ON FILA-KING CONJECTURE IN DIMENSION FOUR

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ABSTRACT. We consider the following Cauchy problem for the four-dimensional energy critical heat equation

$$\begin{cases} u_t = \Delta u + u^3 & \text{in } \mathbb{R}^4 \times (0, \infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^4. \end{cases}$$

We construct a positive infinite time blow-up solution $u(x,t)$ with the blow-up rate $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^4)} \sim \ln t$ as $t \to \infty$ and show the stability of the infinite time blow-up. This gives a rigorous proof of a conjecture by Fila and King [15, Conjecture 1.1].

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

Since the seminal work of Fujita [19], the following nolinear heat equation

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

with $p > 1$, $n \geq 3$ has been extensively studied. The energy functional for (1.1) is

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1},$$

and for classical solution $u(x,t)$ with sufficient spatial decay, one has

$$\frac{d}{dt}E(u(\cdot,t)) = -\int_{\mathbb{R}^n} |u_t|^2.$$

Many literatures have been devoted to studying problem (1.1) about the singularity formation, especially the blow-up rates, profiles and sets. We refer the readers to the book of Quittner and Souplet [38] for comprehensive survey and also recent developments.

For the finite time blow-up, it is said to be of

- *type I* if
$$\limsup_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_\infty < \infty;$$
- *type II* if
$$\limsup_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_\infty = \infty.$$

Type I blow-up is more "generic", while type II blow-up is much more difficult to detect. In particular, two different types of blow-up phenomena in problem (1.1) depend sensitively on the value of the exponent p. In this setting, the critical Sobolev exponent

$$p_s = \begin{cases} \frac{n+2}{n-2} & \text{for } n \geq 3 \\ \infty & \text{for } n = 1, 2 \end{cases}$$

is special in various ways. Giga, Matsui and Sasayama [22, 23] proved that for $p < p_s$, only type I blow-up can occur in the case that Ω is $\mathbb{R}^n$ or a convex domain. For the energy critical case $p = p_s$, in the positive radial and monotonically decreasing class, Filippas, Herrero and Velázquez [18] excluded the possibility of type II blow-up for $n \geq 3$, and Matano and Merle [28, Theorem 1.7] removed the monotone assumption and obtained the same result. Wang and Wei [43] proved the same result to the non-radial positive class in higher dimensions $n \geq 7$. For $p < p_s$, finite time type I blow-up solution was found and its stability was studied in [32]. For the critical case $p = p_s$ in $\mathbb{R}^n$ with $n \geq 7$, classification results were proved near the ground state of the energy critical heat equation in [4]. On the other hand, sign-changing type II blow-up solutions to the energy critical heat equation in dimensions $n = 3, 4, 5, 6$ were first conjectured to exist by [18] and have been rigorously constructed recently in [39, 9, 14, 24, 25, 12, 27]. In the supercritical case, classification of type I and type II solutions in radially symmetric class have been studied in [29, 30, 31] and the references therein, and the construction of Type II blow-up was first established in the radial case by Herrero and Velázquez [26] and in the non-radial case (under some restrictions of the exponent p) by Collot [3].

Concerning infinite time blow-up for $p = p_s$, Galaktionov and King [20] investigated positive, radially symmetric, global unbounded solutions for problem (1.1) in the case of unit ball with Dirichlet boundary condition in dimensions $n \geq 3$. See also [42, Theorem 1.4] for the case that the domain is symmetric and convex. In the non-radial setting, positive infinite time blow-up solution for problem (1.1) with Dirichlet boundary condition and $n \geq 5$ was constructed by Cortazar, del Pino and Musso in [5]. The solution constructed in [5] takes the profile of sharply scaled Aubin-Talenti bubbles

$$U_{\mu,\xi}(x) = \mu^{-\frac{n-2}{2}}U\left(\frac{x-\xi}{\mu}\right) = (n(n-2))^{\frac{n-2}{4}}\left(\frac{\mu}{\mu^2+|x-\xi|^2}\right)^{\frac{n-2}{2}},$$

which solve the Yamabe problem

$$\Delta U + U^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbb{R}^n.$$

Moreover, the blow-up location for the solution is determined by the Green's function of $-\Delta$ in Ω, while for elliptic problems, the role of the Green's function in bubbling phenomena has been known for a long time since the works [1] and [2]. In [13], non-radial and sign-changing solution which blows up at infinite time has been constructed. Bubble towers at infinite time and backward time infinity have been constructed in [11] and [41], respectively.

In a very interesting paper [15], Fila and King studied problem (1.1) in the whole space $\mathbb{R}^n$ with the critical exponent $p = p_s$ and gave insight on the infinite time blow-up in the case of a radially symmetric, positive initial condition with an exact power decay rate. By formal matched asymptotic analysis, they demonstrated that the blow-up rate is determined by the power decay in a precise manner. Intriguingly enough, their analysis leads them to conjecture that infinite time blow-up should only happen in low dimensions 3 and 4, see Conjecture 1.1 in [15]. Recently, this has been confirmed and rigorously proved by del Pino, Musso and the first author in [10] for $n = 3$, where the leading part of the scaling parameter is achieved by asymptotic analysis. For the case $n = 4$, Fila and King conjectured that infinite time blow-up only exists when $\ell > 2$ for radial solutions, where

$$\lim_{|x|\to\infty} |x|^\ell u_0(|x|) = A$$

for some $A > 0$.

In other contexts, for instance, Liouville-type theorems for Fujita equation, in parallel with the seminal work of Gidas and Spruck [21] in the elliptic setting, and long-time behaviors for the solutions to Fujita equation with supercritical exponent have been studied in [35, 16, 34, 17, 36, 37, 33] and the references therein.

In this paper, we are concerned with the following Cauchy problem for the Fujita equation with critical exponent in dimension $n = 4$

$$\begin{cases} u_t = \Delta u + u^3 & \text{in } \mathbb{R}^4 \times (t_0, \infty), \\ u(x,t_0) = u_0(x) & \text{in } \mathbb{R}^4. \end{cases} \tag{1.2}$$

The aim of this paper is to construct infinite time blow-up solution, confirming the conjecture by [15, Conjecture 1.1], and further investigate the stability of the infinite time blow-up. Throughout this paper, η is a smooth cut-off function which satisfies that $\eta(s) = 1$ for $s \le 1$ and $\eta(s) = 0$ for $s \ge \frac{3}{2}$. Our main results are stated as follows.

Theorem 1.1. *For t_0 sufficiently large, there exists initial value $u_0 > 0$ with exponential decay such that the positive solution $u(x,t)$ to (1.2) blows up at infinite time. More precisely, the solution takes the form of the sharply scaled bubble*

$$u(x,t) = \eta\left(\frac{x-\xi}{\sqrt{t}}\right)\mu^{-1}(t)w\left(\frac{x-\xi(t)}{\mu(t)}\right) + O\big((\ln t)^{-1}\min\{t^{-1}, |x|^{-2}\}\big)$$

where $w(y) = 2^{\frac{3}{2}}\frac{1}{1+|y|^2}$. The blow-up rate and location are given by

$$\mu(t) = \frac{1}{\ln t}\Big(1 + O\big(\frac{\ln\ln t}{\ln t}\big)\Big), \quad \xi(t) = O(t^{-1}).$$

More precisely, the positive initial value of the solution constructed above is

$$u(x,t_0) = \mu^{-1}(t_0)w\big(\frac{x-\xi(t_0)}{\mu(t_0)}\big)\eta\big(\frac{x-\xi(t_0)}{\sqrt{t_0}}\big) + 2^{\frac{3}{2}}\mu(t_0)|x-\xi(t_0)|^{-2}\Big(e^{-\frac{|x-\xi(t_0)|^2}{4t_0}} - \eta\big(\frac{x-\xi(t_0)}{\sqrt{t_0}}\big)\Big)$$
$$+ \bar{\mu}_0^{-1}(t_0)\Phi_0\big(\frac{x-\xi(t_0)}{\bar{\mu}_0(t_0)}, t_0\big)\eta\big(\frac{4(x-\xi(t_0))}{\sqrt{t_0}}\big) + \eta\big(\frac{x-\xi(t_0)}{\mu_0(t_0)R(t_0)}\big)e_0\mu^{-1}(t_0)Z_0\big(\frac{x-\xi(t_0)}{\mu(t_0)}\big),$$

where μ_0, $\bar{\mu}_0$ are the leading order of μ, and $\bar{\mu}_0 \sim \mu_0 = (\ln t)^{-1}$; Φ_0 is a global correction function given in Section 2.3; e_0 is a constant and Z_0 is the eigenfunction with respect to the first eigenvalue for the linearized operator, which has exponential decay, see (7.3).

We further investigate the stability of the blow-up solution constructed in Theorem 1.1 and obtain the stability in the following sense.

Theorem 1.2. *For any g_0, not necessarily radially symmetric, satisfying $|g_0(x)| \le C_g t_0^{-\frac{\min\{\ell,4\}}{2}}\langle x\rangle^{-\ell}$, $\ell > 3$, and for t_0 sufficiently large, there exists a solution $u[g_0](x,t)$ to (1.2) blowing up at infinite time with the rate*

$$\mu[g_0](t) = \frac{1}{\ln t}\Big(1 + O\big(\frac{\ln\ln t}{\ln t}\big)\Big), \quad \xi[g_0](t) = O(t^{-1}).$$

The initial value is given by

$$\begin{aligned} u(x,t_0) = {} & \big(\mu[g_0](t_0)\big)^{-1}w\big(\frac{x-\xi[g_0](t_0)}{\mu[g_0](t_0)}\big)\eta\big(\frac{x-\xi[g_0](t_0)}{\sqrt{t_0}}\big) \\ & + 2^{\frac{3}{2}}\mu[g_0](t_0)\big|x-\xi[g_0](t_0)\big|^{-2}\Big(e^{-\frac{|x-\xi[g_0](t_0)|^2}{4t_0}} - \eta\big(\frac{x-\xi[g_0](t_0)}{\sqrt{t_0}}\big)\Big) \\ & + \big(\bar{\mu}_0(t_0)\big)^{-1}\Phi_0\big(\frac{x-\xi[g_0](t_0)}{\bar{\mu}_0(t_0)}, t_0\big)\eta\big(\frac{4(x-\xi[g_0](t_0))}{\sqrt{t_0}}\big) \\ & + \eta\big(\frac{x-\xi[g_0](t_0)}{\mu_0(t_0)R(t_0)}\big)e_0[g_0]\big(\mu[g_0](t_0)\big)^{-1}Z_0\big(\frac{x-\xi[g_0](t_0)}{\mu[g_0](t_0)}\big) + g_0 \end{aligned}$$

where $\mu[g_0] \to \mu$, $\xi[g_0] \to \xi$, $e_0[g_0] \to e_0$ in some topology as $C_g \to 0$. In the radial setting, the same conclusion holds for $\ell > 2$ with $\xi[g_0] \equiv 0$ and $\bar{\mu}_0[g_0] \to \bar{\mu}_0$ as $C_g \to 0$ additionally.

Remark 1.2.1.

- *Indeed, the initial value of the infinite time blow-up solution in Theorem 1.1 has exponential decay at space infinity. By Theorem 1.2, we can add suitable perturbation for the initial value to achieve that*

$$\lim_{|x|\to\infty}|x|^{\ell}u(x,t_0) = A$$

for any $|A|$ small enough, recovering the assumption on the initial value in the conjecture by Fila and King [15].

- *It is very possible to generalize the stability result for all $\ell > 2$ in the non-radial setting, see Remark 6.0.1.*
- *We do not know if the solution we construct is threshold solution or not.*

Our construction is based on the *inner–outer gluing method* developed recently in [5, 8] for parabolic problems, and the gluing method has been a powerful tool to investigate the singularity formation for various nonlinear PDEs such as parabolic equations and systems, fluid equations, geometric flows and others. See [10, 7, 40] and the references therein. The parabolic gluing method is much more different from the asymptotic analysis given in [15]. Some essential new features and difficulties in this paper are listed below.

One key feature and difficulty is the non-local dynamics for the scaling parameter $\mu(t)$. It turns out that the dynamics for $\mu(t)$ is governed by an integro-differential operator, which is a natural consequence of the fact that the linear generator of dilations of the Aubin-Talenti bubble is of slow decay in lower dimensions. This non-local phenomenon has also been observed in [8, 10, 14, 6] for lower dimensional problems. In our case here, neither the usual Laplace transform nor Riemann-Louville type method is applicable since the integro-differential equation is not in the class of Abel-type integral equations. The non-local operator here is the threshold/endpoint case in certain sense, and one needs to carry out much more delicate analysis to investigate its solvability.

Our strategy is to decompose the non-local equation for $\mu(t)$ into two parts: the dominating term and the remainder term. The dominating term will be solved by contraction mapping theorem, while the remainder term will leave a much smaller error. To be more precise, the desired blow-up rate is determined at leading order. However, due to the way that we handle the non-local operator, the time decay is not fast enough for the remainder in the gluing procedure, and we will iterate this process finitely many times to make the remainder term have faster time decay than the one provided by the outer problem. This smaller remainder will be handled when solving the next order of $\mu(t)$.

After getting the leading order of $\mu(t)$, we need to solve the corresponding linearized elliptic equation to improve the time decay of the error term, which is essential for finding suitable weighted topologies ensuring the implementation of the gluing procedure. When solving the next order $\mu_1(t)$, we still need to decompose the non-local equation into two parts. The main difference is that the involved outer problem in the equation of $\mu_1(t)$ only has Hölder continuity in t variable. The derivative of $\mu_1(t)$ will inherit Hölder continuity from the outer problem, which will be used to control the remainder term.

On the other hand, the rather slow logarithmic blow-up rate produces following difficulties. There are several slow decaying linear terms which involves the inner part cannot be controlled as the right hand side of the inner or outer problem. Instead, we regard these slow decaying terms as part of the linearization of the inner problem and develop a new linear theory. See Remark 3.0.1. The dealing of these terms is in a similar spirit as in [6], where the logarithmic blow-up speed also appears.

Thanks to the generality for the gluing method, we are able to study the stability for the solution constructed in Theorem 1.1 with both radial and non-radial perturbations, and non-radial infinite time blow-up solutions are easily found by suitable perturbation for the initial value.

Before carrying out the construction, we list several commonly used notations throughout the paper as follows.

Notations:

- We write $a \lesssim b$ ($a \gtrsim b$) if there exists a constant $C > 0$ such that $a \le Cb$ ($a \ge Cb$) where C is independent of t, t_0. Set $a \sim b$ if $b \lesssim a \lesssim b$.
- In general, the letter $C(a, b, \dots)$ stands for a positive constant depending on parameters $a, b \dots$ that might change its actual value at each occurrence.
- The symbol $f[g_1, g_2, \dots]$ means that the function f depends on some functions $g_1, g_2, \dots$.
- $f \approx g$ means that $|f - g| \to 0$ as $t \to \infty$.
- The symbol $O(f(x))$ is used to denote a real-valued function that satisfies $|O(f(x))| \lesssim |f(x)|$ in a domain of x that is either specified explicitly or follows from the context.
- For any fixed real number x, the symbol $x-$ denotes a number which is less than x and can be chosen close to x arbitrarily.
- Denote $\langle y \rangle = \sqrt{1 + |y|^2}$ for any $y \in \mathbb{R}^n$.
- Denote $\mathbf{1}_{\{x\in\Omega\}}$ as the characteristic function with $\mathbf{1}_{\{x\in\Omega\}} = 1$ if $x \in \Omega$ and $\mathbf{1}_{\{x\in\Omega\}} = 0$ if $x \notin \Omega$.

2. Approximate solution and improvement

2.1. First approximate solution.

We consider the energy critical heat equation in dimension 4

$$\begin{cases} u_t = \Delta u + u^3 & \text{in } \mathbb{R}^4 \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^4. \end{cases} \tag{2.1}$$

Since changing the initial time will not change the structure of the nonlinear heat equation, we assume the initial time is $t = t_0$ and t_0 is sufficiently large.

We use the steady state solution

$$w(y) = 2^{\frac{3}{2}} \frac{1}{1+|y|^2}$$

as the building block of construction. It is known that all the bounded kernels of the corresponding linearized operator $\Delta + 3w^2$ are given by

$$Z_i(y) = \partial_{y_i} w = -2^{\frac{5}{2}} \frac{y_i}{(1+|y|^2)^2}, \text{ for } i = 1, \dots, 4, \quad Z_5(y) = w + y \cdot \nabla w = 2^{\frac{3}{2}} \frac{1-|y|^2}{(1+|y|^2)^2}.$$

We take the leading profile of the infinite time blow-up solution as

$$u_1(x,t) = \mu^{-1}(t) w\left(\frac{x-\xi(t)}{\mu(t)}\right) \eta\left(\frac{x-\xi(t)}{\sqrt{t}}\right)$$

where $\mu(t), \xi(t) \in C^1(t_0, \infty)$. Throughout this paper, we make the following ansatz

$$\frac{1}{C_\mu \ln t} \le |\mu| + t \ln t |\mu_t| \le \frac{C_\mu}{\ln t}, \tag{2.2}$$

$$\xi(t) \to 0 \text{ as } t \to \infty$$

where $C_\mu \ge 1$ is a large constant. Later we shall rigorously justify the above ansatz about the asymptotics for the scaling and translation parameters.

Denote the error function as

$$S[g] := -\partial_t g + \Delta g + g^3.$$

Then the error produced by the first approximate solution u_1 is given by

$$S[u_1] = \mu^{-2}\mu_t Z_5\left(\frac{x-\xi}{\mu}\right)\eta\left(\frac{x-\xi}{\sqrt{t}}\right) + E[\mu] + \mu^{-2}\xi_t \cdot \nabla w\left(\frac{x-\xi}{\mu}\right)\eta\left(\frac{x-\xi}{\sqrt{t}}\right)$$
$$+ \mu^{-1}t^{-\frac{1}{2}}\xi_t \cdot \nabla\eta\left(\frac{x-\xi}{\sqrt{t}}\right) w\left(\frac{x-\xi}{\mu}\right)$$

where

$$E[\mu] := 2^{-1}\mu^{-1}t^{-1} w\left(\frac{x-\xi}{\mu}\right)\nabla\eta\left(\frac{x-\xi}{\sqrt{t}}\right) \cdot \frac{x-\xi}{\sqrt{t}} + 2\mu^{-2}t^{-\frac{1}{2}}\nabla w\left(\frac{x-\xi}{\mu}\right)\cdot\nabla\eta\left(\frac{x-\xi}{\sqrt{t}}\right)$$
$$+ \mu^{-1}t^{-1} w\left(\frac{x-\xi}{\mu}\right)\Delta\eta\left(\frac{x-\xi}{\sqrt{t}}\right) + \mu^{-3}w^3\left(\frac{x-\xi}{\mu}\right)\left[\eta^3\left(\frac{x-\xi}{\sqrt{t}}\right) - \eta\left(\frac{x-\xi}{\sqrt{t}}\right)\right].$$

In next section, we shall add two global corrections to improve the slow decaying error.

2.2. **Transferring slow decaying terms by heat equations.**

For some admissible function $f(x,t)$, denote

$$\mathcal{T}_n^{out}[f](x,t) := \int_{t_0}^{t}\int_{\mathbb{R}^n} (4\pi(t-s))^{-\frac{n}{2}} e^{-\frac{|x-z|^2}{4(t-s)}} f(z,s)dzds. \tag{2.3}$$

In the rest of the paper, we will use Lemma A.1 and Lemma A.2 in the appendix to estimate $\mathcal{T}_n^{out}$ frequently and sometimes will not state repeatedly.

Set $y = \frac{x-\xi}{\mu}$. A term is said to be of slow decay if its spatial decay is equal to or slower than $\langle y\rangle^{-2}$. Otherwise, it is of fast decay. Fast decay is necessary for the gluing procedure. For this reason we will transfer the slow decaying terms in $S[u_1]$ by heat equations. We now introduce the correction function φ to improve the error. For

$$S[u_1 + \varphi] = -\partial_t\varphi + \Delta\varphi + S[u_1] + (u_1+\varphi)^3 - u_1^3,$$

we set $\bar{x} = x - \xi$ and choose $\varphi(\bar{x},t) = \varphi_1(\bar{x},t) + \varphi_2(\bar{x},t)$ such that

$$\partial_t\varphi_1 = \Delta_{\bar{x}}\varphi_1 + E[\mu], \quad \partial_t\varphi_2 = \Delta_{\bar{x}}\varphi_2 + \mu^{-2}\mu_t Z_5(\frac{\bar{x}}{\mu})\eta(\frac{\bar{x}}{\sqrt{t}}).$$

The properties of φ_1 and φ_2 are given in the following two lemmas.

Lemma 2.1. *Assume that μ satisfies (2.2) and μ_1 satisfies $|\mu_1| \le \frac{\mu}{2}$. Consider*

$$\partial_t \varphi_1 = \Delta_{\bar{x}} \varphi_1 + E[\mu]. \tag{2.4}$$

There exists a solution $\varphi_1 = \varphi_1[\mu]$ satisfying the following pointwise estimates

$$|\varphi_1[\mu]| \lesssim (t \ln t)^{-1} \mathbf{1}_{\{|\bar{x}| \le 2t^{\frac{1}{2}}\}} + t^2 (\ln t)^{-1} |\bar{x}|^{-6} \mathbf{1}_{\{|\bar{x}| > 2t^{\frac{1}{2}}\}},$$

$$|\nabla_{\bar{x}} \varphi_1| \lesssim t^{-\frac{3}{2}} (\ln t)^{-1} \mathbf{1}_{\{|\bar{x}| \le 2t^{\frac{1}{2}}\}} + t^{\frac{3}{2}} (\ln t)^{-1} |\bar{x}|^{-6} \mathbf{1}_{\{|\bar{x}| > 2t^{\frac{1}{2}}\}}.$$

More precisely,

$$\begin{aligned}
\varphi_1[\mu] = & \left[-2^{-\frac{1}{2}} \mu t^{-1} + O(\mu t^{-2} |\bar{x}|^2) + O\Big(t^{-2} \int_{t_0/2}^{t} (s^{-1} \mu^3(s) + s|\mu_t(s)|) ds \Big) \right] \mathbf{1}_{\{|\bar{x}| \le 2t^{\frac{1}{2}}\}} \\
& + O\left(\mu |\bar{x}|^{-2} e^{-\frac{|\bar{x}|^2}{16t}} + |\bar{x}|^{-6} \int_{t_0/2}^{t} s^2 |\mu_t(s)| ds + t^{-2} e^{-\frac{|\bar{x}|^2}{16t}} \int_{t_0/2}^{t/2} (s^{-1} \mu^3(s) + s|\mu_t(s)|) ds \right) \mathbf{1}_{\{|\bar{x}| > 2t^{\frac{1}{2}}\}},
\end{aligned}$$

$$\begin{aligned}
& \varphi_1[\mu + \mu_1] - \varphi_1[\mu] = \Big[-2^{-\frac{1}{2}} \mu_1 t^{-1} + O(|\mu_1| t^{-2} |\bar{x}|^2) \\
& + O\Big(t^{-2} \mu^2 \sup_{t_1 \in [t/2, t]} |\mu_1(t_1)| + \sup_{t_1 \in [t/2, t]} |\mu_{1t}(t_1)| + t^{-2} \int_{t_0/2}^{t/2} \left(s^{-1} |\mu_1(s)| \mu^2(s) + s|\mu_{1t}(s)| \right) ds \Big) \Big] \mathbf{1}_{\{|\bar{x}| \le 2t^{\frac{1}{2}}\}} \\
& + O\Big(\sup_{t_1 \in [t/2, t]} |\mu_1(t_1)| |\bar{x}|^{-2} e^{-\frac{|\bar{x}|^2}{16t}} + |\bar{x}|^{-6} \Big(t^3 \sup_{t_1 \in [t/2, t]} |\mu_{1t}(t_1)| + \int_{t_0/2}^{t/2} s^2 |\mu_{1t}(s)| ds \Big) \\
& + t^{-2} e^{-\frac{|\bar{x}|^2}{16t}} \int_{t_0/2}^{t/2} \left(s^{-1} |\mu_1(s)| \mu^2(s) + s|\mu_{1t}(s)| \right) ds \Big) \mathbf{1}_{\{|\bar{x}| > 2t^{\frac{1}{2}}\}}.
\end{aligned}$$

Proof. The support of E is in $\{t^{\frac{1}{2}} \le |\bar{x}| \le 2t^{\frac{1}{2}}\}$. In this region, by (2.2), $\mu^{-1}|\bar{x}| \gg 1$, which implies

$$w(\frac{|\bar{x}|}{\mu}) = 2^{\frac{3}{2}} \mu^2 |\bar{x}|^{-2} + O(\mu^4 |\bar{x}|^{-4}), \quad \frac{|\bar{x}|}{\mu} w'(\frac{|\bar{x}|}{\mu}) = -2^{\frac{5}{2}} \mu^2 |\bar{x}|^{-2} + O(\mu^4 |\bar{x}|^{-4}).$$

Then the leading term of E denoted by $\tilde{E}$ is given by

$$\tilde{E} = 2^{\frac{3}{2}} \mu t^{-2} \left(2^{-1} \zeta^{-1} \eta'(\zeta) - \zeta^{-3} \eta'(\zeta) + \zeta^{-2} \eta''(\zeta) \right), \quad \zeta = \frac{|\bar{x}|}{\sqrt{t}}.$$

Take $\tilde{\varphi}_1$ as the approximate solution to (2.4). Set $\tilde{\varphi}_1 = \mu \hat{\varphi}_1$, $\tilde{E} = \mu \hat{E}$ and $\hat{\varphi}_1$ satisfies

$$\partial_t \hat{\varphi}_1 = \Delta_{\bar{x}} \hat{\varphi}_1 + \hat{E}.$$

We take $\hat{\varphi}_1 = t^{-1} A\left(\frac{|\bar{x}|}{\sqrt{t}} \right)$ in the self-similar form. Then

$$A'' + \left(\frac{3}{\zeta} + \frac{\zeta}{2} \right) A' + A + h(\zeta) = 0, \tag{2.5}$$

where

$$h(\zeta) = 2^{\frac{3}{2}} \zeta^{-2} \left(\eta''(\zeta) - \frac{1}{\zeta} \eta'(\zeta) + \frac{\zeta}{2} \eta'(\zeta) \right).$$

Observe that ζ^{-2}, $\zeta^{-2}(1 - e^{-\frac{\zeta^2}{4}})$ are linearly independent kernels to the homogeneous part of (2.5). And (2.5) has a particular solution

$$A_p(\zeta) = -\zeta^{-2} \int_0^{\zeta} a e^{-\frac{a^2}{4}} \int_0^{a} h(b) b e^{\frac{b^2}{4}} db da = -\zeta^{-2} \int_0^{\zeta} 2^{\frac{3}{2}} \eta'(a) da = 2^{\frac{3}{2}} \zeta^{-2} (1 - \eta(\zeta)),$$

where we have used $h(b) b e^{\frac{b^2}{4}} = 2^{\frac{3}{2}} (b^{-1} e^{\frac{b^2}{4}} \eta'(b))'$.

In order to find a solution with fast spatial decay, we take

$$A(\zeta) = A_p(\zeta) - 2^{\frac{3}{2}} \zeta^{-2} (1 - e^{-\frac{\zeta^2}{4}}) = 2^{\frac{3}{2}} \zeta^{-2} (e^{-\frac{\zeta^2}{4}} - \eta(\zeta))$$

which implies that

$$\hat{\varphi}_1(\bar{x}, t) = 2^{\frac{3}{2}} |\bar{x}|^{-2} \Big(e^{-\frac{|\bar{x}|^2}{4t}} - \eta(\frac{|\bar{x}|}{\sqrt{t}}) \Big), \quad \tilde{\varphi}_1[\mu] = 2^{\frac{3}{2}} \mu |\bar{x}|^{-2} \Big(e^{-\frac{|\bar{x}|^2}{4t}} - \eta(\frac{|\bar{x}|}{\sqrt{t}}) \Big). \tag{2.6}$$

It is straightforward to see $\tilde{\varphi}_1[\mu](0,t) = -2^{-\frac{1}{2}}\mu t^{-1}$,

$$\tilde{\varphi}_1[\mu] = \left(-2^{-\frac{1}{2}}\mu t^{-1} + O(\mu t^{-2}|\bar{x}|^2)\right)\mathbf{1}_{\{|\bar{x}|\le 2t^{\frac{1}{2}}\}} + O\left(\mu|\bar{x}|^{-2}e^{-\frac{|\bar{x}|^2}{4t}}\right)\mathbf{1}_{\{|\bar{x}|>2t^{\frac{1}{2}}\}},$$

$$\tilde{\varphi}_1[\mu+\mu_1] - \tilde{\varphi}_1[\mu] = \left(-2^{-\frac{1}{2}}\mu_1 t^{-1} + O(|\mu_1|t^{-2}|\bar{x}|^2)\right)\mathbf{1}_{\{|\bar{x}|\le 2t^{\frac{1}{2}}\}} + O\left(|\mu_1||\bar{x}|^{-2}e^{-\frac{|\bar{x}|^2}{4t}}\right)\mathbf{1}_{\{|\bar{x}|>2t^{\frac{1}{2}}\}},$$

and

$$\begin{aligned}\partial_{|\bar{x}|}\hat{\varphi}_1 =& -2^{\frac{5}{2}}|\bar{x}|^{-3}\left(e^{-\frac{|\bar{x}|^2}{4t}} - \eta(\frac{|\bar{x}|}{\sqrt{t}})\right) - 2^{\frac{3}{2}}|\bar{x}|^{-2}\left(e^{-\frac{|\bar{x}|^2}{4t}}\frac{|\bar{x}|}{2t} + \eta'(\frac{|\bar{x}|}{\sqrt{t}})t^{-\frac{1}{2}}\right)\\ =& O\left(|\bar{x}|t^{-2}\mathbf{1}_{\{|\bar{x}|\le t^{\frac{1}{2}}\}} + |\bar{x}|^{-1}t^{-1}e^{-\frac{|\bar{x}|^2}{4t}}\mathbf{1}_{\{|\bar{x}|>t^{\frac{1}{2}}\}}\right),\\ |\nabla_{\bar{x}}\tilde{\varphi}_1[\mu]| =& |\mu\nabla_{\bar{x}}\hat{\varphi}_1| \lesssim |\bar{x}||\mu|t^{-2}\mathbf{1}_{\{|\bar{x}|\le 2t^{\frac{1}{2}}\}} + |\bar{x}|^{-1}|\mu|t^{-1}e^{-\frac{|\bar{x}|^2}{4t}}\mathbf{1}_{\{|\bar{x}|>2t^{\frac{1}{2}}\}}.\end{aligned} \tag{2.7}$$

Take $\varphi_1 = \tilde{\varphi}_1 + \tilde{\varphi}_{1b}$. Then $\tilde{\varphi}_{1b}$ satisfies

$$\partial_t\tilde{\varphi}_{1b} = \Delta_{\bar{x}}\tilde{\varphi}_{1b} - \mu_t\hat{\varphi}_1 + E - \tilde{E},$$

where $\tilde{\varphi}_{1b}$ is given by

$$\tilde{\varphi}_{1b}[\mu](\bar{x},t) = \mathcal{T}_4^{out}[-\mu_t\hat{\varphi}_1 + E - \tilde{E}](\bar{x},t) \tag{2.8}$$

with

$$\begin{aligned}E - \tilde{E} =& \frac{1}{2}\mu^{-1}t^{-1}\left(w(\frac{|\bar{x}|}{\mu}) - 2^{\frac{3}{2}}\mu^2|\bar{x}|^{-2}\right)\eta'(\frac{|\bar{x}|}{\sqrt{t}})\frac{|\bar{x}|}{\sqrt{t}} + 2\mu^{-1}t^{-1}\left(\frac{|\bar{x}|}{\mu}w'(\frac{|\bar{x}|}{\mu}) + 2^{\frac{5}{2}}\mu^2|\bar{x}|^{-2}\right)\frac{1}{|\bar{x}|t^{-\frac{1}{2}}}\eta'(\frac{|\bar{x}|}{\sqrt{t}})\\ &+ \mu^{-1}t^{-1}\left(w(\frac{|\bar{x}|}{\mu}) - 2^{\frac{3}{2}}\mu^2|\bar{x}|^{-2}\right)\Delta\eta(\frac{|\bar{x}|}{\sqrt{t}}) + \mu^{-3}w^3(\frac{|\bar{x}|}{\mu})\left(\eta^3(\frac{|\bar{x}|}{\sqrt{t}}) - \eta(\frac{|\bar{x}|}{\sqrt{t}})\right) = O(\mu^3t^{-3}\mathbf{1}_{\{\sqrt{t}\le|\bar{x}|\le 2\sqrt{t}\}}).\end{aligned} \tag{2.9}$$

Similarly, we evaluate

$$(E-\tilde{E})[\mu+\mu_1] - (E-\tilde{E})[\mu] = O\left(|\mu_1|\mu^2t^{-3}\mathbf{1}_{\{\sqrt{t}\le|\bar{x}|\le 2\sqrt{t}\}}\right).$$

