partial z_1}\Big) - \frac{\partial^2\bar\psi_{H,j}}{\partial z_1\partial z_2}U_j\\ &- \Big(\frac{\partial(\phi_{H,j}\eta_{H,j})}{\partial z_1}\frac{\partial\Gamma_j}{\partial z_2} + \frac{\partial\Gamma_j}{\partial z_2}\frac{\partial(\phi_{H,j}\eta_{H,j})}{\partial z_1}\Big) - \frac{\partial^2\Gamma_j}{\partial z_1\partial z_2}(\phi_{H,j}\eta_{H,j})\Big]\\ &- \varepsilon\rho''(\varepsilon z_1)\Big[\frac{\partial(\phi_{H,j}\eta_{H,j})}{\partial z_2} - U_j\frac{\partial\bar\psi_{H,j}}{\partial z_2} - (\phi_{H,j}\eta_{H,j})\frac{\partial\Gamma_j}{\partial z_2}\Big],\end{aligned} \tag{5.6}$$

where $\bar\psi_{H,j} := -(\Delta+\varepsilon^2)^{-1}(\phi_{H,j}\eta_{H,j})$. We further define $\hat{\mathbf{P}}_1$ and $\hat{\mathbf{P}}_2$ as the first and second coordinates of ξ, then set parameter vector $\mathbf{P}_H$ as

$$\mathbf{P}_H = \big(\mathbf{c}, \hat{\mathbf{P}}_{H1}, \hat{\mathbf{P}}_{H2}\big) = (\mathbf{c}, \hat{\mathbf{P}}_1, \hat{\mathbf{P}}_2 - \rho(\hat{\mathbf{P}}_1)). \tag{5.7}$$

With the definitions of (5.5), (5.6) and (5.7), we find transformation (5.2) changes the forms of $L_j(\phi)$, $j = 1, \cdots, m$, then the inner equation becomes

$$L_j[\phi_j] = \Big(N_{\rho,j} + \varepsilon^{-\gamma_1}(F_j(\bar P_{\rho,\xi_j'}(\cdot), \mathbf{P}_H)\Big)\eta_{H,j} := F_{H,j}(\bar P_{\rho,\xi_j'}(\cdot), \mathbf{P}_H)\eta_{H,j}, \tag{5.8}$$

where $(z_1, z_2) \in \mathbb{R}^2_+$, $F_j(\vec\phi, \mathbf{p})$ is given by (4.12) and all functions in $F_j(\cdot)$ are replaced by $\bar P_{\rho,\xi_j'}(\cdot)$. It is worthwhile to mention that there does not exist leading order terms involving with the mean curvature $\rho''(0)$ in $F_j(\phi, \mathbf{P}_H)$ since $U_j = e^{\Gamma_j}$, $j = 1, \cdots .m$, pointwisely. Next, we consider the outer problem and find that the local transformation (5.1) can not influence the outer operator since it only straightens the boundary near the centres of spikes.

Our next goal is to establish the estimate of $F_{H,j}$ shown in (5.8). Before that, we define $\xi'_{H,j} := \big(\xi'_{j,(1)}, \xi'_{j,(2)} - \frac{1}{\varepsilon}\rho\big(\varepsilon\xi'_{j,(1)}\big)\big)$ and the inner norm in the half space as

$$\|h\|_{\nu,H,j} := \sup_{z\in\mathbb{R}^2_+} |h|(1+|z|)^\nu.$$

Moreover, spaces $X_{H,j}$, $X_{o,H}$ and $X_{p,H}$ are given the same as X_j, X_o and X_p except that $\mathbb{R}^2$ and $\|\cdot\|_{2+\sigma,j}$ are replaced by $\mathbb{R}^2_+$ and $\|\cdot\|_{2+\sigma,H,j}$, respectively. Similarly as shown in Section 4, we denote the norm and inner solutions for boundary spikes as $\|\cdot\|_{\bar{X}}$ and $\vec{\phi}_{H,j}$, next discuss the new error N_ρ and focus on the following worse term:

