ON SINGULAR SOLUTIONS OF LANE-EMDEN EQUATION ON THE HEISENBERG GROUP

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Abstract. By applying gluing method, we construct infinitely many axial symmetric singular positive solutions to the Lane-Emden equation

$$\Delta_{\mathbb{H}} u + u^p = 0, \quad \text{in} \quad \mathbb{H}^n \backslash \{0\}$$

on the Heisenberg group H^n, where $n > 1, Q/(Q-4) < p < p_{JL}(Q-2)$ and $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$.

Dedicated to David Jerison on the occasion of his 70th birthday, with admiration

1. Introduction

Let $n > 1$ and $\mathbb{H}^n$ be the Heisenberg group $(\mathbb{R}^{2n+1}, \circ)$ equipped with the group action

$$\xi_0 \circ \xi = (x + x_0, y + y_0, t + t_0 + 2\sum_{i=1}^{n}(x_i y_{0i} - y_i x_{0i}))$$

for

$$\xi = (x_1, x_2, \cdots, y_1, y_2, \cdots, y_n, t) := (x, y, t) \in \mathbb{R}^{2n+1}.$$

Let $\Delta_{\mathbb{H}}$ be the subelliptic Laplacian defined by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{n}(X_i^2 + Y_i^2).$$

A direct calculation shows that

$$\Delta_{\mathbb{H}} := \sum_{i=1}^{n}[\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2)\frac{\partial^2}{\partial t^2}].$$

Let $Q = 2n + 2$ denote the homogeneous dimension of $\mathbb{H}^n$. In a seminal work, Jerison and Lee [9] proved the following celebrated classification result.

Theorem 1.1. *All positive solutions of the equation*

$$\Delta_{\mathbb{H}} u + u^{\frac{Q+2}{Q-2}} = 0, \quad in \quad \mathbb{H}^n \tag{1.1}$$

satisfying the integrability condition

$$\int_{\mathbb{H}^n} u^{\frac{2Q}{Q-2}} < +\infty \tag{1.2}$$

can be written as $\omega_{\lambda,\xi}$ for some $\lambda > 0$ and $\xi \in \mathbb{H}^n$, where

$$\omega_{\lambda,\xi} = \lambda^{\frac{2-Q}{2}} \omega \circ \delta_{\lambda^{-1}} \circ \tau_{\xi^{-1}}$$

and

$$\omega(x, y, t) = c_0 \frac{1}{(t^2 + (1 + x^2 + y^2)^2)^{\frac{Q-2}{4}}} \tag{1.3}$$

with c_0 being a suitable positive constant. (Here $\tau_\xi(\xi') = \xi \circ \xi'$ is the left translation on $\mathbb{H}^n$ and $\delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t)$ is the natural dilation.)

The work of Jerison-Lee completely solved the so-called CR-Yamabe problem in $\mathbb{H}^n$ and it opened door in the study of more general Lane-Emden equations on $\mathbb{H}^n$:

$$\Delta_{\mathbb{H}} u + u^p = 0, \quad \text{in} \quad \mathbb{H}^n. \tag{1.4}$$

In [12], Malchiodi and Uguzzoni proved that the unique positive solution classified in [9] is nondegenerate in the sense that $\psi \in S_0^1(\mathbb{H}^n)$ is a solution of the linearized equation

$$\Delta_{\mathbb{H}} \psi + \frac{Q+2}{Q-2} \omega^{\frac{4}{Q-2}} \psi = 0 \tag{1.5}$$

if and only if there exist coefficients $\mu, \nu_1, \nu_2, \cdots, \nu_{2n}, \nu_{2n+1}$ such that

$$\psi = \mu \frac{\partial \omega_{\lambda,\xi}}{\partial \lambda}\bigg|_{(\lambda,\xi)=(1,0)} + \sum_{\nu=1}^{2n+1} \nu_i \frac{\partial \omega_{\lambda,\xi}}{\partial \xi_i}\bigg|_{(\lambda,\xi)=(1,0)}, \tag{1.6}$$

where $S_0^1(\mathbb{H}^n)$ is the Folland-Stein Sobolev space (see [12] for the details of the definition).

In the subcritical case $1 < p < (Q+2)/(Q-2)$, the equation (1.4) was first considered by Birindelli-Capuzzo Dolcetta-Cutrà in [2]. It is proved in [2] that if $1 < p \le Q/(Q-2)$ and if u is a nonnegative solution of (1.4), then $u \equiv 0$. In [10], Lu and the first author considered Lane-Emden equations in more general stratified groups and the existence and non-existence of solutions were obtained. By applying the moving plane method, Birindelli and Prajapat proved in [3] that if $1 < p < (Q+2)/(Q-2)$ and if u is a nonnegative solution of the equation (1.4) such that $u(x, y, t) = u(r, t)$ with $r = \sqrt{x^2 + y^2}$, then $u \equiv 0$. In [14], Yu generalized the method in [3] to some semilinear elliptic equations in the Heisenberg group with general nonlinearities. In [13], Xu improved the result in [2] to the range $n > 1, 1 < p < (Q(Q+2))/(Q-1)^2$. Since the proof in [13] is based on integration by part, it is not necessary to assume that solutions satisfy any symmetry. Recently, in a interesting paper [11], Ma and Ou give a complete classification of nonnegative solutions to the equation (1.7) when p is subcritical. The proof in [11] is based on a generalized Obata type formula found by Jerison and Lee [9].

In this paper, we consider positive solutions of the following Lane-Emden equation on $\mathbb{H}^n$ with a singularity

$$\Delta_{\mathbb{H}} u + u^p = 0, \quad \text{in} \quad \mathbb{H}^n \backslash \{0\}. \tag{1.7}$$

Compared with the equation (1.4), the results concerning (1.7) are less known. In [11], a pointwise estimate for the positive solutions near the isolated singularity was proved when p is subcritical. In [1], Afeltra constructed a family of positive singular solutions to the equation

$$\Delta_{\mathbb{H}} u + u^{\frac{Q+2}{Q-2}} = 0, \quad \text{in} \quad \mathbb{H}^n \backslash \{0\}. \tag{1.8}$$

Similar to the Fowler solutions of the Yamabe equations on $\mathbb{R}^n$, the positive singular solutions constructed in [1] satisfy the homogeneity property

$$u \circ \delta_T = T^{-\frac{Q-2}{2}} u \tag{1.9}$$

for some T large enough. It will be an interesting problem to prove whether any singular positive solution of (1.8) satisfies the homogeneity property (1.9).

In this paper, we apply the gluing method in [5] to construct positive singular solutions to the equation (1.7) in the supercritical case $p > Q/(Q-4)$. In order to give the statement of our main result, we introduce the Joseph-Lundgren exponent

$$p_{JL}(n) = \begin{cases} \infty, & \text{if} \quad 3 \le n \le 10, \\ \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)}, & \text{if} \quad n \ge 11. \end{cases} \tag{1.10}$$

The exponent (1.10) is closely related to the classification of stable solutions of the Lane Emden equation

$$\Delta u + u^p = 0, \quad \text{in} \quad \mathbb{R}^n. \tag{1.11}$$

Here, a solution of (1.11) is called stable if

$$\int_{\mathbb{R}^n} |\nabla \psi|^2 dx - p \int_{\mathbb{R}^n} u^{p-1}\psi^2 dx \ge 0, \quad \forall \psi \in C_0^\infty(\mathbb{R}^n).$$

Indeed, it was proved in [6] that if $u \in C^2(\mathbb{R}^n)$ is a stable solution of (1.11) with $1 < p < p_{JL}(n)$, then $u \equiv 0$. Moreover, (1.11) admits a smooth positive, bounded, stable and radial solution for $n \ge 11, p > p_{JL}(n)$.

The main result in this paper is the following.

Theorem 1.2. *Assume that $n > 1$ and*

$$\frac{Q}{Q-4} < p < p_{JL}(Q-2), \tag{1.12}$$

then the equation (1.7) admits infinitely many singular solutions.

Remark 1.3. *Since $Q = 2n+2$, then $Q/(Q-4) = (n+1)/(n-1)$ is the critical exponent of the Hardy equation*

$$\Delta u + |x|^{-1} u^p = 0, \quad in \quad \mathbb{R}^{n+1}.$$

Remark 1.4. *If $p > Q/(Q-4)$, then*

$$n - 1 - \frac{1}{p-1} > \frac{n-1}{2}. \tag{1.13}$$

Moreover, since $p < p_{JL}(Q-2)$, then

$$4(n-1-\frac{1}{p-1}) - (n-1-\frac{2}{p-1})^2 > 0. \tag{1.14}$$

Indeed, by the properties of the Joseph-Lundgren exponent, we can check that if $p < p_{JL}(N)$, then

$$\frac{2p}{p-1}(N-2-\frac{2}{p-1}) > \frac{(N-2)^2}{4}.$$

Therefore, if $p < p_{JL}(Q-2) = p_{JL}(2n)$, then

$$\frac{2p}{p-1}(2n-2-\frac{2}{p-1}) > \frac{(2n-2)^2}{4}.$$

But this is equivalent to (1.14).