By Lemma A.1, one has

$$\mathcal{T}_4^{out}\left[\mu^3t^{-3}\mathbf{1}_{\{\sqrt{t}\le|\bar{x}|\le 2\sqrt{t}\}}\right] \lesssim t^{-2}e^{-\frac{|\bar{x}|^2}{16t}}\int_{t_0/2}^{t/2}\mu^3(s)s^{-1}ds + \begin{cases}\mu^3t^{-2} & \text{if } |\bar{x}|\le t^{\frac{1}{2}}\\ \mu^3t^{-1}|\bar{x}|^{-2}e^{-\frac{|\bar{x}|^2}{16t}} & \text{if } |\bar{x}|>t^{\frac{1}{2}}\end{cases},$$

$$\begin{aligned}\mathcal{T}_4^{out}\left[|\mu_1|\mu^2t^{-3}\mathbf{1}_{\{\sqrt{t}\le|\bar{x}|\le 2\sqrt{t}\}}\right] \lesssim& t^{-2}e^{-\frac{|\bar{x}|^2}{16t}}\int_{t_0/2}^{t/2}|\mu_1(s)|\mu^2(s)s^{-1}ds\\ &+\begin{cases}\sup_{t_1\in[t/2,t]}|\mu_1(t_1)|\mu^2t^{-2} & \text{if } |\bar{x}|\le t^{\frac{1}{2}}\\ \sup_{t_1\in[t/2,t]}|\mu_1(t_1)|\mu^2t^{-1}|\bar{x}|^{-2}e^{-\frac{|\bar{x}|^2}{16t}} & \text{if } |\bar{x}|>t^{\frac{1}{2}}\end{cases}.\end{aligned}$$

Notice that

$$|\mu_t\hat{\varphi}_1| \lesssim |\mu_t|t^{-1}\mathbf{1}_{\{|\bar{x}|\le t^{\frac{1}{2}}\}} + |\mu_t||\bar{x}|^{-2}e^{-\frac{|\bar{x}|^2}{4t}}\mathbf{1}_{\{|\bar{x}|>t^{\frac{1}{2}}\}}. \tag{2.10}$$

Therefore, by Lemma A.1, we obtain

$$\mathcal{T}_4^{out}\left[|\mu_t|t^{-1}\mathbf{1}_{\{|\bar{x}|\le t^{\frac{1}{2}}\}}\right] \lesssim t^{-2}e^{-\frac{|\bar{x}|^2}{16t}}\int_{t_0/2}^{t/2}s|\mu_t(s)|ds + \begin{cases}|\mu_t| & \text{if } |\bar{x}|\le t^{\frac{1}{2}}\\ |\mu_t|t|x|^{-2}e^{-\frac{|\bar{x}|^2}{16t}} & \text{if } |\bar{x}|>t^{\frac{1}{2}}\end{cases}.$$

By Lemma A.2, we have

$$\begin{aligned}&\mathcal{T}_4^{out}\left[|\mu_t||\bar{x}|^{-2}e^{-\frac{|\bar{x}|^2}{4t}}\mathbf{1}_{\{|\bar{x}|>t^{\frac{1}{2}}\}}\right] \lesssim \mathcal{T}_4^{out}\left[|\mu_t|t^2|\bar{x}|^{-6}\mathbf{1}_{\{|\bar{x}|>t^{\frac{1}{2}}\}}\right]\\ \lesssim& t^{-2}e^{-\frac{|\bar{x}|^2}{16t}}\int_{t_0/2}^{t/2}s|\mu_t(s)|ds + \begin{cases}|\mu_t| & \text{if } |\bar{x}|\le t^{\frac{1}{2}}\\ |\bar{x}|^{-6}(t^3|\mu_t| + \int_{t_0/2}^{t/2}|\mu_t(s)|s^2ds) & \text{if } |\bar{x}|>t^{\frac{1}{2}}\end{cases},\end{aligned}$$

and thus

$$|\mathcal{T}_4^{out}[\mu_t\hat{\varphi}_1]| \lesssim t^{-2}e^{-\frac{|\bar{x}|^2}{16t}}\int_{t_0/2}^{t/2}|\mu_t(s)|sds + \begin{cases}|\mu_t| & \text{if } |\bar{x}|\le t^{\frac{1}{2}}\\ |\bar{x}|^{-6}\int_{t_0/2}^{t}|\mu_t(s)|s^2ds & \text{if } |\bar{x}|>t^{\frac{1}{2}}\end{cases}.$$

It then follows that

$$|\tilde{\varphi}_{1b}[\mu]| \lesssim t^{-2} e^{-\frac{|\bar{x}|^2}{16t}} \int_{t_0/2}^{t/2} (s^{-1}\mu^3(s) + s|\mu_t(s)|) ds + \begin{cases} \mu^3 t^{-2} + |\mu_t| & \text{if } |\bar{x}| \le t^{\frac{1}{2}} \\ \mu^3 t^{-1} |\bar{x}|^{-2} e^{-\frac{|\bar{x}|^2}{16t}} + |\bar{x}|^{-6} \int_{t_0/2}^{t} s^2 |\mu_t(s)| ds & \text{if } |\bar{x}| > t^{\frac{1}{2}} \end{cases},$$

$$\begin{aligned} \left|\tilde{\varphi}_{1b}[\mu+\mu_1] - \tilde{\varphi}_{1b}[\mu]\right| &\lesssim t^{-2} e^{-\frac{|\bar{x}|^2}{16t}} \int_{t_0/2}^{t/2} \left(s^{-1}|\mu_1(s)|\mu^2(s) + s|\mu_{1t}(s)|\right) ds \\ &+ \begin{cases} t^{-2}\mu^2 \sup_{t_1\in[t/2,t]} |\mu_1(t_1)| + \sup_{t_1\in[t/2,t]} |\mu_{1t}(t_1)| & \text{if } |\bar{x}| \le t^{\frac{1}{2}} \\ \sup_{t_1\in[t/2,t]} |\mu_1(t_1)|\mu^2 t^{-1} |\bar{x}|^{-2} e^{-\frac{|\bar{x}|^2}{16t}} + |\bar{x}|^{-6}\Big(t^3 \sup_{t_1\in[t/2,t]} |\mu_{1t}(t_1)| + \int_{t_0/2}^{t/2} s^2 |\mu_{1t}(s)| ds\Big) & \text{if } |\bar{x}| > t^{\frac{1}{2}} \end{cases}. \end{aligned} \tag{2.11}$$

In particular, for $|\mu| \lesssim (\ln t)^{-1}$, $|\mu_t| \lesssim t^{-1}(\ln t)^{-2}$, one has

$$|\tilde{\varphi}_{1b}[\mu]| \lesssim t^{-1}(\ln t)^{-2} \mathbf{1}_{\{|\bar{x}|\le t^{\frac{1}{2}}\}} + t^2 (\ln t)^{-2} |\bar{x}|^{-6} \mathbf{1}_{\{|\bar{x}|> t^{\frac{1}{2}}\}}, \tag{2.12}$$

$$|E - \tilde{E}| \lesssim (t\ln t)^{-3} \mathbf{1}_{\{\sqrt{t}\le|\bar{x}|\le 2\sqrt{t}\}}, \quad |\mu_t \hat{\varphi}_1| \lesssim (t\ln t)^{-2} \mathbf{1}_{\{|\bar{x}|\le t^{\frac{1}{2}}\}} + t^{-1}(\ln t)^{-2} |\bar{x}|^{-2} e^{-\frac{|\bar{x}|^2}{4t}} \mathbf{1}_{\{|\bar{x}|> t^{\frac{1}{2}}\}}.$$

Then by scaling argument, we have

$$|\nabla_{\bar{x}} \tilde{\varphi}_{1b}[\mu]| \lesssim t^{-\frac{3}{2}} (\ln t)^{-2} \mathbf{1}_{\{|\bar{x}|\le t^{\frac{1}{2}}\}} + t^{\frac{3}{2}} (\ln t)^{-2} |\bar{x}|^{-6} \mathbf{1}_{\{|\bar{x}|> t^{\frac{1}{2}}\}}.$$

Combining above estimates with (2.7), we have

$$|\nabla_{\bar{x}} \varphi_1| \lesssim t^{-\frac{3}{2}} (\ln t)^{-1} \mathbf{1}_{\{|\bar{x}|\le 2t^{\frac{1}{2}}\}} + t^{\frac{3}{2}} (\ln t)^{-1} |\bar{x}|^{-6} \mathbf{1}_{\{|\bar{x}|> 2t^{\frac{1}{2}}\}}.$$

□

Lemma 2.2. *Assume that μ satisfies (2.2) and μ_1 satisfies $|\mu_1| \le \frac{\mu}{2}$, $|\mu_{1t}| \le \frac{|\mu_t|}{2}$. Consider*

$$\partial_t \varphi_2 = \Delta_{\bar{x}} \varphi_2 + \mu^{-2} \mu_t Z_5(\frac{\bar{x}}{\mu}) \eta(\frac{\bar{x}}{\sqrt{t}}),$$

where φ_2 is given by $\varphi_2 = \varphi_2[\mu] = \mathcal{T}_4^{out}\left[\mu^{-2}\mu_t Z_5(\frac{\bar{x}}{\mu})\eta(\frac{\bar{x}}{\sqrt{t}})\right]$. Then the following estimates hold

$$\begin{aligned} |\varphi_2[\mu]| &\lesssim t^{-2} e^{-\frac{|\bar{x}|^2}{16t}} \int_{t_0/2}^{t/2} s|\mu_t(s)| ds + \begin{cases} |\mu_t|(\ln(\mu^{-1}t^{\frac{1}{2}}) + 1) & \text{if } |\bar{x}| \le \mu \\ |\mu_t|(\ln(|\bar{x}|^{-1}t^{\frac{1}{2}}) + 1) & \text{if } \mu < |\bar{x}| \le t^{\frac{1}{2}} \\ |\mu_t| t |\bar{x}|^{-2} e^{-\frac{|\bar{x}|^2}{16t}} & \text{if } |\bar{x}| > t^{\frac{1}{2}} \end{cases} \\ &\lesssim \begin{cases} (t\ln t)^{-1} & \text{if } |\bar{x}| \le \mu \\ t^{-1}(\ln t)^{-2}(\ln(|\bar{x}|^{-1}t^{\frac{1}{2}}) + 1) & \text{if } \mu < |\bar{x}| \le t^{\frac{1}{2}} \\ t^{-1}(\ln t)^{-2} e^{-\frac{|\bar{x}|^2}{16t}} & \text{if } |\bar{x}| > t^{\frac{1}{2}} \end{cases}, \end{aligned} \tag{2.13}$$

$$|\nabla_{\bar{x}} \varphi_2[\mu]| \lesssim \begin{cases} t^{-1} & \text{if } |\bar{x}| \le \mu \\ t^{-1}(\ln t)^{-2}(\ln(|\bar{x}|^{-1}t^{\frac{1}{2}}) + 1)|\bar{x}|^{-1} & \text{if } \mu < |\bar{x}| \le t^{\frac{1}{2}} \\ t^{-\frac{3}{2}}(\ln t)^{-2} e^{-\frac{|\bar{x}|^2}{30t}} & \text{if } |\bar{x}| > t^{\frac{1}{2}} \end{cases}, \tag{2.14}$$

$$\begin{aligned} \left|\varphi_2[\mu+\mu_1] - \varphi_2[\mu]\right| &\lesssim t^{-2} e^{-\frac{|\bar{x}|^2}{16t}} \int_{t_0/2}^{t/2} s|\mu_t(s)| \Big(\frac{|\mu_1(s)|}{\mu(s)} + \frac{|\mu_{1t}(s)|}{|\mu_t(s)|}\Big) ds \\ &+ |\mu_t| \sup_{t_1\in[t/2,t]} \Big(\frac{|\mu_1(t_1)|}{\mu(t)} + \frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big) \begin{cases} \ln(\mu^{-1}t^{\frac{1}{2}}) + 1 & \text{if } |\bar{x}| \le \mu \\ \ln(|\bar{x}|^{-1}t^{\frac{1}{2}}) + 1 & \text{if } \mu < |\bar{x}| \le t^{\frac{1}{2}} \\ t|\bar{x}|^{-2} e^{-\frac{|\bar{x}|^2}{16t}} & \text{if } |\bar{x}| > t^{\frac{1}{2}} \end{cases}. \end{aligned} \tag{2.15}$$

More precisely,

$$\begin{aligned} \varphi_2[\mu] = &\left[-2^{-\frac{1}{2}} \int_{t/2}^{t-\mu_0^2} \frac{\mu_t(s)}{t-s} ds + O\Big(t^{-2}\int_{t_0/2}^{t} s|\mu_t(s)| ds + \min\{\mu^{-1}|\mu_t||\bar{x}|, \ln t|\mu_t|\}\Big)\right] \mathbf{1}_{\{|\bar{x}|\le 2t^{\frac{1}{2}}\}} \\ &+ O\Big(t^{-2}\int_{t_0/2}^{t} s|\mu_t(s)| ds\Big) e^{-\frac{|\bar{x}|^2}{16t}} \mathbf{1}_{\{|\bar{x}|> 2t^{\frac{1}{2}}\}}, \end{aligned}$$

$$
\begin{aligned}
&\varphi_2[\mu+\mu_1]-\varphi_2[\mu]\\
=&\Bigg[-2^{-\frac12}\int_{t/2}^{t-\mu_0^2}\frac{\mu_{1t}(s)}{t-s}ds+O\Big(|\mu_t|\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)}+\frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big)+t^{-2}\int_{t_0/2}^{t/2}s|\mu_t(s)|\Big(\frac{|\mu_1(s)|}{\mu(s)}+\frac{|\mu_{1t}(s)|}{|\mu_t(s)|}\Big)ds\\
&+|\mu_t|\ln t\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)}+\frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big)^2+\mu^{-1}|\mu_t|\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)}+\frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big)|\bar x|\Big)\Bigg]\mathbf{1}_{\{|\bar x|\le 2t^{\frac12}\}}\\
&+O\Big(|\mu_t|\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)}+\frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big)+t^{-2}\int_{t_0/2}^{t/2}s|\mu_t(s)|\Big(\frac{|\mu_1(s)|}{\mu(s)}+\frac{|\mu_{1t}(s)|}{|\mu_t(s)|}\Big)ds\Big)e^{-\frac{|\bar x|^2}{16t}}\mathbf{1}_{\{|\bar x|>2t^{\frac12}\}}.
\end{aligned}
$$

Proof. Since

$$
|\mu^{-2}\mu_t Z_5(\frac{\bar x}{\mu})\eta(\frac{\bar x}{\sqrt t})|\lesssim \mu^{-2}\mu_t\mathbf{1}_{\{|\bar x|\le\mu\}}+\mu_t|\bar x|^{-2}\mathbf{1}_{\{\mu<|\bar x|\le 2t^{\frac12}\}},\tag{2.16}
$$

by Lemma A.1 and (2.2), we conclude the validity of (2.13). By scaling argument, (2.14) follows.

For μ_1 satisfying $|\mu_1|\le\frac{\mu}{2}$ and $|\mu_{1t}|\le\frac{|\mu_t|}{2}$, we have

$$
\begin{aligned}
&(\mu+\mu_1)^{-2}(\mu_t+\mu_{1t})Z_5(\frac{\bar x}{\mu+\mu_1})-\mu^{-2}\mu_tZ_5(\frac{\bar x}{\mu})\\
=&\mu^{-2}\mu_{1t}Z_5(\frac{\bar x}{\mu})-\mu^{-3}\mu_1\mu_t\Big(2Z_5(\frac{\bar x}{\mu})+\frac{\bar x}{\mu}\cdot\nabla Z_5(\frac{\bar x}{\mu})\Big)+\big(\mu^{-3}\mu_1\mu_t+\mu^{-2}\mu_{1t}\big)\langle\frac{\bar x}{\mu}\rangle^{-2}O\Big(\frac{|\mu_1|}{\mu}+\frac{|\mu_{1t}|}{|\mu_t|}\Big)\\
=&O\Big(\mu^{-2}|\mu_t|\Big(\frac{|\mu_1|}{\mu}+\frac{|\mu_{1t}|}{|\mu_t|}\Big)\langle\frac{\bar x}{\mu}\rangle^{-2}\Big).
\end{aligned}\tag{2.17}
$$

Then by Lemma A.1, one gets (2.15).

In order to extract the dominating part of φ_2 for the preparation of solving the orthogonal equation, we split φ_2 into several parts to estimate. Set $\mu_0(t)=(\ln t)^{-1}$ and consider

$$
\begin{aligned}
\varphi_2=&\Big(\int_{t_0}^{t/2}+\int_{t/2}^{t-\mu_0^2(t)}+\int_{t-\mu_0^2(t)}^{t}\Big)\int_{\mathbb R^4}(4\pi(t-s))^{-2}e^{-\frac{|x-z|^2}{4(t-s)}}\mu^{-2}(s)\mu_t(s)Z_5(\frac{|z|}{\mu(s)})\eta(\frac{|z|}{\sqrt s})dzds\\
:=&I_1+I_2+I_3.
\end{aligned}
$$

For I_1, by rearrangement inequality, we have

$$
\begin{aligned}
|I_1|\lesssim&\int_{t_0/2}^{t/2}\int_{\mathbb R^4}(t-s)^{-2}e^{-\frac{|x-z|^2}{4(t-s)}}\mu^{-2}(s)|\mu_t(s)|\langle\frac{|z|}{\mu(s)}\rangle^{-2}\eta(\frac{|z|}{\sqrt s})dzds\\
\lesssim&t^{-2}\int_{t_0/2}^{t/2}\int_{\mathbb R^4}e^{-\frac{|z|^2}{4t}}\mu^{-2}(s)|\mu_t(s)|\langle\frac{|z|}{\mu(s)}\rangle^{-2}\eta(\frac{|z|}{\sqrt s})dzds\lesssim t^{-2}\int_{t_0/2}^{t/2}\mu^{-2}(s)|\mu_t(s)|\int_0^{2\sqrt s}\langle\frac{r}{\mu(s)}\rangle^{-2}r^3drds\\
\lesssim&t^{-2}\int_{t_0/2}^{t/2}s|\mu_t(s)|ds
\end{aligned}
$$

since $\int_0^{\mu(s)}\langle\frac{r}{\mu(s)}\rangle^{-2}r^3dr\sim\mu^4(s)$, $\quad\int_{\mu(s)}^{2\sqrt s}\langle\frac{r}{\mu(s)}\rangle^{-2}r^3dr\lesssim s\mu^2(s)$.

Using (2.17) and similar calculations above, one has

$$
\big|I_1[\mu+\mu_1]-I_1[\mu]\big|\lesssim t^{-2}\int_{t_0/2}^{t/2}s|\mu_t(s)|\Big(\frac{|\mu_1(s)|}{\mu(s)}+\frac{|\mu_{1t}(s)|}{|\mu_t(s)|}\Big)ds.
$$

For I_3, we have

$$
\begin{aligned}
|I_3|\lesssim&\int_{t-\mu_0^2(t)}^{t}\int_{\mathbb R^4}(t-s)^{-2}e^{-\frac{|z|^2}{4(t-s)}}\mu^{-2}(s)|\mu_t(s)|\langle\frac{|z|}{\mu(s)}\rangle^{-2}\eta(\frac{|z|}{\sqrt s})dzds\\
\lesssim&\mu^{-2}|\mu_t|\int_{t-\mu_0^2(t)}^{t}(t-s)^{-2}\int_0^{2\sqrt t}e^{-\frac{r^2}{4(t-s)}}\langle\frac{r}{\mu(t)}\rangle^{-2}r^3drds\lesssim|\mu_t|
\end{aligned}
$$

since for $s\in(t-\mu_0^2(t),t)$,

$$
\begin{aligned}
&\int_0^{\mu(t)}e^{-\frac{r^2}{4(t-s)}}\langle\frac{r}{\mu(t)}\rangle^{-2}r^3dr\sim\int_0^{\mu(t)}e^{-\frac{r^2}{4(t-s)}}r^3dr\sim(t-s)^2\int_0^{\frac{\mu^2(t)}{4(t-s)}}e^{-z}zdz\sim(t-s)^2,\\
&\int_{\mu(t)}^{2\sqrt t}e^{-\frac{r^2}{4(t-s)}}\langle\frac{r}{\mu(t)}\rangle^{-2}r^3dr\sim(t-s)\mu^2(t)\int_{\frac{\mu^2(t)}{4(t-s)}}^{\frac{t}{t-s}}e^{-z}dz\lesssim(t-s)\mu^2(t)e^{-\frac{\mu^2(t)}{4(t-s)}}\lesssim(t-s)^2.
\end{aligned}
$$

Similarly, using (2.17), one has

$$\big|I_3[\mu+\mu_1]-I_3[\mu]\big|\lesssim|\mu_t|\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)}+\frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big).$$

For I_2, more delicate calculations are needed to single out the leading term. Set

$$I_2=I_{02}+(I_2-I_{02}),$$

where

$$I_{02}=\int_{t/2}^{t-\mu_0^2(t)}\int_{\mathbb{R}^4}(4\pi(t-s))^{-2}e^{-\frac{|z|^2}{4(t-s)}}\mu^{-2}(s)\mu_t(s)Z_5(\frac{|z|}{\mu(s)})\eta(\frac{|z|}{\sqrt{s}})dzds:=I_*+I_{021}+I_{022}$$

and

$$\begin{aligned}I_*&=-2^{\frac32}\int_{t/2}^{t-\mu_0^2(t)}\int_{\mathbb{R}^4}(4\pi(t-s))^{-2}e^{-\frac{|z|^2}{4(t-s)}}\mu^{-2}(s)\mu_t(s)\frac{\mu^2(s)}{|z|^2}dzds,\\ I_{021}&=2^{\frac32}\int_{t/2}^{t-\mu_0^2(t)}\int_{\mathbb{R}^4}(4\pi(t-s))^{-2}e^{-\frac{|z|^2}{4(t-s)}}\mu^{-2}(s)\mu_t(s)\frac{\mu^2(s)}{|z|^2}\Big(1-\eta(\frac{|z|}{\sqrt{s}})\Big)dzds,\\ I_{022}&=\int_{t/2}^{t-\mu_0^2(t)}\int_{\mathbb{R}^4}(4\pi(t-s))^{-2}e^{-\frac{|z|^2}{4(t-s)}}\mu^{-2}(s)\mu_t(s)\Big(Z_5(\frac{|z|}{\mu(s)})+2^{\frac32}\frac{\mu^2(s)}{|z|^2}\Big)\eta(\frac{|z|}{\sqrt{s}})dzds.\end{aligned}$$

For I_*, we evaluate

$$I_*=-2^{\frac32}|S^3|\int_{t/2}^{t-\mu_0^2(t)}\mu_t(s)\int_0^\infty(4\pi(t-s))^{-2}e^{-\frac{r^2}{4(t-s)}}rdrds=-2^{-\frac12}\int_{t/2}^{t-\mu_0^2(t)}\frac{\mu_t(s)}{t-s}ds.$$

In the same way, one has

$$I_*[\mu+\mu_1]-I_*[\mu]=-2^{-\frac12}\int_{t/2}^{t-\mu_0^2(t)}\frac{\mu_{1t}(s)}{t-s}ds.$$

For I_{021}, we get

$$|I_{021}|\lesssim|\mu_t|\int_{\frac t2}^{t-\mu_0^2(t)}\int_{\frac{\sqrt t}{2}}^\infty(t-s)^{-2}e^{-\frac{r^2}{4(t-s)}}rdrds\lesssim|\mu_t|,\quad \big|I_{021}[\mu+\mu_1]-I_{021}[\mu]\big|\lesssim\sup_{t_1\in[t/2,t]}|\mu_{1t}(t_1)|.$$

For I_{022}, we have

$$\begin{aligned}|I_{022}|&\lesssim\int_{t/2}^{t-\mu_0^2}\int_{\mathbb{R}^4}(t-s)^{-2}e^{-\frac{|z|^2}{4(t-s)}}\mu^{-2}(s)|\mu_t(s)|(\frac{|z|}{\mu(s)})^{-2}\langle\frac{|z|}{\mu(s)}\rangle^{-2}\eta(\frac{|z|}{\sqrt{s}})dzds\\ &\lesssim|\mu_t|\int_{t/2}^{t-\mu_0^2}\int_{\mathbb{R}^4}(t-s)^{-2}e^{-\frac{|z|^2}{4(t-s)}}|z|^{-2}\langle\frac{|z|}{\mu(t)}\rangle^{-2}\eta(\frac{|z|}{2\sqrt{t}})dzds\\ &\lesssim|\mu_t|\int_{t/2}^{t-\mu_0^2}(t-s)^{-2}\int_0^{4\sqrt t}e^{-\frac{r^2}{4(t-s)}}\langle\frac{r}{\mu(t)}\rangle^{-2}rdrds\lesssim|\mu_t|\mu^2\int_{t/2}^{t-\mu_0^2}(t-s)^{-2}\Big(1-\ln(\frac{\mu^2(t)}{4(t-s)})\Big)ds\lesssim|\mu_t|\end{aligned}$$

since for $\frac t2\le s\le t-\mu^2(t)$,

$$\int_0^{\mu(t)}e^{-\frac{r^2}{4(t-s)}}\langle\frac{r}{\mu(t)}\rangle^{-2}rdr\lesssim\mu^2(t),$$

$$\int_{\mu(t)}^{4\sqrt t}e^{-\frac{r^2}{4(t-s)}}\langle\frac{r}{\mu(t)}\rangle^{-2}rdr\sim\mu^2(t)\int_{\mu(t)}^{4\sqrt t}e^{-\frac{r^2}{4(t-s)}}r^{-1}dr\sim\mu^2(t)\int_{\frac{\mu^2(t)}{4(t-s)}}^{\frac{4t}{t-s}}e^{-z}z^{-1}dz\lesssim\mu^2(t)\Big(1-\ln(\frac{\mu^2(t)}{4(t-s)})\Big).$$

Next, we estimate $I_{022}[\mu+\mu_1]-I_{022}[\mu]$. By (2.17), we have

$$\begin{aligned}&(\mu+\mu_1)^{-2}(\mu_t+\mu_{1t})\left(Z_5(\frac{|z|}{\mu+\mu_1})+2^{\frac32}\frac{(\mu+\mu_1)^2}{|z|^2}\right)-\mu^{-2}\mu_t\left(Z_5(\frac{|z|}{\mu})+2^{\frac32}\frac{\mu^2}{|z|^2}\right)\\ &=\mu^{-2}\mu_{1t}Z_5(\frac{z}{\mu})+2^{\frac32}\frac{\mu_{1t}}{|z|^2}-\mu^{-3}\mu_1\mu_t\Big(2Z_5(\frac{z}{\mu})+\frac{z}{\mu}\cdot\nabla Z_5(\frac{z}{\mu})\Big)+\big(\mu^{-3}\mu_1\mu_t+\mu^{-2}\mu_{1t}\big)\langle\frac{z}{\mu}\rangle^{-2}O\Big(\frac{|\mu_1|}{\mu}+\frac{|\mu_{1t}|}{|\mu_t|}\Big)\\ &=O\Big(\mu^{-2}|\mu_{1t}|\frac{|z|^{-2}}{\mu^{-2}}\langle\frac{z}{\mu}\rangle^{-2}\Big)-\mu^{-3}\mu_1\mu_t\Big(2Z_5(\frac{z}{\mu})+\frac{z}{\mu}\cdot\nabla Z_5(\frac{z}{\mu})\Big)+\big(\mu^{-3}\mu_1\mu_t+\mu^{-2}\mu_{1t}\big)\langle\frac{z}{\mu}\rangle^{-2}O\Big(\frac{|\mu_1|}{\mu}+\frac{|\mu_{1t}|}{|\mu_t|}\Big)\\ &=O\Big(\mu^{-2}|\mu_t|\Big(\frac{|\mu_1|}{\mu}+\frac{|\mu_{1t}|}{|\mu_t|}\Big)\frac{|z|^{-2}}{\mu^{-2}}\langle\frac{z}{\mu}\rangle^{-2}\Big)+O\Big(\mu^{-2}|\mu_t|\Big(\frac{|\mu_1|}{\mu}+\frac{|\mu_{1t}|}{|\mu_t|}\Big)^2\langle\frac{z}{\mu}\rangle^{-2}\Big).\end{aligned}$$

Similar to the estimates of I_{022}, we then have

$$\int_{t/2}^{t-\mu_0^2}\int_{\mathbb{R}^4}(4\pi(t-s))^{-2}e^{-\frac{|z|^2}{4(t-s)}}\mu^{-2}(s)|\mu_t(s)|\Big(\frac{|\mu_1(s)|}{\mu(s)}+\frac{|\mu_{1t}(s)|}{|\mu_t(s)|}\Big)\frac{|z|^{-2}}{\mu^{-2}(s)}\langle\frac{z}{\mu(s)}\rangle^{-2}\eta(\frac{|z|}{\sqrt{s}})dzds$$
$$\lesssim|\mu_t|\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)}+\frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big),$$

$$\int_{t/2}^{t-\mu_0^2}\int_{\mathbb{R}^4}(4\pi(t-s))^{-2}e^{-\frac{|z|^2}{4(t-s)}}\mu^{-2}(s)|\mu_t(s)|\Big(\frac{|\mu_1(s)|}{\mu(s)}+\frac{|\mu_{1t}(s)|}{|\mu_t(s)|}\Big)^2\langle\frac{z}{\mu(s)}\rangle^{-2}\eta(\frac{|z|}{\sqrt{s}})dzds$$
$$\lesssim\mu^{-2}|\mu_t|\Big(\frac{|\mu_1|}{\mu}+\frac{|\mu_{1t}|}{|\mu_t|}\Big)^2\int_{t/2}^{t-\mu_0^2}(t-s)^{-2}\int_0^{2t^{\frac12}}e^{-\frac{r^2}{4(t-s)}}\langle\frac{r}{\mu(t)}\rangle^{-2}r^3drds$$
$$\lesssim|\mu_t|\ln t\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)}+\frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big)^2$$

since

$$\int_0^{\mu(t)}e^{-\frac{r^2}{4(t-s)}}\langle\frac{r}{\mu(t)}\rangle^{-2}r^3dr\lesssim\mu^4,\quad\int_{\mu(t)}^{2t^{\frac12}}e^{-\frac{r^2}{4(t-s)}}\langle\frac{r}{\mu(t)}\rangle^{-2}r^3dr\sim\mu^2\int_{\mu(t)}^{2t^{\frac12}}e^{-\frac{r^2}{4(t-s)}}rdr\lesssim\mu^2(t-s).\tag{2.18}$$

Therefore, one has

$$\big|I_{022}[\mu+\mu_1]-I_{022}[\mu]\big|\lesssim|\mu_t|\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)}+\frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big)+|\mu_t|\ln t\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)}+\frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big)^2.$$

Let us now estimate I_2-I_{02}

$$|I_2-I_{02}|=\Big|\int_{t/2}^{t-\mu_0^2(t)}\int_{\mathbb{R}^4}\int_0^1(4\pi(t-s))^{-2}e^{-\frac{|\theta x-z|^2}{4(t-s)}}\frac{\theta x-z}{2(t-s)}\cdot x\mu^{-2}(s)\mu_t(s)Z_5(\frac{|z|}{\mu(s)})\eta(\frac{|z|}{\sqrt{s}})d\theta dzds\Big|$$
$$\lesssim\mu^{-2}|\mu_t||x|\int_{t/2}^{t-\mu_0^2(t)}\int_{\mathbb{R}^4}\int_0^1(t-s)^{-\frac52}e^{-\frac{|\theta x-z|^2}{8(t-s)}}\langle\frac{|z|}{\mu(s)}\rangle^{-2}\eta(\frac{|z|}{\sqrt{s}})d\theta dzds$$
$$\lesssim\mu^{-2}|\mu_t||x|\int_{t/2}^{t-\mu_0^2(t)}\int_{\mathbb{R}^4}(t-s)^{-\frac52}e^{-\frac{|z|^2}{8(t-s)}}\langle\frac{|z|}{\mu(s)}\rangle^{-2}\eta(\frac{|z|}{\sqrt{s}})dzds$$
$$\lesssim\mu^{-2}|\mu_t||x|\int_{t/2}^{t-\mu_0^2(t)}(t-s)^{-\frac52}\int_0^{2\sqrt{t}}e^{-\frac{r^2}{8(t-s)}}\langle\frac{r}{\mu(t)}\rangle^{-2}r^3drds\lesssim\mu^{-1}|\mu_t||\bar{x}|$$

since $\int_0^{2\sqrt{t}}e^{-\frac{r^2}{8(t-s)}}\langle\frac{r}{\mu(t)}\rangle^{-2}r^3dr\lesssim\mu^2(t-s)$ by similar estimate in (2.18).

Using rearrangement inequality, one has another upper bound for $|I_2-I_{02}|$,

$$|I_2-I_{02}|\lesssim\int_{t/2}^{t-\mu_0^2(t)}\int_{\mathbb{R}^4}(t-s)^{-2}e^{-\frac{|z|^2}{4(t-s)}}\mu^{-2}(s)|\mu_t(s)|\langle\frac{|z|}{\mu(s)}\rangle^{-2}\eta(\frac{|z|}{\sqrt{s}})dzds$$
$$\lesssim\mu^{-2}|\mu_t|\int_{t/2}^{t-\mu_0^2(t)}(t-s)^{-2}\int_0^{2t^{\frac12}}e^{-\frac{r^2}{4(t-s)}}\langle\frac{r}{\mu(t)}\rangle^{-2}r^3drds\lesssim|\mu_t(t)|\int_{t/2}^{t-\mu_0^2(t)}(t-s)^{-1}ds\lesssim\ln t|\mu_t|.$$

Thus

$$|I_2-I_{02}|\lesssim\min\big\{\mu^{-1}|\mu_t||\bar{x}|,\ln t|\mu_t|\big\}.$$

Using (2.17) and similar calculations, one has

$$\big|(I_2-I_{02})[\mu+\mu_1]-(I_2-I_{02})[\mu]\big|\lesssim\mu^{-1}|\mu_t|\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)}+\frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big)|\bar{x}|.$$

Combining all the estimates above, we conclude the validity of Lemma 2.2.

