Global minimizers of the Allen-Cahn equation in dimension $n \geq 8$

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Abstract

We prove the existence of global minimizers of Allen-Cahn equation in dimensions 8 and above. More precisely, given any strictly area-minimizing Lawson's cones, there are global minimizers whose nodal sets are asymptotic to the cones. As a consequence of Jerison-Monneau's program we establish the existence of many counter-examples to the De Giorgi's conjecture whose nodal sets are different from the Bombierie-De Giorgi-Giusti minimal graph.

1 Introduction and main results

The bounded entire solutions of the Allen-Cahn equation

$$-\Delta u = u - u^3 \quad \text{in } \mathbb{R}^n, \quad |u| < 1, \tag{1}$$

has attracted a lot of attentions in recent years, partly due to its intricate connection to the minimal surface theory. For $n = 1$, (1) has a heteroclinic solution $H(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)$. Up to a translation, this is the unique monotone increasing solution in $\mathbb{R}$. De Giorgi ([8]) conjectured that for $n \leq 8$, if a solution to (1) is monotone in one direction, then up to translation and rotation it must

be one dimensional and hence equals H in certain coordinate. De Giorgi's conjecture is parallel to the Bernstein conjecture in minimal surface theory, which states that if $F : \mathbb{R}^n \to \mathbb{R}$ is a solution to the minimal surface equation

$$\operatorname{div} \frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} = 0,$$

then F must be a linear function in its variables. The Bernstein conjecture has been proved to be true for $n \leq 7$. The famous Bombieri-De Giorgi-Giusti minimal graph ([4]) gives a counter-example for $n = 8$, which also disproves the Bernstein conjecture for all $n \geq 8$. As for the De Giorgi conjecture, it has been proved to be true for $n = 2$ (Ghoussoub-Gui [14]), $n = 3$ (Ambrosio-Cabre [3]), and for $4 \leq n \leq 8$ (Savin [22]), under an additional limiting condition

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1.$$

This condition together with the monotone property ensures that u is a global minimizer in the sense that, for any bounded domain $\Omega \subset \mathbb{R}^n$,

$$J(u) \leq J(u + \phi), \text{ for all } \phi \in C_0^\infty(\Omega),$$

where

$$J(u) := \int_\Omega \left[\frac{1}{2} |\nabla u|^2 + \frac{\left(u^2 - 1\right)^2}{4} \right],$$

see Alberti-Ambrosio-Cabre [1] and Savin [22]. We also refer to Farina-Valdinoci[13] for discussion on related results.

On the other hand, it turns out that for $n \geq 9$, there indeed exist monotone solutions which are not one dimensional. These nontrivial examples have been constructed in [11] using the machinery of infinite dimensional Lyapunov-Schmidt reduction. The nodal set of these solutions are actually close to the Bombieri-De Giorgi-Giusti minimal graph. Indeed, it is also proved in [12] that for any nondegenerate minimal surfaces with finite total curvature in $\mathbb{R}^3$, one could construct family of solutions for the Allen-Cahn equation which "follow" these minimal surfaces. These results provide us with further indication that there is a deep relation between the minimal surface theory and the Allen-Chan equation.

Regarding solutions which are not necessary monotone, in [22], Savin also proved that if u is a global minimizer and $n \leq 7$, then u is one dimensional. While the monotone solutions of Del Pino-Kowalczyk-Wei provides examples of nontrivial global minimizers in dimension $n \geq 9$, it is not known whether there are nontrivial global minimizers for $n = 8$. Due to the connection with minimal surface theory, these global minimizers have long been conjectured to exist in dimension 8 and higher. The existence of these global minimizers will be our main focus in this paper.

To state our results, let us recall some basic facts from the minimal surface theory. It is known that in $\mathbb{R}^8$, there is a minimal cone with one singularity at

the origin which minimizes the area, called Simons cone. It is given explicitly by:

$$\left\{x_1^2+...+x_4^2=x_5^2+...+x_8^2\right\}.$$

The minimality of this cone is proved in [4]. More generally, if we consider the so-called Lawson's cone $(2\leq i\leq j)$

$$C_{i,j}:=\left\{(x,y)\in\mathbb{R}^i\oplus\mathbb{R}^j:|x|^2=\frac{i-1}{j-1}|y|^2\right\},$$

then it has mean curvature zero except at the origin and hence is a minimal hypersurface with one singularity. For $i+j\leq 7$, the cone is unstable (Simons [24]). Indeed, it is now known that for $i+j\geq 8$, and $(i,j)\neq(2,6)$, $C_{i,j}$ are area minimizing, and $C_{2,6}$ is not area minimizing but it is one sided minimizer. (See [2], [9], [18], [20]...). Note that the cone $C_{i,j}$ has the $O(i)\times O(j)$ symmetry, that is, it is invariant under the natural group actions of $O(i)$ on the first i variables and $O(j)$ on the last j variables. We also refer to [19] and references therein for more complete history and details on related subjects.

It turns out there are analogous objects as the cone $C_{i,i}$ in the theory of Allen-Cahn equation. They are the so-called saddle-shaped solutions, which are solutions in $\mathbb{R}^{2i}$ of (1) vanishes exactly on the cone $C_{i,i}$ (Cabre-Terra [5, 6] and Cabre [7]). We denote them by $D_{i,i}$. It has been proved in [5] that these solutions are unique in the class of symmetric functions. Furthermore in [5, 6] it is proved that for $2\leq i\leq 3$, the saddle-shaped solution is unstable, while for $i\geq 7$, they are stable([7]). It is conjectured that for $i\geq 4$, $D_{i,i}$ should be a global minimizer. This turns out to be a difficult problem. We show however in this paper the following

Theorem 1 *Suppose either (1) $i+j\geq 9$ or (2) $i+j=8$, $|i-j|\leq 4$. Then there is a family of global minimizers of the Allen-Cahn equation in $\mathbb{R}^{i+j}$ having $O(i)\times O(j)$ symmetry and are not one dimensional. The zero level set of these solutions converge to the cone $C_{i,j}$ at infinity.*

More detailed asymptotic behavior of this family of solutions could be found in Proposition 7 below. This family of solutions could be parametrized by the closeness of its zero-level set to the minimizing cones. Recall that a result of Jerison and Monneau ([16]) proved that the existence of a nontrivial global minimizer in $\mathbb{R}^8$ which is even in all of its variables implies the existence of a family of counter-examples for the De Giorgi conjecture in $\mathbb{R}^9$. Hence an immediate corollary of Theorem 1 is the following

Corollary 2 *Suppose either $i+j\geq 9$ or $i+j=8$ with $|i-j|\leq 4$. There is a family of monotone solutions to the Allen-Cahn equation (1) in $\mathbb{R}^{i+j+1}$, which is not one-dimensional and having $O(i)\times O(j)$ symmetry in the first $i+j$ variables.*

This corollary could be regarded as a parallel result due to Simon [23] on the existence of entire minimal graphs. Our idea of the proof is quite straightforward. We shall firstly construct minimizers on bounded domains, with suitable

boundary conditions. As we enlarge the domain, we will see that a subsequence of solutions on these bounded domains will converge to a global minimizer, as one expected. To ensure that the solutions converge, we will use the family of solutions constructed by Pacard-Wei[21] as barriers. The condition that the cone we start with is strict area minimizing is used to ensure that the solutions of Pacard-Wei are ordered. To show the compactness and precise asymptotic behavior we use the convenient tool of Fermi coordinate. The rest of the paper is devoted to the proof of Theorem 1.