The content of this paper will be organized as follows. In Section 2, we present some preliminary results. In Section 3, we construct inner solutions by studying an initial value problem. In Section 4, we study the asymptotic behavior of the outer problem. In Section 5, we match the inner solutions and the outer solutions constructed in Section 3 and Section 4 to obtain solutions of the equation (1.7).

2. Preliminaries

We will say that u is cylindrical in $\mathbb{H}^n$ if for all $(x, y, t) \in \mathbb{H}^n$,

$$u(x, y, t) = u(r, t)$$

with $r = \sqrt{x^2 + y^2}$. If u is a cylindrical solution of the equation (2.1), then

$$u_{rr} + \frac{2n-1}{r} u_r + 4r^2 u_{tt} + u^p = 0. \tag{2.1}$$

Let us consider the transform

$$\rho^2 = \sqrt{r^4 + t^2}, \quad \theta = \arctan \frac{r^2}{t}, \quad \theta \in (0, \pi). \tag{2.2}$$

By applying this new coordinates, then u satisfies the equation

$$\frac{r^2}{\rho^2} u_{\rho\rho} + 4\frac{r^2}{\rho^4} u_{\theta\theta} + (2n+1)\frac{r^2}{\rho^3} u_\rho + 4n\frac{t}{\rho^4} u_\theta + u^p = 0. \tag{2.3}$$

We want to find a solution of the form

$$u(\rho, \theta) = \rho^{-\frac{2}{p-1}} \Phi(\theta).$$

After some computations, we can check that Φ satisfies the equation

$$4\sin\theta \Phi_{\theta\theta} + 4n\cos\theta \Phi_\theta - \beta \sin\theta \Phi + \Phi^p = 0, \tag{2.4}$$

where

$$\beta = \frac{4}{p-1}(n - \frac{1}{p-1}). \tag{2.5}$$

If

$$\Phi(\theta) = \Phi(\pi - \theta), \quad \text{for} \quad 0 \le \theta < \frac{\pi}{2},$$

then Φ satisfies the equation

$$\begin{cases} 4\sin\theta\Phi_{\theta\theta} + 4n\cos\theta\Phi_\theta - \beta\sin\theta\Phi + \Phi^p = 0, & \text{in} \quad (0, \frac{\pi}{2}), \\ \Phi(\theta) > 0, & \text{in} \quad (0, \frac{\pi}{2}), \\ \Phi'(0) \text{ exists}, \ \Phi'(\frac{\pi}{2}) = 0. \end{cases} \tag{2.6}$$

Remark 2.1. *It is an important to observe that the equation (2.4) has a explicit singular solution. Indeed, the function*

$$\Phi_*(\theta) = A_p[\sin\theta]^{-\frac{1}{p-1}} \tag{2.7}$$

with

$$A_p = [\frac{4}{p-1}(n - 1 - \frac{1}{p-1})]^{\frac{1}{p-1}} \tag{2.8}$$

is a singular solution of the equation (2.4) with a singular point at $\theta = 0$ and $\theta = \pi$.

3. Inner solutions

In this section, we study solutions $\Phi(\theta)$ of (2.6) with $\Phi(0) = \Lambda$ and analyze their behaviors near $\theta = 0$, where Λ is a sufficiently large number. Since Λ is sufficiently large, it is convenient to set

$$\Lambda = \epsilon^{-\alpha}, \quad \alpha = \frac{1}{p-1}$$

with ϵ sufficiently small. Let

$$\Phi(\theta) = \epsilon^{-\alpha} v(s), \quad s = \frac{\theta}{\epsilon}, \tag{3.1}$$

we obtain from (2.4) that $v(s)$ satisfies the initial value problem

$$\begin{cases} v_{ss} + n\epsilon\cot(\epsilon s)v_s - \frac{\beta}{4}\epsilon^2 v + \frac{\epsilon}{4\sin(\epsilon s)}v^p = 0, \\ v(0) = 1. \end{cases} \tag{3.2}$$

Since for $\epsilon > 0$ sufficiently small,

$$\cot(\epsilon s) = \frac{\cos(\epsilon s)}{\sin(\epsilon s)} = \frac{1}{\epsilon s} - \frac{1}{3}(\epsilon s) + \sum_{k=1}^{\infty}\alpha_k(\epsilon s)^{2k+1},$$

$$\csc(\epsilon s) = \frac{1}{\sin(\epsilon s)} = \frac{1}{\epsilon s} + \frac{1}{6}(\epsilon s) + \sum_{k=1}^{\infty}\beta_k(\epsilon s)^{2k+1},$$

we have

$$\begin{cases} v_{ss} + \frac{n}{s}v_s - \frac{n}{3}(\epsilon^2 s)v_s + (\sum\limits_{k=1}^{\infty} n\alpha_k\epsilon^{2(k+1)}s^{2k+1})v_s - \frac{\beta}{4}\epsilon^2 v \\ +\frac{1}{4s}v^p + \frac{1}{24}\epsilon^2 s v^p + \frac{1}{4}\sum\limits_{k=1}^{\infty}\beta_k\epsilon^{2(k+1)}s^{2k+1}v^p = 0, \\ v(0) = 1. \end{cases} \tag{3.3}$$

The first approximation to the solution of (3.3) is the radial solution of the Hardy equation

$$\begin{cases} \Delta v + \frac{1}{4|x|}v^p = 0, \quad \text{in} \quad \mathbb{R}^{n+1}, \\ v(0) = 1. \end{cases} \tag{3.4}$$

Since $p > Q/(Q-4) = (n+1)/(n-1)$, it is proved in [4] (Lemma 4.1) and [8] that the equation (3.4) has a unique positive radial solution.

Our first objective in this section is to characterize the asymptotic behavior of the unique positive radial solution of (3.4). More precisely, we have the following result.

Lemma 3.1. *Let $Q/(Q-4) < p < p_{JL}(Q-2)$, then there exist constants a_0, b_0 and S_0 such that for $s \geq S_0$, the unique positive radial solution $v_0(s)$ of (3.4) satisfies*

$$v_0(s) = A_p s^{-\alpha} + \frac{a_0\cos(\omega\ln s) + b_0\sin(\omega\ln s)}{s^{\frac{n-1}{2}}} + O(s^{-(n-1-\frac{1}{p-1})}), \tag{3.5}$$

where

$$A_p^{p-1} = \frac{4}{p-1}(n-1-\frac{1}{p-1}) \tag{3.6}$$

and

$$\omega = \frac{1}{2}\sqrt{4(n-1-\frac{1}{p-1}) - (n-1-\frac{2}{p-1})^2}. \tag{3.7}$$

Moreover, in (3.5), we have $a_0^2 + b_0^2 \neq 0$.

Proof. Let

$$v_0(s) = s^{-\frac{1}{p-1}}w(\tau), \quad \tau = \ln s.$$

It follows from (3.4) that w satisfies the equation

$$w''(\tau) + (n-1-\frac{2}{p-1})w'(\tau) - \frac{1}{p-1}(n-1-\frac{1}{p-1})w(\tau) + \frac{1}{4}w^p(\tau) = 0. \tag{3.8}$$

By Lemma 4.1 in [4], we know that the unique positive radial solution of (3.4) satisfies

$$\lim_{|x|\to\infty}|x|^{\frac{1}{p-1}}v_0(x) = [\frac{4}{p-1}(n-1-\frac{1}{p-1})]^{\frac{1}{p-1}}. \tag{3.9}$$

(3.9) is equivalent to

$$\lim_{\tau\to\infty} w(\tau) = A_p,$$

where A_p is defined in (3.6). Let

$$w(\tau) = A_p + V(\tau),$$

then V satisfies the equation

$$V''(\tau) + (n-1-\frac{2}{p-1})V'(\tau) + \frac{p-1}{4}A_p^{p-1}V(\tau) + g(V(\tau)) = 0 \tag{3.10}$$

with

$$g(V(\tau)) = \frac{1}{4}[(A_p+V(\tau))^p - A_p^p - pA_p^{p-1}V(\tau)].$$

We notice that

$$\frac{p-1}{4}A_p^{p-1} = n-1-\frac{1}{p-1},$$

then (3.10) can also be written as

$$V_{\tau\tau} + (n-1-\frac{2}{p-1})V_\tau + (n-1-\frac{1}{p-1})V + g(V) = 0. \tag{3.11}$$

Fix a constant T, by the method of variation of constants, we have

$$V(\tau) = e^{\sigma\tau}[a\cos(\omega\tau) + b\sin(\omega\tau)] + \frac{1}{\omega}\int_T^\tau e^{\sigma(\tau-\tau')}\sin\omega(\tau-\tau')g(V(\tau'))d\tau', \tag{3.12}$$

where

$$\sigma = -\frac{1}{2}(n-1-\frac{2}{p-1})$$

and ω is given by (3.7). Set

$$\tilde{V}(\tau) = e^{-\sigma\tau}V(\tau)$$

and let $T_0 = \ln S_0$ be a sufficiently large constant, then

$$\tilde{V}(\tau) = a\cos(\omega\tau) + b\sin(\omega\tau) + \frac{1}{\omega}\int_{T_0}^\tau e^{-\sigma\tau'}\sin\omega(\tau-\tau')g(e^{\sigma\tau'}\tilde{V}(\tau'))d\tau'. \tag{3.13}$$