□

Recalling $\varphi[\mu]=\varphi_1[\mu]+\varphi_2[\mu]$ and combining Lemma 2.1 and Lemma 2.2, one has

Corollary 2.3. *Assume that μ satisfies (2.2) and μ_1 satisfies $|\mu_1| \le \frac{\mu}{2}$, $|\mu_{1t}| \le \frac{|\mu_t|}{2}$. We have*

$$
\begin{aligned}
|\varphi[\mu]| &\lesssim (\mu t^{-1} + g[\mu])\mathbf{1}_{\{|\bar{x}|\le 2t^{\frac12}\}} + \begin{cases} |\mu_t|(\ln(\mu^{-1}t^{\frac12})+1) & \text{if } |\bar{x}|\le \mu \\ |\mu_t|(\ln(|\bar{x}|^{-1}t^{\frac12})+1) & \text{if } \mu<|\bar{x}|\le t^{\frac12} \\ t|\mu_t||\bar{x}|^{-2}e^{-\frac{|\bar{x}|^2}{16t}} & \text{if } |\bar{x}|>t^{\frac12}\end{cases} \\
&\quad + O\Big(\mu|\bar{x}|^{-2}e^{-\frac{|\bar{x}|^2}{16t}} + |\bar{x}|^{-6}\int_{t_0/2}^t s^2|\mu_t(s)|ds + g[\mu]e^{-\frac{|\bar{x}|^2}{16t}}\Big)\mathbf{1}_{\{|\bar{x}|>2t^{\frac12}\}} \\
&\lesssim (t\ln t)^{-1}\mathbf{1}_{\{|\bar{x}|\le 2t^{\frac12}\}} + O(t^2(\ln t)^{-1}|\bar{x}|^{-6})\mathbf{1}_{\{|\bar{x}|>2t^{\frac12}\}}
\end{aligned}
$$

where

$$
g[\mu] = O\Big(t^{-2}\int_{t_0/2}^t (s^{-1}\mu^3(s) + s|\mu_t(s)|)ds\Big).
$$

$$
|\nabla_{\bar{x}}\varphi[\mu]| \lesssim \begin{cases} t^{-1} & \text{if } |\bar{x}|\le\mu \\ t^{-1}(\ln t)^{-2}(\ln(|\bar{x}|^{-1}t^{\frac12})+1)|\bar{x}|^{-1} + t^{-\frac32}(\ln t)^{-1} & \text{if } \mu<|\bar{x}|\le t^{\frac12} \\ t^{\frac32}(\ln t)^{-1}|\bar{x}|^{-6} & \text{if } |\bar{x}|>t^{\frac12}\end{cases}.
$$

$$
\begin{aligned}
&\big|\varphi[\mu+\mu_1]-\varphi[\mu]\big| \lesssim \big(O(|\mu_1|t^{-1}) + \tilde{g}[\mu,\mu_1]\big)\,\mathbf{1}_{\{|\bar{x}|\le 2t^{\frac12}\}} \\
&+ \sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)} + \frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big)\begin{cases} |\mu_t|(\ln(\mu^{-1}t^{\frac12})+1) & \text{if } |\bar{x}|\le\mu \\ |\mu_t|(\ln(|\bar{x}|^{-1}t^{\frac12})+1) & \text{if } \mu<|\bar{x}|\le t^{\frac12} \\ t|\mu_t||\bar{x}|^{-2}e^{-\frac{|\bar{x}|^2}{16t}} & \text{if } |\bar{x}|>t^{\frac12}\end{cases} \\
&+ O\Big(\sup_{t_1\in[t/2,t]}|\mu_1(t_1)||\bar{x}|^{-2}e^{-\frac{|\bar{x}|^2}{16t}} + |\bar{x}|^{-6}\Big(t^3\sup_{t_1\in[t/2,t]}|\mu_{1t}(t_1)| + \int_{t_0/2}^t s^2|\mu_{1t}(s)|ds\Big) + \tilde{g}[\mu,\mu_1]e^{-\frac{|\bar{x}|^2}{16t}}\Big)\mathbf{1}_{\{|\bar{x}|>2t^{\frac12}\}}
\end{aligned}
$$

where

$$
\begin{aligned}
&\tilde{g}[\mu,\mu_1] = O\Big(|\mu_t|\ln t\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)} + \frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big)^2\Big) \\
&\quad + O\Big(|\mu_t|\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)} + \frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big) + t^{-2}\int_{t_0/2}^t\Big(s^{-1}|\mu_1(s)|\mu^2(s) + s|\mu_t(s)|\Big(\frac{|\mu_1(s)|}{\mu(s)} + \frac{|\mu_{1t}(s)|}{|\mu_t(s)|}\Big)\Big)ds\Big).
\end{aligned}
$$

More precisely,

$$
\begin{aligned}
\varphi[\mu] = &\Big[-2^{-\frac12}\Big(\mu t^{-1} + \int_{t/2}^{t-\mu_0^2}\frac{\mu_t(s)}{t-s}ds\Big) + O\big(\mu t^{-2}|\bar{x}|^2 + |\mu_t|\min\{\frac{|\bar{x}|}{\mu},\ln t\}\big) + g[\mu]\Big]\mathbf{1}_{\{|\bar{x}|\le 2t^{\frac12}\}} \\
&+ O\Big(\mu|\bar{x}|^{-2}e^{-\frac{|\bar{x}|^2}{16t}} + |\bar{x}|^{-6}\int_{t_0/2}^t s^2|\mu_t(s)|ds + g[\mu]e^{-\frac{|\bar{x}|^2}{16t}}\Big)\mathbf{1}_{\{|\bar{x}|>2t^{\frac12}\}},
\end{aligned}
$$

$$
\begin{aligned}
&\varphi[\mu+\mu_1]-\varphi[\mu] = \Big[-2^{-\frac12}\Big(\mu_1 t^{-1} + \int_{t/2}^{t-\mu_0^2}\frac{\mu_{1t}(s)}{t-s}ds\Big) \\
&+ O\Big(|\mu_1|t^{-2}|\bar{x}|^2 + |\mu_t|\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)} + \frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big)\frac{|\bar{x}|}{\mu}\Big) + \tilde{g}[\mu,\mu_1]\Big]\mathbf{1}_{\{|\bar{x}|\le 2t^{\frac12}\}} \\
&+ O\Big(\sup_{t_1\in[t/2,t]}|\mu_1(t_1)||\bar{x}|^{-2}e^{-\frac{|\bar{x}|^2}{16t}} + |\bar{x}|^{-6}\Big(t^3\sup_{t_1\in[t/2,t]}|\mu_{1t}(t_1)| + \int_{t_0/2}^{t/2}s^2|\mu_{1t}(s)|ds\Big) + e^{-\frac{|\bar{x}|^2}{16t}}\tilde{g}[\mu,\mu_1]\Big)\mathbf{1}_{\{|\bar{x}|>2t^{\frac12}\}}.
\end{aligned}
$$

In order to extract the leading term, we will use the precise version of $\varphi[\mu]$ and $\varphi[\mu+\mu_1]-\varphi[\mu]$ when calculating the orthogonal equation. In other cases, we are inclined to adopt the rougher upper bound.

With introduction of the correction term φ, the new error is given by

$$
\begin{aligned}
S[u_1+\varphi[\mu]] = &\, 3u_1^2\varphi[\mu] + 3u_1\varphi^2[\mu] + \varphi^3[\mu] + \xi_t\cdot\nabla_{\bar{x}}\varphi[\mu](x-\xi,t) \\
&+ \mu^{-2}\xi_t\cdot\nabla w(\frac{x-\xi}{\mu})\eta(\frac{x-\xi}{\sqrt{t}}) + \mu^{-1}t^{-\frac12}\xi_t\cdot\nabla\eta(\frac{x-\xi}{\sqrt{t}})w(\frac{x-\xi}{\mu}).
\end{aligned}
$$

2.3. **Further improvement by solving an elliptic equation.** In order to find suitable parameters to design the topology for solving inner-outer gluing system and the orthogonal equation, we will use the corresponding linearized elliptic equation to cut off the error term so that the time decay rate will be improved.

Set the correction term as

$$\bar{\mu}_0^{-1}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)$$

where $\bar{\mu}_0$ is the leading order of μ to be determined later. Formally speaking, Φ_0 will be chosen to satisfy the following equation

$$\begin{aligned}\Delta_y\Phi_0+3w^2(y)\Phi_0\approx&-\mu^3\Big(3u_1^2\varphi[\mu](\bar{x},t)+3u_1\varphi^2[\mu](\bar{x},t)\Big)\\=&-3\mu\Big(w^2(y)\eta^2(\frac{\mu y}{\sqrt{t}})\varphi[\mu](\mu y,t)+\mu w(y)\eta(\frac{\mu y}{\sqrt{t}})\varphi^2[\mu](\mu y,t)\Big).\end{aligned}\tag{2.19}$$

Set

$$\begin{aligned}\mathcal{M}[\mu]:=&\int_{\mathbb{R}^4}\Big(w^2(y)\eta^2(\frac{\mu y}{\sqrt{t}})\varphi[\mu](\mu y,t)+\mu w(y)\eta(\frac{\mu y}{\sqrt{t}})\varphi^2[\mu](\mu y,t)\Big)Z_5(y)dy\\=&\mu^{-4}\int_{\mathbb{R}^4}\Big(w^2(\frac{\bar{x}}{\mu})Z_5(\frac{\bar{x}}{\mu})\eta^2(\frac{\bar{x}}{\sqrt{t}})\varphi[\mu](\bar{x},t)+\mu w(\frac{\bar{x}}{\mu})Z_5(\frac{\bar{x}}{\mu})\eta(\frac{\bar{x}}{\sqrt{t}})\varphi^2[\mu](\bar{x},t)\Big)d\bar{x}.\end{aligned}$$

In order to find Φ_0 with fast spatial decay, we aim to find $\bar{\mu}_0$ as the leading order of μ such that $\mathcal{M}[\mu]\approx 0$. In other words, above orthogonality condition is satisfied at leading order for careful choice of $\bar{\mu}_0$, which will be adjusted and corrected several times in order to further improve the time decay, and we shall see that

$$\bar{\mu}_0\sim(\ln t)^{-1}.$$

The iteration of finding proper $\bar{\mu}_0$ consists of three steps:

- the first step is to single out the leading part in above orthogonal equation, and this results in the blow-up rate predicted in [15],
- the second step is to add next-order correction of the scaling parameter,
- the last step is to iterate the second step finitely many times such that the new error has sufficiently fast time decay.

We now start the iteration.

Step 1. Finding the leading part μ_0.

Using the precise expression of $\varphi[\mu]$ in Corollary 2.3, one has

$$\begin{aligned}&\int_{\mathbb{R}^4}w^2(y)Z_5(y)\eta^2(\frac{\mu y}{\sqrt{t}})\varphi(\mu y,t)dy\\=&-2^{-\frac12}\Big(\mu t^{-1}+\int_{t/2}^{t-\mu_0^2}\frac{\mu_t(s)}{t-s}ds\Big)\int_{\mathbb{R}^4}w^2(y)Z_5(y)\eta^2(\frac{\mu y}{\sqrt{t}})dy+O\big(\mu^3t^{-2}\ln(\mu^{-1}t^{\frac12})\big)+O(|\mu_t|)+g[\mu],\end{aligned}$$

and

$$\begin{aligned}\mu\int_{\mathbb{R}^4}w(y)\eta(\frac{\mu y}{\sqrt{t}})\varphi^2(\mu y,t)Z_5(y)dy=&\mu\int_{\mathbb{R}^4}w(y)\eta(\frac{\mu y}{\sqrt{t}})Z_5(y)O\big(\mu^2t^{-2}+g^2[\mu]+|\mu_t|^2(\ln(\mu^{-1}t^{\frac12}))^2\big)dy\\=&\mu\ln(\mu^{-1}t^{\frac12})O\big(\mu^2t^{-2}+g^2[\mu]+|\mu_t|^2(\ln(\mu^{-1}t^{\frac12}))^2\big).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\mathcal{M}[\mu]=&-2^{-\frac12}\int_{\mathbb{R}^4}w^2(y)Z_5(y)\eta^2(\frac{\mu y}{\sqrt{t}})dy\Big(\mu t^{-1}+\int_{t/2}^{t-\mu_0^2}\frac{\mu_t(s)}{t-s}ds\\&+O(|\mu_t|)+g[\mu]+\mu\ln(\mu^{-1}t^{\frac12})O\big(\mu^2t^{-2}+g^2[\mu]+|\mu_t|^2(\ln(\mu^{-1}t^{\frac12}))^2\big)\Big)\end{aligned}$$

where $\int_{\mathbb{R}^4}w^2(y)Z_5(y)\eta^2(\frac{\mu y}{\sqrt{t}})dy<0$ when t is large. Balancing the following two leading terms

$$\mu t^{-1}+\int_{t/2}^{t-\mu_0^2}\frac{\mu_t(s)}{t-s}ds\sim\mu t^{-1}+\mu_t(t)\int_{t/2}^{t-\mu_0^2}\frac{1}{t-s}ds\sim\mu t^{-1}+\mu_t\ln t=0,$$

one gets $\mu_0=(\ln t)^{-1}$ as the leading order of μ. Notice that

$$\mu_0 t^{-1}+\int_{t/2}^{t-\mu_0^2}\frac{\mu_{0t}(s)}{t-s}ds=(t\ln t)^{-1}-t^{-1}\int_{1/2}^{1-t^{-1}(\ln t)^{-2}}\frac{(\ln t+\ln z)^{-2}}{z(1-z)}dz$$
$$=(t\ln t)^{-1}-t^{-1}(\ln t)^{-2}(1+O((\ln t)^{-1}))\int_{1/2}^{1-t^{-1}(\ln t)^{-2}}\frac{1}{z(1-z)}dz$$
$$=(t\ln t)^{-1}-t^{-1}(\ln t)^{-2}(1+O((\ln t)^{-1}))\ln(t(\ln t)^2-1)=O(t^{-1}(\ln t)^{-2}\ln\ln t),$$

$$O(|\mu_{0t}|)+g[\mu_0]+\mu_0\ln(\mu_0^{-1}t^{\frac12})O\big(\mu_0^2t^{-2}+|\mu_{0t}|^2(\ln(\mu_0^{-1}t^{\frac12}))^2+g^2[\mu_0]\big)=O(t^{-1}(\ln t)^{-2}),$$

and thus

$$\mathcal{M}[\mu_0]=O(t^{-1}(\ln t)^{-2}\ln\ln t).$$

Step 2. Finding the corrected term μ_{01}.

In order to improve the time decay of the error, we introduce the next order term μ_{01} and make the ansatz $|\mu_{01}|\ll\mu_0$, $|\mu_{01t}|\ll|\mu_{0t}|$.

Then by Corollary 2.3, we estimate

$$\begin{aligned}
&\mathcal{M}[\mu_0+\mu_{01}]\\
=&\int_{\mathbb{R}^4}\Big((\mu_0+\mu_{01})^{-4}w^2(\frac{\bar x}{\mu_0+\mu_{01}})Z_5(\frac{\bar x}{\mu_0+\mu_{01}})\eta^2(\frac{\bar x}{\sqrt t})\varphi[\mu_0+\mu_{01}](\bar x,t)\\
&+(\mu_0+\mu_{01})^{-3}w(\frac{\bar x}{\mu_0+\mu_{01}})Z_5(\frac{\bar x}{\mu_0+\mu_{01}})\eta(\frac{\bar x}{\sqrt t})\varphi^2[\mu_0+\mu_{01}](\bar x,t)\Big)d\bar x\\
=&\int_{\mathbb{R}^4}\Big\{\Big[(\mu_0+\mu_{01})^{-4}w^2(\frac{\bar x}{\mu_0+\mu_{01}})Z_5(\frac{\bar x}{\mu_0+\mu_{01}})-\mu_0^{-4}w^2(\frac{\bar x}{\mu_0})Z_5(\frac{\bar x}{\mu_0})\Big]\eta^2(\frac{\bar x}{\sqrt t})\varphi[\mu_0+\mu_{01}](\bar x,t)\\
&+\mu_0^{-4}w^2(\frac{\bar x}{\mu_0})Z_5(\frac{\bar x}{\mu_0})\eta^2(\frac{\bar x}{\sqrt t})(\varphi[\mu_0+\mu_{01}]-\varphi[\mu_0])(\bar x,t)+\mu_0^{-4}w^2(\frac{\bar x}{\mu_0})Z_5(\frac{\bar x}{\mu_0})\eta^2(\frac{\bar x}{\sqrt t})\varphi[\mu_0](\bar x,t)\\
&+\Big[(\mu_0+\mu_{01})^{-3}w(\frac{\bar x}{\mu_0+\mu_{01}})Z_5(\frac{\bar x}{\mu_0+\mu_{01}})-\mu_0^{-3}w(\frac{\bar x}{\mu_0})Z_5(\frac{\bar x}{\mu_0})\Big]\eta(\frac{\bar x}{\sqrt t})\varphi^2[\mu_0+\mu_{01}](\bar x,t)\\
&+\mu_0^{-3}w(\frac{\bar x}{\mu_0})Z_5(\frac{\bar x}{\mu_0})\eta(\frac{\bar x}{\sqrt t})(\varphi^2[\mu_0+\mu_{01}]-\varphi^2[\mu_0])(\bar x,t)+\mu_0^{-3}w(\frac{\bar x}{\mu_0})Z_5(\frac{\bar x}{\mu_0})\eta(\frac{\bar x}{\sqrt t})\varphi^2[\mu_0](\bar x,t)\Big\}d\bar x\\
=&\int_{\mathbb{R}^4}\Big\{O\Big(\frac{|\mu_{01}|}{\mu_0}\mu_0^{-4}\langle\frac{\bar x}{\mu_0}\rangle^{-6}\Big)\eta^2(\frac{\bar x}{\sqrt t})\Big[-2^{-\frac12}\Big((\mu_0+\mu_{01})t^{-1}+\int_{t/2}^{t-\mu_0^2}\frac{\mu_{0t}(s)+\mu_{01t}(s)}{t-s}ds\Big)\\
&+O\Big((\mu_0+\mu_{01})t^{-2}|\bar x|^2+|\mu_{0t}+\mu_{01t}|\frac{|\bar x|}{\mu_0+\mu_{01}}\Big)+g[\mu_0+\mu_{01}]\Big]\\
&+\mu_0^{-4}w^2(\frac{\bar x}{\mu_0})Z_5(\frac{\bar x}{\mu_0})\eta^2(\frac{\bar x}{\sqrt t})\Big[-2^{-\frac12}\Big(\mu_{01}t^{-1}+\int_{t/2}^{t-\mu_0^2}\frac{\mu_{01t}(s)}{t-s}ds\Big)\\
&+O\Big(|\mu_{01}|t^{-2}|\bar x|^2+|\mu_{0t}|\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_{01}(t_1)|}{\mu_0}+\frac{|\mu_{01t}(t_1)|}{|\mu_{0t}|}\Big)\frac{|\bar x|}{\mu_0}\Big)+\tilde g[\mu_0,\mu_{01}]\Big]\\
&+\frac{|\mu_{01}|}{\mu_0}\mu_0^{-3}\langle\frac{\bar x}{\mu_0}\rangle^{-4}\eta(\frac{\bar x}{\sqrt t})O((t\ln t)^{-2})+\mu_0^{-3}w(\frac{\bar x}{\mu_0})Z_5(\frac{\bar x}{\mu_0})\eta(\frac{\bar x}{\sqrt t})O((t\ln t)^{-2})\Big\}d\bar x+\mathcal{M}[\mu_0]\\
=&\int_{\mathbb{R}^4}\Big\{O\Big(\frac{|\mu_{01}|}{\mu_0}\langle\frac{\bar x}{\mu_0}\rangle^{-6}\Big)\eta^2(\frac{\bar x}{\sqrt t})\Big[-2^{-\frac12}\Big(\mu_{01}t^{-1}+\int_{t/2}^{t-\mu_0^2}\frac{\mu_{01t}(s)}{t-s}ds\Big)+O(t^{-1}(\ln t)^{-2}\ln\ln t)\\
&+O\Big(\mu_0^3t^{-2}\frac{|\bar x|^2}{\mu_0^2}+|\mu_{0t}|\frac{|\bar x|}{\mu_0}\Big)\Big]+w^2(\frac{\bar x}{\mu_0})Z_5(\frac{\bar x}{\mu_0})\eta^2(\frac{\bar x}{\sqrt t})\Big[-2^{-\frac12}\Big(\mu_{01}t^{-1}+\int_{t/2}^{t-\mu_0^2}\frac{\mu_{01t}(s)}{t-s}ds\Big)\\
&+O\Big(\mu_0^2|\mu_{01}|t^{-2}\frac{|\bar x|^2}{\mu_0^2}+|\mu_{0t}|\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_{01}(t_1)|}{\mu_0}+\frac{|\mu_{01t}(t_1)|}{|\mu_{0t}|}\Big)\frac{|\bar x|}{\mu_0}\Big)+\tilde g[\mu_0,\mu_{01}]\Big]\\
&+\frac{|\mu_{01}|}{\mu_0^2}\langle\frac{\bar x}{\mu_0}\rangle^{-4}\eta(\frac{\bar x}{\sqrt t})O((t\ln t)^{-2})+\mu_0\langle\frac{\bar x}{\mu_0}\rangle^{-4}\eta(\frac{\bar x}{\sqrt t})O((t\ln t)^{-2})\Big\}d(\frac{\bar x}{\mu_0})+\mathcal{M}[\mu_0]
\end{aligned}$$

$$
\begin{aligned}
&= O(\frac{|\mu_{01}|}{\mu_0})\Big(\mu_{01}t^{-1} + \int_{t/2}^{t-\mu_0^2} \frac{\mu_{01t}(s)}{t-s}ds\Big) - 2^{-\frac{1}{2}} \int_{\mathbb{R}^4} w^2(y) Z_5(y) \eta^2(\frac{\mu_0 y}{\sqrt{t}}) dy \Big(\mu_{01}t^{-1} + \int_{t/2}^{t-\mu_0^2} \frac{\mu_{01t}(s)}{t-s}ds\Big) \\
&\quad + O(t^{-1}(\ln t)^{-2}\ln\ln t)\frac{|\mu_{01}|}{\mu_0} + O\Big(\mu_0^2|\mu_{01}|t^{-2}\ln t + |\mu_{0t}| \sup_{t_1\in[t/2,t]} \Big(\frac{|\mu_{01}(t_1)|}{\mu_0} + \frac{|\mu_{01t}(t_1)|}{|\mu_{0t}|}\Big)\Big) \\
&\quad + \tilde{g}[\mu_0,\mu_{01}] + \frac{|\mu_{01}|}{\mu_0^2}\ln t O((t\ln t)^{-2}) + \mu_0 \ln t O((t\ln t)^{-2}) + \mathcal{M}[\mu_0] \\
&= \Big(O(\frac{|\mu_{01}|}{\mu_0}) - 2^{-\frac{1}{2}} \int_{\mathbb{R}^4} w^2(y) Z_5(y) \eta^2(\frac{\mu_0 y}{\sqrt{t}}) dy\Big)\Big(\mu_{01}t^{-1} + \int_{t/2}^{t-\mu_0^2} \frac{\mu_{01t}(s)}{t-s}ds\Big) \\
&\quad + O\big((t\ln t)^{-1} \sup_{t_1\in[t/2,t]} |\mu_{01}(t_1)| + \sup_{t_1\in[t/2,t]} |\mu_{01t}(t_1)|\big) \\
&\quad + O(t^{-1}(\ln t)^{-1}\ln\ln t)|\mu_{01}| + \tilde{g}[\mu_0,\mu_{01}] + O((t\ln t)^{-2}) + \mathcal{M}[\mu_0] \\
&= \Big(O(\frac{|\mu_{01}|}{\mu_0}) - 2^{-\frac{1}{2}} \int_{\mathbb{R}^4} w^2(y) Z_5(y) \eta^2(\frac{\mu_0 y}{\sqrt{t}}) dy\Big)\Big[\mu_{01}t^{-1}(1 + O((\ln t)^{-\frac{1}{2}})) + \int_{t/2}^{t-t^{1-\nu_1}} \frac{\mu_{01t}(s)}{t-s}ds \\
&\quad + \int_{t-t^{1-\nu_1}}^{t-\mu_0^2(t)} \frac{\mu_{01t}(t)}{t-s}ds + \mathcal{E}_{\nu_1}[\mu_{01}] + O\big((t\ln t)^{-1} \sup_{t_1\in[t/2,t]} |\mu_{01}(t_1)| + \sup_{t_1\in[t/2,t]} |\mu_{01t}(t_1)|\big) \\
&\quad + \tilde{g}[\mu_{01},\mu_0] + O((t\ln t)^{-2}) + \mathcal{M}[\mu_0]\Big] \\
&= \Big(O(\frac{|\mu_{01}|}{\mu_0}) - 2^{-\frac{1}{2}} \int_{\mathbb{R}^4} w^2(y) Z_5(y) \eta^2(\frac{\mu_0 y}{\sqrt{t}}) dy\Big)\Big[\mu_{01}t^{-1}\big(1 + O((\ln t)^{-\frac{1}{2}})\big) + \int_{t/2}^{t-t^{1-\nu_1}} \frac{\mu_{01t}(s)}{t-s}ds \\
&\quad + \mu_{01t}((1-\nu_1)\ln t + 2\ln\ln t) + O\big((t\ln t)^{-1} \sup_{t_1\in[t/2,t]} |\mu_{01}(t_1)| + \sup_{t_1\in[t/2,t]} |\mu_{01t}(t_1)|\big) \\
&\quad + \mathcal{E}_{\nu_1}[\mu_{01}] + \tilde{g}[\mu_{01},\mu_0] + O((t\ln t)^{-2}) + \mathcal{M}[\mu_0]\Big],
\end{aligned}
$$

where

$$
\mathcal{E}_{\nu_1}[\mu_{01}] = \int_{t-t^{1-\nu_1}}^{t-\mu_0^2(t)} \frac{\mu_{01t}(s) - \mu_{01t}(t)}{t-s}ds.
$$

Since it is too difficult to solve the nonlocal equation about μ_{01} thoroughly, we put $\mathcal{E}_{\nu_1}[\mu_{01}]$ aside as the new error term and consider the following equation

$$
\mu_{01t} + \beta_{\nu_1}(t)\mu_{01} = f_{\nu_1}[\mu_{01}], \tag{2.20}
$$

where

$$
\begin{aligned}
\beta_{\nu_1}(t) &= t^{-1}(1 + O((\ln t)^{-\frac{1}{2}}))[(1-\nu_1)\ln t + 2\ln\ln t]^{-1}, \\
f_{\nu_1}[\mu_{01}] &= \chi(t)\big[(1-\nu_1)\ln t + 2\ln\ln t\big]^{-1}\Big(- \int_{t/2}^{t-t^{1-\nu_1}} \frac{\mu_{01t}(s)}{t-s}ds - \tilde{g}[\mu_{01},\mu_0] \\
&\quad + O\big((t\ln t)^{-1} \sup_{t_1\in[t/2,t]} |\mu_{01}(t_1)| + \sup_{t_1\in[t/2,t]} |\mu_{01t}(t_1)|\big) + O((t\ln t)^{-2}) - \mathcal{M}[\mu_0]\Big),
\end{aligned}
$$

$\chi(t)$ is a smooth cut-off function such that $\chi(t) = 0$ for $t < \frac{3}{4}t_0$ and $\chi(t) = 1$ for $t \geq t_0$. Since $\mu_{01}(t)$ will be defined in $(\frac{t_0}{4},\infty)$, the introduction of $\chi(t)$ is used to avoid the occurrence of $\mu_{01}(t)$ for t beyond $(\frac{t_0}{4},\infty)$ in the terms like $\int_{t/2}^{t-t^{1-\nu_1}} \frac{\mu_{01t}(s)}{t-s}ds$. After all, the original orthogonal equation is only required to hold in (t_0,∞). For technical reasons, we extend the domain of μ_{01} to $(\frac{t_0}{4},\infty)$.

It then suffices to consider the following fixed point problem:

$$
\begin{aligned}
A_{\nu_1}[\mu_{01}](t) &= -\int_t^\infty \partial_t A_{\nu_1}[\mu_{01}](s)ds = -e^{-\int^t \beta_{\nu_1}(u)du} \int_t^\infty e^{\int^s \beta_{\nu_1}(u)du} f_{\nu_1}[\mu_{01}](s)ds, \\
\partial_t A_{\nu_1}[\mu_{01}](t) &= \beta_{\nu_1}(t) e^{-\int^t \beta_{\nu_1}(u)du} \int_t^\infty e^{\int^s \beta_{\nu_1}(u)du} f_{\nu_1}[\mu_{01}](s)ds + f_{\nu_1}[\mu_{01}](t),
\end{aligned} \tag{2.21}
$$

where $\nu_1 \in (0,\frac{1}{2})$ will be determined later.

Since

$$
O((t\ln t)^{-2}) + |\mathcal{M}[\mu_0]| \leq C_0 t^{-1}(\ln t)^{-2}\ln\ln t
$$

where $C_0 \geq 1$ is a large constant independent of t_0, we have

$$\left|[(1-\nu_1)\ln t + 2\ln\ln t]^{-1}(O((t\ln t)^{-2}) + |\mathcal{M}[\mu_0]|)\right| \leq (1-\nu_1)^{-1}C_0 t^{-1}(\ln t)^{-3}\ln\ln t.$$

By L'Hôpital's rule,

$$\begin{aligned}
&\lim_{t\to\infty} \frac{-\int_t^\infty e^{\int^s \beta_{\nu_1}(u)du} s^{-1}(\ln s)^{-3}\ln\ln s ds}{e^{\int^t \beta_{\nu_1}(u)du}(\ln t)^{-2}\ln\ln t} \\
= &\lim_{t\to\infty} \frac{e^{\int^t \beta_{\nu_1}(u)du} t^{-1}(\ln t)^{-3}\ln\ln t}{\beta_{\nu_1}(t)e^{\int^t \beta_{\nu_1}(u)du}(\ln t)^{-2}\ln\ln t + e^{\int^t \beta_{\nu_1}(u)du}(-2)t^{-1}(\ln t)^{-3}\ln\ln t + e^{\int^t \beta_{\nu_1}(u)du}t^{-1}(\ln t)^{-3}} \\
= &[(1-\nu_1)^{-1} - 2]^{-1}.
\end{aligned}$$

Notice $\nu_1 < \frac{1}{2}$ implies $e^{\int^t \beta_{\nu_1}(u)du} \ll (\ln t)^2$ so that $\int_t^\infty e^{\int^s \beta_{\nu_1}(u)du} s^{-1}(\ln s)^{-3}\ln\ln s ds$ is well defined. Thus we have

$$\frac{-\int_t^\infty e^{\int^s \beta_{\nu_1}(u)du} s^{-1}(\ln s)^{-3}\ln\ln s ds}{e^{\int^t \beta_{\nu_1}(u)du}(\ln t)^{-2}\ln\ln t} = [(1-\nu_1)^{-1} - 2]^{-1} + o(1)$$

where $o(1) \to 0$ as $t_0 \to \infty$. Then

$$\begin{aligned}
&\left|e^{-\int^t \beta_{\nu_1}(u)du}\int_\infty^t e^{\int^s \beta_{\nu_1}(u)du}[(1-\nu_1)\ln s + 2\ln\ln s]^{-1}s^{-1}(\ln s)^{-2}\ln\ln s ds\right| \\
\leq &(1-\nu_1)^{-1}\left|e^{-\int^t \beta_{\nu_1}(u)du}\int_\infty^t e^{\int^s \beta_{\nu_1}(u)du}s^{-1}(\ln s)^{-3}\ln\ln s ds\right| = |(2\nu_1 - 1)^{-1} + o(1)|(\ln t)^{-2}\ln\ln t,
\end{aligned}$$

$$\begin{aligned}
&\left|\beta_{\nu_1}(t)e^{-\int^t \beta_{\nu_1}(u)du}\int_\infty^t e^{\int^s \beta_{\nu_1}(u)du}[(1-\nu_1)\ln s + 2\ln\ln s]^{-1}s^{-1}(\ln s)^{-2}\ln\ln s ds\right| \\
&+ \left|[(1-\nu_1)\ln t + 2\ln\ln t]^{-1}t^{-1}(\ln t)^{-2}\ln\ln t\right| \\
\leq &t^{-1}(1-\nu_1)^{-1}(\ln t)^{-1}|(2\nu_1-1)^{-1} + o(1)|(\ln t)^{-2}\ln\ln t + (1-\nu_1)^{-1}t^{-1}(\ln t)^{-3}\ln\ln t \\
= &(1-\nu_1)^{-1}(1 + |(2\nu_1-1)^{-1} + o(1)|)t^{-1}(\ln t)^{-3}\ln\ln t.
\end{aligned}$$

From the estimates above, for $\mu_{01} \in C^1(t_0/4, \infty)$ and $\mu_{01}(t) \to 0$ as $t \to \infty$, we set the norm as

$$\|\mu_{01}\|_{01} = \sup_{t \geq t_0/4} t(\ln t)^3(\ln\ln t)^{-1}|\mu_{01t}(t)|$$

and will solve the fixed point problem (2.21) in the space

$$B_{01} = \left\{g(t) \in C^1(t_0/4, \infty), \quad g(t) \to 0 \text{ as } t \to \infty \ : \ \|g\|_{01} \leq 2C_0 C(\nu_1)\right\}$$

where $C(\nu_1) = (1-\nu_1)^{-1}(1 + |(2\nu_1-1)^{-1} + o(1)|)$. We take $\nu_1 < \frac{1}{2}$ and t_0 large enough to guarantee $C(\nu_1) < \infty$. Let us estimate other terms for $\partial_t A_{\mu_1}[\mu_{01}]$ in (2.21).