$$\begin{aligned}
&(\rho'(\varepsilon z_1))^2\frac{\partial^2(\phi_{H,j}\eta_{H,j})}{\partial z_2^2} - \rho'(\varepsilon z_1)\frac{\partial^2(\phi_{H,j}\eta_{H,j})}{\partial z_1\partial z_2} - \varepsilon\frac{\partial(\phi_{H,j}\eta_{H,j})}{\partial z_2}\rho''(\varepsilon z_1)\\
=&(\rho''(0))^2(\varepsilon z_1)^2\frac{\partial^2(\phi_{H,j}\eta_{H,j})}{\partial z_2^2} - \rho''(0)\varepsilon z_1\frac{\partial^2(\phi_{H,j}\eta_{H,j})}{\partial z_1\partial z_2}\\
&-\varepsilon\rho''(0)\frac{\partial(\phi_{H,j}\eta_{H,j})}{\partial z_2} + O(\varepsilon^2).
\end{aligned}\tag{5.9}$$

Since z satisfies $|z| < \delta$ for some constant $\delta > 0$, we can similarly choose $\delta > 0$ small enough such that (5.9) satisfies

$$\left\|(\rho'(\varepsilon z_1))^2\frac{\partial^2(\phi_{H,j}\eta_{H,j})}{\partial z_2^2} - \rho'(\varepsilon z_1)\frac{\partial^2(\phi_{H,j}\eta_{H,j})}{\partial z_1\partial z_2} - \varepsilon\frac{\partial(\phi_{H,j}\eta_{H,j})}{\partial z_2}\rho''(\varepsilon z_1)\right\|_{4+\sigma,H,j} < \sigma_1,$$

where $\sigma_1 > 0$ is a small constant. Proceeding the other term in N_ρ with the similar discussion, one can show that

$$\|N_{\rho,j}\eta_{H,j}\|_{4+\sigma,H,j} \le \sigma_2, \tag{5.10}$$

where $\sigma_2 > 0$ is a small constant.

The next step is to estimate (5.8) in $F_j(\cdot)$ then prove

$$\|F_j(\bar{P}_{\rho,\xi'_j}(\cdot), \mathbf{P}_H)\|_{4+\sigma,H,j} < \sigma_3 \tag{5.11}$$

for some small constant $\sigma_3 > 0$. To this end, we repeat the argument what we have used for (4.23) and decompose F_j into III_1, III_2, III_3 and III_4. It is easy to establish the estimate for the four terms and we only exhibit the discussion for one error in $\nabla_y \bar{P}_{\rho,\xi'_j} U_j \cdot \nabla_y \bar{P}_{\rho,\xi'_j}\Gamma_l$ since the other ones can be similarly tackled. In fact, thanks to (5.4), we find for $|z| \le 2\delta$, there exists the following term:

$$\left|(\rho'(\varepsilon z_1))^2\frac{\partial U_j(z)}{\partial z_2}\cdot\frac{\partial \Gamma_l(z-(\xi'_{H,l}-\xi'_{H,j}))}{\partial z_2}\right|,$$

where $\xi'_{H,l}$ and $\xi'_{H,j}$ denote the location of spikes in the z-variable. Since $\rho'(\varepsilon z_1) = \rho''(0)\varepsilon z_1$ and U_j, $\frac{\partial\Gamma}{\partial z_2}$ have decay properties, one can choose $\delta > 0$ small enough such that $\left\|(\rho'(\varepsilon z_1))^2\frac{\partial U_j}{\partial z_2}\cdot\frac{\partial\Gamma_l}{\partial z_2}\right\|_{4+\sigma,H,j} \le \sigma_4$ for some small constant $\sigma_4 > 0$. Now, we have obtained $\|III_i\|_{4+\sigma,H,j} = o(1)$, $i = 1,2,3,4$, which implies $\|F_{H,j}\eta_{H,j}\|_{4+\sigma,H,j}$ is bounded. It immediately follows from Lemma 3.2 that (5.8) admits the solution $\phi_{H,j}$ satisfying