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2 Solutions on bounded domains and their asymptotic behavior

Let us first of all deal with the case that the cone is the Simons cone in $\mathbb{R}^8$. The starting point of our construction of global minimizers will be the solutions of Pacard-Wei ([21]) which we describe below.

Let $\nu(\cdot)$ be a choice of the unit normal of the Simons cone $C_{4,4}$. Since we are interested in solutions with $O(4)\times O(4)$ symmetry, let us introduce

$$r=\sqrt{x_1^2+...+x_4^2},\quad s=\sqrt{x_5^2+...+x_8^2},\quad l=\sqrt{x_1^2+...+x_8^2}. \tag{2}$$

There is a smooth minimal surface Γ^+ lying in one side of the Simons cone which is asymptotic to this cone and has the following properties (see [15]). Γ^+ is invariant under the group of action of $O(4)\times O(4)$. Outside of a ball, Γ^+ is a graph over $C_{4,4}$ and

$$\Gamma^+=\left\{X+\left[r^{-2}+O\left(r^{-3}\right)\right]\nu(X):X\in C_{4,4}\right\},\ \text{as } r\to+\infty.$$

Similarly, there is a smooth minimal hypersurface Γ^- in the other side of the cone. For $\lambda\geq 0$, let $\Gamma_\lambda^\pm=\lambda\Gamma^\pm$ be the family of homotheties of $\Gamma^\pm$. Then it is known that $\Gamma_\lambda^\pm$ forms a foliation of $\mathbb{R}^8$. We use $s=f_\lambda(r)$ to denote these minimal surfaces.

For λ sufficiently large, say $\lambda\geq\lambda_0$, by a construction of Pacard-Wei ([21]), there exist solutions $U_\lambda^\pm$ whose zero level set is close to $\Gamma_\lambda^\pm$. Moreover, they depend continuously on the parameter λ and are ordered. That is,

$$\begin{aligned}&U_{\lambda_1}^+(X)<U_{\lambda_2}^+(X),\lambda_1<\lambda_2.\\&U_{\lambda_1}^-(X)<U_{\lambda_2}^-(X),\lambda_1>\lambda_2,\\&U_{\lambda_0}^-(X)<U_{\lambda_0}^+(X).\end{aligned}$$

We use N_u to denote the zero level set of a function u. Suppose that in the r-s plane we have

$$N_{U_\lambda^\pm} = \left\{(r,s) : s = F_\lambda^\pm(r)\right\}.$$

Then we have the following asymptotic behavior:

$$F_\lambda^\pm(r) = r \pm \frac{\lambda^3}{r^2} + o\left(r^{-2}\right), \text{ as } r \to +\infty.$$

It should be emphasized that the construction in [21] only gives us these solutions when λ is sufficiently large.

Proposition 3 *As $\lambda \to +\infty$, $U_\lambda^\pm \to \pm 1$ uniformly on any compact set of $\mathbb{R}^8$.*

Proof. Denote $\varepsilon = \lambda^{-1}$ and $u_\varepsilon^\pm(X) := U_\lambda^\pm(\lambda X)$. Then $u_\varepsilon^\pm$ are solutions to the singularly perturbed Allen-Cahn equation

$$-\varepsilon \Delta u_\varepsilon = \frac{1}{\varepsilon}\left(u_\varepsilon - u_\varepsilon^3\right).$$

Moreover, the construction of [21] implies that $\{u_\varepsilon^\pm = 0\}$ lies in an $O(\varepsilon)$ neighborhood of $\Gamma^\pm$. Because the distance from the origin to $\Gamma^\pm$ is positive, by the equation, we see $u_\varepsilon^\pm$ is close to ± 1 in a fixed ball around the origin. Rescaling back we finish the proof. ■

2.1 Minimizing arguments and solutions with $O(4) \times O(4)$ symmetry

For each $a \in \mathbb{R}$, we would like to construct a solution whose zero level set in the r-s plane is asymptotic to the curve

$$s = r + ar^{-2}$$

at infinity. Without loss of generality, let us assume $a \geq 0$.

Consider the first quadrant of the r-s plane. Let (l,t) be the Fermi coordinate around the minimal surface Γ_a^+, where t is the signed distance to Γ_a^+. Keep in mind that this Fermi coordinate is not smoothly defined on the whole space $\mathbb{R}^8$. (It may not be smooth around the axes.) For each d large, let L_d be the line orthogonal to Γ_a^+ at the point $(d, f_a(d))$. Denote the domain enclosed by L_d and the r, s axes by Ω_d. (This domain should be considered as a domain in $\mathbb{R}^8$. We still denote it by Ω_d for notational simplicity). We shall impose Neumann boundary condition on r, s axes and suitable Dirichlet boundary condition on L_d, to get a minimizer for the energy functional.

Let $H_d^*(\cdot)$ be a smooth function defined on L_d, equal to $H(t)$ away from the r-s axes, where H is the one dimensional heteroclinic solution. Since $\int_{\mathbb{R}} tH'(t)^2\, dt = 0$, there exists a unique solution of the problem

$$\begin{cases} -\eta'' + \left(3H^2 - 1\right)\eta = -tH', \\ \int_{\mathbb{R}} \eta H' = 0. \end{cases}$$

This solution will be denoted by $\eta(\cdot)$. (There is an explicit form for η, see [11], but we will not use this fact.)