We consider

$$\mathcal{N}\tilde{V}(\tau) = a\cos(\omega\tau) + b\sin(\omega\tau) + \frac{1}{\omega}\int_{T_0}^\tau e^{-\sigma\tau'}\sin\omega(\tau-\tau')g(e^{\sigma\tau'}\tilde{V}(\tau'))d\tau'$$

as a map on $C[T_0,\infty)$, then $\mathcal{N}\tilde{V}(\tau)$ is a map from $C[T_0,\infty)$ into itself. Let

$$\mathcal{B} = \{\tilde{V}\in C[T_0,\infty) : \|\tilde{V}\|_0 = \sup_{T_0<\tau<\infty}|\tilde{V}(\tau)| \le 2C_1\},$$

where C_1 is a positive constant. If $\tilde{V}\in\mathcal{B}$, then

$$|g(e^{\sigma\tau}\tilde{V}(\tau))| = e^{2\sigma\tau}O(1) \tag{3.14}$$

and

$$\|\mathcal{N}\tilde{V} - (a\cos(\omega\tau) + b\sin(\omega\tau))\|_0 \le C'e^{\sigma T_0} \tag{3.15}$$

for some positive constant C' independent of T_0. Since $\sigma < 0$, we can get that if we choose $T_0 > 1$ suitable large, then

$$\|\mathcal{N}\tilde{V} - (a\cos(\omega\tau) + b\sin(\omega\tau))\|_0 \le C_1. \tag{3.16}$$

If we choose C_1 large, then

$$\|\mathcal{N}\tilde{V}\|_0 \le 2C_1.$$

In particular, $\mathcal{N}\tilde{V}$ is a map from $\mathcal{B}$ into itself. Similarly, we can prove that

$$\|\mathcal{N}\tilde{V}_1 - \mathcal{N}\tilde{V}_2\|_0 \le C_1 e^{\sigma T_0} \|\tilde{V}_1 - \tilde{V}_2\|_0. \tag{3.17}$$

Therefore, it is possible to choose two constants C_1 and T_0 such that $\mathcal{N}\tilde{V}$ is a contraction mapping from $\mathcal{B}$ to itself. The contraction mapping theorem then ensures that (3.13) has a fixed point in $\mathcal{B}$. This fixed point yields a solution of (3.10). By the above analysis, we conclude that

$$V(\tau) = e^{\sigma\tau} O(1). \tag{3.18}$$

By (3.18), we know that there exist two constants a_1' and a_2' such that

$$\begin{aligned} &\frac{1}{\omega}\int_T^\tau e^{\sigma(\tau-\tau')} \sin\omega(\tau-\tau') g(V(\tau'))d\tau' \\ =& a_1' e^{\sigma\tau}\sin(\omega\tau) - \frac{1}{\omega} e^{\sigma\tau}\sin(\omega\tau)\int_\tau^\infty e^{-\sigma\tau'}\cos(\omega\tau')g(V(\tau'))d\tau' \\ &+ b_1' e^{\sigma\tau}\cos(\omega\tau) + \frac{1}{\omega}e^{\sigma\tau}\cos(\omega\tau)\int_\tau^\infty e^{-\sigma\tau'}\sin(\omega\tau')g(V(\tau'))d\tau' \\ =& a_1' e^{\sigma\tau}\sin(\omega\tau) + b_1' e^{\sigma\tau}\cos(\omega\tau) + O(e^{2\sigma\tau}). \end{aligned} \tag{3.19}$$

By (3.12), (3.19) and the definition of $\omega(\tau)$, we have

$$V(\tau) = e^{\sigma\tau}[(a+a_1')\sin(\omega\tau) + (b+b_1')\cos(\omega\tau)] + O(e^{2\sigma\tau}). \tag{3.20}$$

Take

$$a_0 = a + a_1', \qquad b_0 = b + b_1',$$

then (3.20) implies that for $s \in (S_0, \infty)$,

$$v_0(s) = A_p s^{-\alpha} + \frac{a_0\cos(\omega\ln s) + b_0\sin(\omega\ln s)}{s^{\frac{n-1}{2}}} + O(s^{-(n-1-\frac{1}{p-1})}). \tag{3.21}$$

This is exactly (3.5). Next, we show that $a_0^2 + b_0^2 \neq 0$. If it is false, then (3.12) and (3.19) imply

$$\begin{aligned} V(\tau) =& -\frac{1}{\omega}e^{\sigma\tau}\sin(\omega\tau)\int_\tau^\infty e^{-\sigma\tau'}\cos(\omega\tau')g(V(\tau'))d\tau' \\ &+ \frac{1}{\omega}e^{\sigma\tau}\cos(\omega\tau)\int_\tau^\infty e^{-\sigma\tau'}\sin(\omega\tau')g(V(\tau'))d\tau'. \end{aligned} \tag{3.22}$$

Since $V(\tau) = O(e^{2\sigma\tau})$, we have $V(\tau) = O(e^{4\sigma\tau})$. By repeating this argument, we conclude that $V \equiv 0$. Since $v_0 \neq A_p s^{-\alpha}$, this is a contradiction. Hence we have proved that $a_0^2 + b_0^2 \neq 0$. The proof of the lemma is completed. □

Lemma 3.2. *Let $Q/(Q-4) < p < p_{JL}(Q-2)$ and let $v_1(r)$ be the unique solution of the initial value problem*

$$\begin{cases} v_1''(s) + \frac{n}{s}v_1'(s) + \frac{p}{4s}v_0^{p-1}(s)v_1(s) - \frac{\beta}{4}v_0(s) - \frac{n}{3}sv_0'(s) + \frac{1}{24}sv_0^p(s) = 0, \\ v_1(0) = 0, \\ v_1'(0) = 0. \end{cases} \tag{3.23}$$

Then for $s \in [S_0, \infty)$,

$$v_1(s) = C_p s^{2-\alpha} + s^{2-\frac{n-1}{2}}(a_1\cos(\omega\ln s) + b_1\sin(\omega\ln s)) + o(s^{2-\frac{n-1}{2}}) \tag{3.24}$$

with C_p satisfies

$$[(2-\alpha)(n+1-\alpha)+\frac{p}{4}A_p^{p-1}]C_p = A_p[\frac{\beta}{4}-\frac{n}{3(p-1)}-\frac{1}{24}A_p^{p-1}]. \tag{3.25}$$

Moreover, (a_1, b_1) is the solution of

$$\begin{cases} D_1 a_1 + 4\omega b_1 = \beta a_0 + \frac{n}{3}b_0\omega - \frac{n(n-1)}{6}a_0 - E_1 a_0, \\ -4\omega a_1 + D_1 b_1 = \beta b_0 - \frac{n}{3}a_0\omega - \frac{n(n-1)}{6}b_0 - E_1 b_0, \end{cases} \tag{3.26}$$

where

$$D_1 = \frac{(5-n)(n+3)}{4} - \omega^2 + pA_p^{p-1},$$
$$E_1 = \frac{p}{4}(p-1)A_p^{p-2}C_p - \frac{1}{24}pA_p^{p-1},$$

a_0, b_0 and ω are given by Lemma 3.1.

Proof. The existence and the uniqueness of solutions of (3.23) follows from standard ordinary differential equation theory. Analyzing the terms which contain v_0 in (3.23) and using the Taylor expansion, we can find that the leading terms are of the forms

$$s^{-\alpha}, \quad s^{-\frac{n-1}{2}}\cos(\omega \ln s), \quad s^{-\frac{n-1}{2}}\sin(\omega \ln s).$$

We also notice that

$$O(s^{-(n-1-\frac{1}{n-1})}) = o(s^{-\frac{n-1}{2}})$$

provided that $p > Q/(Q-4)$. Hence it is natural to assume that $v_1(s)$ can be written as

$$v_1(s) = C_p s^{2-\alpha} + s^{2-\frac{n-1}{2}}(a_1\cos(\omega\ln s) + b_1\sin(\omega\ln s)) + o(s^{2-\frac{n-1}{2}}).$$

With the help of this explicit form, (3.25) and (3.26) can be obtained by direct calculation. □

Remark 3.3. *Since $Q/(Q-4) < p < p_{JL}(Q-2)$, then $\omega \neq 0$. Therefore, the matrix*

$$J = \begin{bmatrix} D_1 & 4\omega \\ -4\omega & D_1 \end{bmatrix}$$

is invertible. In particular, (3.26) is solvable. Moreover, since

$$b_0[\beta a_0 + \frac{n}{3}b_0\omega - \frac{n(n-1)}{6}a_0 - E_1 a_0] - a_0[\beta b_0 - \frac{n}{3}a_0\omega - \frac{n(n-1)}{6}b_0 - E_1 b_0] \neq 0,$$

we conclude that $a_1^2 + b_1^2 \neq 0$.