For any $\bar{\mu}_{01} \in B_{01}$,

$$\begin{aligned}
&\chi(t)\left|\int_{t/2}^{t-t^{1-\nu_1}} \frac{\bar{\mu}_{01t}(s)}{t-s}ds\right| \leq \|\bar{\mu}_{01}\|_{01}\int_{t/2}^{t-t^{1-\nu_1}} \frac{s^{-1}(\ln s)^{-3}\ln\ln s}{t-s}ds \\
= &\|\bar{\mu}_{01}\|_{01}t^{-1}\int_{1/2}^{1-t^{-\nu_1}} \frac{(\ln t + \ln z)^{-3}\ln(\ln t + \ln z)}{z(1-z)}dz \\
\leq &\|\bar{\mu}_{01}\|_{01}(1 + O((\ln t)^{-\frac{1}{2}}))t^{-1}(\ln t)^{-3}\ln\ln t\int_{1/2}^{1-t^{-\nu_1}} \frac{1}{z(1-z)}dz \\
= &\|\bar{\mu}_{01}\|_{01}(1 + O((\ln t)^{-\frac{1}{2}}))t^{-1}(\ln t)^{-3}\ln\ln t \ln(t^{\nu_1} - 1) \\
\leq &\|\bar{\mu}_{01}\|_{01}\nu_1(1 + O((\ln t)^{-\frac{1}{2}}))t^{-1}(\ln t)^{-2}\ln\ln t \\
\leq &2C_0 C(\nu_1)\nu_1(1 + O((\ln t)^{-\frac{1}{2}}))t^{-1}(\ln t)^{-2}\ln\ln t
\end{aligned} \tag{2.22}$$

which implies

$$\left|\left[(1-\nu_1)\ln t + 2\ln\ln t + O(1)\right]^{-1}\chi(t)\int_{t/2}^{t-t^{1-\nu_1}} \frac{\bar{\mu}_{01t}(s)}{t-s}ds\right| \leq 2C_0 C(\nu_1)\nu_1(1-\nu_1)^{-1}(1 + O((\ln t)^{-\frac{1}{2}}))t^{-1}(\ln t)^{-3}\ln\ln t.$$

We take $\nu_1 \in (0, \frac{1}{4})$ to make $\nu_1(1-\nu_1)^{-1}(1 + |(2\nu_1-1)^{-1}|) < 1$.

Since $\bar{\mu}_{01} \in B_{01}$, one has $|\bar{\mu}_{01}(t)| = |\int_t^\infty \bar{\mu}_{01t}(s)ds| \lesssim \|\bar{\mu}_{01}\|_{01}(\ln t)^{-2}\ln\ln t$. Then

$$\begin{aligned}\chi(t)\tilde{g}[\bar{\mu}_{01},\mu_0] &= O\Big(t^{-2}\int_{t_0/2}^{t}(s^{-1}\|\bar{\mu}_{01}\|_{01}(\ln s)^{-4}\ln\ln s + (\ln s)^{-2}\|\bar{\mu}_{01}\|_{01}(\ln s)^{-1}\ln\ln s)ds \\ &\quad + (t\ln t)^{-1}\|\bar{\mu}_{01}\|_{01}^2((\ln t)^{-1}\ln\ln t)^2\Big) \\ &= (\ln t_0)^{-\frac{1}{2}}O\big((\|\bar{\mu}_{01}\|_{01} + \|\bar{\mu}_{01}\|_{01}^2)t^{-1}(\ln t)^{-2}\ln\ln t\big),\end{aligned}$$

$$\chi(t)O\big((t\ln t)^{-1}\sup_{t_1\in[t/2,t]}|\mu_{01}(t_1)| + \sup_{t_1\in[t/2,t]}|\mu_{01t}(t_1)|\big) \lesssim t^{-1}(\ln t)^{-3}\ln\ln t \lesssim (\ln t_0)^{-1}t^{-1}(\ln t)^{-2}\ln\ln t.$$

Then for any fixed $\nu_1 \in (0, \frac{1}{4})$ and t_0 large enough, one sees that $A_{\nu_1}[\bar{\mu}_{01}] \in B_{01}$.

The contraction property can be derived similarly. Indeed, for any $\mu_{01a}, \mu_{01b} \in B_{01}$, similar to (2.22), we have

$$\begin{aligned}&\chi(t)\Big|\int_{t/2}^{t-t^{1-\nu_1}}\frac{\partial_t\bar{\mu}_{01a}(s)}{t-s}ds - \int_{t/2}^{t-t^{1-\nu_1}}\frac{\partial_t\bar{\mu}_{01b}(s)}{t-s}ds\Big| \le \|\bar{\mu}_{01a}-\bar{\mu}_{01b}\|_{01}\int_{t/2}^{t-t^{1-\nu_1}}\frac{s^{-1}(\ln s)^{-3}\ln\ln s}{t-s}ds \\ &\le \|\bar{\mu}_{01a}-\bar{\mu}_{01b}\|_{01}\nu_1(1+O((\ln t)^{-\frac{1}{2}}))t^{-1}(\ln t)^{-2}\ln\ln t,\end{aligned}$$

$$\begin{aligned}&\chi(t)|\tilde{g}[\mu_{01a},\mu_0] - \tilde{g}[\mu_{01b},\mu_0]| = (\ln t_0)^{-\frac{1}{2}}O(C_0C(\nu_1)t^{-1}(\ln t)^{-2}\ln\ln t)\|\bar{\mu}_{01a}-\bar{\mu}_{01b}\|_{01}, \\ &\chi(t)\Big|O((t\ln t)^{-1}\sup_{t_1\in[t/2,t]}|\mu_{01a}(t_1)| + \sup_{t_1\in[t/2,t]}|\partial_t\mu_{01a}(t_1)|) \\ &\quad - O((t\ln t)^{-1}\sup_{t_1\in[t/2,t]}|\mu_{01b}(t_1)| + \sup_{t_1\in[t/2,t]}|\partial_t\mu_{01b}(t_1)|)\Big| \\ &\lesssim \chi(t)\Big|O((t\ln t)^{-1}\sup_{t_1\in[t/2,t]}|\mu_{01a}(t_1)-\mu_{01b}(t_1)| + \sup_{t_1\in[t/2,t]}|\partial_t\mu_{01a}(t_1)-\partial_t\mu_{01b}(t_1)|)\Big| \\ &\lesssim \|\bar{\mu}_{01a}-\bar{\mu}_{01b}\|_{01}t^{-1}(\ln t)^{-3}\ln\ln t \lesssim (\ln t_0)^{-1}\|\bar{\mu}_{01a}-\bar{\mu}_{01b}\|_{01}t^{-1}(\ln t)^{-2}\ln\ln t\end{aligned} \tag{2.23}$$

by the estimate of $\varphi[\mu_0+\mu_{01a}] - \varphi[\mu_0+\mu_{01b}]$ in Corollary 2.3.

Due to the choice of ν_1 and t_0 above, the contraction property is achieved. By contraction mapping theorem, there exists a unique solution $\mu_{01} \in B_{01}$ for (2.21).

From now on, ν_1 will be regarded as a general constant unless otherwise stated. For notational simplicity, ∂_t is denoted by "$\,'\,$". Once we have solved μ_{01}, the regularity of μ_{01} can be improved by the equation of μ_{01} and μ_{01}'' decays to 0 as $t\to\infty$. For the purpose of finding a better decay estimate of μ_{01}'', we take derivative on both sides of (2.20). Then

$$\mu_{01}'' + (\beta_{\nu_1}(t))'\mu_{01} + \beta_{\nu_1}(t)\mu_{01}' = (f_{\nu_1}[\mu_{01}])',$$

where we can evaluate

$$|(\beta_{\nu_1}(t))'\mu_{01} + \beta_{\nu_1}(t)\mu_{01}'| \lesssim t^{-2}(\ln t)^{-3}\ln\ln t,$$

$$\begin{aligned}&\big(\chi(t)[(1-\nu_1)\ln t + 2\ln\ln t]^{-1}\big)'\Big(-\int_{t/2}^{t-t^{1-\nu_1}}\frac{\mu_{01t}(s)}{t-s}ds - \tilde{g}[\mu_{01},\mu_0] \\ &+ O((t\ln t)^{-1}\sup_{t_1\in[t/2,t]}|\mu_{01}(t_1)| + \sup_{t_1\in[t/2,t]}|\mu_{01t}(t_1)|) + O((t\ln t)^{-2}) - \mathcal{M}[\mu_0]\Big) \lesssim t^{-2}(\ln t)^{-3}\ln\ln t,\end{aligned}$$

$$\chi(t)\big[(1-\nu_1)\ln t + 2\ln\ln t\big]^{-1}\big(-\tilde{g}[\mu_{01},\mu_0] + O((t\ln t)^{-2}) - \mathcal{M}[\mu_0]\big)' \lesssim t^{-2}(\ln t)^{-3}\ln\ln t,$$

where we used similar calculation in (C.2) for $(\tilde{g}[\mu_{01},\mu_0])'$.

$$\begin{aligned}&\chi(t)\big[(1-\nu_1)\ln t + 2\ln\ln t\big]^{-1}\Big(-\int_{t/2}^{t-t^{1-\nu_1}}\frac{\mu_{01t}(s)}{t-s}ds\Big)' \\ &= \chi(t)\big[(1-\nu_1)\ln t + 2\ln\ln t\big]^{-1}\Big[-\frac{\mu_{01}'(t-t^{1-\nu_1})}{t^{1-\nu_1}}(1-(1-\nu_1)t^{-\nu_1}) + \frac{\mu_{01}'(\frac{t}{2})}{t} + \int_{t/2}^{t-t^{1-\nu_1}}\frac{\mu_{01}'(s)}{(t-s)^2}ds)\Big] \\ &= \chi(t)\big[(1-\nu_1)\ln t + 2\ln\ln t\big]^{-1}\Big[(1-\nu_1)\frac{\mu_{01}'(t-t^{1-\nu_1})}{t} - \frac{\mu_{01}'(\frac{t}{2})}{t} - \int_{t/2}^{t-t^{1-\nu_1}}\frac{\mu_{01}''(s)}{t-s}ds\Big] \\ &= -\chi(t)\big[(1-\nu_1)\ln t + 2\ln\ln t\big]^{-1}\int_{t/2}^{t-t^{1-\nu_1}}\frac{\mu_{01}''(s)}{t-s}ds + O(t^{-2}(\ln t)^{-4}\ln\ln t).\end{aligned}$$

Revisiting the process of proving Corollary 2.3, we have

$$\begin{aligned}&\chi(t)\big[(1-\nu_1)\ln t+2\ln\ln t\big]^{-1}\big(O((t\ln t)^{-1}\sup_{t_1\in[t/2,t]}|\mu_{01}(t_1)|+\sup_{t_1\in[t/2,t]}|\mu_{01t}(t_1)|)\big)'\\&=\chi(t)\big[(1-\nu_1)\ln t+2\ln\ln t\big]^{-1}O\big(\sup_{t_1\in[t/2,t]}|\mu_{01}''(t_1)|\big)+O(t^{-2}(\ln t)^{-4}\ln\ln t).\end{aligned}\tag{2.24}$$

Using the estimates above, we have

$$\mu_{01}''=\chi(t)\big[(1-\nu_1)\ln t+2\ln\ln t\big]^{-1}\Big(-\int_{t/2}^{t-t^{1-\nu_1}}\frac{\mu_{01}''(s)}{t-s}ds+O\big(\sup_{t_1\in[t/2,t]}|\mu_{01}''(t_1)|\big)\Big)+\tilde f(t)$$

where $|\tilde f(t)|\le C_1t^{-2}(\ln t)^{-3}\ln\ln t$. For this reason, we solve μ_{01}'' in the following space

$$B_{2,2}=\{g\in C(t_0/4,\infty),g(t)\to 0\ \text{ as }\ t\to\infty\ :\ \|g\|_{2,2}\le 2C_1\},$$

where for $g\in C(t_0/4,\infty)$, we define

$$\|g\|_{a,b}:=\sup_{t\ge t_0/4}t^a(\ln t)^b|g(t)|.$$

For any $g_1,g_2\in B_{2,2}$, similar to (2.22), we have

$$\begin{aligned}&\chi(t)\big[(1-\nu_1)\ln t+2\ln\ln t\big]^{-1}\Big|\int_{t/2}^{t-t^{1-\nu_1}}\frac{g_1(s)}{t-s}ds-\int_{t/2}^{t-t^{1-\nu_1}}\frac{g_2(s)}{t-s}ds\Big|\\&\le(1-\nu_1)^{-1}(\ln t)^{-1}\|g_1-g_2\|_{2,2}\int_{t/2}^{t-t^{1-\nu_1}}\frac{s^{-2}(\ln s)^{-2}}{t-s}ds\\&\le\nu_1(1-\nu_1)^{-1}\|g_1-g_2\|_{2,2}t^{-2}(\ln t)^{-2}(1+(\ln t)^{-1})(1+(\nu_1\ln t)^{-1}).\end{aligned}$$

Similar to (2.24), one has

$$\begin{aligned}&\chi(t)\big[(1-\nu_1)\ln t+2\ln\ln t\big]^{-1}\Big|O\big(\sup_{t_1\in[t/2,t]}|g_1(t_1)|\big)-O\big(\sup_{t_1\in[t/2,t]}|g_2(t_1)|\big)\Big|\\&\lesssim\chi(t)\big[(1-\nu_1)\ln t+2\ln\ln t\big]^{-1}O\big(\sup_{t_1\in[t/2,t]}|g_1(t_1)-g_2(t_1)|\big)\lesssim t^{-2}(\ln t)^{-3}\|g_1-g_2\|_{2,2}.\end{aligned}$$

For any $g\in B_{2,2}$, we have

$$\begin{aligned}&\Big|\chi(t)[(1-\nu_1)\ln t+2\ln\ln t]^{-1}\int_{t/2}^{t-t^{1-\nu_1}}\frac{g(s)}{t-s}ds\Big|\le(1-\nu_1)^{-1}(\ln t)^{-1}\|g\|_{2,2}\int_{t/2}^{t-t^{1-\nu_1}}\frac{s^{-2}(\ln s)^{-2}}{t-s}ds\\&\le\nu_1(1-\nu_1)^{-1}\|g\|_{2,2}t^{-2}(\ln t)^{-2}(1+(\ln t)^{-1})(1+(\nu_1\ln t)^{-1}).\end{aligned}$$

Since $\nu_1\in(0,\frac14)$, when t_0 is large enough, the contraction property follows and then $\mu_{01}''\in B_{2,2}$. Thus the improved error is given by

$$\mathcal{M}[\mu_0+\mu_{01}]=\mathcal{E}_{\nu_1}[\mu_{01}]=\int_{t-t^{1-\nu_1}}^{t-\mu_0^2(t)}\frac{\mu_{01}'(s)-\mu_{01}'(t)}{t-s}ds=O(t^{-1-\nu_1}(\ln t)^{-2}).$$

Step 3. Further improvement by iteration.

Repeating Step 2 finitely many times, we can find μ_{0i}, $i=1,\dots,k_0$ such that

$$M\Big[\mu_0+\sum_{i=1}^{k_0}\mu_{0i}\Big]=O(t^{-2}).\tag{2.25}$$

Denote

$$\bar\mu_0=\mu_0+\sum_{i=1}^{k_0}\mu_{0i}.$$

From the construction above, we see that $\bar\mu_0\sim\mu_0=(\ln t)^{-1}$, $\bar\mu_{0t}\sim\mu_{0t}$.

Since $\bar\mu_0$ is determined, we are now able to describe Φ_0 rigorously. Set $\bar y=\frac{x}{\bar\mu_0}$ and consider

$$\Delta_{\bar y}\Phi_0+3w^2(\bar y)\Phi_0=\tilde H(|\bar y|,t),$$

where

$$\tilde H(|\bar y|,t)=-3\bar\mu_0\left(w^2(\bar y)\eta^2(\frac{\bar\mu_0\bar y}{\sqrt t})\varphi[\bar\mu_0](\bar\mu_0\bar y,t)+\bar\mu_0w(\bar y)\eta(\frac{\bar\mu_0\bar y}{\sqrt t})\varphi^2[\bar\mu_0](\bar\mu_0\bar y,t)\right)+3\bar\mu_0\mathcal{M}[\bar\mu_0]\frac{\eta(\bar y)Z_5(\bar y)}{\int_{B_2}\eta(z)Z_5^2(z)dz}.$$

Then $\Phi_0(\bar{y},t)$ is given by

$$\Phi_0(\bar{y},t)=\tilde{Z}_5(\bar{y})\int_0^{|\bar{y}|}\tilde{H}(s,t)Z_5(s)s^3ds-Z_5(\bar{y})\int_0^{|\bar{y}|}\tilde{H}(s,t)\tilde{Z}_5(s)s^3ds,$$

where $\tilde{Z}_5(r)$ is the other linearly independent kernel of the homogeneous equation, which satisfies that the Wronskian $W[Z_5,\tilde{Z}_5]=r^{-3}$, so $\tilde{Z}_5(r)\sim r^{-2}$ if $r\to 0$ and $\tilde{Z}_5(r)\sim 1$ if $r\to\infty$.

By the definition of $\mathcal{M}[\bar{\mu}_0]$, it is easy to have

$$\int_{\mathbb{R}^4}\tilde{H}(z,t)Z_5(z)dz=0. \tag{2.26}$$

By Corollary 2.3, $|\tilde{H}|\lesssim t^{-1}(\ln t)^{-2}\langle\bar{y}\rangle^{-4}$. Due to the special choice of $\bar{\mu}_0$, one can get better time decay for $\tilde{H}$. Indeed, we have

$$\begin{aligned}
&w^2(\bar{y})\eta^2(\frac{\bar{\mu}_0\bar{y}}{\sqrt{t}})\varphi[\bar{\mu}_0](\bar{\mu}_0\bar{y},t)\\
&=w^2(\bar{y})\eta^2(\frac{\bar{\mu}_0\bar{y}}{\sqrt{t}})\Big[-2^{-\frac{1}{2}}\Big(\bar{\mu}_0t^{-1}+\int_{t/2}^{t-\mu_0^2}\frac{\bar{\mu}_{0t}(s)}{t-s}ds\Big)+O(\bar{\mu}_0^3t^{-2}|\bar{y}|^2+|\bar{\mu}_{0t}||\bar{y}|)+O(t^{-1}(\ln t)^{-2})\Big]\\
&=\Big[O(t^{-1}(\ln t)^{-2}|\bar{y}|)+O(t^{-1}(\ln t)^{-2}\ln\ln t)\Big]\langle\bar{y}\rangle^{-4}\mathbf{1}_{\{|\bar{y}|\le 2\bar{\mu}_0^{-1}t^{\frac{1}{2}}\}}\\
&=O(t^{-1}(\ln t)^{-2}\ln\ln t)\langle\bar{y}\rangle^{-3}\mathbf{1}_{\{|\bar{y}|\le 2\bar{\mu}_0^{-1}t^{\frac{1}{2}}\}},
\end{aligned}$$

$$\bar{\mu}_0w(\bar{y})\eta(\frac{\bar{\mu}_0\bar{y}}{\sqrt{t}})\varphi^2[\bar{\mu}_0](\bar{\mu}_0\bar{y},t)=O(t^{-2}(\ln t)^{-3})\langle\bar{y}\rangle^{-2}\mathbf{1}_{\{|\bar{y}|\le 2\bar{\mu}_0^{-1}t^{\frac{1}{2}}\}}=O(t^{-1}(\ln t)^{-2}\ln\ln t)\langle\bar{y}\rangle^{-3}\mathbf{1}_{\{|\bar{y}|\le 2\bar{\mu}_0^{-1}t^{\frac{1}{2}}\}},$$

which implies $|\tilde{H}|\lesssim t^{-1}(\ln t)^{-3}\ln\ln t\langle\bar{y}\rangle^{-3}$. As a result, one has

$$|\tilde{H}|\lesssim\min\big\{t^{-1}(\ln t)^{-2}\langle\bar{y}\rangle^{-4},t^{-1}(\ln t)^{-3}\ln\ln t\langle\bar{y}\rangle^{-3}\big\}. \tag{2.27}$$

Claim:

$$\begin{aligned}
|\Phi_0(\bar{y},t)|&\lesssim\min\big\{t^{-1}(\ln t)^{-2}\langle\bar{y}\rangle^{-2}\ln(2+|\bar{y}|),t^{-1}(\ln t)^{-3}\ln\ln t\langle\bar{y}\rangle^{-1}\big\},\\
|\nabla_{\bar{y}}\Phi_0(\bar{y},t)|&\lesssim\min\big\{t^{-1}(\ln t)^{-2}\langle\bar{y}\rangle^{-3}\ln(2+|\bar{y}|),t^{-1}(\ln t)^{-3}\ln\ln t\langle\bar{y}\rangle^{-2}\big\},\\
|\partial_t\Phi_0(\bar{y},t)|&\lesssim t^{-2}(\ln t)^{-1}\langle\bar{y}\rangle^{-2}\ln(2+|\bar{y}|).
\end{aligned} \tag{2.28}$$

Indeed, the estimate about $\Phi_0(\bar{y},t)$ is derived from (2.26) (2.27). The upper bound of $\nabla_{\bar{y}}\Phi_0(\bar{y},t)$ follows by scaling argument. In order to estimate $\partial_t\Phi_0(\bar{y},t)$, we need to take a closer look at $\partial_t\tilde{H}(z,t)$. By the definition of $\tilde{H}$, it is straightforward to have

$$\int_{\mathbb{R}^4}\partial_t\tilde{H}(z,t)Z_5(z)dz=0.$$

$$\begin{aligned}
\partial_t\tilde{H}(|\bar{y}|,t)&=-3\bar{\mu}_{0t}\Big(w^2(\bar{y})\eta^2(\frac{\bar{\mu}_0\bar{y}}{\sqrt{t}})\varphi[\bar{\mu}_0](\bar{\mu}_0\bar{y},t)+\bar{\mu}_0w(\bar{y})\eta(\frac{\bar{\mu}_0\bar{y}}{\sqrt{t}})\varphi^2[\bar{\mu}_0](\bar{\mu}_0\bar{y},t)-\mathcal{M}[\bar{\mu}_0]\frac{\eta(\bar{y})Z_5(\bar{y})}{\int_{B_2}\eta Z_5^2dz}\Big)\\
&-3\bar{\mu}_0\Bigg[w^2(\bar{y})2\eta(\frac{\bar{\mu}_0\bar{y}}{\sqrt{t}})\nabla\eta(\frac{\bar{\mu}_0\bar{y}}{\sqrt{t}})\cdot\frac{\bar{\mu}_0\bar{y}}{\sqrt{t}}(\frac{\bar{\mu}_0}{\sqrt{t}})^{-1}\partial_t(\frac{\bar{\mu}_0}{\sqrt{t}})\varphi[\bar{\mu}_0](\bar{\mu}_0\bar{y},t)\\
&+w^2(\bar{y})\eta^2(\frac{\bar{\mu}_0\bar{y}}{\sqrt{t}})\big(\nabla_{\bar{x}}\varphi[\bar{\mu}_0](\bar{\mu}_0\bar{y},t)\cdot\bar{\mu}_{0t}\bar{y}+\partial_t\varphi[\bar{\mu}_0](\bar{\mu}_0\bar{y},t)\big)\\
&+\bar{\mu}_{0t}w(\bar{y})\eta(\frac{\bar{\mu}_0\bar{y}}{\sqrt{t}})\varphi^2[\bar{\mu}_0](\bar{\mu}_0\bar{y},t)+\bar{\mu}_0w(\bar{y})\nabla\eta(\frac{\bar{\mu}_0\bar{y}}{\sqrt{t}})\cdot\frac{\bar{\mu}_0\bar{y}}{\sqrt{t}}(\frac{\bar{\mu}_0}{\sqrt{t}})^{-1}\partial_t(\frac{\bar{\mu}_0}{\sqrt{t}})\varphi^2[\bar{\mu}_0](\bar{\mu}_0\bar{y},t)\\
&+\bar{\mu}_0w(\bar{y})\eta(\frac{\bar{\mu}_0\bar{y}}{\sqrt{t}})2\varphi[\bar{\mu}_0](\bar{\mu}_0\bar{y},t)\big(\nabla_{\bar{x}}\varphi[\bar{\mu}_0](\bar{\mu}_0\bar{y},t)\cdot\bar{\mu}_{0t}\bar{y}+\partial_t\varphi[\bar{\mu}_0](\bar{\mu}_0\bar{y},t)\big)-\partial_t(\mathcal{M}[\bar{\mu}_0])\frac{\eta(\bar{y})Z_5(\bar{y})}{\int_{B_2}\eta Z_5^2dz}\Bigg].
\end{aligned}$$

Using (C.1) in Appendix, one has

$$\begin{aligned}
&\Big|\nabla_{\bar{x}}\varphi[\bar{\mu}_0](\bar{\mu}_0\bar{y},t)\cdot\bar{\mu}_{0t}\bar{y}+\partial_t\varphi[\bar{\mu}_0](\bar{\mu}_0\bar{y},t)\Big|\mathbf{1}_{\{\bar{\mu}_0|\bar{y}|\le 2t^{\frac{1}{2}}\}}\\
&\lesssim\Big|\Big[(t\ln t)^{-1}\mathbf{1}_{\{|\bar{x}|\le\mu_0\}}+(t^{-1}(\ln t)^{-2}|\bar{x}|^{-1}+t^{-\frac{3}{2}}(\ln t)^{-1})\mathbf{1}_{\{|\bar{x}|>\mu_0\}}\Big]\bar{\mu}_0^{-1}|\bar{\mu}_{0t}||\bar{x}|+t^{-2}\Big|\mathbf{1}_{\{|\bar{x}|\le 2t^{\frac{1}{2}}\}}\lesssim t^{-2}.
\end{aligned}$$

Thus

$$\begin{aligned}&|\partial_t \tilde{H}(|\bar{y}|,t)| \lesssim t^{-2}(\ln t)^{-3}\langle\bar{y}\rangle^{-4} + (\ln t)^{-1}\Big[t^{-2}(\ln t)^{-1}\langle\bar{y}\rangle^{-4} + t^{-2}\langle\bar{y}\rangle^{-4} + t^{-3}(\ln t)^{-4}\langle\bar{y}\rangle^{-2}\\&\quad + t^{-3}(\ln t)^{-3}\langle\bar{y}\rangle^{-2} + t^{-3}(\ln t)^{-2}\langle\bar{y}\rangle^{-2} + t^{-3}\mathbf{1}_{\{|\bar{y}|\le 3\}}\Big]\mathbf{1}_{\{\bar{\mu}_0|\bar{y}|\le 2t^{\frac12}\}}\\&\lesssim t^{-2}(\ln t)^{-3}\langle\bar{y}\rangle^{-4} + (\ln t)^{-1}\Big(t^{-2}\langle\bar{y}\rangle^{-4} + t^{-3}(\ln t)^{-2}\langle\bar{y}\rangle^{-2}\Big)\mathbf{1}_{\{\bar{\mu}_0|\bar{y}|\le 2t^{\frac12}\}} \lesssim t^{-2}(\ln t)^{-1}\langle\bar{y}\rangle^{-4}.\end{aligned}$$

Therefore, we have the estimate about $\partial_t\Phi_0(\bar{y},t)$ in (2.28).

In order to avoid the influence in the remote region $|\bar{x}| \gtrsim t^{\frac12}$, we add cut-off function and set $\bar{\mu}_0^{-1}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\eta(\frac{4(x-\xi)}{\sqrt{t}})$ as the correction term. It is easy to check

$$\partial_t\Big(\bar{\mu}_0^{-1}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\Big) = -\bar{\mu}_0^{-2}\bar{\mu}_{0t}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t) - \bar{\mu}_0^{-2}\nabla_{\bar{y}}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\cdot\big(\bar{\mu}_{0t}\frac{x-\xi}{\bar{\mu}_0} + \xi_t\big) + \bar{\mu}_0^{-1}\partial_t\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t).$$

Set $\mu = \bar{\mu}_0 + \mu_1$ where $|\mu_1| \le \frac{\bar{\mu}_0}{2}$, $|\mu_{1t}| \le \frac{|\bar{\mu}_{0t}|}{2}$. Let us estimate the new error

$$\begin{aligned}&S\Big[u_1 + \varphi[\mu] + \bar{\mu}_0^{-1}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\eta(\frac{4(x-\xi)}{\sqrt{t}})\Big] \qquad (2.29)\\
=&-\partial_t\Big(\bar{\mu}_0^{-1}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\eta(\frac{4(x-\xi)}{\sqrt{t}})\Big) + \Delta_x\Big(\bar{\mu}_0^{-1}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\eta(\frac{4(x-\xi)}{\sqrt{t}})\Big) + S[u_1+\varphi[\mu]]\\
&+\Big(u_1+\varphi[\mu]+\bar{\mu}_0^{-1}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\eta(\frac{4(x-\xi)}{\sqrt{t}})\Big)^3 - (u_1+\varphi[\mu])^3\\
=&\,\bar{\mu}_0^{-1}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\nabla\eta(\frac{4(x-\xi)}{\sqrt{t}})\cdot\big(4t^{-\frac12}\xi_t + 2t^{-\frac32}(x-\xi)\big)\\
&-\eta(\frac{4(x-\xi)}{\sqrt{t}})\Big(-\bar{\mu}_0^{-2}\bar{\mu}_{0t}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t) - \bar{\mu}_0^{-2}\nabla_{\bar{y}}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\cdot\big(\bar{\mu}_{0t}\frac{x-\xi}{\bar{\mu}_0}+\xi_t\big) + \bar{\mu}_0^{-1}\partial_t\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\Big)\\
&+\bar{\mu}_0^{-3}\Delta_{\bar{y}}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\eta(\frac{4(x-\xi)}{\sqrt{t}}) + 8t^{-\frac12}\bar{\mu}_0^{-2}\nabla_{\bar{y}}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\cdot\nabla\eta(\frac{4(x-\xi)}{\sqrt{t}})\\
&+16t^{-1}\bar{\mu}_0^{-1}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\Delta\eta(\frac{4(x-\xi)}{\sqrt{t}}) + S[u_1+\varphi[\mu]] + 3\bar{\mu}_0^{-3}w^2(\frac{x-\xi}{\bar{\mu}_0})\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\eta(\frac{4(x-\xi)}{\sqrt{t}})\\
&+\Big(u_1+\varphi[\mu]+\bar{\mu}_0^{-1}\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\eta(\frac{4(x-\xi)}{\sqrt{t}})\Big)^3 - (u_1+\varphi[\mu])^3 - 3\bar{\mu}_0^{-3}w^2(\frac{x-\xi}{\bar{\mu}_0})\Phi_0(\frac{x-\xi}{\bar{\mu}_0},t)\eta(\frac{4(x-\xi)}{\sqrt{t}})\\
=&\,\bar{\mu}_0^{-1}\Phi_0(\frac{\bar{x}}{\bar{\mu}_0},t)\nabla\eta(\frac{4\bar{x}}{\sqrt{t}})\cdot\big(4t^{-\frac12}\xi_t + 2t^{-\frac32}\bar{x}\big)\\
&+8t^{-\frac12}\bar{\mu}_0^{-2}\nabla_{\bar{y}}\Phi_0(\frac{\bar{x}}{\bar{\mu}_0},t)\cdot\nabla\eta(\frac{4\bar{x}}{\sqrt{t}}) + 16t^{-1}\bar{\mu}_0^{-1}\Phi_0(\frac{\bar{x}}{\bar{\mu}_0},t)\Delta\eta(\frac{4\bar{x}}{\sqrt{t}})\\
&-\eta(\frac{4\bar{x}}{\sqrt{t}})\Big[-\bar{\mu}_0^{-2}\bar{\mu}_{0t}\Phi_0(\frac{\bar{x}}{\bar{\mu}_0},t) - \bar{\mu}_0^{-2}\nabla_{\bar{y}}\Phi_0(\frac{\bar{x}}{\bar{\mu}_0},t)\cdot\big(\bar{\mu}_{0t}\frac{\bar{x}}{\bar{\mu}_0}+\xi_t\big) + \bar{\mu}_0^{-1}\partial_t\Phi_0(\frac{\bar{x}}{\bar{\mu}_0},t)\Big]\\
&+3\Big(\eta^2(\frac{\bar{x}}{\sqrt{t}}) - \eta(\frac{4\bar{x}}{\sqrt{t}})\Big)\mu^{-2}w^2(\frac{\bar{x}}{\mu})\varphi[\mu](\bar{x},t) + 3\Big(\eta(\frac{\bar{x}}{\sqrt{t}}) - \eta(\frac{4\bar{x}}{\sqrt{t}})\Big)\mu^{-1}w(\frac{\bar{x}}{\mu})\varphi^2[\mu](\bar{x},t)\\
&+3\eta(\frac{4\bar{x}}{\sqrt{t}})\Big(\mu^{-2}w^2(\frac{\bar{x}}{\mu})\varphi[\mu](\bar{x},t) - \bar{\mu}_0^{-2}w^2(\frac{\bar{x}}{\bar{\mu}_0})\varphi[\bar{\mu}_0](\bar{x},t)\Big)\\
&+3\eta(\frac{4\bar{x}}{\sqrt{t}})\Big(\mu^{-1}w(\frac{\bar{x}}{\mu})\varphi^2[\mu](\bar{x},t) - \bar{\mu}_0^{-1}w(\frac{\bar{x}}{\bar{\mu}_0})\varphi^2[\bar{\mu}_0](\bar{x},t)\Big)\\
&+\xi_t\cdot\nabla_{\bar{x}}\varphi[\mu](\bar{x},t) + \mu^{-2}\xi_t\cdot\nabla w(\frac{\bar{x}}{\mu})\eta(\frac{\bar{x}}{\sqrt{t}}) + \mu^{-1}t^{-\frac12}w(\frac{\bar{x}}{\mu})\xi_t\cdot\nabla\eta(\frac{\bar{x}}{\sqrt{t}})\\
&+\varphi^3[\mu] + 3\bar{\mu}_0^{-2}\mathcal{M}[\bar{\mu}_0]\Big(\int_{B_2}\eta(z)Z_5^2(z)dz\Big)^{-1}\eta(\frac{\bar{x}}{\bar{\mu}_0})Z_5(\frac{\bar{x}}{\bar{\mu}_0})\\
&+\Big(u_1+\varphi[\mu]+\bar{\mu}_0^{-1}\Phi_0(\frac{\bar{x}}{\bar{\mu}_0},t)\eta(\frac{4\bar{x}}{\sqrt{t}})\Big)^3 - (u_1+\varphi[\mu])^3 - 3\bar{\mu}_0^{-3}w^2(\frac{\bar{x}}{\bar{\mu}_0})\Phi_0(\frac{\bar{x}}{\bar{\mu}_0},t)\eta(\frac{4\bar{x}}{\sqrt{t}}).\end{aligned}$$