Let $\varepsilon > 0$ be a small constant. Let ρ be a cut-off function defined outside the unit ball, equal to 1 in the region $\left\{\varepsilon s < r < \varepsilon^{-1} s\right\}$, equal to 0 near the r, s axes. It is worth pointing out that the Fermi coordinate is smoothly defined in the region $\left\{\varepsilon s < r < \varepsilon^{-1} s\right\} \backslash B_R(0)$, for R sufficiently large. We seek a minimizer of the function J within the class of functions

$$S_d := \left\{\phi \in H^{1,2}(\Omega_d) : \phi = H_d^* + \rho\eta(t)|A|^2 \text{ on } L_d\right\}.$$

Here $|A|^2$ is the squared norm of the second fundamental form of the minimal surface Γ_a^+ and hence it decays like $O\left(r^{-2}\right)$ as r tends to infinity. Slightly modifying the function H_d^* near the axes if necessary, using the asymptotic expansion of the solutions $U_\lambda^\pm$, we could assume that

$$U_{\lambda_0}^- < H_d^* + \rho\eta(t)|A|^2 < U_{\lambda_0}^+ \text{ on } L_d. \tag{3}$$

Let $u = u_d$ be a minimizer of the functional J over S_d. The existence of u follows immediately from standard arguments. But in principle, we may not have uniqueness. Intuitively speaking, the uniqueness of minimizer should be an issue related to minimizing property of the saddle-shaped solutions.

Proposition 4 *u_d is invariant under the natural action of $O(4) \times O(4)$.*

Proof. Let $e \in O(4) \times O(4)$. Then due to the invariance of the energy functional and the boundary condition, $u(e\cdot)$ is still a minimizer. By elliptic regularity, u is smooth.

Suppose e is given by: $(x_1, x_2, ..., x_8) \to (-x_1, x_2, ..., x_8)$. We first show that $u(x) = u(ex)$ for any $x \in \Omega_d$. Assume to the contrary that this is not true. Let us consider the functions

$$w_1(x) := \min\{u(x), u(ex)\},$$
$$w_2(x) := \max\{u(x), u(ex)\}.$$

Since $u(\cdot)$ and $u(e\cdot)$ have the same boundary condition, w_1 and w_2 are also minimizers of the functional J. Hence they are solutions of the Allen-Cahn equation. Since $w_1 \le w_2$ and $w_1(0, x_2, ..., x_8) = w_2(0, x_2, ..., x_8)$, by the strong maximum principle, $w_1 = w_2$. It follows that $u(x) = u(ex)$.

Let us use y_1 to denote the first four coordinates $(x_1, ..., x_4)$ and y_2 denote the last four coordinates $(x_5, ..., x_8)$. Suppose $\bar{e}$ is a reflection across a three dimensional hyperplane L in $\mathbb{R}^4$ which passes through origin. This gives us a corresponding element in $O(4) \times O(4)$, still denoted by $\bar{e}$,

$$\bar{e}(y_1, y_2) := (\bar{e}(y_1), y_2).$$

Similar arguments as above tell us that $u(x) = u(\bar{e}x)$ for any $x \in \Omega_d$. Because $O(4) \times O(4)$ is generated by reflections, this implies that u is invariant under the group of action of $O(4) \times O(4)$. ■

Let us now fix a number $\lambda^* > \max\left\{|a|^{\frac{1}{3}}, \lambda_0\right\}$.

Lemma 5 *For any d large, there holds*

$$U_{\lambda^*}^- < u_d < U_{\lambda^*}^+. \tag{4}$$

Proof. By Proposition 3, for λ sufficiently large, $u_d < U_\lambda^+$ on Ω_d. Now let us continuously decrease the value of λ. Since we have a continuous family of solutions U_λ^+, and for $\lambda > \lambda^*$, each of them is greater than u_d on the boundary of Ω_d (recalling (3) and the ordering properties of $U_\lambda^\pm$), hence by the strong maximum principle, $u_d < U_{\lambda^*}^+$. Similarly, one could show that $U_{\lambda^*}^- < u_d$. This finishes the proof. ■

2.2 Asymptotic analysis of the solutions

The minimizers u_d are invariant under $O(4) \times O(4)$ action. By (4), we know that as d goes to infinity, $u_d \to U$ in C_{loc}^2 for a nontrivial entire solution U to the Allen-Cahn equation.

We then claim

Lemma 6 *U is a global minimizer.*

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be any fixed function. Then for $d > 2R_0$, where $supp(\phi) \subset B_{R_0}(0)$, we have

$$J(u_d) \leq J(u_d + \phi) \tag{5}$$

Letting $d \to +\infty$ we arrive at the conclusion. ■

By Lemma 5, the nodal set N_U of U must lie between $N_{U_{\lambda^*}^+}$ and $N_{U_{\lambda^*}^-}$. We use $s = F(r)$ to denote the nodal set of U.

The main result of this section is the following

Proposition 7 *The zero level set of U has the following asymptotic behavior:*

$$F(r) - f_a(r) = o\left(r^{-2}\right), \text{ as } r \to +\infty. \tag{6}$$

The proof of this proposition relies on detailed analysis of the asymptotic behavior of the sequence of solutions u_d.

To begin with, let us define an approximate solution as

$$\bar{H}(l,t) = \rho H(t - h_d(l)) + (1-\rho)\frac{H(t - h_d(l))}{|H(t - h_d(l))|},$$

where ρ is the cutoff function introduced before. We shall write the solution u_d in the Fermi coordinate as

$$u_d(l,t) = \bar{H} + \phi_d, \tag{7}$$

for some small function h_d. Introduce the notation $\bar{H}' := H'(t-h)$. We require the following orthogonality condition on ϕ_d :

$$\int_{\mathbb{R}} \phi_d \rho \bar{H}' dt = 0.$$

Since u_d is close to $H(t)$, we could find a unique small function h_d satisfying (7) using implicit function theorem for each fixed l.

Our starting point for the asymptotic analysis is the following estimate:

Lemma 8 *The function h_d and ϕ_d satisfy*

$$|h_d| + |\phi_d| + |h_d'| + |h_d''| \leq Cl^{-2},$$

where C does not depend on d.

Proof. We first prove $|h_d| \leq Cl^{-2}$. By the orthogonal condition,

$$\int_{\mathbb{R}} \left(u_d - \bar{H}\right) \rho \bar{H}'(t)\, dt = 0. \tag{8}$$

Hence

$$\int_{\mathbb{R}} \left[u_d - H(t)\right] \rho \bar{H}' dt = \int_{\mathbb{R}} \left[\bar{H} - H(t)\right] \rho \bar{H}' dt = h_d \int_{\mathbb{R}} \bar{H}'^2 dt + o(h_d).$$

Since

$$U_{\lambda^*}^- \leq u_d \leq U_{\lambda^*}^+,$$

and

$$\left|U_{\lambda^*}^{\pm} - H(t)\right| \leq Cl^{-2},$$

then it holds that

$$|h_d| \leq C \frac{\left|\int_{\mathbb{R}} \left[u_d - H(t)\right] \rho \bar{H}' dt\right|}{\int_{\mathbb{R}} \bar{H}'^2 dt} \leq Cl^{-2}.$$