Lemma 3.4. *Let $Q/(Q-4) < p < p_{JL}(Q-2)$, then for $\epsilon > 0$ sufficiently small, the equation (3.2) has a solution $v(s)$ such that*

$$v(s) = v_0(s) + \sum_{k=1}^{\infty}\epsilon^{2k}v_k(s). \tag{3.27}$$

Moreover, for $s \in [S_0, \infty)$,

$$v_k(s) = \sum_{j=1}^{k} d_j^k s^{2j-\alpha} + \sum_{j=1}^{k} e_j^k s^{2j-\frac{n-1}{2}}\sin(\omega\ln s + E_j^k) + o(s^{2k-\frac{n-1}{2}}), \tag{3.28}$$

where d_j^k, e_j^k, E_j^k $(j = 1, 2, \cdots, k)$ *are constants. Moreover,*

$$d_1^1 = C_p, \quad e_1^1 = \sqrt{a_1^2 + b_1^2}, \quad \sin E_1^1 = \frac{a_1}{e_1^1}, \quad \cos E_1^1 = \frac{b_1}{e_1^1}. \tag{3.29}$$

Proof. We take (3.27) into (3.3) and we expand (3.3) according to the order of ϵ. By calculation, we can check that for $k \geq 2$, v_k satisfies the equation

$$\begin{cases} v_k''(s) + \frac{n}{3}v_k'(s) - \frac{n}{3}sv_{k-1}'(s) + \sum\limits_{j=1}^{k-1} n\alpha_j v_{k-j-1}'(s) - \frac{\beta}{4}v_{k-1}(s) \\ +\frac{1}{4s}\frac{d^k}{dt^k}(\sum\limits_{l=0}^{k} t^l v_l)^p|_{t=0} + \frac{1}{24}s\frac{d^{k-1}}{dt^{k-1}}(\sum\limits_{l=0}^{k-1} t^l v_l)^p|_{t=0} \\ +\frac{1}{4}\sum\limits_{j=1}^{k-1}\beta_j s^{2j+1}\frac{d^{k-j-1}}{dt^{k-j-1}}(\sum\limits_{l=0}^{k-j-1} t^l v_l)^p|_{t=0}, \\ v_k(0) = 0, \\ v_k'(0) = 0. \end{cases} \tag{3.30}$$

Similar to the proof of Lemma 3.2, we can find that the leading order of the terms involve only $v_0, v_1, \cdots, v_{k-1}$. Since we have obtained the expansion of v_0 and v_1. Then the expansion of v_k can be derived by using the Taylor expansion of v^p and the induction argument. □

By Lemma 3.4 and the definition of the function v, we can obtain the following proposition.

Proposition 3.5. *Let* $Q/(Q-4) < p < p_{JL}(Q-2)$ *and let* $\Phi_\epsilon^{inn}(\theta)$ *be an inner solution of the equation* (2.6) *with* $\Phi_\epsilon^{inn}(0) = \epsilon^{-\alpha}$*. Then for any sufficiently small* $\epsilon > 0$ *and* $\theta > S_0\epsilon$ *but* θ *is also sufficiently small,*

$$\begin{aligned} \Phi_\epsilon^{inn}(\theta) =& \frac{A_p}{\theta^\alpha} + \frac{C_p}{\theta^{\alpha-2}} + \sum_{k=2}^{\infty}\sum_{j=1}^{k} d_j^k \epsilon^{2(k-j)}\theta^{2j-\alpha} \\ &+ \epsilon^{\frac{n-1}{2}-\alpha}[\frac{a_0\cos(\omega\ln\frac{\theta}{\epsilon}) + b_0\cos(\omega\ln\frac{\theta}{\epsilon})}{\theta^{\frac{n-1}{2}}}] \\ &+ \epsilon^{\frac{n-1}{2}-\alpha}[\frac{a_1\cos(\omega\ln\frac{\theta}{\epsilon}) + b_1\cos(\omega\ln\frac{\theta}{\epsilon})}{\theta^{\frac{n-1}{2}-2}}] \\ &+ \epsilon^{\frac{n-1}{2}-\alpha}\sum_{k=2}^{\infty}(\sum_{j=1}^{k} e_j^k \epsilon^{2(k-j)}\theta^{2j-\frac{n-1}{2}}\sin(\omega\ln\frac{\theta}{\epsilon} + E_j^k)) \\ &+ \epsilon^{\frac{n-1}{2}-\alpha}[\epsilon^{\frac{n-1}{2}-\alpha}O(\theta^{\sigma-\frac{1}{2}}) + \sum_{k=1}^{\infty} o(\theta^{2k-\frac{n-1}{2}})]. \end{aligned} \tag{3.31}$$

Proof. Since

$$\Phi_\epsilon^{inn}(\theta) = \epsilon^{-\alpha}v(\frac{\theta}{\epsilon}) = \epsilon^{-\alpha}(v_0(\frac{\theta}{\epsilon}) + \sum_{k=1}^{\infty}\epsilon^{2k}v_k(\frac{\theta}{\epsilon})),$$

then (3.31) is a direct consequence of Lemma 3.4 by setting $s = \theta/\epsilon$. □

The results obtained above can be summarized as the following theorem.

Theorem 3.6. *Let $Q/(Q-4) < p < p_{JL}(Q-2)$ and let $\Phi_\Lambda^{inn}(\theta)$ be an inner solution of the equation (2.6) with $\Phi_\Lambda(0) = \Lambda$. Then for any sufficiently large $\Lambda > 0$,*

$$\begin{aligned}\Phi_\Lambda^{inn}(\theta) =& \frac{A_p}{\theta^\alpha} + \frac{C_p}{\theta^{\alpha-2}} + \sum_{k=2}^{\infty}\sum_{j=1}^{k} d_j^k \Lambda^{-2(p-1)(k-j)}\theta^{2j-\alpha} \\ &+ \Lambda^{\frac{\sigma}{\alpha}}[\frac{a_0\cos(\omega\ln(\Lambda^{p-1}\theta)) + b_0\sin(\omega\ln(\Lambda^{p-1}\theta))}{\theta^{\frac{n-1}{2}}}] \\ &+ \Lambda^{\frac{\sigma}{\alpha}}[\frac{a_1\cos(\omega\ln(\Lambda^{p-1}\theta)) + b_1\sin(\omega\ln(\Lambda^{p-1}\theta))}{\theta^{\frac{n-1}{2}-2}}] \\ &+ \sum_{k=2}^{\infty}(\sum_{j=1}^{k} e_j^k \Lambda^{\frac{\sigma}{\alpha}-2(p-1)(k-j)}\theta^{2j-\frac{n-1}{2}}\sin(\omega\ln(\Lambda^{p-1}\theta) + E_j^k)) \\ &+ \Lambda^{\frac{\sigma}{\alpha}}[\Lambda^{\frac{\sigma}{\alpha}}O(\theta^{\sigma-\frac{n-1}{2}}) + \sum_{k=1}^{\infty}\Lambda^{\frac{\sigma}{\alpha}}o(\theta^{2k-\frac{n-1}{2}})]\end{aligned} \tag{3.32}$$

provided that $\theta = |O(\Lambda^{\frac{\sigma}{(2-\sigma)\alpha}})|$.

Finally, we prove two lemmas which will be useful in the proof of the main theorem.

Lemma 3.7. *Let $Q/(Q-4) < p < p_{JL}(Q-2)$ and let v_0 be the unique positive radial solution of the equation (3.4). We define*

$$v(\Lambda,\theta) = \Lambda v_0(\Lambda^{p-1}\theta), \tag{3.33}$$

then for $\Lambda^{p-1}\theta \geq S_0, v(\Lambda,\theta)$ satisfies

(i) *For $k = 0, 1, 2$,*

$$\begin{aligned}\frac{\partial^k}{\partial\Lambda^k}(v(\Lambda,\theta)) =& \frac{\partial^k}{\partial\Lambda^k}(\frac{A_p}{\theta^\alpha}) \\ &+ C\frac{\partial^k}{\partial\Lambda^k}\{\theta^{-\frac{n-1}{2}}\Lambda^{-(\frac{(n-1)(p-1)}{2}-1)}\sin(\omega\ln(\Lambda^{p-1}\theta) + D)\} \\ &+ \Lambda^{-k-[(p-1)(n-1-\frac{1}{p-1})-1]}O(\theta^{-(n-1-\frac{1}{p-1})}).\end{aligned}$$

(ii) *For $k = 0, 1, 2$,*

$$\begin{aligned}\frac{\partial^k}{\partial\Lambda^k}(v_\theta(\Lambda,\theta)) =& -\alpha\frac{\partial^k}{\partial\Lambda^k}(\frac{A_p}{\theta^{\alpha+1}}) \\ &+ C\frac{\partial^{k+1}}{\partial\Lambda^k\partial\theta}\{\theta^{-\frac{n-1}{2}}\Lambda^{-(\frac{(n-1)(p-1)}{2}-1)}\sin(\omega\ln(\Lambda^{p-1}\theta) + D)\} \\ &+ \Lambda^{-k-[(n-1-\frac{1}{p-1})(p-1)-1]}O(\theta^{-(n-\frac{1}{p-1})}),\end{aligned}$$

where

$$C = \sqrt{a_0^2 + b_0^2}, \quad D = \tan^{-1}(\frac{b_0}{a_0}). \tag{3.34}$$

Proof. We know from Lemma 3.1 that

$$\begin{aligned}v_0(s) &= A_p s^{-\alpha} + \frac{a_0\cos(\omega\ln s) + b_0\sin(\omega\ln s)}{s^{\frac{n-1}{2}}} + O(s^{-(n-1-\frac{1}{p-1})}) \\ &= A_p s^{-\alpha} + Cs^{-\frac{n-1}{2}}\sin(\omega\ln s + D) + O(s^{-(n-1-\frac{1}{p-1})}),\end{aligned}$$

where C and D are given by (3.34). Then

$$v(\Lambda,\theta) = \frac{A_p}{\theta^\alpha} + C\Lambda^{-(\frac{(n-1)(p-1)}{2}-1)}\theta^{-\frac{n-1}{2}}\sin(\omega\ln(\Lambda^{p-1}\theta)+D) + \Lambda^{-[(n-1-\frac{1}{p-1})(p-1)-1]}O(\theta^{-(n-1-\frac{1}{p-1})}). \tag{3.35}$$