$$\|\phi_{H,j}\|_{2+\sigma,H,j} \leq C\|F_{H,j}(z,\mathbf{P}_H)\eta_{H,j}\|_{4+\sigma,H,j}$$

for some constant $C > 0$. On the other hand, we note the local transformation does not change the structure of the outer operator given in (4.12), thereby obtain that there exists bounded linear operator $\mathcal{A}_H(\vec{\phi}_H)$ such that $\vec{\phi}_H = \mathcal{A}_H(\vec{\phi}_H)$ with the help of Lemma 3.3. Here $\vec{\phi}_H$ and $\mathcal{A}_H$ are defined the same as (4.17) and (4.18) except that ϕ_j and $\mathbf{p}$ are replaced by $\phi_{H,j}$ and $\mathbf{P}_H$, respectively.

We would like to point out that the orthogonality condition stated in Lemma 3.2 should be checked and it plays the vital role on determining $\xi'_{H,j}$. Indeed, we claim $\mathbf{P}_{1H} = o(1)$, where $\mathbf{P}_{1H} := \mathbf{P}_H - \mathbf{P}_{0H}$ and $\mathbf{P}_{0H}$ is given the same as (4.15) except that ξ is replaced by ξ_H. Combining this claim with (5.10) and (5.11), we similarly obtain that if $\|\vec{\phi}_H\|_{\bar{X}} < 1$, then $\|\mathcal{A}_H(\vec{\phi}_H)\|_{\bar{X}} < 1$. By using the same argument shown in Section 4, one has $\mathcal{A}_{\bar{H}}$ is a contraction mapping. To be more precisely, for any $\vec{\phi}^1_H$, $\vec{\phi}^2_H \in X_{H,j}$,

$$\|\mathcal{A}_H(\vec{\phi}^1_H) - \mathcal{A}_H(\vec{\phi}^2_H)\|_{\bar{X}} \leq \bar{\alpha}_3\|\vec{\phi}^1_H - \vec{\phi}^2_H\|_{\bar{X}}, \ \ 0 < \bar{\alpha}_3 < 1.$$

Now, we shall finish the proof of our claim. For the mass orthogonality condition shown in (3.29), one has the leading term in $F_{H,j}$ is $\int_{\mathbb{R}^+_2} f(U)\eta_{H,j}dz$. It is similar to calculate then obtain from (2.35) that

$$\varepsilon^2\int_{\mathbb{R}^2_+} f(U)\eta_{H,j}dz = O(\varepsilon^2)c_{1j} + O(\varepsilon^4), \tag{5.12}$$

since $c_{j0} = \frac{3\mu_j^2\bar{u}}{8}$. While checking the mass of divergence form operator, we find it is necessary to analyze the effect of Neumann boundary operator on the error. We only exhibit the analysis of the leading term $\nabla_z\cdot(u\nabla H)$ since the other ones can be similarly treated. To estimate it, we similarly decompose H_j into $H_{j1} + H_{j2}$, $j = 1,\cdots,m$, where $H_{j1} := -|x-\xi_j|^2\log|x-\xi_j|$. After transforming H_{j1} into the z-variable and rewriting z_1 as $z_1 := y_1$ without translation, we consider H_{j2} and calculate to find

$$\varepsilon\sum_{l=1}^m\int_{\mathbb{R}^2_+}\nabla_z\cdot(U_l\nabla H_{l2})\eta_{H,j}dz = \varepsilon\int_{\xi'^{(1)}_{H,j}-\frac{2\delta}{\varepsilon}}^{\xi'^{(1)}_{H,j}+\frac{2\delta}{\varepsilon}} U_j\frac{\partial H_{j2}}{\partial z_2}\eta_{H,j}\mathrm{d}z_1 + O(\varepsilon^4). \tag{5.13}$$