This also implies that

$$|\phi_d| = \left|u_d - \bar{H}\right| \leq Cl^{-2}.$$

Next we show $|h_d'| \leq Cl^{-2}$. To see this, we differentiate equation (8) with respect to l. This yields

$$\int_{\mathbb{R}} \left(\partial_l u_d - \partial_l \bar{H}\right) \rho \bar{H}'(t)\, dt = -\int_{\mathbb{R}} \left(u_d - \bar{H}\right) \left[\partial_l \rho \bar{H}'(t) + \rho \partial_l \bar{H}'(t)\right] dt.$$

As a consequence,

$$\left|h_d' \int_{\mathbb{R}} \rho \bar{H}'^2(t)\, dt\right| \leq C \left|\int \partial_l u_d \rho \bar{H}' dt\right| + O\left(l^{-2}\right). \tag{9}$$

Observe that u_d is a solution trapped between $U_{\lambda^*}^+$ and $U_{\lambda^*}^-$. Hence elliptic regularity tells us $\partial_l u_d = O\left(l^{-2}\right)$. This together with (9) yield $|h_d'| \leq Cl^{-2}$. Similarly, we can prove that $|h_d''| \leq Cl^{-2}$. ■

The Laplacian operator Δ has the following expansion in the Fermi coordinate (l, t):

$$\Delta = \Delta_{\Gamma^t} + \partial_t^2 - M_t \partial_t.$$

Here M_t is the mean curvature of the surface

$$\Gamma^t := \left\{ X + t\nu\left(X\right), X \in \Gamma_a^+ \right\}.$$

We use $g_{i,j}^t$ to denote the induced metric on the surface Γ^t. Then

$$\Delta_{\Gamma^t}\varphi = \frac{\partial_i\left(g^{i,j,t}\sqrt{|g^t|}\partial_j\varphi\right)}{\sqrt{|g^t|}}. \tag{10}$$

Here we have used $|g^t|$ to denote the determinant of the metric tensor. For $t = 0$, $g_{i,j}^0 := g_{ij}$ is close to the metric on the Simons cone, which has the form

$$dl^2 + l^2 ds^2,$$

where ds^2 is the metric on $S^3 \times S^3$. In general, when $t \neq 0$, the metric on Γ^t and Γ^0 is in (l,t) coordinate is related by

$$g^t = g^0\left(I - tA\right)^2.$$

In particular,

$$g_{i,j}^t = g_{i,j} + O\left(l^{-1}\right).$$

Lemma 9 *The Laplacian-Beltrami operator Δ_{Γ^t} on the hypersurface Γ^t has the form*

$$\Delta_{\Gamma^t}\varphi = \Delta_{\Gamma^0}\varphi + P_1\left(l,t\right)D\varphi + P_2\left(l,t\right)D^2\varphi,$$

where

$$\left|P_1\left(l,t\right)\right| \leq Cl^{-2}, \left|P_2\left(l,t\right)\right| \leq Cl^{-1}.$$

Proof. Using (10), we get

$$\Delta_{\Gamma^t}\varphi = \frac{\partial_i\left(g^{i,j,t}\sqrt{|g^t|}\partial_j\varphi\right)}{\sqrt{|g^t|}} = \partial_i g^{i,j,t}\partial_j\varphi + g^{i,j,t}\partial_{ij}\varphi + g^{i,j,t}\partial_j\varphi\partial_i\ln\sqrt{|g^t|}.$$

We compute

$$\begin{aligned}\Delta_{\Gamma^t}\varphi - \Delta_{\Gamma^0}\varphi &= \partial_i\left(g^{i,j,t} - g^{i,j}\right)\partial_j\varphi + \left(g^{i,j,t} - g^{i,j}\right)\partial_{ij}\varphi \\ &\quad + g^{i,j,t}\partial_j\varphi\partial_i\ln\frac{\sqrt{|g^t|}}{\sqrt{|g|}} + \left(g^{i,j,t} - g^{i,j}\right)\partial_j\varphi\partial_i\ln\sqrt{|g|}.\end{aligned}$$

The estimate follows from this formula. ■

One main step of our analysis will be the estimate of the approximate solution.

Lemma 10 *The error of the approximate solution $\bar{H}$ has the following estimate:*

$$\begin{aligned}\Delta\bar{H} + \bar{H} - \bar{H}^3 &= \left(h_d'' + \frac{6}{l}h_d'\right)\bar{H}' + O\left(h_d'^2\right) + O\left(l^{-2}\right)h_d' + O\left(l^{-1}\right)h_d'' \\ &\quad + \left(t\left|A\right|^2 + t^3\sum_{i=1}^{7}k_i^4\right)\bar{H}' + O\left(l^{-5}\right).\end{aligned}$$

Proof. Computing the Laplacian in the Fermi coordinate, we obtain, up to an exponential decay term introduced by the cutoff function ρ,

$$\begin{aligned}\Delta\bar{H}+\bar{H}-\bar{H}^3 &= \Delta_{\Gamma^t}\bar{H}+\partial_t^2\bar{H}-M_t\partial_t\bar{H}+\bar{H}-\bar{H}^3\\ &= \Delta_{\Gamma^t}\bar{H}-M_t\partial_t\bar{H}.\end{aligned}$$

Let k_i be the principle curvatures of Γ_a^+. Since Γ_a^+ is a minimal surface,

$$\begin{aligned}M_t &= \sum_{i=1}^{7}\frac{k_i}{1-tk_i}\\ &= t\left|A\right|^2+t^2\sum_{i=1}^{7}k_i^3+t^3\sum_{i=1}^{7}k_i^4+O\left(k_i^5\right).\end{aligned}$$

Observe that k_i^3 decays like $O\left(r^{-3}\right)$ at infinity. However, we would like to show that $\sum_{i=1}^{7}k_i^3$ actually has a faster decay. Indeed, recall that(see for example [**?**]), in a parametrization of the curve Γ_a^+,

$$\begin{aligned}k_1 &= \frac{-r''s'+r's''}{\left(r'^2+s'^2\right)^{\frac{3}{2}}},\\ k_2=k_3=k_4 &= \frac{s'}{r\sqrt{r'^2+s'^2}},\\ k_5=k_6=k_7 &= \frac{-r'}{s\sqrt{r'^2+s'^2}}.\end{aligned}$$

In particular, using the fact that along Γ_a^+, $s=r+ar^{-2}+O\left(r^{-3}\right)$,

$$k_1=O\left(r^{-4}\right),\ \text{and}\ \left|k_2\right|=...=\left|k_7\right|=\frac{1}{\sqrt{2}r}+O\left(r^{-4}\right).$$