With the help of (3.35), (i) and (ii) can be obtained directly. □

Lemma 3.8. *In the region $\theta = |O(\Lambda^{\frac{\sigma}{(2-\sigma)\alpha}})|$, the solution $\Phi(\Lambda,\theta)$ of (2.6) with*

$$\Phi(\Lambda,0)=\Lambda, \quad \Phi_\theta(\Lambda,0)=0$$

satisfies

(i) $|\frac{\partial\Phi}{\partial\Lambda}(\Lambda,\theta) - \frac{\partial v}{\partial\Lambda}(\Lambda,\theta)| = \Lambda^{-\frac{(p-1)(n-1)}{2}}|o(\theta^{-\frac{n-1}{2}})|$;
(ii) $|\frac{\partial\Phi_\theta}{\partial\Lambda}(\Lambda,\theta) - \frac{\partial v_\theta}{\partial\Lambda}(\Lambda,\theta)| = \Lambda^{-\frac{(p-1)(n-1)}{2}}|o(\theta^{-\frac{n+1}{2}})|$;
(iii) $|\frac{\partial^2\Phi}{\partial\Lambda^2}(\Lambda,\theta) - \frac{\partial^2 v}{\partial\Lambda^2}(\Lambda,\theta)| = \Lambda^{-(\frac{(p-1)(n-1)}{2}+1)}|o(\theta^{-\frac{n-1}{2}})|$;
(iv) $|\frac{\partial^2\Phi_\theta}{\partial\Lambda^2}(\Lambda,\theta) - \frac{\partial^2 v_\theta}{\partial\Lambda^2}(\Lambda,\theta)| = \Lambda^{-(\frac{(p-1)(n-1)}{2}+1)}|o(\theta^{-\frac{n+1}{2}})|$.

Proof. By (3.1), we deduce that

$$\begin{aligned}\Phi(\Lambda,\theta) = \Lambda v(\Lambda^{p-1}\theta) &= \Lambda(v_0(\Lambda^{p-1}\theta) + \sum_{k=1}^{\infty}\Lambda^{-\frac{2k}{\alpha}}v_k(\Lambda^{p-1}\theta)) \\ &= v(\Lambda,\theta) + \Lambda\sum_{k=1}^{\infty}\Lambda^{-\frac{2k}{\alpha}}v_k(\Lambda^{p-1}\theta).\end{aligned}$$

Since $\theta = |O(\Lambda^{\frac{\sigma}{(2-\sigma)\alpha}})|$, then

$$\Lambda^{p-1}\theta = |O(\Lambda^{\frac{2(p-1)}{2-\sigma}})| > S_0$$

provided that Λ is sufficiently large. Note that

$$\epsilon = \Lambda^{-\frac{1}{\alpha}}, \quad \frac{-\sigma}{\alpha} = \frac{(p-1)(n-1)}{2} - 1.$$

Then this lemma can be obtained from Lemma 3.7 and Proposition 3.5. □

4. Outer solutions

In this section, we study the asymptotic behavior of solutions $\Phi(\theta)$ of (2.6) far from $\theta=0$. Let $\Phi_*(\theta)$ be the singular solution in Remark 2.1, we first obtain the following lemma.

Lemma 4.1. *The ordinary differential equation*

$$4\sin\theta\phi''(\theta) + 4n\cos\theta\phi'(\theta) - \beta\sin\theta\phi + pA_p^{p-1}[\sin\theta]^{-1}\phi = 0 \tag{4.1}$$

admits two fundamental solutions $\phi_1(\theta)$ and $\phi_2(\theta)$ such that any solution $\phi(\theta)$ of (4.1) can be written in the form

$$\phi(\theta) = c_1\phi_1(\theta) + c_2\phi_2(\theta),$$

where c_1, c_2 are two constants. Moreover, as $\theta \to 0$, there exists two constants c_1', c_2' such that

$$\phi(\theta) = \theta^{-\frac{n-1}{2}}[c_1'\cos(\omega\ln\frac{\theta}{2}) + c_2'\sin(\omega\ln\frac{\theta}{2})] + O(\theta^{2-\frac{n-1}{2}}). \tag{4.2}$$

If $\phi \neq 0$, then $c_1'^2 + c_2'^2 \neq 0$.

Proof. Let

$$\phi(\theta) = [\sin\theta]^{-\frac{1}{p-1}}\tilde{\phi}(\theta).$$

We know from (4.1) that $\tilde{\phi}(\theta)$ satisfies the equation

$$4\sin^2\theta\tilde{\phi}''(\theta) + 4(n - \frac{2}{p-1})\sin\theta\cos\theta\tilde{\phi}'(\theta) + (p-1)A_p^{p-1}\tilde{\phi} = 0. \tag{4.3}$$

Under the Emden-Fowler transformations:

$$\psi(\tau) = \tilde{\phi}(\theta), \quad \tau = \ln\tan\frac{\theta}{2},$$

we obtain that $\psi(\tau)$ satisfies

$$\psi''(\tau) + (n - 1 - \frac{2}{p-1})(1 - \frac{2e^{2\tau}}{1+e^{2\tau}})\psi'(\tau) + (n - 1 - \frac{1}{p-1})\psi = 0. \tag{4.4}$$

By the standard ordinary differential equation theories, we know that for every a, (4.4) has a unique solution such that $\psi(0) = a, \psi'(0) = 0$. Moreover, (4.4) admits two fundamental solutions $\psi_1, \psi_2 \in C^2(-\infty, 0)$ such that any solution $\psi(\tau)$ of (4.4) can be written as

$$\psi(\tau) = c_1\psi_1(\tau) + c_2\psi_2(\tau).$$

By the method of variation of constant, we can obtain that

$$\psi(\tau) = e^{\sigma\tau}[\ell_3\cos(\omega\tau) + \ell_4\sin(\omega\tau)] + \frac{1}{\omega}\int_T^\tau e^{\sigma(\tau-\tau')}\sin\omega(\tau-\tau')j(\psi)(\tau')d\tau', \tag{4.5}$$

where $T \in (-\infty, 0)$ and

$$j(\psi)(\tau') = -(n - 1 - \frac{2}{p-1})\frac{2e^{2\tau'}}{1+e^{2\tau'}}\psi'(\tau').$$

Let

$$\hat{\psi}(\tau) = e^{-\sigma\tau}\psi(\tau),$$

then

$$\hat{\psi}(\tau) = [\ell_3\cos(\omega\tau) + \ell_4\sin(\omega\tau)] + \frac{1}{\omega}\int_T^\tau \sin\omega(\tau-\tau')j(\hat{\psi})(\tau')d\tau' \tag{4.6}$$

with

$$j(\hat{\psi})(\tau') = -(n - 1 - \frac{2}{p-1})\frac{2e^{2\tau'}}{1+e^{2\tau'}}(\sigma\hat{\psi}(\tau') + \hat{\psi}'(\tau')). \tag{4.7}$$

We claim that by choosing $|T|$ suitable large, there exists a constant c depends only on p, n, T, c_1, c_2 such that

$$\|\hat{\psi}\|_0 \le c, \quad \|\hat{\psi}'\|_0 \le c, \tag{4.8}$$

where $\|\hat{\psi}\|_0 = \sup_{\tau<\tau'<T}|\hat{\psi}(\tau')|$ and $\|\hat{\psi}'\|_0 = \sup_{\tau<\tau'<T}|\hat{\psi}'(\tau')|$. Indeed, it follows from (4.6) and (4.7) that

$$\|\hat{\psi} - [\ell_3\cos(\omega\tau) + \ell_4\sin(\omega\tau)]\|_0 \le c_0e^{2T}(|\sigma|\|\hat{\psi}\|_0 + \|\hat{\psi}'\|_0), \tag{4.9}$$

where c_0 is a positive constant independent of τ. On the other hand, we can check that $z(\tau) := \psi'(\tau)$ satisfies the equation

$$z''(\tau) + (n - 1 - 2\alpha)z'(\tau) + (n - 1 - \alpha)z(\tau) + h(\tau, \psi(\tau), \psi'(\tau)) = 0, \tag{4.10}$$

where

$$\begin{aligned} h(\tau,\psi(\tau),\psi'(\tau)) =&(n-1-2\alpha)^2\frac{2e^{2\tau}}{1+e^{2\tau}}(1-\frac{2e^{2\tau}}{1+e^{2\tau}})\psi'(\tau) \\ &+2(n-1-\alpha)(n-1-2\alpha)\frac{2e^{2\tau}}{1+e^{2\tau}}\psi(\tau) \\ &-2(n-1-2\alpha)\frac{2e^{2\tau}}{(1+e^{2\tau})^2}\psi'(\tau). \end{aligned} \tag{4.11}$$