Moreover, the corresponding boundary term satisfies

$$-U_j\frac{\partial H_{j2}}{\partial\boldsymbol{\nu}} = U_j\frac{\partial H_{j2}}{\partial z_2} - U_j\frac{\partial H_{j2}}{\partial z_1}\rho''(0)\varepsilon z_1 + O(\varepsilon^2) = U_j\frac{\partial H_{j2}}{\partial z_2} + O(\varepsilon^2). \tag{5.14}$$

By collecting (5.13) and (5.14), one has for $j = 1,2,\cdots,m$,

$$\varepsilon\sum_{l=1}^m\int_{\mathbb{R}^2_+}\nabla_z\cdot(U_l\nabla H_{l2})\eta_{H,j}dz + \varepsilon\sum_{l=1}\int_{\partial\mathbb{R}^2_+}U_l\frac{\partial H_{l2}}{\partial\boldsymbol{\nu}}\eta_{H,j}\mathrm{d}S = O(\varepsilon^3). \tag{5.15}$$

Proceeding the other terms in the divergence form operator with the same argument, we obtain from (5.12) and (5.15) that $c_{j1} = O(\varepsilon)$. Focusing on the first moment orthogonality condition (3.29), we similarly check the effect of boundary operator on the error, then use (2.13) in the z-variable to show $\xi^{(1)}_{H,j1} = O(\varepsilon^{\bar{\alpha}})$, where $\bar{\alpha} < 1$ but $\bar{\alpha} \approx 1$. This completes the proof of our claim, then we obtain $\mathcal{A}_H$ is a contraction mapping. Thus, one has there exists the unique solution $\vec{\phi}_H$ such that $\vec{\phi}_H = \mathcal{A}_H(\vec{\phi}_H)$.

Combining all results obtained in Section 4 and Section 5, we give the rigorous proof of Theorem 1.1. This Theorem demonstrates that there are finite many multi-spikes to (1.3) with the heights of cellular density u and chemical concentration v being $O(1)$ and $O(\varepsilon^2)$, respectively.

6. Numerical Studies and Discussion

In this section, we shall present several set of numerical simulations to illustrate and highlight our theoretical results. The FLEXPDE [7] will be used to study system (1.1) numerically. By setting the error is 10^{-4} and the maximal running time is $3500s$, we investigate the effect of parameters on the dynamics of system (1.1).

Fig. 1 presents the evolution of the cellular density u and chemical concentration v within system (1.1) when the chemotactic rate is sufficiently large. In particular, when $t = 670s$, the profiles of (u, v) can represent the stable steady states (u_s, v_s) of (1.1). It can be seen that u_s is located at the corner with the concentrated structure and v_s possesses the positive lower bound and grows towards the same corner slowly. The numerical results implies that system (1.1) admits the stable boundary spike, which is identical with our theoretical analysis.

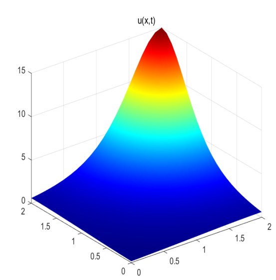
We next increase μ from 0.2 to 3 and fix the other parameters to plot Fig. 2. It is shown that when μ is large, the solution (u, v) converges to stripes rather than boundary spikes even though the initial data is the small perturbation of some boundary spike. Indeed, Theorem 1.1 states that only under the assumption $\mu\bar{u}$ is small, we are able to construct the multi-spikes with the form of (1.11) and (1.12).

Fig. 3 is presented to show the temporal-spatial dynamics of time-dependent system (1.1) when chemical diffusion rate d_2 is small. It is surprising that system (1.1) admits the stable interior spike, which can not be observed in classical Keller–Segel models. [6] The numerical simulations illustrate that the logistic source $f(u)$ plays the critical role on the formation of complex patterns.

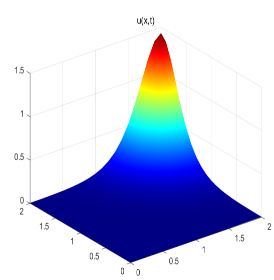
6.1. *Discussion*

We have employed the gluing method to study the existence of non-constant stationary solutions of (1.1) in the asymptotically limit of $\chi \gg 1$. The idea shown in [5] has been developed and applied on establishing the linear theories and analyzing the inner-outer gluing system. Our main work is the construction of multi-spiky solutions to (1.3) with cellular density u and chemical concentration v being $O(1)$ and $O(\varepsilon^2)$, respectively.