Therefore we obtain

$$\sum_{i=1}^{7}k_i^3=O\left(r^{-5}\right).$$

It follows that,

$$M_t=t\left|A\right|^2+t^3\sum_{i=1}^{7}k_i^4+O\left(r^{-5}\right).$$

Next we compute $\Delta_{\Gamma^t}\bar{H}$. By Lemma 9, in the Fermi coordinate,

$$\begin{aligned}\Delta_{\Gamma^t}\bar{H} &= \frac{\partial_i\left(g^{i,j,t}\sqrt{\left|g^t\right|}\partial_j\left[H\left(t-h_d\left(l\right)\right)\right]\right)}{\sqrt{\left|g^t\right|}}\\ &= \Delta_{\Gamma^0}\bar{H}+O\left(l^{-2}\right)h_d'+O\left(l^{-1}\right)h_d''\\ &= -\left(h_d''+\frac{6}{l}h_d'\right)\bar{H}'+O\left(h_d'^2\right)+O\left(l^{-2}\right)h_d'+O\left(l^{-1}\right)h_d''+O\left(l^{-5}\right).\end{aligned}$$

Hence

$$\Delta\bar{H}+\bar{H}-\bar{H}^3=-\left(h_d''+\frac{6}{l}h_d'\right)\bar{H}'+O\left(h_d'^2\right)+O\left(l^{-2}\right)h_d'+O\left(l^{-1}\right)h_d''$$
$$-\left(t\left|A\right|^2+t^3\sum_{i=1}^{7}k_i^4\right)\bar{H}'+O\left(l^{-5}\right).$$

■

Let us set

$$J\left(h_d\right)=h_d''+\frac{6}{l}h_d'+\frac{6}{l^2}h_d.$$

With all these understood, we are ready to prove the following

Proposition 11 *The function h_d satisfies*

$$\left|J\left(h_d\right)\right|\leq Cl^{-5},$$

where C is a constant independent of d.

Proof. Frequently, we drop the subscript d for notational simplicity.

Since $\phi+\bar{H}$ solves the Allen-Cahn equation, ϕ satisfies

$$-\Delta\phi+\left(3\bar{H}^2-1\right)\phi=\Delta\bar{H}+\bar{H}-\bar{H}^3-3\bar{H}\phi^2-\phi^3.$$

By Lemma 10,

$$\begin{aligned}&-\Delta\phi+\left(3\bar{H}^2-1\right)\phi\\&=-J\left(h\right)\bar{H}'-\left(t-h\right)\bar{H}'\left|A\right|^2-\left(t-h\right)^3\bar{H}'\sum_{i=1}^{7}k_i^4\\&+O\left(h'^2\right)+O\left(l^{-2}h'\right)+O\left(l^{-1}h''\right)\\&-3\phi^2\bar{H}-\phi^3+O\left(l^{-5}\right).\end{aligned}$$

The function $\left(t-h\right)\left|A\right|^2\bar{H}'$ is orthogonal to $\bar{H}'$ and decays like $O\left(l^{-2}\right)$. This is a slow decaying term. Recall that we defined a function η satisfying

$$-\eta''+\left(3H^2-1\right)\eta=-tH'\left(t\right).$$

We introduce $\bar{\eta}=\eta\left(t-h\right)$. Straightforward computation yields

$$\begin{aligned}&-\Delta\left(\bar{\eta}\left|A\right|^2\right)+\left(3\bar{H}^2-1\right)\bar{\eta}\left|A\right|^2\\&=-\Delta_{\Gamma^t}\left(\bar{\eta}\left|A\right|^2\right)-\partial_t^2\left(\bar{\eta}\left|A\right|^2\right)+M_t\partial_t\left(\bar{\eta}\left|A\right|^2\right)+\left(3\bar{H}^2-1\right)\bar{\eta}\left|A\right|^2\\&=-\Delta_{\Gamma^t}\left(\bar{\eta}\left|A\right|^2\right)+M_t\partial_t\left(\bar{\eta}\left|A\right|^2\right)-\left(t-h\right)\bar{H}'\left|A\right|^2\\&=-\bar{\eta}\Delta_{\Gamma}\left|A\right|^2+\left(t-h\right)\partial_t\bar{\eta}\left(\left|A\right|^2\right)^2-\left(t-h\right)\bar{H}'\left|A\right|^2+O\left(l^{-5}\right).\end{aligned}$$

Due to the fact that η is odd, $-\bar{\eta}\Delta_\Gamma |A|^2+(t-h)\,\partial_t\bar{\eta}\left(|A|^2\right)^2$ decays like $O\left(l^{-4}\right)$ but is orthogonal to $\bar{H}'$.

Let $\phi=\rho\bar{\eta}\,|A|^2+\bar{\phi}$. Then the new function $\bar{\phi}$ should satisfy

$$\begin{aligned}-\Delta\bar{\phi}+\left(3\bar{H}^2-1\right)\bar{\phi}=&-J\left(h\right)\bar{H}'+\left(t-h\right)^3H'\sum_{i=1}^{7}k_i^4\\&+\bar{\eta}\Delta_\Gamma|A|^2-\left(t-h\right)\partial_t\bar{\eta}\left(|A|^2\right)^2\\&+O\left(h'^2\right)+O\left(l^{-2}\right)h'+O\left(l^{-1}\right)h''\\&+3\left(\bar{\eta}\,|A|^2+\bar{\phi}\right)^2\bar{H}+\left(\bar{\eta}\,|A|^2+\bar{\phi}\right)^3+O\left(l^{-5}\right).\end{aligned}\tag{11}$$

Denote the right hand by E. Then multiplying both sides with $\bar{H}'$ and integrating in t, we get

$$\int_{\mathbb{R}}E\bar{H}'=o\left(\bar{\phi}\right).$$

¿From this estimate, we get

$$-\Delta\bar{\phi}+\left(3\bar{H}^2-1\right)\bar{\phi}=E-\frac{\int_{\mathbb{R}}E\bar{H}'}{\int_{\mathbb{R}}\bar{H}'^2}\bar{H}'+o\left(\bar{\phi}\right).\tag{12}$$

Note that

$$\begin{aligned}E-\frac{\int_{\mathbb{R}}E\bar{H}'}{\int_{\mathbb{R}}\bar{H}'^2}\bar{H}'=&\left(t-h\right)^3\bar{H}'\sum_{i=1}^{7}k_i^4\\&+\bar{\eta}\Delta_\Gamma|A|^2-\left(t-h\right)\partial_t\bar{\eta}\left(|A|^2\right)^2\\&+O\left(h'^2\right)+O\left(l^{-2}\right)h'+O\left(l^{-1}\right)h''\\&+3\bar{\eta}^2\left(|A|^2\right)^2\bar{H}+o\left(\bar{\phi}\right)+O\left(l^{-5}\right).\end{aligned}$$

By the estimate of h,h',h'', we get the following estimate(non optimal):

$$E-\frac{\int_{\mathbb{R}}EH'}{\int_{\mathbb{R}}H'^2}H'=O\left(l^{-3}\right)+o\left(\bar{\phi}\right).$$

Since $\bar{\phi}=0$ on L_d (This follows from the boundary condition of u_d. Actually $\bar{\phi}$ is not identically zero on L_d, but of the order $O\left(e^{-d}\right)$), we could use the well known estimate of the linear theory for the operator

$$-\Delta+\left(3H^2-1\right),$$

to deduce that

$$\left|\bar{\phi}\right|+|D\phi|+\left|D^2\phi\right|\leq Cl^{-3},\tag{13}$$

with C independent of d.