Claim:

$$\begin{aligned}&\Big|S\Big[u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt{t}})\Big]\Big|\lesssim t^{-2}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)\mathbf{1}_{\{|x|\le 9t^{\frac12}\}}\\&+\Big[t^{-1}(\ln t)^2|\mu_1|+(\ln t)^2\Big(|\tilde g[\bar\mu_0,\mu_1]|+|\bar\mu_{0t}|\ln t\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\bar\mu_0(t)}+\frac{|\mu_{1t}(t_1)|}{|\bar\mu_{0t}(t)|}\Big)\Big)\Big]\langle y\rangle^{-4}\mathbf{1}_{\{|x|\le 9t^{\frac12}\}}\\&+|\xi_t|(\ln t)^2\langle y\rangle^{-3}\mathbf{1}_{\{|x|\le 2t^{\frac12}\}}+\Big[|\xi_t|t^{\frac32}(\ln t)^{-1}|x|^{-6}+(t^2(\ln t)^{-1}|x|^{-6})^3\Big]\mathbf{1}_{\{|x|>2t^{\frac12}\}}+t^{-2}(\ln t)^2\eta(\bar y).\end{aligned}\tag{2.30}$$

We need to estimate term by term. Indeed, by (2.28), one has

$$\begin{aligned}&\Big|\bar\mu_0^{-1}\Phi_0(\frac{\bar x}{\bar\mu_0},t)\nabla\eta(\frac{4\bar x}{\sqrt t})\cdot(4t^{-\frac12}\xi_t+2t^{-\frac32}\bar x)\Big|\lesssim(t\ln t)^{-1}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)(t^{-\frac12}|\xi_t|+t^{-1})\mathbf{1}_{\{9^{-1}t^{\frac12}\le|x|\le 9t^{\frac12}\}}\\&\sim\Big[t^{-\frac52}(\ln t)^{-2}|\xi_t|+t^{-3}(\ln t)^{-2}\Big]\mathbf{1}_{\{9^{-1}t^{\frac12}\le|x|\le 9t^{\frac12}\}}.\end{aligned}$$

Also, we have

$$\begin{aligned}&\Big|8t^{-\frac12}\bar\mu_0^{-2}\nabla_{\bar y}\Phi_0(\frac{\bar x}{\bar\mu_0},t)\cdot\nabla\eta(\frac{4\bar x}{\sqrt t})+16t^{-1}\bar\mu_0^{-1}\Phi_0(\frac{\bar x}{\bar\mu_0},t)\Delta\eta(\frac{4\bar x}{\sqrt t})\Big|\\&\lesssim\Big|t^{-\frac32}\langle\bar y\rangle^{-3}\ln(2+|\bar y|)+t^{-2}(\ln t)^{-1}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)\Big|\mathbf{1}_{\{9^{-1}t^{\frac12}\le|x|\le 9t^{\frac12}\}}\sim t^{-3}(\ln t)^{-2}\mathbf{1}_{\{9^{-1}t^{\frac12}\le|x|\le 9t^{\frac12}\}},\end{aligned}$$

$$\begin{aligned}&\Big|\bar\mu_0^{-2}\bar\mu_{0t}\Phi_0(\frac{\bar x}{\bar\mu_0},t)+\bar\mu_0^{-2}\nabla_{\bar y}\Phi_0(\frac{\bar x}{\bar\mu_0},t)\cdot(\bar\mu_{0t}\frac{\bar x}{\bar\mu_0}+\xi_t)-\bar\mu_0^{-1}\partial_t\Phi_0(\frac{\bar x}{\bar\mu_0},t)\Big|\eta(\frac{4\bar x}{\sqrt t})\\&\lesssim\Big[(t\ln t)^{-2}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)+t^{-1}|\xi_t|\langle\bar y\rangle^{-3}\ln(2+|\bar y|)+t^{-2}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)\Big]\eta(\frac{4\bar x}{\sqrt t})\\&\sim\Big(t^{-1}|\xi_t|\langle\bar y\rangle^{-3}\ln(2+|\bar y|)+t^{-2}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)\Big)\eta(\frac{4\bar x}{\sqrt t}).\end{aligned}$$

By Corollary 2.3, we have the following estimates

$$\begin{aligned}&\Big|3\big(\eta^2(\frac{\bar x}{\sqrt t})-\eta(\frac{4\bar x}{\sqrt t})\big)\mu^{-2}w^2(\frac{\bar x}{\mu})\varphi[\mu](\bar x,t)+3\big(\eta(\frac{\bar x}{\sqrt t})-\eta(\frac{4\bar x}{\sqrt t})\big)\mu^{-1}w(\frac{\bar x}{\mu})\varphi^2[\mu](\bar x,t)\Big|\\&\lesssim\Big[(\ln t)^2\langle y\rangle^{-4}(t\ln t)^{-1}+\ln t\langle y\rangle^{-2}(t\ln t)^{-2}\Big]\mathbf{1}_{\{9^{-1}t^{\frac12}\le|\bar x|\le 9t^{\frac12}\}}\sim(t\ln t)^{-3}\mathbf{1}_{\{9^{-1}t^{\frac12}\le|\bar x|\le 9t^{\frac12}\}},\end{aligned}$$

$$\begin{aligned}&\Big|\eta(\frac{4\bar x}{\sqrt t})(\mu^{-2}w^2(\frac{\bar x}{\mu})\varphi[\mu](\bar x,t)-\bar\mu_0^{-2}w^2(\frac{\bar x}{\bar\mu_0})\varphi[\bar\mu_0](\bar x,t))\Big|\\&=\Big|\eta(\frac{4\bar x}{\sqrt t})\Big[\big(\mu^{-2}w^2(\frac{\bar x}{\mu})-\bar\mu_0^{-2}w^2(\frac{\bar x}{\bar\mu_0})\big)\varphi[\mu]+\bar\mu_0^{-2}w^2(\frac{\bar x}{\bar\mu_0})(\varphi[\mu]-\varphi[\bar\mu_0])\Big]\Big|\\&\lesssim\eta(\frac{4\bar x}{\sqrt t})\Big[t^{-1}(\ln t)^2|\mu_1|\langle y\rangle^{-4}+(\ln t)^2\langle y\rangle^{-4}\Big(|\tilde g[\bar\mu_0,\mu_1]|+(t\ln t)^{-1}\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\bar\mu_0(t)}+\frac{|\mu_{1t}(t_1)|}{|\bar\mu_{0t}(t)|}\Big)\Big)\Big],\end{aligned}$$

$$\begin{aligned}&\eta(\frac{4\bar x}{\sqrt t})\Big|\big(\mu^{-1}w(\frac{\bar x}{\mu})\varphi^2[\mu](\bar x,t)-\bar\mu_0^{-1}w(\frac{\bar x}{\bar\mu_0})\varphi^2[\bar\mu_0](\bar x,t)\big)\Big|\\&=\eta(\frac{4\bar x}{\sqrt t})\Big|\big(\mu^{-1}w(\frac{\bar x}{\mu})-\bar\mu_0^{-1}w(\frac{\bar x}{\bar\mu_0})\big)\varphi^2[\mu]+\bar\mu_0^{-1}w(\frac{\bar x}{\bar\mu_0})(\varphi[\mu]-\varphi[\bar\mu_0])(\varphi[\mu]+\varphi[\bar\mu_0])\Big|\\&\lesssim\eta(\frac{4\bar x}{\sqrt t})\Big[\bar\mu_0^{-2}|\mu_1|\langle y\rangle^{-2}(t\ln t)^{-2}+\bar\mu_0^{-1}\langle y\rangle^{-2}(t\ln t)^{-1}\Big(|\mu_1|t^{-1}+|\tilde g[\bar\mu_0,\mu_1]|\\&\quad+|\mu_t|\ln t\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)}+\frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big)\Big)\Big]\\&\sim\eta(\frac{4\bar x}{\sqrt t})\Big[t^{-2}|\mu_1|\langle y\rangle^{-2}+t^{-1}\langle y\rangle^{-2}\Big(|\tilde g[\bar\mu_0,\mu_1]|+(t\ln t)^{-1}\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)}+\frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big)\Big)\Big]\\&\lesssim\eta(\frac{4\bar x}{\sqrt t})\Big[t^{-1}(\ln t)^2|\mu_1|\langle y\rangle^{-4}+(\ln t)^2\langle y\rangle^{-4}\Big(|\tilde g[\bar\mu_0,\mu_1]|+(t\ln t)^{-1}\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\mu(t)}+\frac{|\mu_{1t}(t_1)|}{|\mu_t(t)|}\Big)\Big)\Big].\end{aligned}$$

Using Corollary 2.3, one has

$$\begin{aligned}
&\Big|\xi_t\cdot\nabla_{\bar x}\varphi[\mu](\bar x,t)+\mu^{-2}\xi_t\cdot\nabla w(\frac{\bar x}{\mu})\eta(\frac{\bar x}{\sqrt t})+\mu^{-1}t^{-\frac12}w(\frac{\bar x}{\mu})\xi_t\cdot\nabla\eta(\frac{\bar x}{\sqrt t})\Big|\\
\lesssim\,&|\xi_t|\Big(t^{-1}\langle y\rangle^{-1}\mathbf{1}_{\{|\bar x|\le 2t^{\frac12}\}}+t^{\frac32}(\ln t)^{-1}|\bar x|^{-6}\mathbf{1}_{\{|\bar x|>2t^{\frac12}\}}+(\ln t)^2\langle y\rangle^{-3}\mathbf{1}_{\{|\bar x|\le 2t^{\frac12}\}}\Big)\\
\sim\,&|\xi_t|\Big((\ln t)^2\langle y\rangle^{-3}\mathbf{1}_{\{|x|\le 2t^{\frac12}\}}+t^{\frac32}(\ln t)^{-1}|x|^{-6}\mathbf{1}_{\{|x|>2t^{\frac12}\}}\Big).
\end{aligned}$$

$$|\varphi^3[\mu]|\lesssim(t\ln t)^{-3}\mathbf{1}_{\{|\bar x|\le 2t^{\frac12}\}}+(t^2(\ln t)^{-1}|\bar x|^{-6})^3\mathbf{1}_{\{|\bar x|>2t^{\frac12}\}},\quad \Big|\bar\mu_0^{-2}\mathcal{M}[\bar\mu_0]\frac{\eta(\bar y)Z_5(\bar y)}{\int_{B_2}\eta(z)Z_5^2(z)dz}\Big|\lesssim t^{-2}(\ln t)^2\eta(\bar y).$$

$$\begin{aligned}
&\Big|\Big(u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{\bar x}{\bar\mu_0},t)\eta(\frac{4\bar x}{\sqrt t})\Big)^3-(u_1+\varphi[\mu])^3-3\bar\mu_0^{-3}w^2(\frac{\bar x}{\bar\mu_0})\Phi_0(\frac{\bar x}{\bar\mu_0},t)\eta(\frac{4\bar x}{\sqrt t})\Big|\\
=\,&3\Big|\Big(\mu^{-1}w(\frac{\bar x}{\mu})-\bar\mu_0^{-1}w(\frac{\bar x}{\bar\mu_0})+\varphi[\mu]+\theta\bar\mu_0^{-1}\Phi_0(\frac{\bar x}{\bar\mu_0},t)\eta(\frac{4\bar x}{\sqrt t})\Big)\\
&\times\Big(u_1+\varphi[\mu]+\theta\bar\mu_0^{-1}\Phi_0(\frac{\bar x}{\bar\mu_0},t)\eta(\frac{4\bar x}{\sqrt t})+\bar\mu_0^{-1}w(\frac{\bar x}{\bar\mu_0})\Big)\bar\mu_0^{-1}\Phi_0(\frac{\bar x}{\bar\mu_0},t)\eta(\frac{4\bar x}{\sqrt t})\Big|\\
\lesssim\,&\Big(|\mu_1|\mu_0^{-2}\langle y\rangle^{-2}+(t\ln t)^{-1}+t^{-1}(\ln t)^{-1}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)\Big)\\
&\times\big((t\ln t)^{-1}+\ln t\langle y\rangle^{-2}\big)(t\ln t)^{-1}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)\eta(\frac{4\bar x}{\sqrt t})\\
\lesssim\,&\big(|\mu_1|(\ln t)^2\langle y\rangle^{-2}+(t\ln t)^{-1}\big)t^{-1}\langle y\rangle^{-4}\ln(2+|y|)\eta(\frac{4\bar x}{\sqrt t})\\
=\,&|\mu_1|t^{-1}(\ln t)^2\langle y\rangle^{-6}\ln(2+|y|)\eta(\frac{4\bar x}{\sqrt t})+t^{-2}(\ln t)^{-1}\langle y\rangle^{-4}\ln(2+|y|)\eta(\frac{4\bar x}{\sqrt t}).
\end{aligned}$$

We have completed the proof of claim (2.30).

3. Gluing system and solving the outer problem

In this section, we formulate the inner–outer gluing system such that an infinite time blow-up solution to (2.1) with desired asymptotics can be found. We look for solution of the form

$$u(x,t)=u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})+\psi(x,t)+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)$$

with

$$\eta_R(x,t)=\eta(\frac{x-\xi}{\mu_0R(t)}),\quad R(t)=t^\gamma,\quad 0<\gamma<\frac12,$$

where ψ, ϕ are perturbations in the outer region and inner region, respectively. In order for the following to hold

$$\begin{aligned}
0=\,&S\Big[u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})+\psi+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\Big]\\
=\,&-\partial_t\psi-\partial_t\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)+\eta_R\mu^{-2}\mu_t\big(\phi(\frac{x-\xi}{\mu},t)+\frac{x-\xi}{\mu}\cdot\nabla_y\phi(\frac{x-\xi}{\mu},t)\big)\\
&+\eta_R\mu^{-2}\xi_t\cdot\nabla_y\phi(\frac{x-\xi}{\mu},t)-\eta_R\mu^{-1}\partial_t\phi(\frac{x-\xi}{\mu},t)\\
&+\Delta_x\psi+\Delta_x\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)+2\nabla_x\eta_R\cdot\mu^{-2}\nabla_y\phi(\frac{x-\xi}{\mu},t)+\eta_R\mu^{-3}\Delta_y\phi(\frac{x-\xi}{\mu},t)\\
&+3\big(\mu^{-1}w(\frac{x-\xi}{\mu})\big)^2\big(\psi+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\big)+S\big[u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})\big]\\
&+\Big(u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})+\psi+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\Big)^3\\
&-\Big(u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})\Big)^3-3\big(\mu^{-1}w(\frac{x-\xi}{\mu})\big)^2\big(\psi+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\big),
\end{aligned}$$

it suffices to solving the following inner-outer gluing system for (ψ,ϕ).

The outer problem for ψ:

$$\partial_t\psi = \Delta_x\psi + \mathcal{G}[\psi,\phi,\mu_1,\xi] \quad \text{in } \mathbb{R}^4\times(t_0,\infty),$$

where

$$\begin{aligned}
&\mathcal{G}[\psi,\phi,\mu_1,\xi] := 3\mu^{-2}w^2(\frac{x-\xi}{\mu})\psi(1-\eta_R) + \eta_R\mu^{-2}\xi_t\cdot\nabla_y\phi(\frac{x-\xi}{\mu},t)\\
&+\Delta_x\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t) + 2\nabla_x\eta_R\cdot\mu^{-2}\nabla_y\phi(\frac{x-\xi}{\mu},t) - \partial_t\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\\
&+(1-\eta_R)S\big[u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})\big]\\
&+\Big(u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})+\psi+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\Big)^3\\
&-\Big(u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})\Big)^3 - 3\big(\mu^{-1}w(\frac{x-\xi}{\mu})\big)^2\big(\psi+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\big)\\
&-\Big[3\big(u_1+\varphi[\mu]-\mu^{-1}w(\frac{x-\xi}{\mu})\big)\big(u_1+\varphi[\mu]+\mu^{-1}w(\frac{x-\xi}{\mu})\big)\\
&+6(u_1+\varphi[\mu])\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})\Big]\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t).
\end{aligned}\tag{3.1}$$

The inner problem for ϕ:

$$\mu^2\partial_t\phi(y,t) = \Delta_y\phi(y,t) + 3w^2(y)\phi(y,t) + f_1(y,t)\phi(y,t) + f_2(t)y\cdot\nabla_y\phi(y,t) + \mathcal{H}[\psi,\mu_1,\xi](y,t) \quad \text{in } \mathcal{D}_{4R} \tag{3.2}$$

where $y=\frac{x-\xi}{\mu}$, $\mathcal{D}_{4R}=\{(y,t)\ :\ y\in B_{4R(t)},\ t\in(t_0,\infty)\}$, and

$$f_1(y,t)=\mu\mu_t+\mu^2\Big[3\big(u_1+\varphi[\mu]-\mu^{-1}w(y)\big)\big(u_1+\varphi[\mu]+\mu^{-1}w(y)\big)+6(u_1+\varphi[\mu])\bar\mu_0^{-1}\Phi_0(\frac{\mu y}{\bar\mu_0},t)\Big],\ f_2(t)=\mu\mu_t,$$

$$\mathcal{H}[\psi,\mu_1,\xi](y,t) := \mu^3\Big(3\mu^{-2}w^2(y)\psi(\mu y+\xi,t)+S\big[u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{\mu y}{\bar\mu_0},t)\eta(\frac{4\mu y}{\sqrt t})\big]\Big). \tag{3.3}$$

By (2.30), one has

$$\begin{aligned}
&\big|\mathcal{H}[\psi,\mu_1,\xi](y,t)\big|\\
&\lesssim (\ln t)^{-1}\langle y\rangle^{-4}|\psi(\mu y+\xi,t)| + t^{-2}(\ln t)^{-3}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)\\
&\quad+(t\ln t)^{-1}|\mu_1|\langle y\rangle^{-4}+(\ln t)^{-1}\Big(|\tilde g[\bar\mu_0,\mu_1]|+|\bar\mu_{0t}|\ln t\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\bar\mu_0(t)}+\frac{|\mu_{1t}(t_1)|}{|\bar\mu_{0t}(t)|}\Big)\Big)\langle y\rangle^{-4}\\
&\quad+|\xi_t|(\ln t)^{-1}\langle y\rangle^{-3}+t^{-2}(\ln t)^{-1}\eta(\bar y)\\
&\lesssim (\ln t)^{-1}\langle y\rangle^{-4}|\psi(\mu y+\xi,t)| + t^{a_1\gamma-2}(\ln t)^{-2}\langle y\rangle^{-2-a_1}\\
&\quad+(t\ln t)^{-1}|\mu_1|\langle y\rangle^{-4}+(\ln t)^{-1}\Big(|\tilde g[\bar\mu_0,\mu_1]|+|\bar\mu_{0t}|\ln t\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\bar\mu_0(t)}+\frac{|\mu_{1t}(t_1)|}{|\bar\mu_{0t}(t)|}\Big)\Big)\langle y\rangle^{-4}\\
&\quad+|\xi_t|(\ln t)^{-1}\langle y\rangle^{-3}
\end{aligned}\tag{3.4}$$

where we have used $|y|\le 4R=4t^\gamma$, and for later purpose, we require that

$$a_1\gamma-2<5\delta-\kappa-a\gamma,\ 0<a_1\le 1. \tag{3.5}$$

Here above constants are those which measure the weighted topology for the inner problem (see (3.8)). Notice in $\mathcal{D}_{4R}$, we have

$$\begin{aligned}
|f_1(y,t)|+|f_2(t)| &\lesssim t^{-1}(\ln t)^{-3}+\Big[(t\ln t)^{-1}\ln t\langle y\rangle^{-2}+\ln t\langle y\rangle^{-2}t^{-1}(\ln t)^{-1}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)\Big](\ln t)^{-2}\\
&\sim t^{-1}(\ln t)^{-3}+t^{-1}(\ln t)^{-2}\langle y\rangle^{-2}.
\end{aligned}$$

Remark 3.0.1. *Due to the time decay rate of $f_1(y,t)$, $f_2(t)$, we are forced to put $f_1(y,t)\phi(y,t)+f_2(t)y\cdot\nabla_y\phi(y,t)$ in the linear part of the inner problem. We can not put this term in the right hand side of the outer problem since this will influence the Hölder continuity of ψ about t variable. Besides, we can not use the inner linear theory in [5] since $f_1(y,t)\phi(y,t)+f_2(t)y\cdot\nabla_y\phi(y,t)$ will influence the Hölder about μ_{1t} through the orthogonal equation, which will result in failure to choose suitable topology for solving the inner–outer gluing system. Instead, we rebuild a new inner linear theory in Section 7 to avoid including $f_1(y,t)\phi(y,t)+f_2(t)y\cdot\nabla_y\phi(y,t)$ in the orthogonal equation about μ_1.*

We decompose the inner problem (3.2) into two parts. Set $\phi = \phi_1 + \phi_2$, then it suffices to consider

$$\begin{aligned}\mu^2\partial_t\phi_1(y,t) &= \Delta_y\phi_1(y,t) + 3w^2(y)\phi_1(y,t) + f_1(y,t)\phi_1(y,t) + f_2(t)y\cdot\nabla_y\phi_1(y,t) + \mathcal{H}[\psi,\mu_1,\xi](y,t)\\ &+ \Big(\int_{B_2}\eta(z)Z_5^2(z)dz\Big)^{-1}\Big(-2^{-\frac12}3\int_{B_{2R_0}} w^2(z)Z_5(z)dz + O((t\ln t)^{-1})\Big)\mu\mathcal{E}_\nu[\mu_1]\eta(y)Z_5(y) \quad \text{in } \mathcal{D}_{4R},\end{aligned} \tag{3.6}$$

$$\begin{aligned}\mu^2\partial_t\phi_2(y,t) &= \Delta_y\phi_2(y,t) + 3w^2(y)\phi_2(y,t) + f_1(y,t)\phi_2(y,t) + f_2(t)y\cdot\nabla_y\phi_2(y,t)\\ &- \Big(\int_{B_2}\eta(z)Z_5^2(z)dz\Big)^{-1}\Big(-2^{-\frac12}3\int_{B_{2R_0}} w^2(z)Z_5(z)dz + O((t\ln t)^{-1})\Big)\mu\mathcal{E}_\nu[\mu_1]\eta(y)Z_5(y) \quad \text{in } \mathcal{D}_{4R},\end{aligned} \tag{3.7}$$

where $R_0(\tau) = \tau^\delta$ with $\delta > 0$ very small and

$$\mathcal{E}_\nu[\mu_1] = \int_{t-t^{1-\nu}}^{t-\mu_0^2(t)} \frac{\mu_{1t}(s)-\mu_{1t}(t)}{t-s}ds.$$

Set

$$\tau(t) = \int_{t_0}^t \mu^{-2}(s)ds + t_0(\ln t_0)^2, \quad \tau_0 = t_0(\ln t_0)^2.$$

Then $\tau(t)\sim t(\ln t)^2$ for all $t\ge t_0$. In τ variable, $\mathcal{D}_{4R} = \{(y,\tau)\ :\ y\in B_{4R(t(\tau))},\ t\in(\tau_0,\infty)\}$. It is easy to rewrite (3.6) and (3.7) in the form as in Proposition 7.1 and Lemma 7.5, respectively.

The reason for decomposing the inner problem into above two parts is that the orthogonal equation involving μ_1 is too difficult to solve. More detailed explanations will be given in Section 4.1.

Before stating the solvability of the outer problem, let us first fix the inner solution ϕ to the inner problem, the next order of scaling parameter μ_1 and translating parameter ξ in the spaces with the following norms

$$\|\phi\|_{i,\kappa-5\delta,a} := \sup_{(y,\tau)\in\mathcal{D}_{4R}} \tau^{\kappa-5\delta}\langle y\rangle^a\big(\langle y\rangle|\nabla\phi(y,t(\tau))| + |\phi(y,t(\tau))|\big) \tag{3.8}$$

where κ, a are some positive constants to be determined later.

For $\mu_1(t)\in C^1(\frac{t_0}{4},\infty)$, $\mu_1(t)\to 0$ as $t\to\infty$, denote

$$\|\mu_1\|_{*1} := \sup_{t\ge t_0/4}\big[\ln t(t(\ln t)^2)^{5\delta-\kappa}R(t)^{-a}\big]^{-1}|\mu_{1t}|. \tag{3.9}$$

For $\xi(t) = (\xi_1(t),\dots,\xi_4(t))\in C^1(t_0,\infty)$, $\xi(t)\to 0$ as $t\to\infty$, denote

$$\|\xi\|_{*2} := \max_{1\le j\le 4}\sup_{t\ge t_0}\big[(\ln t)^2(t(\ln t)^2)^{5\delta-\kappa}R(t)^{-a}\big]^{-1}|\xi_{jt}|. \tag{3.10}$$

The outer problem is solved in the following Proposition.

Proposition 3.1. *Consider*

$$\partial_t\psi(x,t) = \Delta\psi(x,t) + \mathcal{G}[\psi,\phi,\mu_1,\xi] \ \text{ in } \mathbb{R}^4\times(t_0,\infty), \quad \psi(x,t_0) = 0 \ \text{ in } \mathbb{R}^4 \tag{3.11}$$

where $\mathcal{G}[\psi,\phi,\mu_1,\xi]$ is given in (3.1). Assume ϕ, μ_1, ξ satisfy $\|\phi\|_{i,\kappa-5\delta,a}, \|\mu_1\|_{*1}, \|\xi\|_{*2} < \Lambda_1$ where $\Lambda_1 > 1$ is a constant and the parameters satisfy

$$5\delta-\kappa-a\gamma > -2,\ 5\delta-\kappa < -1,\ 0<a<2,\ 0<\gamma<\frac12, \tag{3.12}$$

then there exists a solution $\psi = \psi[\phi,\mu_1,\xi]$ with the following estimates:

$$\begin{aligned}&|\psi(x,t)| \le C(\Lambda_1)\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}\left(\mathbf{1}_{\{|x|\le t^{\frac12}\}} + t|x|^{-2}\mathbf{1}_{\{|x|>t^{\frac12}\}}\right),\\ &|\nabla\psi(x,t)| \le C(\Lambda_1)\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a},\\ &\sup_{s_1,s_2\in(t-\frac{\lambda^2(t)}{4},t)}\frac{|\psi(x,s_1)-\psi(x,s_2)|}{|s_1-s_2|^\alpha} \le C(\Lambda_1,\alpha)\Big\{\lambda^{-2\alpha}(t)\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}\\ &\qquad + \lambda^{2-2\alpha}(t)\left[(\mu_0R)^{-2}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a} + (\ln t)^3(t(\ln t)^2)^{10\delta-2\kappa}\right]\Big\}\end{aligned}$$

where $0<\lambda(t)\le t^{\frac12}$.

The proof is postponed to Section B.

4. ORTHOGONAL EQUATIONS FOR μ_1, ξ

4.1. **Solving μ_1 and ξ.** In order to utilize Proposition 7.1 with $R_0 = \tau^\delta \sim (t(\ln t)^2)^\delta$ where $\delta > 0$ is small, one needs to adjust μ_1, ξ such that $c_i[\mathcal{H}] = 0$, $i = 1, \dots, 5$ in Proposition 7.1 with $\mathcal{H}$ given in (3.3).

However, for $i = 5$, it is too difficult to solve $c_5[\mathcal{H}] = 0$ thoroughly. We are only able to make $c_5[\mathcal{H}] \approx 0$ and leave smaller remainder to be solved by the non-orthogonal linear theory of the inner problem.

In this section, we only care about the estimate in $|y| \le 4R$ since this is served for the inner problem. Set

$$\mathcal{M}_i[\psi, \mu_1, \xi] = \int_{B_{2R_0}} \mathcal{H}[\psi, \mu_1, \xi](y,t) Z_i(y) dy, \quad i = 1, \dots, 5,$$

$$\mathcal{H}_5(|y|,t) = \int_{S^3} \mathcal{H}[\psi, \mu_1, \xi](|y|\theta, t)\Upsilon_0(\theta) d\theta, \ \mathcal{H}_i(|y|,t) = \int_{S^3} \mathcal{H}[\psi, \mu_1, \xi](|y|\theta, t)\Upsilon_i(\theta) d\theta, \quad i = 1, \dots, 4$$

where Υ_i are spherical harmonic functions, which are given in Section 7.