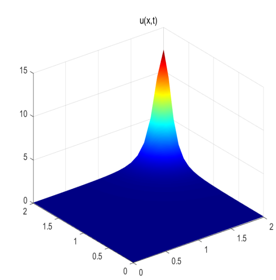
We would like to discuss some open problems that deserve further explorations. We have shown that there exist the multi-spikes with the form of (1.11) and (1.12) to system (1.3). However, we do not investigate their stability properties which is another interesting but complicated problem. In fact, the numerical results illustrate that when $\chi \gg 1$, system (1.1) only admits stable boundary spikes. In this paper, we set chemotactic rate χ be large and keep other parameters are $O(1)$ to seek the non-trivial patterns. However, we can find when chemical diffusion rate d_2 is small, there exists other type of steady states shown in Fig. 3. This result implies that there exist other regimes such that (1.1) admits different types of spiky steady states. Some formal computations in this regime for one-dimensional Keller-Segel with logistic growth were done in Ref. 16. The relevant theoretical analysis is challenging but deserves exploring in the future.



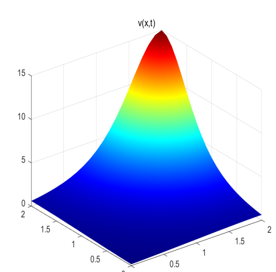
(a) u at t=0s



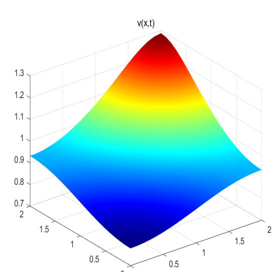
(b) u at t=1s



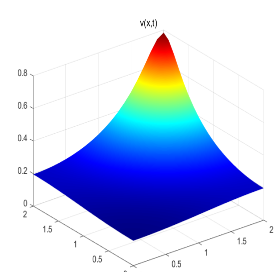
(c) u at t=670s



(d) v at t=0s



(e) v at t=1s



(f) v at t=670s

Fig. 1. *Dynamics of the cellular density u and the chemical concentration v to system (1.1) in a 2-D rectangle with length $L = 2$, intrinsic rate $\mu = 0.2$, carrying capacity $\bar{u} = 3$, diffusion rate $d_1 = d_2 = 1$, chemotactic rate $\chi = 10$ and initial data $u_0 = v_0 = \frac{15}{[1+(x-2)^2+(y-2)^2]^2}$.* We can see that the solution (u, v) tends to be located at the corner in which the heights of u and v are $O(1)$ and $O(1/\chi)$, respectively.

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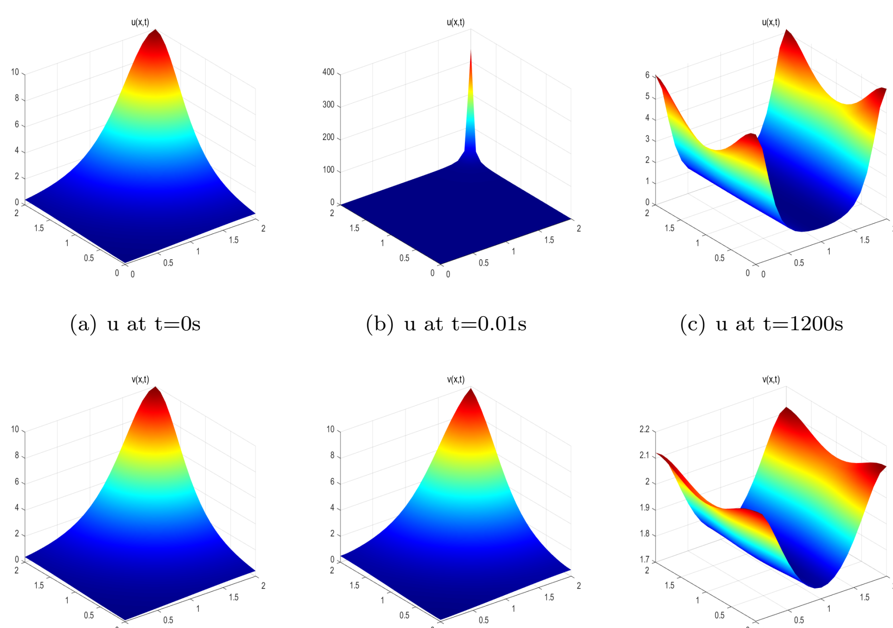