Now multiplying both sides of (11) by $\bar{H}'$ again and integrating in t, using the estimate (13) of $\bar{\phi}$, we get

$$J(h)+O\left(h'^2\right)+O\left(l^{-2}\right)h'+O\left(l^{-1}h''\right)=\int_{\mathbb{R}}\left[-\Delta\bar{\phi}+\left(3\bar{H}^2-1\right)\bar{\phi}\right]\bar{H}'dt+O\left(l^{-5}\right). \tag{14}$$

We compute,

$$\begin{aligned}\int_{\mathbb{R}}\left[-\Delta\bar{\phi}+\left(3\bar{H}^2-1\right)\bar{\phi}\right]\bar{H}'dt &= \int_{\mathbb{R}}\left[\Delta_{\Gamma^t}\bar{\phi}+\partial_t^2\bar{\phi}-M_t\partial_t\bar{\phi}+\left(3\bar{H}^2-1\right)\bar{\phi}\right]\bar{H}'dt\\ &= \int_{\mathbb{R}}\Delta_{\Gamma^t}\bar{\phi}\bar{H}'dt+O\left(l^{-5}\right).\end{aligned}$$

¿From this equation, we get

$$J(h)+O\left(h'^2\right)+O\left(l^{-2}\right)h'+O\left(l^{-1}\right)h''=\int_{\mathbb{R}}\Delta_{\Gamma^t}\bar{\phi}\bar{H}'dt+O\left(l^{-5}\right). \tag{15}$$

Using the fact that $|h'|,|h''|\leq Cl^{-2}$ and

$$\int_{\mathbb{R}}\Delta_{\Gamma^t}\bar{\phi}\bar{H}'dt=O\left(l^{-4}\right),$$

We deduce from (15) that

$$|h'|\leq Cl^{-3},|h''|\leq Cl^{-4}. \tag{16}$$

Insert this estimate back into (12), we get an improved estimate for $\bar{\phi}$:

$$\left|\bar{\phi}\right|\leq Cl^{-4}.$$

It follows that

$$J(h)=O\left(l^{-5}\right). \tag{17}$$

This finishes the proof. ■

Next we would like to use Proposition 11 to analyze the behavior of the nodal curve of the limiting solution U.

Lemma 12 *There exists a constant $b\in[-\lambda^*,\lambda^*]$ such that*

$$F(r)=r+br^{-2}+O\left(r^{-3}\right).$$

Proof. For the limiting solution U, we write it in the Fermi coordinate as

$$U=\bar{H}\left(t-h^*(l)\right)+\phi^*,$$

where ϕ^* is orthogonal to $\bar{H}'$. Since we have the uniform estimate for the function h_d

$$J(h_d)=O\left(l^{-5}\right),$$

We get

$$J(h^*) = O\left(l^{-5}\right).$$

Variation of parameter formula tells us that

$$h^*(l) = k_1 l^{-2} + o\left(l^{-2}\right).$$

This together with the fact that U satisfies

$$U_\lambda^- \leq U \leq U_\lambda^+.$$

completes the proof. ■

Next we show that the solution U has the desired asymptotic behavior as $l \to +\infty$.

Proof of Proposition 7. It suffices to show that $b = a$, where the constant b is derived in Lemma 12.

Let r_0 be a constant large enough but fixed. Since $u_d \to U$,

$$h_d(r_0) = (b-a)\, r_0^{-2} + o\left(r_0^{-2}\right), \text{ for } d \text{ large.} \tag{18}$$

On the other hand, we have

$$\begin{cases} J(h_d) = O\left(l^{-5}\right) \text{ in } (r_0, d), \\ h_d(d) = 0. \end{cases}$$

Variation of parameters formula tells us that

$$h_d = c_1 l^{-2} + c_2 l^{-3} + O\left(l^{-3} \ln l\right),$$

where c_1, c_2 may depend on d. Taking into the boundary condition of $h_d(d) = 0$, we get

$$c_1 + c_2 d^{-1} = O\left(d^{-1} \ln d\right). \tag{19}$$

On the other hand, (18) tells us that

$$c_1 + c_2 r_0^{-1} + O\left(r_0^{-1} \ln r_0\right) + o(1) = b - a. \tag{20}$$

Equation (19) and (20) lead to

$$\left(r_0^{-1} - d^{-1}\right) c_2 = O(1). \tag{21}$$

Let $d \to +\infty$. (21) clearly implies that c_2 is bounded, which in turn tells us that $c_1 \to 0$ as $d \to +\infty$. Hence $b = a$. ■

3 Global minimizers from Lawson's minimizing cones

We have obtained global minimizers from the minimal surfaces asymptotic to the Simons cone. In this section, we will perform similar analysis for more general

strictly area minimizing cone. More precisely, we will consider the Lawson's minimizing cone $C_{i,j}$ mentioned in the first section, where either

$$i+j \geq 9,$$

or

$$i+j=8, \ \ |i-j| \leq 4.$$

Let

$$r=\sqrt{x_1^2+...+x_i^2}, \quad s=\sqrt{x_{i+1}^2+...+x_{i+j}^2}, \quad l=\sqrt{r^2+s^2}.$$

Let $|A|^2$ be the squared norm of the second fundamental form of $C_{i,j}$. Put $i+j=n$. The Jacobi operator of $C_{i,j}$, acting on functions $h(l)$ defined on $C_{i,j}$ which additionally only depends on l, has the form

$$J(h)=h''+\frac{n-2}{l}h'+\frac{n-2}{l^2}h.$$

Solutions of the equation $J(h)=0$ is given by

$$c_1 l^{\alpha^+}+c_2 l^{\alpha^-},$$

where

$$\alpha^{\pm}=\frac{-(n-3)\pm\sqrt{(n-3)^2-4(n-2)}}{2}.$$

One could check that for $n \geq 8$, we always have

$$-2 \leq \alpha^+ < -1.$$

The main result of this section is the following

Proposition 13 *There exists a constant $c_{i,j}$ such that for each $k \in \mathbb{R}$, we could construct a solution U_k to the Allen-Cahn equation such that for r large, the nodal set of U_k has the asymptotic behavior:*

$$f_{U_k}(r)=\sqrt{\frac{j-1}{i-1}}r+c_{i,j}r^{-1}+kr^{\alpha^+}+o\left(r^{\alpha^+}\right).$$