Using (2.29), for $i = 5$, since Z_5 is radially symmetric, one has

$$\begin{aligned}
&\mathcal{M}_5[\psi, \mu_1, \xi] = \int_{B_{2R_0}} 3\mu w^2(y) \psi(\mu y + \xi, t) Z_5(y) dy \\
&+ \int_{B_{2R_0}} \mu^3 \Big(\bar\mu_0^{-2} \bar\mu_{0t} \Phi_0(\frac{\mu y}{\bar\mu_0}, t) + \bar\mu_0^{-2} \nabla_{\bar y}\Phi_0(\frac{\mu y}{\bar\mu_0}, t) \cdot \bar\mu_{0t} \frac{\mu y}{\bar\mu_0} - \bar\mu_0^{-1} \partial_t \Phi_0(\frac{\mu y}{\bar\mu_0}, t) \Big) Z_5(y) dy \\
&+ \int_{B_{2R_0}} 3\mu^3 \Big(\mu^{-2} w^2(y) \varphi[\mu](\mu y, t) - \bar\mu_0^{-2} w^2(\frac{\mu y}{\bar\mu_0}) \varphi[\bar\mu_0](\mu y, t) \Big) Z_5(y) dy \\
&+ \int_{B_{2R_0}} 3\mu^3 \Big(\mu^{-1} w(y) \varphi^2[\mu](\mu y, t) - \bar\mu_0^{-1} w(\frac{\mu y}{\bar\mu_0}) \varphi^2[\bar\mu_0](\mu y, t) \Big) Z_5(y) dy \\
&+ \int_{B_{2R_0}} \mu^3 \varphi^3[\mu](\mu y, t) Z_5(y) dy + \int_{B_{4R}} 3\mu^3 \bar\mu_0^{-2} \mathcal{M}[\bar\mu_0] \frac{\eta(\bar\mu_0^{-1}\mu y) Z_5(\bar\mu_0^{-1}\mu y)}{\int_{B_2} \eta(z) Z_5^2(z) dz} Z_5(y) dy \\
&+ \int_{B_{2R_0}} \mu^3 \Big[(u_1 + \varphi[\mu] + \bar\mu_0^{-1}\Phi_0(\frac{\mu y}{\bar\mu_0}, t)\eta(\frac{4\mu y}{\sqrt{t}}))^3 - (u_1 + \varphi[\mu])^3 - 3\bar\mu_0^{-3} w^2(\frac{\mu y}{\bar\mu_0}) \Phi_0(\frac{\mu y}{\bar\mu_0}, t) \eta(\frac{4\mu y}{\sqrt{t}}) \Big] Z_5(y) dy,
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{H}_5(|y|,t) = \int_{S^3} 3\mu w^2(|y|\theta) \psi(\mu|y|\theta + \xi, t) \Upsilon_0(\theta) d\theta \\
&+ \int_{S^3} \mu^3 \Big(\bar\mu_0^{-2} \bar\mu_{0t} \Phi_0(\frac{\mu|y|\theta}{\bar\mu_0}, t) + \bar\mu_0^{-2} \nabla_{\bar y}\Phi_0(\frac{\mu|y|\theta}{\bar\mu_0}, t) \cdot \bar\mu_{0t} \frac{\mu|y|\theta}{\bar\mu_0} - \bar\mu_0^{-1} \partial_t \Phi_0(\frac{\mu|y|\theta}{\bar\mu_0}, t) \Big) \Upsilon_0(\theta) d\theta \\
&+ \int_{S^3} 3\mu^3 \Big(\mu^{-2} w^2(|y|\theta) \varphi[\mu](\mu|y|\theta, t) - \bar\mu_0^{-2} w^2(\frac{\mu|y|\theta}{\bar\mu_0}) \varphi[\bar\mu_0](\mu|y|\theta, t) \Big) \Upsilon_0(\theta) d\theta \\
&+ \int_{S^3} 3\mu^3 \Big(\mu^{-1} w(|y|\theta) \varphi^2[\mu](\mu|y|\theta, t) - \bar\mu_0^{-1} w(\frac{\mu|y|\theta}{\bar\mu_0}) \varphi^2[\bar\mu_0](\mu|y|\theta, t) \Big) \Upsilon_0(\theta) d\theta \\
&+ \int_{S^3} \mu^3 \varphi^3[\mu](\mu|y|\theta, t) \Upsilon_0(\theta) d\theta + \int_{S^3} 3\mu^3 \bar\mu_0^{-2} \mathcal{M}[\bar\mu_0] \frac{\eta(\bar\mu_0^{-1}\mu|y|\theta) Z_5(\bar\mu_0^{-1}\mu|y|\theta)}{\int_{B_2} \eta(z) Z_5^2(z) dz} \Upsilon_0(\theta) d\theta \\
&+ \int_{S^3} \mu^3 \Big[(u_1 + \varphi[\mu] + \bar\mu_0^{-1}\Phi_0(\frac{\mu|y|\theta}{\bar\mu_0}, t)\eta(\frac{4\mu|y|\theta}{\sqrt{t}}))^3 - (u_1 + \varphi[\mu])^3 \\
&- 3\bar\mu_0^{-3} w^2(\frac{\mu|y|\theta}{\bar\mu_0}) \Phi_0(\frac{\mu|y|\theta}{\bar\mu_0}, t) \eta(\frac{4\mu|y|\theta}{\sqrt{t}}) \Big] \Upsilon_0(\theta) d\theta.
\end{aligned}$$

For $i = 1, \dots, 4$, we have

$$\begin{aligned}
\mathcal{M}_i[\psi, \mu_1, \xi] &= \int_{B_{2R_0}} 3\mu w^2(y) \psi(\mu y + \xi, t) Z_i(y) dy + \xi_{it} \int_{B_{2R_0}} \mu^3 \bar\mu_0^{-2} \partial_{\bar y_i} \Phi_0(\frac{\mu y}{\bar\mu_0}, t) Z_i(y) dy \\
&+ \xi_{it} \int_{B_{2R_0}} \Big(\mu^3 \partial_{\bar x_i} \varphi(\mu y, t) Z_i(y) + \mu Z_i^2(y) \Big) dy \\
&= \int_{B_{2R_0}} 3\mu w^2(y) \psi(\mu y + \xi, t) Z_i(y) dy + \mu \xi_{it} \Big(\int_{B_{2R_0}} Z_i^2(y) dy + O(t^{-\frac{1}{2}}) \Big)
\end{aligned}$$

by Corollary 2.3 and (2.28). Also,

$$\mathcal{H}_i(|y|,t) = \int_{S^3} 3\mu w^2(|y|\theta)\psi(\mu|y|\theta+\xi,t)\Upsilon_i(\theta)d\theta + \xi_{it}\int_{S^3}\mu^3\bar\mu_0^{-2}\partial_{\bar y_i}\Phi_0(\frac{\mu|y|\theta}{\bar\mu_0},t)\Upsilon_i(\theta)d\theta$$
$$+\xi_{it}\int_{S^3}\mu^3\Big(\partial_{\bar x_i}\varphi(\mu|y|\theta,t)+\mu^{-2}\partial_{z_i}w(|y|\theta)\eta(\frac{\mu|y|\theta}{\sqrt t})\Big)\Upsilon_i(\theta)d\theta.$$

Using similar calculations as in (2.30), one has

$$|\mathcal{H}_5(|y|,t)| \lesssim (\ln t)^{-1}\langle y\rangle^{-3}\sup_{z\in B_{4R(t)}}\langle z\rangle^{-1}|\psi(\mu z+\xi,t)| + (\ln t)^{-3}\Big\{t^{-2}\langle \bar y\rangle^{-2}\ln(2+|\bar y|)$$
$$+\Big[t^{-1}(\ln t)^2|\mu_1| + (\ln t)^2\Big(|\tilde g[\bar\mu_0,\mu_1]| + |\bar\mu_{0t}|\ln t\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\bar\mu_0(t)}+\frac{|\mu_{1t}(t_1)|}{|\bar\mu_{0t}(t)|}\Big)\Big)\Big]\langle y\rangle^{-4} + t^{-2}(\ln t)^2\mathbf{1}_{\{|y|\le 4\}}\Big\}$$
$$\lesssim (\ln t)^{-1}\langle y\rangle^{-3}\sup_{z\in B_{4R(t)}}\langle z\rangle^{-1}|\psi(\mu z+\xi,t)| + t^{a_1\gamma-2}(\ln t)^{-2}\langle y\rangle^{-2-a_1}$$
$$+\Big[(t\ln t)^{-1}|\mu_1| + (\ln t)^{-1}\Big(|\tilde g[\bar\mu_0,\mu_1]| + |\bar\mu_{0t}|\ln t\sup_{t_1\in[t/2,t]}\Big(\frac{|\mu_1(t_1)|}{\bar\mu_0(t)}+\frac{|\mu_{1t}(t_1)|}{|\bar\mu_{0t}(t)|}\Big)\Big)\Big]\langle y\rangle^{-4}$$

where $a_1>0$ is chosen such that $a_1\gamma-2<5\delta-\kappa-a\gamma$. It then follows that

$$|\mathcal{H}_i(|y|,t)| \lesssim (\ln t)^{-1}\langle y\rangle^{-3}\sup_{z\in B_{4R(t)}}\langle z\rangle^{-1}|\psi(\mu z+\xi,t)| + |\xi_{it}|(\ln t)^2\langle y\rangle^{-3}, \quad i=1,\dots 4.$$

By Proposition 7.1, the orthogonal equation $c_i[\mathcal{H}][\tau]=0$ $(i=1,\dots,4)$ is equivalent to solving

$$\mathcal{M}_i[\psi,\mu_1,\xi] + (t(\ln t)^2)^{-\delta\epsilon_0}O\big(\sup_{y\in B_{4R(t)}}\langle y\rangle^3|\mathcal{H}_i(y,t)|\big) = \int_{B_{2R_0}}3\mu w^2(y)\psi(\mu y+\xi,t)Z_i(y)dy$$
$$+\mu\xi_{it}\big(\int_{B_{2R_0}}Z_i^2(y)dy + O(t^{-\frac12})\big) + (t(\ln t)^2)^{-\delta\epsilon_0}O\Big((\ln t)^{-1}\sup_{z\in B_{4R(t)}}\langle z\rangle^{-1}|\psi(\mu z+\xi,t)| + |\xi_{it}|(\ln t)^2\Big) = 0$$

where $\epsilon_0>0$ is given in Proposition 7.1. One can write above equation as

$$\xi_{it} = \Pi_i[\mu_1,\xi] \tag{4.1}$$

where

$$\Pi_i[\mu_1,\xi] = \big(\int_{B_{2R_0}}Z_i^2(y)dy + O(t(\ln t)^2)^{-\frac{\delta\epsilon_0}{2}}\big)^{-1}\Big[-\int_{B_{2R_0}}3w^2(y)\psi(\mu y+\xi,t)Z_i(y)dy$$
$$-(t(\ln t)^2)^{-\delta\epsilon_0}O\big(\sup_{z\in B_{4R(t)}}\langle z\rangle^{-1}|\psi(\mu z+\xi,t)|\big)\Big].$$

Let us estimate $\mathcal{M}_5$ term by term. By (2.28), one has

$$\int_{B_{2R_0}}\mu^3\Big[\bar\mu_0^{-2}\bar\mu_{0t}\Phi_0(\frac{\mu y}{\bar\mu_0},t) + \bar\mu_0^{-2}\nabla_{\bar y}\Phi_0(\frac{\mu y}{\bar\mu_0},t)\cdot\bar\mu_{0t}\frac{\mu y}{\bar\mu_0} - \bar\mu_0^{-1}\partial_t\Phi_0(\frac{\mu y}{\bar\mu_0},t)\Big]Z_5(y)dy = O(t^{-2}(\ln t)^{-1}).$$

By Corollary 2.3 and the special choice of $\bar\mu_0$, we have for $|\bar x|\le 2t^{\frac12}$

$$\varphi[\bar\mu_0] = -2^{-\frac12}\Big(\bar\mu_0 t^{-1} + \int_{t/2}^{t-\mu_0^2}\frac{\bar\mu_{0t}(s)}{t-s}ds\Big) + O\Big(\bar\mu_0 t^{-2}|\bar x|^2 + |\bar\mu_{0t}|\frac{|\bar x|}{\bar\mu_0}\Big) + O(t^{-1}(\ln t)^{-2})$$
$$= O(t^{-1}(\ln t)^{-2}\ln\ln t) + O\Big(\bar\mu_0 t^{-2}|\bar x|^2 + |\bar\mu_{0t}|\frac{|\bar x|}{\bar\mu_0}\Big).$$

Notice that

$$\mu^{-2}w^2(y) - \bar\mu_0^{-2}w^2(\frac{\mu y}{\bar\mu_0}) = -2\mu_1(\theta\mu+(1-\theta)\bar\mu_0)^{-3}(w^2(y_\theta)+w(y_\theta)\nabla w(y_\theta)\cdot y_\theta)\Big|_{y_\theta=\frac{x-\xi}{\theta\mu+(1-\theta)\bar\mu_0}}$$
$$= -2\mu_1\bar\mu_0^{-3}(w^2(y)+w(y)\nabla w(y)\cdot y) + O(\mu_1^2\bar\mu_0^{-4}\langle y\rangle^{-4}).$$

Then by Corollary 2.3, it follows that

$$\int_{B_{2R_0}}3\mu^3\Big(\mu^{-2}w^2(y)\varphi[\mu](\mu y,t) - \bar\mu_0^{-2}w^2(\frac{\mu y}{\bar\mu_0})\varphi[\bar\mu_0](\mu y,t)\Big)Z_5(y)dy$$
$$= 3\mu^3\int_{B_{2R_0}}\Big[\mu^{-2}w^2(y)(\varphi[\mu]-\varphi[\bar\mu_0]) + \big(\mu^{-2}w^2(y) - \bar\mu_0^{-2}w^2(\frac{\mu y}{\bar\mu_0})\big)\varphi[\bar\mu_0]\Big]Z_5(y)dy$$

$$
\begin{aligned}
&= 3\mu^3 \int_{B_{2R_0}} \Big\{ \mu^{-2} w^2(y) Z_5(y) \Big[-2^{-\frac{1}{2}} \Big(\mu_1 t^{-1} + \int_{t/2}^{t-\mu_0^2} \frac{\mu_{1t}(s)}{t-s} ds \Big) \\
&\quad + O\Big(|\mu_1| \bar\mu_0^2 t^{-2} \frac{|\bar x|^2}{\bar\mu_0^2} + |\bar\mu_{0t}| \sup_{t_1 \in [t/2,t]} \Big(\frac{|\mu_1(t_1)|}{\bar\mu_0} + \frac{|\mu_{1t}(t_1)|}{|\bar\mu_{0t}|} \Big) \frac{|\bar x|}{\bar\mu_0} \Big) + \tilde g[\bar\mu_0, \mu_1] \Big] \\
&\quad + \Big[-2\mu_1 \bar\mu_0^{-3} (w^2(y) + w(y) \nabla w(y) \cdot y) + O(\mu_1^2 \bar\mu_0^{-4} \langle y \rangle^{-4}) \Big] Z_5(y) \\
&\quad \times \Big(O(t^{-1} (\ln t)^{-2} \ln\ln t) + O(\bar\mu_0 t^{-2} |\bar x|^2 + |\bar\mu_{0t}| \frac{|\bar x|}{\bar\mu_0}) \Big) \Big\} dy \\
&= \mu \Big[-2^{-\frac{1}{2}} 3 \int_{B_{2R_0}} w^2(y) Z_5(y) dy \Big(\mu_1 t^{-1} + \int_{t/2}^{t-\mu_0^2} \frac{\mu_{1t}(s)}{t-s} ds \Big) \\
&\quad + O\Big((t \ln t)^{-1} \sup_{t_1 \in [t/2,t]} |\mu_1(t_1)| + \sup_{t_1 \in [t/2,t]} |\mu_{1t}(t_1)| \Big) + \tilde g[\bar\mu_0, \mu_1] + O(|\mu_1| t^{-1} (\ln t)^{-1} \ln\ln t) \Big].
\end{aligned}
$$

By Corollary 2.3, we have

$$
\begin{aligned}
&\int_{B_{2R_0}} 3\mu^3 \Big(\mu^{-1} w(y) \varphi^2[\mu](\mu y, t) - \bar\mu_0^{-1} w(\frac{\mu y}{\bar\mu_0}) \varphi^2[\bar\mu_0](\mu y, t) \Big) Z_5(y) dy \\
&= 3\mu^3 \int_{B_{2R_0}} \Big[\mu^{-1} w(y) (\varphi[\mu] - \varphi[\bar\mu_0])(\varphi[\mu] + \varphi[\bar\mu_0]) + \big(\mu^{-1} w(y) - \bar\mu_0^{-1} w(\frac{\mu y}{\bar\mu_0}) \big) \varphi^2[\bar\mu_0] \Big] Z_5(y) dy \\
&= 3\mu^3 \int_{B_{2R_0}} \Big\{ \mu^{-1} w(y) Z_5(y) \Big[-2^{-\frac{1}{2}} \Big(\mu_1 t^{-1} + \int_{t/2}^{t-\mu_0^2} \frac{\mu_{1t}(s)}{t-s} ds \Big) \\
&\quad + O\Big(|\mu_1| \bar\mu_0^2 t^{-2} \frac{|\bar x|^2}{\bar\mu_0^2} + |\bar\mu_{0t}| \sup_{t_1 \in [t/2,t]} \Big(\frac{|\mu_1(t_1)|}{\bar\mu_0} + \frac{|\mu_{1t}(t_1)|}{|\bar\mu_{0t}|} \Big) \frac{|\bar x|}{\bar\mu_0} \Big) + \tilde g[\bar\mu_0, \mu_1] \Big] (t \ln t)^{-1} \\
&\quad + (-\mu_1 \bar\mu_0^{-2} (w(y) + y \cdot \nabla w(y)) + O(\mu_1^2 \bar\mu_0^{-3}) \langle y \rangle^{-2}) Z_5(y) (t \ln t)^{-2} \Big\} dy \\
&= \mu^3 \Big\{ \Big[\Big(\mu_1 t^{-1} + \int_{t/2}^{t-\mu_0^2} \frac{\mu_{1t}(s)}{t-s} ds \Big) O(\ln t) + O\Big(|\mu_1| \bar\mu_0^2 t^{-2} R_0^2 + |\bar\mu_{0t}| \sup_{t_1 \in [t/2,t]} \Big(\frac{|\mu_1(t_1)|}{\bar\mu_0} + \frac{|\mu_{1t}(t_1)|}{|\bar\mu_{0t}|} \Big) R_0 \Big) \\
&\quad + O(\ln t) \tilde g[\bar\mu_0, \mu_1] \Big] O(t^{-1}) + \mu_1 \bar\mu_0^{-2} O(t^{-2} (\ln t)^{-1}) \Big\} \\
&= \mu \Big[\Big(\mu_1 t^{-1} + \int_{t/2}^{t-\mu_0^2} \frac{\mu_{1t}(s)}{t-s} ds \Big) O((t \ln t)^{-1}) \\
&\quad + O\Big(t^{-\frac{3}{2}} \sup_{t_1 \in [t/2,t]} |\mu_1(t_1)| + t^{-\frac{1}{2}} \sup_{t_1 \in [t/2,t]} |\mu_{1t}(t_1)| \Big) + O((t \ln t)^{-1}) \tilde g[\bar\mu_0, \mu_1] \Big]
\end{aligned}
$$

since $\delta > 0$ is very small and $\mu^{-1} w(y) - \bar\mu_0^{-1} w(\frac{\mu y}{\bar\mu_0}) = -\mu_1 \bar\mu_0^{-2} (w(y) + y \cdot \nabla w(y)) + O(\mu_1^2 \bar\mu_0^{-3}) \langle y \rangle^{-2}$. Similarly, the following estimates hold

$$
\int_{B_{2R_0}} \mu^3 \varphi^3[\mu](\mu y, t) Z_5(y) dy = O(\mu^3 (t \ln t)^{-3} R_0^2) = O(t^{-\frac{5}{2}})
$$

when δ is small enough.

$$
\Big| \int_{B_{2R_0}} \mu^3 \bar\mu_0^{-2} \mathcal{M}[\bar\mu_0] \frac{\eta(\bar y) Z_5(\bar y)}{\int_{B_2} \eta(z) Z_5^2(z) dz} Z_5(y) dy \Big| \lesssim O(t^{-2} (\ln t)^{-1}),
$$

$$
\begin{aligned}
&\int_{B_{2R_0}} \mu^3 \Big[\big(u_1 + \varphi[\mu] + \bar\mu_0^{-1} \Phi_0(\frac{\mu y}{\bar\mu_0}, t) \eta(\frac{4\mu y}{\sqrt t}) \big)^3 - (u_1 + \varphi[\mu])^3 - 3\bar\mu_0^{-3} w^2(\frac{\mu y}{\bar\mu_0}) \Phi_0(\frac{\mu y}{\bar\mu_0}, t) \eta(\frac{4\mu y}{\sqrt t}) \Big] Z_5(y) dy \\
&= \int_{B_{2R_0}} \mu^3 \Big[\big(u_1 + \varphi[\mu] + \bar\mu_0^{-1} \Phi_0(\frac{\mu y}{\bar\mu_0}, t) \big)^3 - (u_1 + \varphi[\mu])^3 - 3\mu^{-2} w^2(y) \bar\mu_0^{-1} \Phi_0(\frac{\mu y}{\bar\mu_0}, t) \\
&\quad + 3\big(\mu^{-2} w^2(y) - \bar\mu_0^{-2} w^2(\frac{\mu y}{\bar\mu_0}) \big) \bar\mu_0^{-1} \Phi_0(\frac{\mu y}{\bar\mu_0}, t) \Big] Z_5(y) dy = \mu O(|\mu_1| (t \ln t)^{-1} \ln\ln t) + O(t^{-2} (\ln t)^{-4})
\end{aligned}
$$

since by (2.28),

$$
\begin{aligned}
&\int_{B_{2R_0}} 3\mu^3\Big[(\mu^{-1}w(y))^2-(\bar\mu_0^{-1}w(\frac{\mu y}{\bar\mu_0}))^2\Big]\bar\mu_0^{-1}\Phi_0(\frac{\mu y}{\bar\mu_0},t)Z_5(y)dy\\
&=3\mu^3\bar\mu_0^{-1}\int_{B_{2R_0}}\Big[-2\mu_1\bar\mu_0^{-3}(w^2(y)+w(y)\nabla w(y)\cdot y)+O(\mu_1^2\bar\mu_0^{-4}\langle y\rangle^{-4})\Big]O(t^{-1}(\ln t)^{-3}\ln\ln t\langle y\rangle^{-1})Z_5(y)dy\\
&=\mu O(|\mu_1|(t\ln t)^{-1}\ln\ln t),
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_{B_{2R_0}}\mu^3\Big[\big(u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{\mu y}{\bar\mu_0},t)\big)^3-(u_1+\varphi[\mu])^3-3\mu^{-2}w^2(y)\bar\mu_0^{-1}\Phi_0(\frac{\mu y}{\bar\mu_0},t)\Big]Z_5(y)dy\right|\\
&=\left|\mu^3\int_{B_{2R_0}}\Big[3(u_1+\varphi[\mu])\big(\bar\mu_0^{-1}\Phi_0(\frac{\mu y}{\bar\mu_0},t)\big)^2+\big(\bar\mu_0^{-1}\Phi_0(\frac{\mu y}{\bar\mu_0},t)\big)^3\right.\\
&\quad\left.+3(u_1+\varphi[\mu]-\mu^{-1}w(y))(u_1+\varphi[\mu]+\mu^{-1}w(y))\bar\mu_0^{-1}\Phi_0(\frac{\mu y}{\bar\mu_0},t)\Big]Z_5(y)dy\right|\\
&\lesssim\mu(\ln t)^{-2}\int_{B_{2R_0}}\Big[(\ln t\langle y\rangle^{-2}+(t\ln t)^{-1})((t\ln t)^{-1}\langle\bar y\rangle^{-2}\ln(2+|\bar y|))^2+((t\ln t)^{-1}\langle\bar y\rangle^{-2}\ln(2+|\bar y|))^3\\
&\quad+\big(\ln t\langle y\rangle^{-2}\mathbf{1}_{\{|\bar x|\ge t^{\frac12}\}}+(t\ln t)^{-1}\big)(\ln t\langle y\rangle^{-2}+(t\ln t)^{-1})(t\ln t)^{-1}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)\Big]\langle y\rangle^{-2}dy\\
&\lesssim\mu(\ln t)^{-2}\int_{B_{2R_0}}\Big[\ln t\langle y\rangle^{-2}((t\ln t)^{-1}\langle\bar y\rangle^{-2}\ln(2+|\bar y|))^2+((t\ln t)^{-1}\langle\bar y\rangle^{-2}\ln(2+|\bar y|))^3\\
&\quad+(t\ln t)^{-1}\ln t\langle y\rangle^{-2}(t\ln t)^{-1}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)\Big]\langle y\rangle^{-2}dy\lesssim t^{-2}(\ln t)^{-4}.
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
&\mathcal{M}_5[\psi,\mu_1,\xi]=\int_{B_{2R_0}}3\mu w^2(y)\psi(\mu y+\xi,t)Z_5(y)dy+O(t^{-2}(\ln t)^{-1})\\
&+\mu\Big[-2^{-\frac12}3\int_{B_{2R_0}}w^2(y)Z_5(y)dy\Big(\mu_1t^{-1}+\int_{t/2}^{t-\mu_0^2}\frac{\mu_{1t}(s)}{t-s}ds\Big)\\
&+O\big((t\ln t)^{-1}\sup_{t_1\in[t/2,t]}|\mu_1(t_1)|+\sup_{t_1\in[t/2,t]}|\mu_{1t}(t_1)|\big)+\tilde g[\bar\mu_0,\mu_1]+O(|\mu_1|t^{-1}(\ln t)^{-1}\ln\ln t)\Big]\\
&+\mu\Big[\Big(\mu_1t^{-1}+\int_{t/2}^{t-\mu_0^2}\frac{\mu_{1t}(s)}{t-s}ds\Big)O((t\ln t)^{-1})\\
&+O\big(t^{-\frac32}\sup_{t_1\in[t/2,t]}|\mu_1(t_1)|+t^{-\frac12}\sup_{t_1\in[t/2,t]}|\mu_{1t}(t_1)|\big)+O((t\ln t)^{-1})\tilde g[\bar\mu_0,\mu_1]\Big]\\
&+\mu O(|\mu_1|(t\ln t)^{-1}\ln\ln t)\\
&=\mu\Big\{\int_{B_{2R_0}}3w^2(y)\psi(\mu y+\xi,t)Z_5(y)dy+O(t^{-2})\\
&+O\big((t\ln t)^{-1}\sup_{t_1\in[t/2,t]}|\mu_1(t_1)|+\sup_{t_1\in[t/2,t]}|\mu_{1t}(t_1)|\big)+O(\tilde g[\bar\mu_0,\mu_1])+O(|\mu_1|(t\ln t)^{-1}\ln\ln t)\\
&+\Big(-2^{-\frac12}3\int_{B_{2R_0}}w^2(y)Z_5(y)dy+O((t\ln t)^{-1})\Big)\Big(\mu_1t^{-1}+\int_{t/2}^{t-\mu_0^2}\frac{\mu_{1t}(s)}{t-s}ds\Big)\Big\}\\
&=\mu\big(-2^{-\frac12}3\int_{B_{2R_0}}w^2(y)Z_5(y)dy+O((t\ln t)^{-1})\big)
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big\{ \big(-2^{-\frac{1}{2}} 3 \int_{B_{2R_0}} w^2(y) Z_5(y) dy + O((t\ln t)^{-1}) \big)^{-1} \int_{B_{2R_0}} 3w^2(y)\psi(\mu y+\xi,t) Z_5(y) dy \\
&+ O(t^{-2}) + O\big((t\ln t)^{-1} \sup_{t_1\in[t/2,t]} |\mu_1(t_1)| + \sup_{t_1\in[t/2,t]} |\mu_{1t}(t_1)| \big) + O(\tilde g[\bar\mu_0,\mu_1]) \\
&+ \mu_1 t^{-1}\big(1 + O((\ln t)^{-1}\ln\ln t)\big) + \int_{t/2}^{t-t^{1-\nu}} \frac{\mu_{1t}(s)}{t-s} ds + \mu_{1t}\big((1-\nu)\ln t + 2\ln\ln t\big) + \mathcal{E}_\nu[\mu_1] \Big\}
\end{aligned}
$$

where

$$
\mathcal{E}_\nu[\mu_1] = \int_{t-t^{1-\nu}}^{t-\mu_0^2(t)} \frac{\mu_{1t}(s)-\mu_{1t}(t)}{t-s} ds.
$$

By Proposition 7.1, $c_5[\mathcal{H}] = 0$ is equivalent to

$$
\begin{aligned}
&\mathcal{M}_5[\psi,\mu_1,\xi] + (t(\ln t)^2)^{-\delta\epsilon_0} O\big(\sup_{y\in B_{4R(t)}} \langle y\rangle^{2+a_1} |\mathcal{H}_5(y,t)| \big) \\
=\ & \mu\big(-2^{-\frac{1}{2}} 3 \int_{B_{2R_0}} w^2(y) Z_5(y) dy + O((t\ln t)^{-1}) \big) \\
&\times \Big\{ \big(-2^{-\frac{1}{2}} 3 \int_{B_{2R_0}} w^2(y) Z_5(y) dy + O((t\ln t)^{-1}) \big)^{-1} \int_{B_{2R_0}} 3w^2(y)\psi(\mu y+\xi,t) Z_5(y) dy \\
&+ O(t^{-2}) + O((t\ln t)^{-1} \sup_{t_1\in[t/2,t]} |\mu_1(t_1)| + \sup_{t_1\in[t/2,t]} |\mu_{1t}(t_1)|) + O(\tilde g[\bar\mu_0,\mu_1]) \\
&+ \mu_1 t^{-1}(1 + O((\ln t)^{-1}\ln\ln t)) + \int_{t/2}^{t-t^{1-\nu}} \frac{\mu_{1t}(s)}{t-s} ds + \mu_{1t}((1-\nu)\ln t + 2\ln\ln t) + \mathcal{E}_\nu[\mu_1] \Big\} \\
&+ (t(\ln t)^2)^{-\delta\epsilon_0} O\Bigg((\ln t)^{-1} \sup_{y\in B_{4R(t)}} \langle y\rangle^{-1} |\psi(\mu y+\xi,t)| + t^{a_1\gamma-2}(\ln t)^{-2} \\
&+ (t\ln t)^{-1}|\mu_1| + (\ln t)^{-1} \left(|\tilde g[\bar\mu_0,\mu_1]| + |\bar\mu_{0t}| \ln t \sup_{t_1\in[t/2,t]} \left(\frac{|\mu_1(t_1)|}{\bar\mu_0(t)} + \frac{|\mu_{1t}(t_1)|}{|\bar\mu_{0t}(t)|} \right) \right) \Bigg) \\
=\ & \mu\big(-2^{-\frac{1}{2}} 3 \int_{B_{2R_0}} w^2(y) Z_5(y) dy + O((t\ln t)^{-1}) \big) \\
&\times \Big\{ \big(-2^{-\frac{1}{2}} 3 \int_{B_{2R_0}} w^2(y) Z_5(y) dy + O((t\ln t)^{-1}) \big)^{-1} \int_{B_{2R_0}} 3w^2(y)\psi(\mu y+\xi,t) Z_5(y) dy \\
&+ O(t^{-2}) + O((t\ln t)^{-1} \sup_{t_1\in[t/2,t]} |\mu_1(t_1)| + \sup_{t_1\in[t/2,t]} |\mu_{1t}(t_1)|) + O(\tilde g[\bar\mu_0,\mu_1]) \\
&+ \mu_1 t^{-1}(1 + O((\ln t)^{-1}\ln\ln t)) + \int_{t/2}^{t-t^{1-\nu}} \frac{\mu_{1t}(s)}{t-s} ds + \mu_{1t}((1-\nu)\ln t + 2\ln\ln t) + \mathcal{E}_\nu[\mu_1] \\
&+ (t(\ln t)^2)^{-\delta\epsilon_0} O(\sup_{y\in B_{4R(t)}} \langle y\rangle^{-1} |\psi(\mu y+\xi,t)| + t^{a_1\gamma-2}(\ln t)^{-1}) \Big\} = 0
\end{aligned}
$$

where we have used similar calculations as in (3.4), and

$$
\begin{aligned}
&\sup_{y\in B_{4R(t)}} \langle y\rangle^{2+a_1} |\mathcal{H}_5(y,t)| \lesssim (\ln t)^{-1} \sup_{y\in B_{4R(t)}} \langle y\rangle^{-1} |\psi(\mu y+\xi,t)| + t^{a_1\gamma-2}(\ln t)^{-2} \\
&+ (t\ln t)^{-1}|\mu_1| + (\ln t)^{-1} \left(|\tilde g[\bar\mu_0,\mu_1]| + |\bar\mu_{0t}| \ln t \sup_{t_1\in[t/2,t]} \left(\frac{|\mu_1(t_1)|}{\bar\mu_0(t)} + \frac{|\mu_{1t}(t_1)|}{|\bar\mu_{0t}(t)|} \right) \right).
\end{aligned}
$$

Similar to the methodology in Section 2.3, we leave $\mathcal{E}_\nu[\mu_1]$ as the remainder term and consider the following equation about μ_1.

$$
\begin{aligned}
&\big(-2^{-\frac{1}{2}}3\int_{B_{2R_0}} w^2(y)Z_5(y)dy + O((t\ln t)^{-1})\big)^{-1}\int_{B_{2R_0}} 3w^2(y)\psi(\mu y+\xi,t)Z_5(y)dy\\
&+O\big((t\ln t)^{-1}\sup_{t_1\in[t/2,t]}|\mu_1(t_1)| + \sup_{t_1\in[t/2,t]}|\mu_{1t}(t_1)|\big) + O(\tilde g[\bar\mu_0,\mu_1])\\
&+\mu_1 t^{-1}(1+O((\ln t)^{-1}\ln\ln t)) + \int_{t/2}^{t-t^{1-\nu}}\frac{\mu_{1t}(s)}{t-s}ds + \mu_{1t}((1-\nu)\ln t + 2\ln\ln t)\\
&+(t(\ln t)^2)^{-\delta\epsilon_0}O(\sup_{y\in B_{4R(t)}}\langle y\rangle^{-1}|\psi(\mu y+\xi,t)|) + (t(\ln t)^2)^{-\delta\epsilon_0}O(t^{a_1\gamma-2}(\ln t)^{-1}) = 0
\end{aligned}
$$

when $a_1\gamma > \delta\epsilon_0$. That is,

$$
\mu_{1t} + \beta_v(t)\mu_1 = \Pi_5[\mu_1,\xi] \tag{4.2}
$$

where

$$
\begin{aligned}
&\beta_\nu(t) = t^{-1}\left[(1-\nu)\ln t + 2\ln\ln t\right]^{-1}(1+O((\ln t)^{-1}\ln\ln t)),\\
&\Pi_5[\mu_1,\xi] = \chi(t)((1-\nu)\ln t + 2\ln\ln t)^{-1}\Bigg[-\int_{t/2}^{t-t^{1-\nu}}\frac{\mu_{1t}(s)}{t-s}ds - O((t\ln t)^{-1}\sup_{t_1\in[t/2,t]}|\mu_1(t_1)| + \sup_{t_1\in[t/2,t]}|\mu_{1t}(t_1)|)\\
&-O(\tilde g[\bar\mu_0,\mu_1]) + \big(2^{-\frac{1}{2}}3\int_{B_{2R_0}} w^2(y)Z_5(y)dy + O((t\ln t)^{-1})\big)^{-1}\int_{B_{2R_0}} 3w^2(y)\psi(\mu y+\xi,t)Z_5(y)dy\\
&-(t(\ln t)^2)^{-\delta\epsilon_0}O(\sup_{y\in B_{4R(t)}}\langle y\rangle^{-1}|\psi(\mu y+\xi,t)|) - (t(\ln t)^2)^{-\delta\epsilon_0}O(t^{a_1\gamma-2}(\ln t)^{-1})\Bigg].
\end{aligned} \tag{4.3}
$$

Similar to (2.21), in order to solve (4.2) and (4.1), it is sufficient to consider the following fixed point problem:

$$
\begin{aligned}
&\mathcal{S}_5[\mu_1,\xi](t) = \Pi_5[\mu_1,\xi](t) + \beta_\nu(t)e^{-\int^t\beta_\nu(u)du}\int_t^\infty e^{\int^s\beta_\nu(u)du}\Pi_5[\mu_1,\xi](s)ds,\\
&\mathcal{S}_i[\mu_1,\xi](t) = \Pi_i[\mu_1,\xi],\quad i=1,\dots 4.
\end{aligned} \tag{4.4}
$$

Notice that $|\psi|\lesssim \ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}$ and recall the norms (3.9), (3.10) for μ_1, ξ. We will solve (4.4) in the following spaces

$$
B_{\mu_1} = \{\mu_1\in C^1(t_0/4,\infty)\ :\ \|\mu_1\|_{*1}\le 2\},\quad B_\xi = \{\xi\in C^1(t_0,\infty)\ :\ \|\xi\|_{*2}\le 2\}. \tag{4.5}
$$

For any $\mu_{1a},\mu_{1b}\in B_{\mu_1}$ and $\xi_a,\xi_b\in B_\xi$, similar to (2.22), one has

$$
\begin{aligned}
&\chi(t)\Big|\int_{t/2}^{t-t^{1-\nu}}\frac{\mu_{1at}(s)-\mu_{1bt}(s)}{t-s}ds\Big| \le \|\mu_{1a}-\mu_{1b}\|_{*1}\int_{t/2}^{t-t^{1-\nu}}\frac{\ln s(s(\ln s)^2)^{5\delta-\kappa}R^{-a}(s)}{t-s}ds\\
&=\|\mu_{1a}-\mu_{1b}\|_{*1}(1+O((\ln t)^{-1}))\nu(\ln t)^2(t(\ln t)^2)^{5\delta-\kappa}R^{-a}.
\end{aligned}
$$

By gradient estimate in Proposition 3.1, we have

$$
\begin{aligned}
&\Big|\int_{B_{2R_0}} w^2(y)(\psi(\mu_{1a}y+\xi_a,t)-\psi(\mu_{1b}y+\xi_b,t))Z_5(y)dy\Big| \le C\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(|\mu_{1a}-\mu_{1b}|+|\xi_a-\xi_b|)\\
&\le Ct\ln t[\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}]^2(\|\mu_{1a}-\mu_{1b}\|_{*1}+\|\xi_a-\xi_b\|_{*2}).
\end{aligned}
$$

The estimate for $\int_{B_{2R_0}}3w^2(y)\psi(\mu y+\xi,t)Z_i(y)dy$ is the same.

$$
\begin{aligned}
&(t(\ln t)^2)^{-\delta\epsilon_0}\Big|O(\sup_{y\in B_{4R(t)}}\langle y\rangle^{-1}|\psi(\mu_{1a}y+\xi_a,t)|) - O(\sup_{y\in B_{4R(t)}}\langle y\rangle^{-1}|\psi(\mu_{1b}y+\xi_b,t)|)\Big|\\
&\le (t(\ln t)^2)^{-\delta\epsilon_0}\Big|O(\sup_{y\in B_{4R(t)}}\langle y\rangle^{-1}|\psi(\mu_{1a}y+\xi_a,t)-\psi(\mu_{1b}y+\xi_b,t)|)\Big|\\
&\le C(t(\ln t)^2)^{-\delta\epsilon_0}t\ln t[\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}]^2(\|\mu_{1a}-\mu_{1b}\|_{*1}+\|\xi_a-\xi_b\|_{*2})
\end{aligned}
$$

since $O(\sup_{y\in B_{4R(t)}}\langle y\rangle^{-1}|\psi(\mu y+\xi,t)|)$ depends on ψ linearly.