(a) u at t=0s (b) u at t=0.01s (c) u at t=1200s

(d) v at t=0s (e) v at t=0.01s (f) v at t=1200s

Fig. 2. *Dynamics of the cellular density u and the chemical concentration v in a 2-D rectangle with length $L = 2$, intrinsic rate $\mu = 3$, carrying capacity $\bar{u} = 3$, diffusion rate $d_1 = d_2 = 1$, chemotactic rate $\chi = 10$ and initial data $u_0 = v_0 = \frac{10}{[1+(x-2)^2+(y-2)^2]^2}$.* The solution at $t = 1200s$ can represent the stable steady states of (1.1) and we find that the stable boundary spikes disappear.

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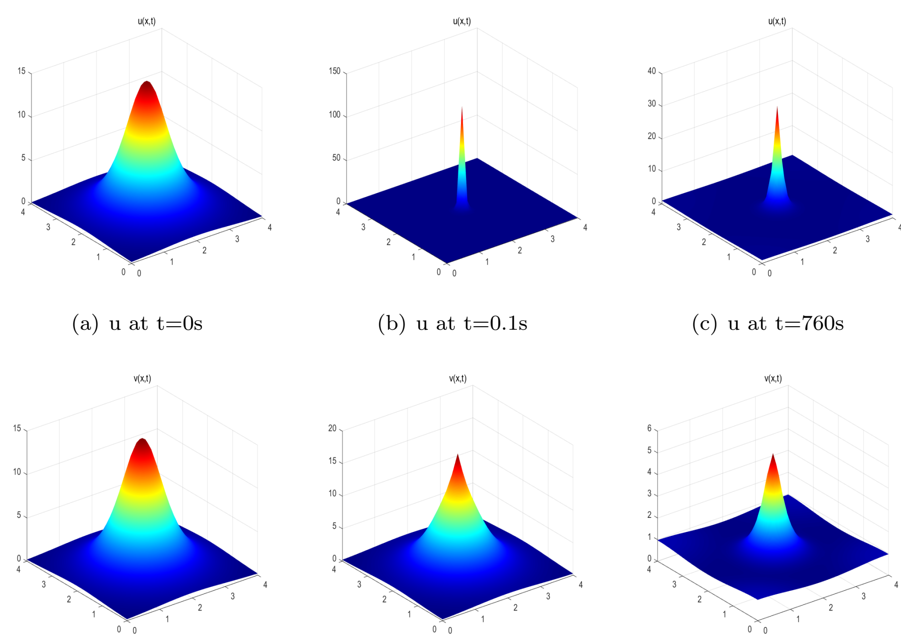

(a) u at t=0s (b) u at t=0.1s (c) u at t=760s

(d) v at t=0s (e) v at t=0.1s (f) v at t=760s

Fig. 3. *Evolution of the population density u and chemical signal concentration v in a 2-D rectangle with length $L = 4$, intrinsic rate $\mu = 0.5$, carrying capacity $\bar{u} = 3$, cellular diffusion rate $d_1 = 5$, chemical diffusion rate $d_2 = 0.1$, chemotactic rate $\chi = 5$ and initial data* $u_0 = v_0 = \frac{15}{[1+(x-2)^2+(y-2)^2]^2}$. It indicates that (1.1) also admits the stable interior spikes with the height of u approximately being $O(1/\sqrt{d_2})$.

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