For notational convenience, let us simply consider the cone $C_{3,5}$ over the product of spheres $S^2\left(\sqrt{\frac{2}{6}}\right)\times S^4\left(\sqrt{\frac{4}{6}}\right)$. The proof for other cases are similar. Under a choice of the unit normal, the principle curvature of $C_{3,5}$ is given by

$$k_1=0, \quad k_2=k_3=\frac{\sqrt{2}}{l}, \quad k_4=k_5=k_6=k_7=-\frac{1}{\sqrt{2}l}.$$

We set $A_m:=\sum_i k_i^m$. In particular, $A_2:=|A|^2=\frac{6}{l^2}$ and $A_3:=\sum k_i^3=\frac{3\sqrt{2}}{l^3}$. It is well known that $C_{3,5}$ is a strict area minimizing cone. There is also a foliation

of $\mathbb{R}^8$ by minimal hypersurfaces asymptotic to $C_{3,5}$. By slightly abusing the notation, we still use $\Gamma_\lambda^\pm$ to denote this foliation. For λ sufficiently large (say $\lambda \geq \lambda_0$), the construction of Pacard-Wei again gives us a family of solutions $U_\lambda^\pm$ whose zero level set is close to $\Gamma_\lambda^\pm$. The strict area minimizing assumption on the cone is actually used to ensure that this family of solutions are ordered.

Lemma 14 *The family of solutions $U_\lambda^\pm$ is ordered. That is,*

$$\begin{aligned}
&U_{\lambda_1}^+(X) < U_{\lambda_2}^+(X), \lambda_1 < \lambda_2. \\
&U_{\lambda_1}^-(X) < U_{\lambda_2}^-(X), \lambda_1 > \lambda_2, \\
&U_{\lambda_0}^-(X) < U_{\lambda_0}^+(X).
\end{aligned}$$

Proof. Since this has not been proven in the paper [21], we give a sketch of the proof.

We only consider the family of solutions U_λ^+, which we simply write it as U_λ. U_λ is obtained from Lyapunov-Schmidt reduction. Adopting the notations of the previous sections, let (l,t) be the Fermi coordinate with respect to $\Gamma_\lambda = \Gamma_\lambda^+$. Then in the Fermi coordinate, U_λ has the form

$$U_\lambda = H(t - h(l)) + \phi := \bar{H} + \phi.$$

where ϕ is orthogonal to $\bar{H}'$.

Similarly as before, we know that ϕ satisfies

$$-\Delta\phi + \left(3\bar{H}^2 - 1\right)\phi = \Delta_{\Gamma^t}\bar{H} - M_t\bar{H}' + o(\phi).$$

For notational convenience, we set $\bar{t} = t - h(l)$. Let us assume for this moment that $|h| \leq Cl^{-1}$. Recall that

$$\begin{aligned}
M_t &= \left(tA_2 - t^2A_3 + t^3A_4\right)\bar{H}' + O\left(l^{-5}\right) \\
&= \left(\bar{t}A_2 - \bar{t}^2A_3 + \bar{t}^3A_4\right)\bar{H}' + h\bar{H}' - 2\bar{t}\bar{H}'hA_3 + O\left(l^{-5}\right).
\end{aligned}$$

Inspecting the projection of these terms onto $\bar{H}'$, we find that the main order term of the projection should be

$$-\bar{t}^2\bar{H}'A_3 + h\bar{H}'.$$

Next let us compute the term $\Delta_{\Gamma^t}\bar{H}$. First of all,

$$\begin{aligned}
\Delta_{\Gamma^0}\bar{H} &= \frac{\partial_i\left(g^{i,j}\sqrt{|g|}\partial_j\left[H(t - h(l))\right]\right)}{\sqrt{|g|}} \\
&= -\left(h'' + \frac{6}{l}h'\right)\bar{H}' + g^{11}H''(\bar{t})h'^2 + O\left(l^{-5}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\Delta_{\Gamma^t}\bar{H} = &-\left(h'' + \frac{6}{l}h'\right)\bar{H}' + H''(\bar{t})O\left(h'^2\right) + O\left(l^{-5}\right) \\
&+ \bar{t}\bar{H}'O\left(h''l^{-1} + h'l^{-2}\right).
\end{aligned}$$

As a consequence, the function ϕ should satisfy

$$\begin{aligned}-\Delta\phi+\left(3\bar{H}^2-1\right)\phi &= \Delta_{\Gamma_t}\bar{H}-M_t\bar{H}'+o\left(\phi\right)\\ &= -J\left(h\right)\bar{H}'+H''\left(\bar{t}\right)O\left(h'^2\right)+\bar{t}\bar{H}'O\left(h''l^{-1}+h'l^{-2}\right)\\ &\quad-\left(\bar{t}A_2-\bar{t}^2A_3+\bar{t}^3A_4\right)\bar{H}'+2\bar{t}\bar{H}'hA_3+O\left(l^{-5}\right)+o\left(\phi\right).\end{aligned}$$

Projecting onto $\bar{H}'$, the main order term at the right hand side is

$$-J\left(h\right)\bar{H}'+\bar{t}^2A_3\bar{H}'.$$

Hence we find that the main order term h_0 of h should satisfy the equation

$$J\left(h_0\right)=c^*A_3,$$

where

$$c^*=\frac{\int_{\mathbb{R}}t^2H'^2dt}{\int_{\mathbb{R}}H'^2dt}>0.$$

Let $\bar{h}_0\left(l\right)=h_0\left(\lambda l\right)$. We find that $\bar{h}_0$ should satisfy

$$\bar{J}\left(\bar{h}_0\right)=\frac{1}{\lambda}c^*A_3.$$

Here $\bar{J},\bar{k}_i$ are the Jacobi operator and principle curvatures corresponding to the rescaled minimal surface Γ_1^+. Using the invertibility of the Jacobi operator $\bar{J}$, we could assume the existence of function ξ solving

$$\bar{J}\left(\xi\right)=c^*A_3,$$

with the asymptotic behavior

$$c_0l^{-1}+o\left(l^{-2}\right). \tag{22}$$

In this way, we deduce that main order of U_λ is $H\left(t-\frac{1}{\lambda}\xi\left(\frac{l}{\lambda}\right)\right)$.