From the same calculations as in (B.1), one has

$$
\chi(t)|\tilde g[\bar\mu_0,\mu_{1a}]-\tilde g[\bar\mu_0,\mu_{1b}]| \le C\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}\|\mu_{1a}-\mu_{1b}\|_{*1}
$$

when $5\delta-\kappa-a\gamma>-2$. Similar to (2.23), one has

$$\begin{aligned}&\chi(t)\Big|O((t\ln t)^{-1}\sup_{t_1\in[t/2,t]}|\mu_{1a}(t_1)|+\sup_{t_1\in[t/2,t]}|\mu_{1at}(t_1)|)-O((t\ln t)^{-1}\sup_{t_1\in[t/2,t]}|\mu_{1b}(t_1)|+\sup_{t_1\in[t/2,t]}|\mu_{1bt}(t_1)|)\Big|\\&\lesssim\chi(t)O\big((t\ln t)^{-1}\sup_{t_1\in[t/2,t]}|\mu_{1a}(t_1)-\mu_{1b}(t_1)|+\sup_{t_1\in[t/2,t]}|\mu_{1at}(t_1)-\mu_{1bt}(t_1)|\big)\\&\lesssim\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}\|\mu_{1a}-\mu_{1b}\|_{*1}.\end{aligned}$$

In conclusion, under the following restrictions

$$a_1>0,\ a_1\gamma-2<5\delta-\kappa-a\gamma,\ a_1\gamma>\delta\epsilon_0,\ 5\delta-\kappa-a\gamma>-2,\ 0<\nu<\frac{1}{2},\tag{4.6}$$

for t_0 is sufficiently large, $(\mathcal{S}_5,\mathcal{S}_i)$ is a contraction mapping in $B_{\mu_1}\times B_\xi$.

Similarly, for $(\mu_1,\xi)\in B_{\mu_1}\times B_\xi$, we have

$$\chi(t)\Big|\int_{t/2}^{t-t^{1-\nu}}\frac{\mu_{1t}(s)}{t-s}ds\Big|\le\|\mu_1\|_{*1}(1+O((\ln t)^{-1}))v(\ln t)^2(t(\ln t)^2)^{5\delta-\kappa}R^{-a},$$

$$|\tilde g[\bar\mu_0,\mu_1]|\le C\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}\|\mu_1\|_{*1},$$

$$\chi(t)\Big|O((t\ln t)^{-1}\sup_{t_1\in[t/2,t]}|\mu_1(t_1)|+\sup_{t_1\in[t/2,t]}|\mu_{1t}(t_1)|)\Big|\lesssim\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}\|\mu_1\|_{*1}.$$

Then

$$(\mathcal{S}_5,\mathcal{S}_i):B_{\mu_1}\times B_\xi\to B_{\mu_1}\times B_\xi.$$

Consequently, by the contraction mapping theorem, we find a unique solution (μ_1,ξ) in $B_{\mu_1}\times B_\xi$.

4.2. **Hölder continuity of μ_{1t} and estimate for $\mu\mathcal{E}_\nu[\mu_1]$.** In order to estimate the left error

$$\mathcal{E}_\nu[\mu_1]=\int_{t-t^{1-\nu}}^{t-\mu_0^2(t)}\frac{\mu_{1t}(s)-\mu_{1t}(t)}{t-s}ds,$$

we need Hölder estimate of μ_{1t}, which satisfies

$$\mu_{1t}=\Pi_5[\mu_1,\xi](t)+\beta_\nu(t)e^{-\int^t\beta_\nu(u)du}\int_t^\infty e^{\int^s\beta_\nu(u)du}\Pi_5[\mu_1,\xi](s)ds.$$

Assume $\frac{3t}{4}\le t_2<t_1\le t$, $\frac{8}{9}<A<1$. A will be chosen to be close to 1 later depending on ν and independent of t_0. We revisit (4.3) term by term.

Notice that ψ only has Hölder continuity in t variable, which restricts the regularity for μ_{1t}. Using Proposition 3.1 with $\lambda^2(t)=t^{\frac12}$, one has

$$\begin{aligned}&|\psi(\mu(t_1)y+\xi(t_1),t_1)-\psi(\mu(t_2)y+\xi(t_2),t_2)|\\&\le|\psi(\mu(t_1)y+\xi(t_1),t_1)-\psi(\mu(t_2)y+\xi(t_2),t_1)|+|\psi(\mu(t_2)y+\xi(t_2),t_1)-\psi(\mu(t_2)y+\xi(t_2),t_2)|\\&\lesssim\big[\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(t)|y|+(\ln t)^2(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(t)\big]\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(t)|t_1-t_2|\\&\quad+C(\alpha)\Big\{\lambda^{-2\alpha}(t)\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(t)\\&\quad+\lambda^{2-2\alpha}(t)[(\mu_0R)^{-2}(t)\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(t)+(\ln t)^3(t(\ln t)^2)^{10\delta-2\kappa}]\Big\}|t_1-t_2|^\alpha\end{aligned}$$

which implies

$$\begin{aligned}&(\ln t_1)^{-1}\Big|\int_{B_{2R_0}}w^2(y)Z_5(y)(\psi(\mu(t_1)y+\xi(t_1),t_1)-\psi(\mu(t_2)y+\xi(t_2),t_2))dy\Big|\\&\lesssim[\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(t)]^2|t_1-t_2|+C(\alpha)\Big\{\lambda^{-2\alpha}(t)(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(t)\\&\quad+\lambda^{2-2\alpha}(t)[(\mu_0R)^{-2}(t)(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(t)+(\ln t)^2(t(\ln t)^2)^{10\delta-2\kappa}]\Big\}|t_1-t_2|^\alpha.\end{aligned}$$

Similarly, $(t(\ln t)^2)^{-\delta\epsilon_0}O(\sup_{y\in B_{4R(t)}}\langle y\rangle^{-1}|\psi(\mu y+\xi,t)|)$ provides the same Hölder estimate as above.

Reviewing the analysis details in solving (4.4), one has

$$|\Pi_5[\mu_1,\xi]| \lesssim \ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}\chi(t).$$

Then

$$\begin{aligned}
&\left|(\beta_\nu(t)e^{-\int^t \beta_\nu(u)du}\int_t^\infty e^{\int^s \beta_\nu(u)du}\Pi_5[\mu_1,\xi](s)ds)'\right| = \Big|\beta_\nu'(t)e^{-\int^t \beta_\nu(u)du}\int_t^\infty e^{\int^s \beta_\nu(u)du}\Pi_5[\mu_1,\xi](s)ds \\
&- \beta_\nu^2(t)e^{-\int^t \beta_\nu(u)du}\int_t^\infty e^{\int^s \beta_\nu(u)du}\Pi_5[\mu_1,\xi](s)ds - \beta_\nu(t)\Pi_5[\mu_1,\xi]\Big| \\
&\lesssim t^{-2}(\ln t)^{-1}e^{-\int^t \beta_\nu(u)du}\int_t^\infty e^{\int^s \beta_\nu(u)du}\ln s(s(\ln s)^2)^{5\delta-\kappa}R^{-a}(s)ds \\
&+ (t\ln t)^{-2}e^{-\int^t \beta_\nu(u)du}\int_t^\infty e^{\int^s \beta_\nu(u)du}\ln s(s(\ln s)^2)^{5\delta-\kappa}R^{-a}(s)ds + (t\ln t)^{-1}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a} \\
&\lesssim (t\ln t)^{-1}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}
\end{aligned}$$

where in the last inequality, we have used $e^{-\int^t \beta_\nu(u)du}\int_t^\infty e^{\int^s \beta_\nu(u)du}\ln s(s(\ln s)^2)^{5\delta-\kappa}R^{-a}(s)ds \lesssim t\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}$ when t_0 is sufficiently large. Also

$$\left|[((1-\nu)\ln t+2\ln\ln t)^{-1}\chi(t)]'[((1-\nu)\ln t+2\ln\ln t)\Pi_5[\mu_1,\xi]]\right| \lesssim t^{-1}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}1_{\{t\ge\frac{3t_0}{4}\}},$$

$$\begin{aligned}
&\left|((1-\nu)\ln t+2\ln\ln t)^{-1}[-O((t\ln t)^{-1}\sup_{\tau_1\in[t/2,t]}|\mu_1(\tau_1)|) - O(\tilde g[\bar\mu_0,\mu_1]) - (t(\ln t)^2)^{-\delta\epsilon_0}O(t^{a_1\gamma-2}(\ln t)^{-1})]'\right| \\
&\lesssim (\ln t)^{-1}(t^{-1}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a} + t^{-1}(t(\ln t)^2)^{-\delta\epsilon_0}O(t^{a_1\gamma-2}(\ln t)^{-1})) \lesssim (t\ln t)^{-1}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}.
\end{aligned}$$

In order to get the estimate

$$\begin{aligned}
&((1-\nu)\ln t+2\ln\ln t)^{-1}\left|O(\sup_{\tau_1\in[t_1/2,t_1]}|\mu_{1t}(\tau_1)|) - O(\sup_{\tau_1\in[t_2/2,t_2]}|\mu_{1t}(\tau_1)|)\right| \\
&\lesssim (\ln t)^{-1}(C(A)t^{-1}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}|t_1-t_2| + [\mu_{1t}]_{C^\alpha(\frac{3At}{4},t)}|t_1-t_2|^\alpha),
\end{aligned}$$

rigorously speaking, we need to estimate all the terms that appeared in the proof of Lemma 2.1 and Lemma 2.2 except the leading term. For simplicity, we take $\tilde\varphi_{1b}[\mu+\mu_1]-\tilde\varphi_{1b}[\mu]$ as an example to illustrate the key idea. We decompose $\tilde\varphi_{1b}[\mu+\mu_1]-\tilde\varphi_{1b}[\mu]$ into two parts to estimate.

$$\begin{aligned}
&(\tilde\varphi_{1b}[\mu+\mu_1]-\tilde\varphi_{1b}[\mu])(\bar x,t) = \mathcal{T}_4^{out}[-\mu_{1t}\hat\varphi_1 + (E-\tilde E)[\mu+\mu_1] - (E-\tilde E)[\mu]](\bar x,t) \\
&= \left(\int_{t_0}^{At} + \int_{At}^t\right)\int_{\mathbb{R}^4}(4\pi(t-s))^{-2}e^{-\frac{|\bar x-z|^2}{4(t-s)}}(-\mu_{1t}\hat\varphi_1 + (E-\tilde E)[\mu+\mu_1] - (E-\tilde E)[\mu])(z,s)dzds.
\end{aligned}$$

Here $\bar x$ is regarded to be independent of t. Then

$$\begin{aligned}
&\left|\partial_t(\int_{t_0}^{At}\int_{\mathbb{R}^4}(4\pi(t-s))^{-2}e^{-\frac{|\bar x-z|^2}{4(t-s)}}(-\mu_{1t}\hat\varphi_1 + (E-\tilde E)[\mu+\mu_1] - (E-\tilde E)[\mu])(z,s)dzds)\right| \\
&= \Big|A\int_{\mathbb{R}^4}(4\pi(t-At))^{-2}e^{-\frac{|\bar x-z|^2}{4(t-At)}}(-\mu_{1t}\hat\varphi_1 + (E-\tilde E)[\mu+\mu_1] - (E-\tilde E)[\mu])(z,At)dz \\
&+ \int_{t_0}^{At}\int_{\mathbb{R}^4}\partial_t((4\pi(t-s))^{-2}e^{-\frac{|\bar x-z|^2}{4(t-s)}})(-\mu_{1t}\hat\varphi_1 + (E-\tilde E)[\mu+\mu_1] - (E-\tilde E)[\mu])(z,s)dzds\Big| \\
&\lesssim C(A)t^{-2}\int_{\mathbb{R}^4}e^{-\frac{|\bar x-z|^2}{4(t-At)}}\Big(|\mu_{1t}(At)|t^{-1}\mathbf{1}_{\{|z|\le t^{\frac12}\}} + |\mu_{1t}(At)||z|^{-2}e^{-\frac{|z|^2}{4t}}\mathbf{1}_{\{|z|>t^{\frac12}\}} \\
&+ |\mu_1(At)|\mu^2(At)t^{-3}\mathbf{1}_{\{2^{-1}\sqrt t\le|z|\le 4\sqrt t\}}\Big)dz \\
&+ \int_{t_0}^{At}\int_{\mathbb{R}^4}(t-s)^{-3}e^{-\frac{|\bar x-z|^2}{8(t-s)}}\left|-\mu_{1t}\hat\varphi_1 + (E-\tilde E)[\mu+\mu_1] - (E-\tilde E)[\mu])(z,s)\right|dzds \\
&\lesssim C(A)t^{-2}\int_{\mathbb{R}^4}e^{-\frac{|z|^2}{4(t-At)}}\Big(|\mu_{1t}(At)|t^{-1}\mathbf{1}_{\{|z|\le t^{\frac12}\}} + |\mu_{1t}(At)||z|^{-2}e^{-\frac{|z|^2}{4t}}\mathbf{1}_{\{|z|>t^{\frac12}\}}
\end{aligned}$$

$$
\begin{aligned}
&+|\mu_1(At)|\mu^2(At)t^{-3}\mathbf{1}_{\{|z|\le 4\sqrt{t}\}}\Big)dz\\
&+C(A)t^{-1}\int_{t_0}^{At}\int_{\mathbb{R}^4}(t-s)^{-2}e^{-\frac{|\bar{x}-z|^2}{8(t-s)}}\Big|(-\mu_{1t}\hat{\varphi}_1+(E-\tilde{E})[\mu+\mu_1]-(E-\tilde{E})[\mu])(z,s)\Big|dzds\\
\lesssim\, & C(A)\Big[t^{-1}|\mu_{1t}(At)|+t^{-3}|\mu_1(At)|\mu^2(At)\\
&+t^{-1}\int_{t_0}^{At}\int_{\mathbb{R}^4}(t-s)^{-2}e^{-\frac{|\bar{x}-z|^2}{8(t-s)}}\Big|(-\mu_{1t}\hat{\varphi}_1+(E-\tilde{E})[\mu+\mu_1]-(E-\tilde{E})[\mu])(z,s)\Big|dzds\Big]\\
\lesssim\, & C(A)t^{-1}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}
\end{aligned}
$$

where the last inequality follows from the same calculations as in (2.11).

For the other part, we have

$$
\begin{aligned}
&\int_{At}^{t}\int_{\mathbb{R}^4}(4\pi(t-s))^{-2}e^{-\frac{|\bar{x}-z|^2}{4(t-s)}}\Big(-\mu_{1t}\hat{\varphi}_1+(E-\tilde{E})[\mu+\mu_1]-(E-\tilde{E})[\mu]\Big)(z,s)dzds\\
=&\int_{A}^{1}\int_{\mathbb{R}^4}t(t-ta)^{-2}e^{-\frac{|\bar{x}-z|^2}{4(t-ta)}}\Big(-\mu_{1t}\hat{\varphi}_1+(E-\tilde{E})[\mu+\mu_1]-(E-\tilde{E})[\mu]\Big)(z,ta)dzda.
\end{aligned}
$$

The terms independent of μ_{1t} are C^1 in time variable t. We only need to focus on the terms including μ_{1t}. By similar calculations in (2.11), we have

$$
\begin{aligned}
&\Big|\int_{A}^{1}\int_{\mathbb{R}^4}\partial_t(t(t-ta)^{-2}e^{-\frac{|\bar{x}-z|^2}{4(t-ta)}})(\mu_{1t}\hat{\varphi}_1)(z,ta)dzda\Big|\lesssim\int_{A}^{1}\int_{\mathbb{R}^4}(t-ta)^{-2}e^{-\frac{|\bar{x}-z|^2}{8(t-ta)}})|\mu_{1t}\hat{\varphi}_1(z,ta)|dzda\\
\lesssim\, & t^{-1}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a},
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\int_{A}^{1}\int_{\mathbb{R}^4}t(t-ta)^{-2}e^{-\frac{|\bar{x}-z|^2}{4(t-ta)}}(\mu_{1t}(t_1a)-\mu_{1t}(t_2a))\hat{\varphi}_1(z,ta)dzda\Big|\\
\lesssim&\int_{A}^{1}\int_{\mathbb{R}^4}t(t-ta)^{-2}e^{-\frac{|\bar{x}-z|^2}{4(t-ta)}}[\mu_{1t}]_{C^\alpha(\frac{3At}{4},t)}|t_1-t_2|^\alpha a^\alpha|\hat{\varphi}_1(z,ta)|dzda\lesssim[\mu_{1t}]_{C^\alpha(\frac{3At}{4},t)}|t_1-t_2|^\alpha.
\end{aligned}
$$

Next, for $\int_{t/2}^{t-t^{1-\nu}}\frac{\mu_{1t}(s)}{t-s}ds=(\int_{t/2}^{At}+\int_{At}^{t-t^{1-\nu}})\frac{\mu_{1t}(s)}{t-s}ds$, we have

$$
\Big|(\int_{t/2}^{At}\frac{\mu_{1t}(s)}{t-s}ds)'\Big|=\Big|\frac{A\mu_{1t}(At)}{t-At}-\frac{\mu_{1t}(\frac{t}{2})}{t}-\int_{t/2}^{At}\frac{\mu_{1t}(s)}{(t-s)^2}ds\Big|\lesssim C(A)t^{-1}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a},
$$

and

$$
\begin{aligned}
&\Big|\int_{At_1}^{t_1-t_1^{1-\nu}}\frac{\mu_{1t}(s)}{t_1-s}ds-\int_{At_2}^{t_2-t_2^{1-\nu}}\frac{\mu_{1t}(s)}{t_2-s}ds\Big|=\Big|\int_{A}^{1-t_1^{-\nu}}\frac{\mu_{1t}(t_1z)}{1-z}dz-\int_{A}^{1-t_2^{-\nu}}\frac{\mu_{1t}(t_2z)}{1-z}dz\Big|\\
=&\Big|\int_{A}^{1-t_1^{-\nu}}\frac{\mu_{1t}(t_1z)-\mu_{1t}(t_2z)}{1-z}dz+\int_{1-t_2^{-\nu}}^{1-t_1^{-\nu}}\frac{\mu_{1t}(t_2z)}{1-z}dz\Big|\\
\le&|t_1-t_2|^\alpha[\mu_{1t}]_{C^\alpha(\frac{3At}{4},t)}\int_{A}^{1-t_1^{-\nu}}\frac{z^\alpha}{1-z}dz+C\int_{1-t_2^{-\nu}}^{1-t_1^{-\nu}}\frac{(t_2z)^{5\delta-\kappa-a\gamma}(\ln(t_2z))^{1+2(5\delta-\kappa)}}{1-z}dz\\
=&|t_1-t_2|^\alpha[\mu_{1t}]_{C^\alpha(\frac{3At}{4},t)}\nu\ln t_1(1+O(|\ln(1-A)|(\ln t_1)^{-1}))\\
&+C(1+O((\ln t_2)^{-1}))t_2^{5\delta-\kappa-a\gamma}(\ln t_2)^{1+2(5\delta-\kappa)}\int_{1-t_2^{-\nu}}^{1-t_1^{-\nu}}\frac{z^{5\delta-\kappa-a\gamma}}{1-z}dz\\
\le&|t_1-t_2|^\alpha[\mu_{1t}]_{C^\alpha(\frac{3At}{4},t)}\nu\ln t_1(1+O(|\ln(1-A)|(\ln t_1)^{-1}))\\
&+C(1+O((\ln t_2)^{-1}))t_2^{5\delta-\kappa-a\gamma}(\ln t_2)^{1+2(5\delta-\kappa)}(1+O(t^{-1}))\nu|\ln t_1-\ln t_2|\\
\le&|t_1-t_2|^\alpha[\mu_{1t}]_{C^\alpha(\frac{3At}{4},t)}\nu\ln t_1(1+O(|\ln(1-A)|(\ln t_1)^{-1}))\\
&+C(1+O((\ln t_2)^{-1}))t_2^{5\delta-\kappa-a\gamma}(\ln t_2)^{1+2(5\delta-\kappa)}(1+O(t^{-1}))\nu t_2^{-1}|t_1-t_2|.
\end{aligned}
$$

Combining the estimates above, one gets that

$$\begin{aligned}
&|\mu_{1t}(t_1)-\mu_{1t}(t_2)|\\
\le\ & C(A)[(t\ln t)^{-1}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(t)+t^{-1}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(t)1_{\{t\ge\frac{3t_0}{4}\}}]|t_1-t_2|\\
&+C(\alpha)\Big\{\lambda^{-2\alpha}(t)(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(t)\\
&+\lambda^{2-2\alpha}(t)[(\mu_0R)^{-2}(t)(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(t)+(\ln t)^2(t(\ln t)^2)^{10\delta-2\kappa}]\Big\}|t_1-t_2|^\alpha\\
&+|t_1-t_2|^\alpha[\mu_{1t}]_{C^\alpha(\frac{3At}{4},t)}[\nu(1-\nu)^{-1}(1+O(|\ln(1-A)|(\ln t)^{-1})+O((\ln t)^{-1}))]
\end{aligned}$$

where we have used $\frac{3t}{4}\le t_2<t_1\le t$. Thus one has

$$[\mu_{1t}]_{C^\alpha(\frac{3t}{4},t)}\le C(A,\alpha)\rho(t)+[\mu_{1t}]_{C^\alpha(\frac{3At}{4},t)}[\nu(1-\nu)^{-1}(1+O(|\ln(1-A)|(\ln t_0)^{-1})+O((\ln t)^{-1}))]\mathbf{1}_{\{t\ge\frac{3t_0}{4}\}}$$

where

$$\rho(t)=t^{-\alpha}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}+t^{1-\alpha}[(\mu_0R)^{-2}(t(\ln t)^2)^{5\delta-\kappa}R^{-a}+(\ln t)^2(t(\ln t)^2)^{10\delta-2\kappa}].$$

Thus

$$\begin{aligned}
&\sup_{\frac{t_0}{2}\le t\le T}\rho^{-1}(t)[\mu_{1t}]_{C^\alpha(\frac{3t}{4},t)}\\
\le\ & C(A,\alpha)+[\nu(1-\nu)^{-1}(1+O(|\ln(1-A)|(\ln t_0)^{-1}))+O((\ln t_0)^{-1})]\sup_{\frac{t_0}{2}\le t\le T}\rho^{-1}(t)[\mu_{1t}]_{C^\alpha(\frac{3At}{4},t)}\mathbf{1}_{\{t\ge\frac{3t_0}{4}\}}\\
=\ & C(A,\alpha)+[\nu(1-\nu)^{-1}(1+O(|\ln(1-A)|(\ln t_0)^{-1}))+O((\ln t_0)^{-1})]\sup_{\frac{3t_0}{4}\le t\le T}\rho^{-1}(t)[\mu_{1t}]_{C^\alpha(\frac{3At}{4},t)}.
\end{aligned}$$

Notice

$$\begin{aligned}
&\rho^{-1}(t)[\mu_{1t}]_{C^\alpha(\frac{3At}{4},t)}=\rho^{-1}(t)\sup_{s_1,s_2\in(\frac{3At}{4},t)}\frac{|\mu_{1t}(s_1)-\mu_{1t}(s_2)|}{|s_1-s_2|^\alpha}\\
=\ &\max\Bigg\{\rho^{-1}(t)\sup_{s_1,s_2\in(\frac{3t}{4},t)}\frac{|\mu_{1t}(s_1)-\mu_{1t}(s_2)|}{|s_1-s_2|^\alpha},\rho^{-1}(t)\sup_{s_1,s_2\in(\frac{3At}{4},At)}\frac{|\mu_{1t}(s_1)-\mu_{1t}(s_2)|}{|s_1-s_2|^\alpha},\\
&\rho^{-1}(t)\sup_{s_2\in(\frac{3At}{4},\frac{3t}{4}),s_1\in(At,t)}\frac{|\mu_{1t}(s_1)-\mu_{1t}(s_2)|}{|s_1-s_2|^\alpha}\Bigg\}\\
\le\ &\max\Bigg\{\rho^{-1}(t)\sup_{s_1,s_2\in(\frac{3t}{4},t)}\frac{|\mu_{1t}(s_1)-\mu_{1t}(s_2)|}{|s_1-s_2|^\alpha},\rho^{-1}(t)\rho(At)\rho^{-1}(At)\sup_{s_1,s_2\in(\frac{3At}{4},At)}\frac{|\mu_{1t}(s_1)-\mu_{1t}(s_2)|}{|s_1-s_2|^\alpha}\Bigg\}+C,
\end{aligned}$$

then one has

$$\sup_{\frac{3t_0}{4}\le t\le T}\rho^{-1}(t)[\mu_{1t}]_{C^\alpha(\frac{3At}{4},t)}\le A^{C(\alpha,\delta,\kappa,a)}(1+O((\ln t_0)^{-1}))\sup_{\frac{t_0}{2}\le t\le T}\rho^{-1}(t)[\mu_{1t}]_{C^\alpha(\frac{3t}{4},t)}+C(A,\alpha).$$

Thus, when $\nu<\frac12$, taking A close to 1 sufficiently, which depends on ν, and then making t_0 large enough, one has $\sup_{\frac{t_0}{2}\le t\le T}\rho^{-1}(t)[\mu_{1t}]_{C^\alpha(\frac{3t}{4},t)}\le C(\nu,\alpha)$. Making $T\to\infty$, one finally gets

$$\sup_{t\ge\frac{t_0}{2}}\rho^{-1}(t)[\mu_{1t}]_{C^\alpha(\frac{3t}{4},t)}\le C(\nu,\alpha). \tag{4.7}$$

Finally, we estimate $\mu\mathcal{E}_\nu[\mu_1]$ as follows

$$\begin{aligned}
&|\mu\mathcal{E}_\nu[\mu_1]|\lesssim C(\nu,\alpha)\{(\ln t)^{-1}t^{-\nu\alpha}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}\\
&+(\ln t)^{-1}t^{1-\nu\alpha}(\mu_0R)^{-2}(t(\ln t)^2)^{5\delta-\kappa}R^{-a}+(\ln t)^{-1}t^{1-\nu\alpha}(\ln t)^2(t(\ln t)^2)^{10\delta-2\kappa}\}.
\end{aligned} \tag{4.8}$$

Although $C_{\nu,\alpha}$ goes to ∞ as $\nu\to\frac12$ and $\alpha\to1$, the smallness is given by $t_0^{-\epsilon}$ where $\epsilon>0$ when solving (3.7). Once ν and α are fixed, we take t_0 large enough.

5. Solving the inner problem

Recalling (3.8), for any fixed $\phi \in B_i$ with

$$B_i = \{\phi \ : \ \|\phi\|_{i,\kappa-5\delta,a} \le 2C_i\} \tag{5.1}$$

where $C_i > 1$ is a constant, we have found $\psi[\phi] \in B_o$, $\mu_1[\phi] \in B_{\mu_1}$ and $\xi[\phi] \in B_\xi$. We abbreviate $\mathcal{H}[\phi] = \mathcal{H}[\psi[\phi], \mu_1[\phi], \xi[\phi]]$. By (3.4), (3.5) and (4.5), we obtain $|\mathcal{H}[\phi](y, t(\tau))| \lesssim \tau^{5\delta-\kappa}(\ln\tau)^4 R^{-a}(t(\tau))\langle y\rangle^{-2-a_1}$. The orthogonal equations of μ_1 and ξ have been solved in Section 4.1, then by Proposition 7.1, one finds a solution for (3.6) satisfying

$$\langle y\rangle|\nabla\phi_1(y,\tau)| + |\phi_1(y,\tau)| \lesssim \tau^{5\delta-\kappa}(\ln\tau)^4 R^{-a}(t(\tau))R_0^5\langle y\rangle^{-a_1} \lesssim \tau_0^{-\epsilon}\tau^{5\delta-\kappa}\langle y\rangle^{-a}$$

with $\epsilon > 0$ sufficiently small provided

$$\gamma\min\{a, a_1\} > 5\delta. \tag{5.2}$$

Combining (4.8) and Lemma 7.5, one can find a solution for (3.7) with the estimate

$$\langle y\rangle|\nabla\phi_2(y,\tau)| + |\phi_2(y,\tau)| \lesssim \tau_0^{-\epsilon}\tau^{5\delta-\kappa}\langle y\rangle^{-a}$$

if

$$-\nu\alpha + (2-a)\gamma < 0,\ 1-\nu\alpha - a\gamma < 0,\ 1-\nu\alpha+5\delta-\kappa+2\gamma < 0,\ 0 < a < 2. \tag{5.3}$$

Combining (3.5), (3.12), (4.6), (5.2), (5.3) and the assumption about parameters in Proposition 7.1 and Lemma 7.5, one needs to choose parameters such that all the inequalities below hold

$$\begin{aligned}
&5\delta-\kappa-a\gamma > -2,\ 5\delta-\kappa < -1,\ 0<a<2,\ 0<\gamma<\frac{1}{2},\ 0<\alpha<1,\ 0<\nu<\frac{1}{2},\\
&a_1\gamma - 2 < 5\delta-\kappa-a\gamma,\ 0<a_1\le 1,\ \gamma\min\{a,a_1\} > 5\delta,\ 6\delta<1,\\
&-\nu\alpha+(2-a)\gamma<0,\ 1-\nu\alpha-a\gamma<0,\ 1-\nu\alpha+5\delta-\kappa+2\gamma<0.
\end{aligned} \tag{5.4}$$

There exists solution given by

$$\begin{aligned}
&1<\kappa\le\frac{5}{4},\ \frac{2-2\alpha\nu}{-1+\kappa+\nu}<a<2,\ \frac{1-\alpha\nu}{a}<\gamma<\frac{-1+\kappa+\nu}{2},\\
&\frac{2-\kappa}{2}<\alpha\nu<\frac{1}{2},\ 0<\alpha<1,\ 0<\nu<\frac{1}{2},\ 6\delta<1,\\
&0<5\delta<\kappa-1,\ a_1\gamma<5\delta+2-\kappa-a\gamma,\ 5\delta<\gamma\min\{a,a_1\},\ 0<a_1\le 1.
\end{aligned} \tag{5.5}$$

Indeed, one may take for example $\kappa = \frac{9}{8}, \nu = \frac{49}{100}, \alpha = \frac{375}{392}, a = \frac{36}{19}, \gamma = \frac{9}{32}, 5\delta = \frac{1}{64}, a_1 = \frac{1}{9}$.

Thanks to (5.4), the desired ϕ_1, ϕ_2 can really be found and then $\phi_1 + \phi_2 \in B_i$ when τ_0 is large enough. The compactness is a consequence of parabolic estimates, so we can find a solution for the inner problem (3.2). Making more efforts to calculate the Lipschitz continuity of $\mathcal{H}[\phi]$ about ϕ, one can prove the existence for the inner problem (3.2) by the contraction mapping theorem.