Therefore,

$$e^{-\sigma\tau}\psi'(\tau)=[\ell_5\cos\omega\tau+\ell_6\sin\omega\tau]+\frac{1}{\omega}\int_T^\tau \sin\omega(\tau-\tau')h(\tau',\hat\psi(\tau'),\hat\psi'(\tau')d\tau' \tag{4.12}$$

with

$$\begin{aligned} h(\tau,\hat\psi(\tau),\hat\psi'(\tau)) =&(n-1-2\alpha)^2\frac{2e^{2\tau}}{1+e^{2\tau}}(1-\frac{2e^{2\tau}}{1+e^{2\tau}})(\sigma\hat\psi(\tau)+\hat\psi'(\tau)) \\ &-2(n-1-2\alpha)\frac{2e^{2\tau}}{(1+e^{2\tau})^2}(\sigma\hat\psi(\tau)+\hat\psi'(\tau)) \\ &+2(n-1-\alpha)(n-1-2\alpha)\frac{2e^{2\tau}}{1+e^{2\tau}}\hat\psi(\tau). \end{aligned} \tag{4.13}$$

Similar to (4.9), we can obtain that

$$\|e^{-\sigma\tau}\psi'(\tau)-[\ell_5\cos(\omega\tau)+\ell_6\sin(\omega\tau)]\|_0\le c_0e^{2T}(|\sigma|\|\hat\psi\|_0+\|\hat\psi'\|_0). \tag{4.14}$$

Since

$$\hat\psi'(\tau)=e^{-\sigma\tau}\psi'(\tau)-\sigma\hat\psi(\tau),$$

then we can get (4.8) by combing (4.9) and (4.14). Both (4.6), (4.7) and (4.8) imply there exist two constants ℓ_3',ℓ_4' such that

$$\begin{aligned} \hat\psi(\tau)=&\ell_3'\cos\omega\tau+\ell_4'\sin\omega\tau+\frac{1}{\omega}\int_{-\infty}^\tau \sin\omega(\tau-\tau')j(\hat\psi)(\tau')d\tau' \\ =&\ell_3'\cos\omega\tau+\ell_4'\sin\omega\tau+O(e^{2\tau}). \end{aligned} \tag{4.15}$$

Therefore, as $\tau\to\infty$,

$$\psi(\tau)=e^{\sigma\tau}[\ell_3'\cos\omega\tau+\ell_4'\sin\omega\tau+O(e^{2\tau})]. \tag{4.16}$$

This implies that as $\theta\to 0$,

$$\begin{aligned} \phi(\theta)=&[\sin\theta]^{-\alpha}(\tan\frac{\theta}{2})^\sigma[\ell_3'\cos(\omega\ln\tan\frac{\theta}{2}) \\ &+\ell_4'\sin(\omega\ln\tan\frac{\theta}{2})+O((\tan\frac{\theta}{2})^2)]. \end{aligned} \tag{4.17}$$

Since

$$\begin{aligned} [\sin\theta]^{-\alpha}=&\frac{1}{\theta^\alpha}+\frac{1}{6(p-1)}\frac{1}{\theta^{\alpha-2}}+O(\frac{1}{\theta^{\alpha-4}}), \\ [\tan\frac{\theta}{2}]^\sigma=&(\frac{\theta}{2})^\sigma+\frac{\sigma}{3}(\frac{\theta}{2})^{\sigma+2}+O(\theta^{\sigma+4}), \end{aligned}$$

then (4.2) follows from (4.17).

Finally, we prove that if $\phi\ne 0$, then $\ell_3'^2+\ell_4'^2\ne 0$. If it is false, we get from (4.5) that

$$\psi(\tau)=\frac{1}{\omega}\int_{-\infty}^\tau e^{\sigma(\tau-\tau')}\sin\omega(\tau-\tau')j(\psi)(\tau')d\tau'=O(e^{(\sigma+2)\tau}). \tag{4.18}$$

Taking the derivative with respect to (4.18), we can get that

$$\psi'(\tau) = O(e^{(\sigma+2)\tau}). \tag{4.19}$$

We take (4.19) into (4.18), then

$$\psi(\tau) = O(e^{(\sigma+4)\tau}). \tag{4.20}$$

By repeating the above arguments, we can get that $\psi \equiv 0$, this is a contradiction. Hence we have finished the proof of Lemma 4.1. □

Remark 4.2. *By the proof of Lemma 4.1, we can get that for any $\delta > 0$, if c_1 and c_2 in (4.2) satisfies*

$$c_1 = \tilde{c}_1\delta, \quad c_2 = \tilde{c}_2\delta,$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are constants, then as $\theta \to 0$,

$$\phi(\theta) := \phi_\delta(\theta) = \delta\theta^{-\frac{n-1}{2}}[\tilde{c}_1' \cos(\omega \ln \frac{\theta}{2}) + \tilde{c}_2' \sin(\omega \ln \frac{\theta}{2})] + O(\delta)\theta^{2-\frac{n-1}{2}}, \tag{4.21}$$

where $\tilde{c}_1', \tilde{c}_2'$ are two constants independent of δ.

For any $\delta > 0$ sufficiently small, if $\Phi \in C^2(0, 2\pi)$ is a solution of the equation (2.6) such that

$$\Phi(\theta) = \Phi_*(\theta) + \delta\phi_\delta(\theta) + \delta^2\psi_\delta(\theta),$$

where

$$\phi_\delta(\theta) = \tilde{c}_1\delta\phi_1(\theta) + \tilde{c}_2\delta\phi_2(\theta)$$

is a solution of (4.1) with

$$c_1 = \tilde{c}_1\delta, \quad c_2 = \tilde{c}_1\delta.$$

Then ψ_δ satisfies the equation

$$\begin{cases} 4\sin\theta\psi''(\theta) + 4n\cos\theta\psi'(\theta) - \beta\sin\theta\psi(\theta) \\ +p\Phi_*^{p-1}\psi(\theta) + \delta^{-2}H(\theta) = 0, \quad \text{in} \quad (0, \frac{\pi}{2}), \\ \psi'(\frac{\pi}{2}) = -(\tilde{c}_1\phi_1'(\frac{\pi}{2}) + \tilde{c}_2\phi_2'(\frac{\pi}{2})), \end{cases} \tag{4.22}$$

where

$$H(\theta) = (\Phi_* + \delta\phi_\delta + \delta^2\psi)^p - \Phi_*^p - p\delta\Phi_*^{p-1}\phi_\delta - p\delta^2\Phi_*^{p-1}\psi.$$

For the equation (4.22), we have the following result.

Lemma 4.3. *For any $\delta > 0$ sufficiently small and each fixed pair $(\tilde{c}_1, \tilde{c}_2)$, the equation (4.22) admits a solution $\psi_\delta \in C^2(0, \frac{\pi}{2})$.*

Proof. We set the initial value conditions of (4.22) at $\theta = \frac{\pi}{2}$: $\psi(\frac{\pi}{2}) = 1$ provided

$$\psi'(\frac{\pi}{2}) = -(\tilde{c}_1\phi_1'(\frac{\pi}{2}) + \tilde{c}_2\phi_2'(\frac{\pi}{2})) = 0;$$

$\psi(\frac{\pi}{2}) = 0$ provided

$$\psi'(\frac{\pi}{2}) = -(\tilde{c}_1\phi_1'(\frac{\pi}{2}) + \tilde{c}_2\phi_2'(\frac{\pi}{2})) \neq 0.$$

Then the standard shooting argument implies that (4.22) admits a unique nontrivial solution ψ_δ in $C^2(0, \frac{\pi}{2})$. □

Proposition 4.4. *Let δ be a sufficiently small constant and let ψ_δ be the function given by Lemma 4.3, then for $\theta = |O(\delta^{\frac{2}{2-\sigma}})|$,*

$$\psi_\delta(\theta) = \theta^{-\frac{n-1}{2}}[\tilde{d}_1 \cos(\omega \ln \frac{\theta}{2}) + \tilde{d}_2 \sin(\omega \ln \frac{\theta}{2})] + O(\theta^{2-\frac{n-1}{2}}), \tag{4.23}$$

where $\tilde{d}_1$ and $\tilde{d}_2$ are constants depending on $\tilde{c}_1$ and $\tilde{c}_2$ but independent of δ.