Fix a λ large. For each δ small, there is a solution $U_{\lambda(1+\delta)}$ of the Allen-Cahn equation associated to the minimal hypersurface $\lambda\left(1+\delta\right)\Gamma_1$. We will denote it by u_δ. To prove the order property of the family of solutions, it will be suffice for us to show that for $0<\delta_1<\delta_2$ sufficiently small,

$$u_{\delta_1}\left(x\right)<u_{\delta_2}\left(x\right),\ x\in\mathbb{R}^8. \tag{23}$$

Let us use t_δ to denote the signed distance of a point to $\Gamma_{\lambda(1+\delta)}$. The previous analysis tells us that the main order of u_δ is

$$H\left(t_\delta-\frac{1}{\lambda\left(1+\delta\right)}\xi\left(\frac{l_\delta}{\lambda\left(1+\delta\right)}\right)\right).$$

Take a large constant k. Let Ξ_k be a radius k tubular neighbourhood of Γ_λ. We claim that for each point $P\in\Xi_k$,

$$t_{\delta_1}-\frac{1}{\lambda\left(1+\delta_1\right)}\xi\left(\frac{l_{\delta_1}}{\lambda\left(1+\delta_1\right)}\right)<t_{\delta_1}-\frac{1}{\lambda\left(1+\delta_2\right)}\xi\left(\frac{l_{\delta_2}}{\lambda\left(1+\delta_2\right)}\right). \tag{24}$$

Indeed, taking into account of the fact that

$$\xi(l) = \frac{1}{1+l} + o\left((1+l)^{-2}\right),$$

we get, for ε small,

$$\varepsilon\xi(\varepsilon l) = \frac{\varepsilon}{1+\varepsilon l} + \varepsilon O\left((1+\varepsilon l)^{-2}\right). \tag{25}$$

On the other hand, for δ_1, δ_2 sufficiently small(depending on λ),

$$t_{\delta_1} - t_{\delta_2} \geq c\lambda^3 (\delta_2 - \delta_1) \frac{1}{1+l^2}, \tag{26}$$

for some constant c. The inequality (24) then follows from (25) and (26) . Once we have (24) , the same argument as in the last section of [21] applies and (23) is proved. ■

By this lemma, the family of solutions $U_\lambda^\pm$ forms a foliation. We could use them as sub and super solutions to obtain solutions between them and we have similar results as in the Simons cone case. However, in the current situation, we show that the nodal set of each solution will be asymptotic to the curve $s = \sqrt{2}r + \frac{c_0}{\sqrt{3}} r^{-1}$, where c_0 is the constant appearing in (22) .

Now we are ready to prove Proposition 13. We still focus on the case $(i,j) = (3,5)$. Since the main steps are same as the case of Simons' cone, we shall only sketch the proof and point out the main difference.

Proof of Proposition 13. Let $k \in \mathbb{R}$ be a fixed real number. Let (l,t) be the Fermi coordinate with respect to the minimal hypersurface asymptotic to the cone $C_{3,5}$ with the asymptotic behavior

$$s = f_k(r) := \sqrt{2}r + kr^{-2} + o\left(r^{-2}\right).$$

We could construct minimizers on a sequences of bounded domain Ω_d. Let L_d be the line orthogonal to the minimal surface at $(d, f_k(d))$. On L_d we impose suitable Dirichlet boundary condition that

$$u_d = H(t - \xi(l)) + \eta(t)|A|^2,$$

at least away from the axes. Recall that we have ordered solutions of Pacard-Wei. We could assume that the boundary function is trapped between two solutions $U_{\lambda*}^+$ and $U_{\lambda^*}^-$ for some fixed λ^*. Then the minimizers u_d on Ω_d will be between $U_{\lambda^*}^+$ and $U_{\lambda^*}^-$. Let us now take the limit for the sequence of solutions $\{u_d\}$ obtained in this way, as $d \to +\infty$.

We need to analyze the asymptotic behavior of $\{u_d\}$. Define the approximate solution $\bar{H}(t-h)$ as before and write $u_d = \bar{H} + \phi$. Then we get

$$\begin{aligned}
-\Delta\phi + \left(3\bar{H}^2 - 1\right)\phi &= \Delta_{\Gamma_t}\bar{H} - M_t\bar{H}' + o(\phi) \\
&= -J(h)\bar{H}' + H''(\bar{t})\, O\left(h'^2\right) + \bar{t}\bar{H}' O\left(h''l^{-1} + h'l^{-2}\right) \\
&\quad - \left(\bar{t}A_2 - \bar{t}^2 A_3 + \bar{t}^3 A_4\right)\bar{H}' + 2\bar{t}\bar{H}' h A_3 + O\left(l^{-5}\right) + o(\phi).
\end{aligned}$$

Write $h(l)$ as $h^*(l) + \xi(l)$. We would like to show that $h^* = kl^{-2} + O\left(l^{-3}\right)$. Indeed,

$$E := \Delta\bar{H} + \bar{H} - \bar{H}^3 = -J(h)\bar{H}' + H''(\bar{t})\,O\left(h'^2\right) + \bar{t}\bar{H}'O\left(h''l^{-1} + h'l^{-2}\right)$$
$$-\left(\bar{t}A_2 - \bar{t}^2A_3 + \bar{t}^3A_4\right)\bar{H}' + 2\bar{t}\bar{H}'hA_3 + O\left(l^{-5}\right).$$

Inspecting each term in this error, it turns out that the projection of $E\left(\bar{H}\right)$ onto $\bar{H}'$ is

$$J(h^*)\bar{H}' + O\left(l^{-5}\right).$$

On the other hand, since $u = \bar{H} + \phi$ and ϕ satisfies

$$-\Delta\phi + \left(3\bar{H}^2 - 1\right)\phi = E - \frac{\int_{\mathbb{R}} E\bar{H}'}{\int_{\mathbb{R}} \bar{H}'^2}\bar{H}' + o(\phi),$$

we find that

$$\phi = \eta(\bar{t})\,|A|^2 + \phi^*,$$

with

$$|\phi^*| \leq Cl^{-3}.$$

A refined analysis shows that the $O\left(r^{-3}\right)$ term could be written as $\eta_2(\bar{t})A_3$, where η_2 satisfies

$$-\eta_2''(t) + \left(3H^2 - 1\right)\eta_2 = t^2H' - \frac{\int_{\mathbb{R}} t^2H'^2dt}{\int_{\mathbb{R}} H'^2dt}H'.$$

It then follows from similar arguments as in the previous sections that

$$J(h^*) = O\left(l^{-5}\right).$$

This provides us sufficient estimate to prove our result, proceeding similarly as in Section 2.2. ■

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