Proof. We set

$$\psi_\delta(\theta) = [\sin\theta]^{-\alpha}\tilde{\psi}_\delta(\theta),$$

then $\tilde{\psi}_\delta$ satisfies the equation

$$4\sin^2\theta\tilde{\psi}''(\theta) + 4(n - \frac{2}{p-1})\sin\theta\cos\theta\tilde{\psi}'(\theta) + (p-1)A_p^{p-1}\tilde{\psi} + G(\tilde{\psi}(\theta)) = 0, \quad (4.24)$$

where

$$\begin{aligned} G(\theta) =& [\sin\theta]^{1+\alpha}\delta^{-2}[(\Phi_* + \delta\phi_\delta + \delta^2[\sin\theta]^{-\alpha}\tilde{\psi})^p \\ & - \Phi_*^p - p\Phi_*^{p-1}\delta\phi_\delta - \delta^2 p\Phi_*^{p-1}[\sin\theta]^{-\alpha}\tilde{\psi}]. \end{aligned}$$

Consider the Emden-Fowler transformations

$$z(\tau) = \tilde{\phi}(\theta), \quad \tau = \ln\tan\frac{\theta}{2},$$

then for $\tau \in (-\infty, 0)$, $z(\tau)$ satisfies the equation

$$z''(\tau) + (n-1-\frac{2}{p-1})(1-\frac{2e^{2\tau}}{1+e^{2\tau}})z'(\tau) + (n-1-\frac{1}{p-1})z + G(z(\tau)) = 0. \quad (4.25)$$

Let

$$\tilde{\phi}_1(\tau) = [\sin\theta]^\alpha\phi_1(\theta), \quad \tilde{\phi}_2(\tau) = [\sin\theta]^\alpha\phi_2(\theta).$$

By the method of variation of constants, we know that for $T \in (-\infty, 0)$ and $|T|$ suitable large,

$$\begin{aligned} z(\tau) =& \vartheta_1\tilde{\phi}_1(\tau) + \vartheta_2\tilde{\phi}_2(\tau) + \int_T^\tau \frac{-\tilde{\phi}_1(\tau)\tilde{\phi}_2(\tau') + \tilde{\phi}_2(\tau)\tilde{\phi}_1(\tau')}{\tilde{\phi}_1(\tau')\tilde{\phi}_2'(\tau') - \tilde{\phi}_1'(\tau')\tilde{\phi}_2(\tau')}G(z(\tau'))d\tau' \\ =& e^{\sigma\tau}[\vartheta_1\cos\omega\tau + \vartheta_2\sin\omega\tau] + O(e^{(\sigma+2)\tau}) \\ & + \frac{p(p-1)}{2\omega}\int_T^\tau e^{\sigma\tau}\sin(\tau-\tau')[e^{\sigma\tau'}\delta^2][\rho(\tau')]^2 d\tau' \\ & + \frac{1}{\omega}\int_T^\tau e^{\sigma\tau}\sin(\tau-\tau')O([e^{\sigma\tau'}\delta^2]^2[\rho(\tau')]^3)d\tau' \\ & + \frac{1}{\omega}\int_T^\tau e^{\sigma\tau}\sin(\tau-\tau')O(e^{2\sigma\tau'})[e^{\sigma\tau'}\delta^2]^2[\rho(\tau')]^2 d\tau' \\ & + \frac{1}{\omega}\int_T^\tau e^{\sigma\tau}\sin(\tau-\tau')O(e^{2\tau'})O([e^{\sigma\tau'}\delta^2]^2[\rho(\tau')]^3)d\tau', \end{aligned} \quad (4.26)$$

where

$$\rho(\tau') = \tilde{c}_1\cos\omega\tau' + \tilde{c}_2\sin\omega\tau' + e^{-\sigma\tau'}z(\tau').$$

Let

$$\hat{z}(\tau) = e^{-\sigma\tau}z(\tau).$$

Similar to the proof of Lemma 4.1, we know that there exists a positive constant $M := M(n, p, T)$ but independent of δ such that

$$\|\hat{z} - (\vartheta_1\cos\omega\tau + \vartheta_2\sin\omega\tau)\|_0 \le M \quad (4.27)$$

provided that for $\tau \in [10T, 2T]$,

$$\delta^2 = |O(e^{(2-\sigma)\tau}|. \quad (4.28)$$

Therefore,

$$z(\tau) = e^{\sigma\tau}[\vartheta_1\cos\omega\tau + \vartheta_2\sin\omega\tau] + O(e^{(\sigma+2)\tau})$$

provided that (4.28) holds. It follows that Proposition 4.4 holds. □

Theorem 4.5. *For any $\delta > 0$ sufficiently small, the equation (2.6) admits a outer solution $\Phi_\delta^{out} \in C^2(0, \frac{\pi}{2})$ such that*

$$\begin{cases} \Phi_\delta^{out}(\theta) = \Phi_*(\theta) + \delta\phi_\delta(\theta) + \delta^2\psi_\delta(\theta), \quad in \quad (0, \frac{\pi}{2}), \\ (\Phi_\delta^{out})'(\frac{\pi}{2}) = 0. \end{cases} \tag{4.29}$$

Moreover, if

$$\theta = |O(\delta^{\frac{2}{2-\sigma}})|, \tag{4.30}$$

then

$$\begin{aligned} \Phi_\delta^{out}(\theta) =& \frac{A_p}{\theta^\alpha} + \frac{A_p}{6(p-1)}\frac{1}{\theta^{\alpha-2}} \\ &+ \delta^2[\frac{\vartheta_3\cos(\omega\ln\frac{\theta}{2}) + \vartheta_4\sin(\omega\ln\frac{\theta}{2})}{\theta^{\frac{n-1}{2}}}] \\ &+ \delta^2 O(\frac{1}{\theta^{\frac{n-1}{2}-2}}), \end{aligned} \tag{4.31}$$

where ϑ_3 and ϑ_4 are constants which are independent of δ. In particular, if

$$\delta^2 = |O(\theta^{2-\sigma})|, \tag{4.32}$$

then $\Phi_\delta^{out}(\theta)$ can also be written as

$$\begin{aligned} \Phi_\delta^{out}(\theta) =& \frac{A_p}{\theta^\alpha} + \frac{A_p}{6(p-1)}\frac{1}{\theta^{\alpha-1}} \\ &+ \delta^2\theta^{-\frac{n-1}{2}}[\vartheta_3\cos(\omega\ln\frac{\theta}{2}) + \vartheta_4\sin(\omega\ln\frac{\theta}{2})] \\ &+ \delta^4 O(\frac{1}{\theta^{\frac{n-1}{2}-\sigma}}). \end{aligned} \tag{4.33}$$

Proof. It follows from the expression of Φ_*, Lemma 4.1 and Proposition 4.4 that

$$\begin{aligned} \Phi_\delta^{out}(\theta) =& \Phi_*(\theta) + \delta^2(\tilde{c}_1\phi_1(\theta) + \tilde{c}_2\phi_2(\theta)) + \delta^2\psi_\delta(\theta) \\ =& A_p[\sin\theta]^{-\frac{1}{p-1}} + \delta^2\{\theta^{\frac{n-1}{2}}[\tilde{c}_1'\cos(\omega\ln\frac{\theta}{2}) + \tilde{c}_2'\sin(\omega\ln\frac{\theta}{2})] + O(\theta^{2-\frac{n-1}{2}})\} \\ &+ \delta^2\{\theta^{\frac{n-1}{2}}[\tilde{d}_1\cos(\omega\ln\frac{\theta}{2}) + \tilde{d}_2\sin(\omega\ln\frac{\theta}{2})] + O(\theta^{2-\frac{n-1}{2}})\}. \end{aligned}$$

Since for $\delta > 0$ sufficiently small and $\theta = |O(\delta^{\frac{2}{2-\sigma}})|$,

$$O(\theta^{4-\alpha}) = O(\delta^2\theta^{2-\frac{n-1}{2}}), \tag{4.34}$$

then (4.34) follows from the Taylor expansion of $\sin\theta$ and $\tan\frac{\theta}{2}$. If

$$\delta^2 = |O(\theta^{2-\sigma})|,$$

we have

$$\delta^2 O(\frac{1}{\theta^{\frac{n-1}{2}-2}}) = \delta^4 O(\frac{1}{\theta^{\frac{n-1}{2}-\sigma}}).$$

Then (4.33) follows from (4.31). □

Remark 4.6. *Similar to the proof of Lemma 4.1 and Proposition 4.4, we can prove that $\vartheta_3^2 + \vartheta_4^2 \neq 0$. This fact will also be used in the proof of Theorem 1.2. Without loss of generality, we will assume that $\theta_3 \neq 0$.*

5. The proof of the main theorem

In this section, we will construct infinitely many regular solutions of the equation (2.6) by combining the inner and outer solutions. For this purpose, we will construct a solution for the equation

$$\begin{cases} 4\sin\theta\Phi_{\theta\theta}+4n\cos\theta\Phi_\theta-\beta\sin\theta\Phi+\Phi^p=0, & \text{in} \quad (0,\frac{\pi}{2}),\\ \Phi(\theta)>0, & \text{in} \quad (0,\frac{\pi}{2}),\\ \Phi(0)=\Lambda,\ \Phi'(\frac{\pi}{2})=0 \end{cases} \tag{5.1}$$

by matching the inner and outer solutions given by Theorems 3.6 and Theorem 4.5. For this purpose, we will find $\Theta\in(0,\frac{\pi}{2})$ such that the following matching conditions hold:

$$\Theta=O(\Lambda^{\frac{\sigma}{(2-\sigma)\alpha}}), \tag{5.2}$$

$$(\Phi_\Lambda^{inn}(\theta)-\Phi_\delta^{out}(\theta))|_{\theta=\Theta}=0, \tag{5.3}$$

$$(\Phi_\Lambda^{inn}(\theta)-\Phi_\delta^{out}(\theta))'_\theta|_{\theta=\Theta}=0, \tag{5.4}$$

First, we have the following identity.

Lemma 5.1. *A_p and C_p satisfies the condition*

$$\frac{A_p}{6(p-1)}=C_p. \tag{5.5}$$

Proof. It is easy to check that

$$\begin{aligned}&(2-\frac{1}{p-1})(n+1-\frac{1}{p-1})+\frac{p}{4}A_p^{p-1}\\ =&(2-\frac{1}{p-1})(n+1-\frac{1}{p-1})+\frac{p}{p-1}(n-1-\frac{1}{p-1})\\ =&3n+1-\frac{5}{p-1}.\end{aligned} \tag{5.6}$$

On the other hand, we have

$$\begin{aligned}&\frac{\beta}{4}-\frac{n}{3(p-1)}-\frac{1}{24}A_p^{p-1}\\ =&\frac{1}{p-1}(n-\frac{1}{p-1})-\frac{n}{3(p-1)}-\frac{1}{6(p-1)}(n-1-\frac{1}{p-1})\\ =&\frac{1}{6(p-1)}(3n+1-\frac{5}{p-1}).\end{aligned} \tag{5.7}$$

By (3.25), (5.6) and (5.7), we can get (5.5). □

It follow from Lemma 5.1 that the first two terms of $\Phi_\Lambda^{inn}(\theta)$ and $\Phi_\delta^{out}(\theta)$ can be matched. Moreover, we notice that

$$\vartheta_3\cos(\omega\ln\frac{\theta}{2})+\vartheta_4\sin(\omega\ln\frac{\theta}{2})=E\sin(\omega\ln\theta+\omega\ln\frac{1}{2}+\eta),$$

$$a_0\cos(\omega\ln(\Lambda^{p-1}\theta))+b_0\sin(\omega\ln(\Lambda^{p-1}\theta))=C\sin(\omega\ln\theta+\omega\ln\Lambda^{p-1}+D),$$

where

$$\begin{aligned}&C=\sqrt{a_0^2+b_0^2},\quad E=\sqrt{\vartheta_3^2+\vartheta_4^2},\\ &D=\tan^{-1}(\frac{b_0}{a_0}),\quad \eta=\tan^{-1}(\frac{\vartheta_4}{\vartheta_3}).\end{aligned} \tag{5.8}$$

In order to match the next term, we choose Λ_* and δ_*^2 such that

$$\delta_*^2 = \sqrt{\frac{a_0^2+b_0^2}{\vartheta_3^2+\vartheta_4^2}}\Lambda_*^{\frac{\sigma}{\alpha}}, \tag{5.9}$$

$$\omega \ln \Lambda_*^{p-1} + D = \omega \ln \frac{1}{2} + \eta + 2m\pi, \tag{5.10}$$

where m is a large positive integer. Consider small perturbations of Λ_* and δ_* defined in (5.9) and (5.10), i.e.,

$$\Lambda = \Lambda_*(1 + O(\Lambda_*^{\frac{2\sigma}{(2-\sigma)\alpha}})), \tag{5.11}$$

$$\delta^2 = \delta_*^2(1 + O(\Lambda_*^{\frac{2\sigma}{(2-\sigma)\alpha}})), \tag{5.12}$$

We will see that the parameters Λ and δ required to satisfy the matching conditions (5.2), (5.3) and (5.4) can be obtained as the above small perturbations. To show this, we define

$$\mathbf{F}(\Lambda,\delta^2) = \begin{pmatrix} \Theta^{\frac{n-1}{2}}(\Phi_\Lambda^{inn}(\Theta) - \Phi_\delta^{out}(\Theta)) \\ \frac{\Theta}{\omega}[\theta^{\frac{n-1}{2}}(\Phi_\Lambda^{inn}(\theta) - \Phi_\delta^{out}(\theta))]'|_{\theta=\Theta} \end{pmatrix}, \tag{5.13}$$

where we treat δ^2 as a new variable. Taking $\Lambda = \Lambda_*$ and $\delta^2 = \delta_*^2$ in (5.13), then Theorem 3.6 and Theorem 4.5 imply

$$|\Theta^{-\frac{n-1}{2}}\mathbf{F}(\Lambda_*,\delta_*^2)| \le M\delta_*^4\Theta^{\sigma-\frac{n-1}{2}} + \text{small terms}. \tag{5.14}$$

As in [5] and [7], we evaluate the Jacobian of $\mathbf{F}$ at (Λ_*,δ_*^2). By Lemma 3.7, Lemma 3.8, Theorem 3.6 and Theorem 4.5, we can obtain that

$$\frac{\partial \mathrm{F}(\Lambda,\delta^2)}{\partial(\Lambda,\delta^2)} = \begin{bmatrix} C(\frac{\sigma}{\alpha}\sin\tau + \omega(p-1)\cos\tau)\Lambda_*^{\frac{\sigma}{\alpha}-1}, & -E\sin\tau \\ C(\frac{\sigma}{\alpha}\cos\tau - \omega(p-1)\sin\tau)\Lambda_*^{\frac{\sigma}{\alpha}-1}, & -E\cos\tau \end{bmatrix} + \text{small terms}, \tag{5.15}$$

where

$$\tau = \omega\ln\Theta + \omega\ln\Lambda_*^{p-1} + D = \omega\ln\Theta + \omega\ln\frac{1}{2} + \eta + 2m\pi.$$

To simplify this expression, we define

$$\mathbf{G}(x,y) = \mathbf{F}(\Lambda_* + x\Lambda_*^{1-\frac{\sigma}{\alpha}}, \delta_*^2 + y).$$

By (5.14), Theorem 4.5, Lemma 3.7, Lemma 3.8, we can express $\mathbf{G}(x,y)$ in the form

$$\mathbf{G}(x,y) = \mathbf{C} + (\mathbf{L} + \text{small terms})\begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{E}(x^2(\delta_*^2)^{-1} + y^2\Theta^\sigma), \tag{5.16}$$

where $\mathbf{C}$ is a constant vector which is bounded by $M\delta_*^4\Theta^\sigma$ and $\mathbf{L}$ is given by

$$\mathbf{L} = \begin{bmatrix} C(\frac{\sigma}{\alpha}\sin\tau + \omega(p-1)\cos\tau), & -E\sin\tau \\ C(\frac{\sigma}{\alpha}\cos\tau - \omega(p-1)\sin\tau), & -E\cos\tau \end{bmatrix}.$$

Also $|\mathbf{E}|$ is bounded independent of x, y, Λ and δ. Thus

$$\mathbf{G}(x,y) = \mathbf{C} + \mathbf{L}\begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{T}(x,y). \tag{5.17}$$

By Lemma 3.1 and Remark 4.6, we have $C \ne 0, E \ne 0$. It follows that the matrix $\mathbf{L}$ is invertible. Moreover, we have

$$|\mathbf{L}^{-1}| \le \frac{2}{(p-1)\omega CE}.$$

Let $\mathbf{J}$ be the operator defined by

$$\mathbf{J}(x,y) = -(\mathbf{L}^{-1}\mathbf{C} + \mathbf{L}^{-1}\mathbf{T}(x,y))$$

and let

$$B = \{(x,y) : (x^2+y^2)^{\frac{1}{2}} \leq \frac{4\delta_*^4\Theta^\sigma M}{(p-1)\omega CE}\ \}.$$

Since $\mathbf{C}$ is bounded by $M\delta_*^4\Theta^\sigma$ and $|\mathbf{E}|$ is bounded independent of x, y, Λ, δ, it is easy to see that $\mathbf{J}$ maps the ball B into itself. By the Brouwer fixed point Theorem, we conclude that $\mathbf{J}$ has a fixed point in B. This point (x,y) satisfies $\mathbf{G}(x,y) = 0$ and

$$(x^2+y^2)^{\frac{1}{2}} \leq A\delta_*^4\Theta^\sigma,$$

where A is a constant independent of δ_*, Q_* and Θ. By substituting for Q and δ, and then taking Θ to have the upper limiting value of $Q_*^{\overline{\frac{\sigma}{(2-\sigma)\alpha}}}$, we obtain (5.11) and (5.12).

The above arguments yields the following result.

Theorem 5.2. *For $m \gg 1$ large and Λ and δ given in (5.11) and (5.12), problem (5.1) admits a C^2 solution $\Phi_{\Lambda,\delta}(\theta)$. Moreover, there is $\Theta = |O(\Lambda^{\overline{\frac{\sigma}{(2-\sigma)\alpha}}})|$ such that (5.3) and (5.4) holds. As a consequence, the equation (2.6) admits infinitely many nonconstant positive solutions.*

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Conflict of Interest

All authors declare that they have no conflict of interest.

Ethical approval

This article does not contain any studies with human participants or animals performed by the authors.

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