Collecting the estimates in Proposition 3.1, Corollary 2.3, (2.28) and (5.1), one gets

$$\begin{aligned}
&\left|\varphi[\mu] + \bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0}, t)\eta(\frac{4(x-\xi)}{\sqrt{t}}) + \psi + \eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu}, t)\right|\\
\lesssim\ & (t\ln t)^{-1}\mathbf{1}_{\{|\bar x|\le 2t^{\frac12}\}} + O(t^2(\ln t)^{-1}|\bar x|^{-6})\mathbf{1}_{\{|\bar x|>2t^{\frac12}\}} + (\ln t)t^{-1}(\ln t)^{-2}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)\mathbf{1}_{\{|\bar x|\le t^{\frac12}\}}\\
&+\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}\left(\mathbf{1}_{\{|x|\le t^{\frac12}\}} + t|x|^{-2}\mathbf{1}_{\{|x|>t^{\frac12}\}}\right) + (t(\ln t)^2)^{-\kappa+5\delta}\langle y\rangle^{-a}\mathbf{1}_{\{|y|\le 4R\}}\\
\lesssim\ & (t\ln t)^{-1}\mathbf{1}_{\{|\bar x|\le 2t^{\frac12}\}} + O((\ln t)^{-1}|\bar x|^{-2})\mathbf{1}_{\{|\bar x|>2t^{\frac12}\}} \lesssim (\ln t)^{-1}\min\{t^{-1}, |x|^{-2}\}.
\end{aligned}$$

Positivity of the solution u. We will demonstrate that the initial value $u(x, t_0)$ that we take in the construction is positive. For simplicity, we abuse the symbols $\mu = \mu(t_0)$, $\bar\mu_0 = \bar\mu_0(t_0)$ in the remainder of this section. Indeed, recalling (2.6), (2.28) and (5.1), we have

$$u(x,t_0)=\mu^{-1}w\left(\frac{\bar x}{\mu}\right)\eta\left(\frac{\bar x}{\sqrt{t_0}}\right)+\tilde\varphi_1(\bar x,t_0)+\bar\mu_0^{-1}\Phi_0\left(\frac{\bar x}{\bar\mu_0},t_0\right)\eta\left(\frac{4\bar x}{\sqrt{t_0}}\right)+\eta_R\mu^{-1}\phi\left(\frac{\bar x}{\mu},t_0\right)$$

$$=2^{\frac32}\mu|\bar x|^{-2}\Bigg\{\left[e^{-\frac{|\bar x|^2}{4t_0}}-\left(1+\left(\frac{|\bar x|}{\mu}\right)^2\right)^{-1}+2^{-\frac32}\mu^{-1}|\bar x|^2\bar\mu_0^{-1}\Phi_0\left(\frac{\bar x}{\bar\mu_0},t_0\right)\eta\left(\frac{4\bar x}{\sqrt{t_0}}\right)\right]\eta\left(\frac{|\bar x|}{\sqrt{t_0}}\right)$$

$$+e^{-\frac{|\bar x|^2}{4t_0}}\left(1-\eta\left(\frac{|\bar x|}{\sqrt{t_0}}\right)\right)\Bigg\}+\eta_R\mu^{-1}\phi\left(\frac{\bar x}{\mu},t_0\right)$$

$$\geq 2^{\frac32}\mu|\bar x|^{-2}\left[1-\frac{|\bar x|^2}{4t_0}-\left(1+\left(\frac{|\bar x|}{\mu}\right)^2\right)^{-1}-C|\bar x|^2(t_0\ln t_0)^{-1}\ln\ln t_0\eta\left(\frac{4\bar x}{\sqrt{t_0}}\right)\right]\eta\left(\frac{|\bar x|}{\sqrt{t_0}}\right)+\eta_R\mu^{-1}\phi\left(\frac{\bar x}{\mu},t_0\right)$$

$$=2^{\frac32}\mu|\bar x|^{-2}|\bar x|^2\left[(\mu^2+|\bar x|^2)^{-1}-(4t_0)^{-1}-C(t_0\ln t_0)^{-1}\ln\ln t_0\eta\left(\frac{4\bar x}{\sqrt{t_0}}\right)\right]\eta\left(\frac{|\bar x|}{\sqrt{t_0}}\right)+\eta_R\mu^{-1}\phi\left(\frac{\bar x}{\mu},t_0\right)$$

$$\geq 2^{-\frac12}\mu^{-1}(1+|y|^2)^{-1}\eta\left(\frac{|\bar x|}{\sqrt{t_0}}\right)-C_1\mu^{-1}(t_0(\ln t_0)^2)^{5\delta-\kappa}\langle y\rangle^{-a}\eta_R>0$$

where we have used $\eta(s)=0$ for $s\geq\frac32$ to make $[\frac58(\mu^2+|\bar x|^2)^{-1}-(4t_0)^{-1}-C(t_0\ln t_0)^{-1}\ln\ln t_0\eta(\frac{4\bar x}{\sqrt{t_0}})]\eta(\frac{|\bar x|}{\sqrt{t_0}})\geq 0$ when t_0 is large, and $5\delta-\kappa+\gamma(2-a)<0$ is used in the last inequality. Therefore, the solution $u(x,t)$ is positive by maximum principle.

6. Stability of blow-up: proof of Theorem 1.2

In this section, we will analyze the stability of the blow-up solution constructed in Theorem 1.1.

Proof of Theorem 1.2. Consider any perturbation $g_0(x)$ satisfying $|g_0(x)|\lesssim t_0^{-\frac{\min\{\ell,4\}}{2}}\langle x\rangle^{-\ell}$, $\ell>2$. Set

$$\psi_0(x,t)=(4\pi(t-t_0))^{-2}\int_{\mathbb{R}^4}e^{-\frac{|x-z|^2}{4(t-t_0)}}g_0(z)dz$$

which satisfies $\partial_t\psi_0=\Delta\psi_0$ in $\mathbb{R}^4\times(t_0,\infty)$, $\psi(x,t_0)=g_0(x)$ in $\mathbb{R}^4$. Without loss of generality, we only consider the case $2<\ell<4$. By Lemma A.3, one has

$$|\psi_0(x,t)|\lesssim t_0^{-\frac{\ell}{2}}\left(\langle t-t_0\rangle^{-\frac{\ell}{2}}\mathbf{1}_{\{|x|\leq\langle t-t_0\rangle^{\frac12}\}}+|x|^{-\ell}\mathbf{1}_{\{|x|>\langle t-t_0\rangle^{\frac12}\}}\right)\lesssim t^{-\frac{\ell}{2}}\mathbf{1}_{\{|x|\leq t^{\frac12}\}}+|x|^{-\ell}\mathbf{1}_{\{|x|>t^{\frac12}\}}.$$

We modify the proof of Proposition 3.1 slightly in order to match the perturbation g_0. Indeed, we split $\psi=\bar\psi+\psi_0$ and consider

$$\partial_t\bar\psi(x,t)=\Delta\bar\psi(x,t)+\mathcal{G}[\bar\psi+\psi_0,\phi,\mu_1,\xi]\ \text{ in }\ \mathbb{R}^4\times(t_0,\infty),\quad \bar\psi(x,t_0)=0\ \text{ in }\ \mathbb{R}^4.$$

When $\ell>2(\kappa+a\gamma)$, by (5.5), one has $|\psi_0|\lesssim t_0^{-\epsilon}w_o$, and thus $\bar\psi$ can be solved in B_o by the same method in Proposition 3.1. Repeating the rest procedures in the construction of Theorem 1.1, $(\mu_1,\xi,\phi,e_0)=(\mu_1[g_0],\xi[g_0],\phi[g_0],e[g_0])$ can be solved in the same topology that we have used before, and the leading order of blowup rate $\bar\mu_0\sim(\ln t)^{-1}$ remains the same. The perturbed initial value is then given by

$$\Bigg[(\bar\mu_0+\mu_1[g_0])^{-1}w\left(\frac{x-\xi[g_0]}{\bar\mu_0+\mu_1[g_0]}\right)\eta\left(\frac{x-\xi[g_0]}{\sqrt t}\right)+2^{\frac32}(\bar\mu_0+\mu_1[g_0])|x-\xi[g_0]|^{-2}\left(e^{-\frac{|x-\xi[g_0]|^2}{4t}}-\eta(\frac{x-\xi[g_0]}{\sqrt t})\right)$$
$$+\bar\mu_0^{-1}\Phi_0\left(\frac{x-\xi[g_0]}{\bar\mu_0},t\right)\eta\left(\frac{4(x-\xi[g_0])}{\sqrt t}\right)+\eta\left(\frac{x-\xi[g_0]}{\mu_0R}\right)e_0[g_0](\bar\mu_0+\mu_1[g_0])^{-1}Z_0\left(\frac{x-\xi[g_0]}{\bar\mu_0+\mu_1[g_0]}\right)\Bigg]\Bigg|_{t=t_0}+g_0.$$

From (5.5), $\kappa>1$ and $a\gamma>1-\alpha\nu$. So all $\ell>3$ is permitted for κ and $\alpha\nu$ close to 1 and $\frac12$, respectively.

In the radial setting, the translation parameter $\xi\equiv0$ automatically in (1.2). Then for $2<\ell\leq3$, we put $3u_1^2\psi_0$ into the right hand side in the equation (2.19). Since $|\psi_0(x,t)|\lesssim t^{-\frac{\ell}{2}}$, $\ell>2$, the extra term involving $3u_1^2\psi_0$ will not influence the leading order μ_0 and will be absorbed into Φ_0. But recalling the construction of $\bar\mu_0$ in Section 2.3, $\bar\mu_0$ depends on g_0, namely, $\bar\mu_0=\bar\mu_0[g_0]$.

We omit the tedious calculations about the Lipschitz continuity with respect to g_0 for ψ,ϕ,μ_1,ξ here. □

Remark 6.0.1.

- *In general nonradial case and $\ell > 2$, since ψ_0 is not radial about $\bar{x} = x - \xi$, the previous ODE solution about (2.19) is not allowed. Instead, we can expand (2.19) by modes similar to the manipulation in section 7 and solve the leading order of μ and ξ. Since this involves more technicalities, given the length of this paper, we refrain from considering such a generality here.*
- *The borderline $\ell > 2$ is also provided in [15].*
- *The stability result can be expected for $|g_0(x)| \lesssim t_0^{-1}(\ln t_0)^{-b_1}\langle x\rangle^{-2}(\ln(|x|+2))^{-b_2}$ for some $b_1, b_2 > 0$. The proof can be in fact achieved by similar computations as in the proof of Theorem 1.2.*

7. Linear theory for the inner problem

In this section, we develop a linear theory for the associated inner problem. Since the construction is independent of the spatial dimension n, we assume $n \geq 3$ in this section unless specifically stated otherwise. Set

$$\mathcal{D}_R = \{(y,\tau) \mid y \in B_{R(\tau)},\ \tau \in (\tau_0,\infty)\}, \quad \partial\mathcal{D}_R = \{(y,\tau) \mid y \in \partial B_{R(\tau)},\ \tau \in (\tau_0,\infty)\}.$$

We consider the associated linear problem

$$\partial_\tau \phi = \Delta\phi + pU^{p-1}\phi + f_1(y,\tau)\phi + f_2(y,\tau)y\cdot\nabla\phi + h(y,\tau) \ \text{ in } \mathcal{D}_R \tag{7.1}$$

where

$$p = \frac{n+2}{n-2}, \quad U(y) = (n(n-2))^{\frac{n-2}{4}}\left(1+|y|^2\right)^{-\frac{n-2}{2}}.$$

Throughout this section, we always assume that f_1, f_2 satisfy

$$f_i(y,\tau) = f_i(|y|,\tau) \ \text{ are radial in space},\ i=1,2,\ \ |f_1|, |f_2|, |y||\nabla f_2| \leq C_f\tau^{-d},\ d>0,\ C_f \geq 0. \tag{7.2}$$

It is easier to make mode expansion by spherical harmonic functions when f_1 and f_2 are radial. And it is very possible to generalize the linear theory without the assumption that f_1 and f_2 are radially symmetric.

Recall that the linearized operator $\Delta + pU^{p-1}$ has only one positive eigenvalue $\gamma_0 > 0$ such that

$$\Delta Z_0 + pU^{p-1}Z_0 = \gamma_0 Z_0, \tag{7.3}$$

where the corresponding eigenfunction $Z_0 \in L^\infty(\mathbb{R}^n)$ is radially symmetric with the asymptotic behavior

$$Z_0(y) \sim |y|^{-\frac{n-1}{2}}e^{-\sqrt{\gamma_0}|y|} \ \text{ as } \ |y| \to \infty.$$

The bounded kernels of $\Delta + pU^{p-1}$ are given by

$$Z_i(y) = \partial_{y_i}U(y),\ i=1,2,\ldots,n, \quad Z_{n+1}(y) = y\cdot\nabla U(y) + \frac{n-2}{2}U(y).$$

Define the weighted L^∞ norm

$$\|h\|_{v,a} := \sup_{(y,\tau)\in\mathcal{D}_R} v^{-1}(\tau)\langle y\rangle^a|h(y,\tau)|$$

where $a \geq 0$ is a constant. Throughout this section, we assume $R(\tau)$, $v(\tau) \in C^1(\tau_0,\infty)$ with the form

$$v(\tau) = a_0\tau^{a_1}(\ln\tau)^{a_2}(\ln\ln\tau)^{a_3}\cdots, \quad R(\tau) = b_0\tau^{b_1}(\ln\tau)^{b_2}(\ln\ln\tau)^{b_3}\cdots, \quad v(\tau) > 0, \quad 1 \ll R(\tau) \ll \tau^{\frac{1}{2}},$$
$$v'(\tau) = O(\tau^{-1}v(\tau)), \quad R'(\tau) = O(\tau^{-1}R(\tau))$$

where $a_0, b_0 > 0$, $a_i, b_i \in \mathbb{R}$, $i = 1,2,\ldots$. For brevity, we write $v = v(\tau)$, $R = R(\tau)$.

We impose a linear constraint on the initial value $\phi(y,\tau_0)$ to handle the instability caused by Z_0. Consider the associated Cauchy problem

$$\begin{cases} \partial_\tau\phi = \Delta\phi + pU^{p-1}(y)\phi + f_1(y,\tau)\phi + f_2(y,\tau)y\cdot\nabla\phi + h, & \text{in } \mathcal{D}_R, \\ \phi(y,\tau_0) = e_0 Z_0(y), & \text{in } B_{R(\tau_0)}, \end{cases} \tag{7.4}$$

where τ_0 is sufficiently large. Formally speaking, when $R \ll \tau^{\frac{d}{2}-\epsilon}$ for some $\epsilon > 0$, we can expect that $f_1\phi + f_2 y\cdot\nabla\phi$ is a small perturbation since $|f_i| \ll \tau^{-2\epsilon}R^{-2} \lesssim \tau^{-2\epsilon}\langle y\rangle^{-2}$ in $\mathcal{D}_R$.

The construction of solution to (7.4) is achieved by decomposing the equation into different spherical harmonic modes. Consider an orthonormal basis $\{\Upsilon_i\}_{i=0}^\infty$ made up of spherical harmonic functions in $L^2(S^{n-1})$, namely eigenvalues of the problem

$$\Delta_{S^{n-1}}\Upsilon_j + \iota_j\Upsilon_j = 0 \ \text{ in } \ S^{n-1}.$$

where $0 = \iota_0 < \iota_1 = \iota_2 = \cdots = \iota_n = n-1 < \iota_{n+1} \leq \ldots$ and $\int_{S^{n-1}}\Upsilon_i(\theta)\Upsilon_j(\theta)d\theta = \delta_{ij}$. More precisely, $\Upsilon_0(y) = a_0$, $\Upsilon_i(y) = a_1 y_i$, $i = 1,\cdots,n$ for two constants a_0, a_1 and the eigenvalue $\iota_l = l(n-2+l)$ has multiplicity

$$\binom{n+l-1}{l} - \binom{n+l-3}{l-2} \ \text{ for } \ l \geq 2.$$

For $h(\cdot,\tau) \in L^2(B_{R(\tau)})$, we decompose h into the form

$$h(y,\tau) = \sum_{j=0}^{\infty} h_j(r,\tau)\Upsilon_j(y/r),\ r = |y|,\ h_j(r,\tau) = \int_{S^{n-1}} h(r\theta,\tau)\Upsilon_j(\theta)d\theta.$$

Write $h = h^0 + h^1 + h^\perp$ with

$$h^0 = h_0(r,\tau)\Upsilon_0,\ h^1 = \sum_{j=1}^{n} h_j(r,\tau)\Upsilon_j,\ h^\perp = \sum_{j=n+1}^{\infty} h_j(r,\tau)\Upsilon_j.$$

Also, we decompose $\phi = \phi^0 + \phi^1 + \phi^\perp$ in a similar form. Then looking for a solution to problem (7.4) is equivalent to finding the pairs $(\phi^0, h^0),\ (\phi^1, h^1),\ (\phi^\perp, h^\perp)$ in each mode.

The key linear theory for the inner problem is stated as follows.

Proposition 7.1. *Consider*

$$\begin{cases} \partial_\tau \phi = \Delta\phi + pU^{p-1}\phi + f_1\phi + f_2 y\cdot\nabla\phi + h(y,\tau) + \sum\limits_{i=1}^{n+1} c_i(\tau)\eta(y)Z_i(y) & \text{in } \mathcal{D}_R \\ \phi(y,\tau_0) = e_0 Z_0(y) & \text{in } B_{R(\tau_0)} \end{cases}$$

where $n \geq 4$, $\|h\|_{v,2+a} < \infty$, $0 < a < 2$. Suppose that $R^2 \ll \tau^{d-}$, $R_0 = C\tau^\delta \gg 1$, $\delta \geq 0$ and $R_0^{n+2} \ll \tau^{\min\{1,d\}-}$, then for τ_0 sufficiently large, there exists (ϕ, e_0, c_i) solving above equation, and $(\phi, e_0, c_i) = (\mathcal{T}_{3i}[h], \mathcal{T}_{3e}[h], c_i[h])$ defines a linear mapping of h with the estimates

$$\langle y\rangle|\nabla\phi| + |\phi| \lesssim R_0^{n+1} v\langle y\rangle^{-a}\|h\|_{v,2+a}, \quad |e_0| \lesssim v(\tau_0)R_0^{2-a}(\tau_0)\|h\|_{v,2+a},$$

$$c_i[h](\tau) = -\Big(\int_{B_2}\eta(y)Z_i^2(y)dy\Big)^{-1}\left(\int_{B_{2R_0}} h(y,\tau)Z_i(y)dy + R_0^{-\epsilon_0}O(v\|h_i\|_{v,2+a})\right), \quad i = 1,\dots,n,$$

$$c_{n+1}[h](\tau) = -\Big(\int_{B_2}\eta(y)Z_{n+1}^2(y)dy\Big)^{-1}\left(\int_{B_{2R_0}} h(y,\tau)Z_{n+1}(y)dy + R_0^{-\epsilon_0}O(v\|h_0\|_{v,2+a})\right),$$

where $0 < \epsilon_0 < \frac{\min\{a,1\}}{2}$ is a small constant,

$$h(y,\tau) = \sum_{j=0}^{\infty}\Upsilon_j\Big(\frac{y}{|y|}\Big)h_j(|y|,\tau), \quad h_j(|y|,\tau) = \int_{S^{n-1}} h(|y|\theta,\tau)\Upsilon_j(\theta)d\theta,$$

$O(v\|h_i\|_{v,2+a})$ linearly depends on h_i for $i = 0, 1, \dots, n$.

The proof of Proposition 7.1 is achieved by the following Proposition and by another gluing procedure (re-gluing).

Proposition 7.2. *Consider*

$$\begin{cases} \partial_\tau \phi = \Delta\phi + pU^{p-1}\phi + f_1\phi + f_2 y\cdot\nabla\phi + h(y,\tau) & \text{in } \mathcal{D}_R \\ \phi(y,\tau_0) = e_0 Z_0(y) & \text{in } B_R \end{cases}$$

where $\|h\|_{v,2+a} < \infty$, $a > 0$ and h satisfies the orthogonal condition

$$\int_{B_{R(\tau)}} h(y,\tau)Z_i(y)dy = 0 \ \text{for all } \tau > \tau_0, \quad i = 1,\dots,n+1.$$

Assume $R^n\theta^1_{Ra} \ll \tau^{\min\{1,d\}}$. Then for τ_0 sufficiently large, there exists a solution $(\phi, e_0) = (\mathcal{T}_{2i}[h], \mathcal{T}_{2e}[h])$ which is a linear mapping of h with the estimates

$$\begin{aligned}\langle y\rangle|\nabla\phi| + |\phi| \lesssim{}& v\min\{\tau^{\frac12}, \lambda_R^{-\frac12}\}\lambda_R^{-\frac12}\theta^0_{R\hat a_0}\left(\langle y\rangle^{-n} + C_f\tau^{-d}\min\{\tau^{\frac12}, \lambda_R^{-\frac12}\}\lambda_R^{-\frac12}\ln R\langle y\rangle^{2-n}\right)\|h^0\|_{v,2+a}\\ &+ v\left(\Theta^0_{R\hat a_0}(|y|)\langle y\rangle^{-2} + C_f\tau^{-d}\min\{\tau^{\frac12}, \lambda_R^{-\frac12}\}\lambda_R^{-\frac12}\theta^0_{R\hat a_0}\ln R\right)\|h^0\|_{v,2+a}\\ &+ v\theta^1_{R\hat a_1}R^n\left(\langle y\rangle^{-1-n} + C_f\tau^{-d}R^n\langle y\rangle^{1-n}\right)\|h^1\|_{v,2+a}\\ &+ v\left(\Theta^0_{R,2+a}(|y|) + R\langle y\rangle^{2-n}\right)\|h^\perp\|_{v,2+a},\end{aligned}$$

$$|e_0| \lesssim v(\tau_0)\theta^0_{R(\tau_0)\hat a_0}\left(1 + C_f\tau_0^{-d}\min\{\tau_0^{\frac12}, \lambda_{R(\tau_0)}^{-\frac12}\}\lambda_{R(\tau_0)}^{-\frac12}\ln R(\tau_0)\right)\|h^0\|_{v,2+a}$$

where

$$\hat{a}_0 = \begin{cases} a & \text{if } a \neq n-2 \\ (n-2)- & \text{if } a = n-2 \end{cases}, \quad \hat{a}_1 = \begin{cases} a & \text{if } a \neq n-1 \\ (n-1)- & \text{if } a = n-1 \end{cases}, \tag{7.5}$$

$$\theta^0_{Ra} = \begin{cases} R^{2-a} & \text{if } a < 2 \\ \ln R & \text{if } a = 2 \\ 1 & \text{if } a > 2 \end{cases}, \quad \theta^1_{Ra} = \begin{cases} R^{1-a} & \text{if } a < 1 \\ \ln R & \text{if } a = 1 \\ 1 & \text{if } a > 1 \end{cases}, \tag{7.6}$$

$$\lambda_R = \begin{cases} R^{-2} & \text{if } n = 3 \\ (R^2 \ln R)^{-1} & \text{if } n = 4 \\ R^{2-n} & \text{if } n \geq 5 \end{cases}, \quad \Theta^0_{Ra}(r) = \begin{cases} R^{2-a} & \text{if } a < 2 \\ \ln R & \text{if } a = 2 \\ \langle r \rangle^{2-\min\{a,n-\}} & \text{if } a > 2 \end{cases}. \tag{7.7}$$

Before we prove Proposition 7.1, we first use Proposition 7.2 to prove Proposition 7.1.

Proof of Proposition 7.1. Set $\phi(y,\tau) = \eta_{R_0}(y)\phi_i(y,\tau) + \phi_o(y,\tau)$, where $\eta_{R_0}(y) = \eta(\frac{y}{R_0})$. In order to find a solution ϕ, it suffices to consider the following inner–outer gluing system for (ϕ_i, ϕ_o)

$$\begin{cases} \partial_\tau \phi_o = \Delta \phi_o + J[\phi_o, \phi_i] \text{ in } \mathcal{D}_R, \\ \phi_o = 0 \text{ on } \partial \mathcal{D}_R, \quad \phi_o = 0 \text{ in } B_{R(\tau_0)}, \end{cases} \tag{7.8}$$

$$\begin{cases} \partial_\tau \phi_i = \Delta \phi_i + pU^{p-1}\phi_i + f_1 \phi_i + f_2 y \cdot \nabla \phi_i + pU^{p-1}\phi_o + h + \sum_{i=1}^{n+1} c_i(\tau)\eta(y)Z_i(y) & \text{in } \mathcal{D}_{2R_0}, \\ \phi_i = e_0 Z_0(y) & \text{in } B_{2R(\tau_0)}, \end{cases} \tag{7.9}$$

where

$$J[\phi_o, \phi_i] = f_1\phi_o + f_2 y \cdot \nabla\phi_o + pU^{p-1}\phi_o(1-\eta_{R_0}) + A[\phi_i] + h(1-\eta_{R_0}),$$
$$A[\phi_i] = \Delta\eta_{R_0}\phi_i + 2\nabla\eta_{R_0} \cdot \nabla\phi_i + f_2 y \cdot \nabla\eta_{R_0}\phi_i - \partial_\tau\eta_{R_0}\phi_i.$$

Here $c_i(\tau)$ is given by

$$c_i(\tau) = c_i[\phi_o](\tau) = C_i \int_{B_{2R_0}} (pU^{p-1}(z)\phi_o(z,\tau) + h(z,\tau))Z_i(z)dz, \quad C_i = -(\int_{B_2} \eta(y)Z_i^2(y)dy)^{-1}$$

such that the orthogonal conditions

$$\int_{B_{2R_0}} \left(pU^{p-1}(z)\phi_o(z,\tau) + h(z,\tau) + \sum_{i=1}^{n+1} c_i(\tau)\eta(y)Z_i(z) \right) Z_j(z)dy = 0 \text{ for } j = 1, \dots, n+1$$

are satisfied.

We reformulate (7.8) and (7.9) into the following form

$$\begin{aligned} \phi_o(y,\tau) &= \mathcal{T}_o[J[\phi_o,\phi_i]], \quad \phi_i(y,\tau) = \mathcal{T}_{2i}\left[pU^{p-1}(y)\phi_o + h + \sum_{i=1}^{n+1} c_i(\tau)\eta(y)Z_i(y) \right], \\ e_0 &= \mathcal{T}_{2e}\left[pU^{p-1}(y)\phi_o + h + \sum_{i=1}^{n+1} c_i(\tau)\eta(y)Z_i(y) \right], \end{aligned} \tag{7.10}$$

where $\mathcal{T}_o$ is a linear mapping given by the standard parabolic theory, and $\mathcal{T}_{2i}$, $\mathcal{T}_{2e}$ are given by Proposition 7.2. We will solve the system (7.10) by the contraction mapping theorem.

Denote the leading term of the right hand side of (7.9) as $H_1 := h + \sum_{i=1}^{n+1} C_i\eta(y)Z_i(y)\int_{B_{2R_0}} h(z,\tau)Z_i(z)dz$. It is easy to check $\|H_1\|_{v,2+a} \lesssim \|h\|_{v,2+a}$. If H_1 satisfies the orthogonal condition in $\mathcal{D}_{2R_0}$, under the assumption $R_0^{n+2} \ll \tau^{\min\{1,d\}-}$, Proposition 7.2 gives following a priori estimates

$$\langle y \rangle |\nabla \mathcal{T}_{2i}[H_1]| + |\mathcal{T}_{2i}[H_1]| \leq D_i w_i(y,\tau), \quad |\mathcal{T}_{2e}[H_1]| \leq D_i v(\tau_0) R_0^{2-a} \|h\|_{v,2+a},$$

where $D_i \geq 1$ is a constant and

$$w_i(y,\tau) = v\left(\lambda_{R_0}^{-1} R_0^{2-a}\langle y\rangle^{-n} + \theta^1_{R_0 a} R_0^n \langle y\rangle^{-1-n} + \langle y\rangle^{-a} + R_0\langle y\rangle^{2-n}\right)\|h\|_{v,2+a}$$

where $\theta^1_{R_0 a}$ is given in (7.6). For this reason, we will solve the inner part in the space

$$\mathcal{B}_i = \{g(y,\tau) \ : \ \langle y\rangle|\nabla_y g(y,\tau)| + |g(y,\tau)| \leq 2D_i w_i(y,\tau)\}.$$

For any $\tilde{\phi}_i \in \mathcal{B}_i$, we will find a solution $\phi_o = \phi_o[\tilde{\phi}_i]$ of (7.8) by the contraction mapping theorem. Let us estimate $J[0, \tilde{\phi}_i]$ term by term. For $n \geq 4$,

$$|A[\tilde{\phi}_i]| \lesssim D_i v(R_0^{-2} + \tau^{-d})(R_0^{-a} \ln R_0 + R_0^{-1}) \mathbf{1}_{\{R_0 \leq |y| \leq 2R_0\}} \|h\|_{v,2+a} \lesssim D_i v R_0^{-\epsilon_0} \langle y \rangle^{-2-a_1} \|h\|_{v,2+a}$$

where constants $0 < a_1 < \min\{a, 1\}$ and $\epsilon_0 = \frac{\min\{a,1\} - a_1}{2}$. Also we have

$$|h(1 - \eta_{R_0})| \lesssim \mathbf{1}_{\{y \geq R_0\}} v \langle y \rangle^{-2-a} \|h\|_{v,2+a} \lesssim v R_0^{-\epsilon_0} \langle y \rangle^{-2-a_1} \|h\|_{v,2+a}.$$

Consider (7.8) with the right hand side $J[0, \tilde{\phi}_i]$. Using $Cv(-\Delta)^{-1}(\langle y \rangle^{-2-a_1}) R_0^{-\epsilon_0} \|h\|_{v,2+a}$ as the barrier function with a large constant C and then scaling argument, we have

$$\langle y \rangle |\nabla \mathcal{T}_o[J[0, \tilde{\phi}_i]](y, \tau)| + |\mathcal{T}_o[J[0, \tilde{\phi}_i]](y, \tau)| \leq w_o(y, \tau) = D_o D_i v R_0^{-\epsilon_0} \langle y \rangle^{-a_1} \|h\|_{v,2+a}$$

with a large constant $D_o \geq 1$. This suggests us solve ϕ_o in the following space:

$$\mathcal{B}_o = \{ f(y, \tau) \ : \ \langle y \rangle |\nabla f(y, \tau)| + |f(y, \tau)| \leq 2 w_o(y, \tau) \}.$$

For any $\tilde{\phi}_o \in \mathcal{B}_o$, due to $|y| \leq 2R(\tau)$, we have

$$|pU^{p-1} \tilde{\phi}_o (1 - \eta_{R_0})| \lesssim R_0^{-2} D_o D_i v R_0^{-\epsilon_0} \langle y \rangle^{-2-a_1} \|h\|_{v,2+a},$$

$$|f_1 \phi_o + f_2 y \cdot \nabla \phi_o| \lesssim \tau^{-d} R^2(\tau) D_o D_i v R_0^{-\epsilon_0} \langle y \rangle^{-2-a_1} \|h\|_{v,2+a}.$$

Since $\tau^{-d} R^2$, R_0^{-2} provide smallness, by comparison principle, we have

$$\mathcal{T}_o[J[\tilde{\phi}_o, \tilde{\phi}_i]] \in \mathcal{B}_o.$$

The contraction mapping property can be deduced in the same way.

Now we have found a solution $\phi_o = \phi_o[\tilde{\phi}_i] \in \mathcal{B}_o$. It follows that

$$\left\| pU^{p-1}(y) \phi_o[\tilde{\phi}_i] + \sum_{i=1}^{n+1} C_i \int_{B_{2R_0}} pU^{p-1}(z) \phi_o[\tilde{\phi}_i](z, \tau) Z_i(z) dz \eta(y) Z_i(y) \right\|_{v,2+a} \lesssim D_o D_i R_0^{-\epsilon_0} \|h\|_{v,2+a}.$$

Due to the choice of $c_i(\tau)$, $H_2 := pU^{p-1}(y) \phi_o[\tilde{\phi}_i] + h + \sum_{i=1}^{n+1} c_i[\phi_o[\tilde{\phi}_i]](\tau) \eta(y) Z_i(y)$ satisfies the orthogonal condition in $\mathcal{D}_{2R_0}$. By Proposition 7.2, since $R_0^{-\epsilon_0}$ provides smallness, we have

$$\mathcal{T}_{2i}[h_2] \in \mathcal{B}_i$$

The contraction property can be deduced in the same way. Thus we find a solution

$$\phi_i = \phi_i[h] \in \mathcal{B}_i. \tag{7.11}$$

Finally we obtain a solution (ϕ_o, ϕ_i) for (7.8) and (7.9).

From the construction above and the topology of $\mathcal{B}_i$, $\phi_i[h] = 0$ if $h = 0$, which deduces that $\phi_i[h]$ is a linear mapping of h. By the similar argument, $\phi_o[h]$ and $c_i[h]$ are also linear mappings of h, and so does ϕ.

We will regard D_o, D_i as general constants hereafter. Then by Propostition 7.2 and (7.10), we have

$$|e_0| \lesssim v(\tau_0) R_0^{2-a} \|h\|_{v,2+a}.$$

Since $\phi_o[h] \in \mathcal{B}_o$, one has

$$c_i[h](\tau) = C_i \int_{B_{2R_0}} h(y, \tau) Z_i(y) dy + R_0^{-\epsilon_0} O(v) \|h\|_{v,2+a}.$$

Since the above operation is linear about h, we are able to decompose h into

$$h(y, \tau) = \sum_{j=0}^{\infty} \Upsilon_j(\frac{y}{|y|}) h_j(|y|, \tau), \quad h_j(|y|, \tau) = \int_{S^{n-1}} h(|y|\theta, \tau) \Upsilon_j(\theta) d\theta$$

and repeat the construction about $\Upsilon_j(\frac{y}{|y|}) h_j(|y|, \tau)$ separately. Then

$$c_{n+1}[h](\tau) = C_{n+1} \int_{B_{2R_0}} h(y, \tau) Z_{n+1}(y) dy + R_0^{-\epsilon_0} O(v) \|h_0\|_{v,2+a},$$

$$c_i[h](\tau) = C_i \int_{B_{2R_0}} h(y, \tau) Z_i(y) dy + R_0^{-\epsilon_0} O(v) \|h_i\|_{v,2+a} \ \text{ for } \ i = 1, \ldots, n.$$

Reviewing the re-gluing procedure, we have

$$|J[0, \phi_i]| \lesssim R_0 v \langle y \rangle^{-2-a} \|h\|_{v,2+a}.$$

Using comparison principle to (7.8) several times, the upper bound of ϕ_o can be improved to

$$|\phi_o| \lesssim R_0 v\langle y\rangle^{-a}\|h\|_{v,2+a}. \tag{7.12}$$

Combining (7.11), (7.12) and then using scaling argument, we conclude

$$\langle y\rangle|\nabla\phi| + |\phi| \lesssim R_0^{n+1} v\langle y\rangle^{-a}\|h\|_{v,2+a}.$$

□

The rest of this section is devoted to the proof of Proposition 7.2. We first invoke a coercive estimate for the linearized operator

Lemma 7.3. *[5, Lemma 7.2] There exists a constant $c_0 > 0$ such that for all sufficiently large R and all radially symmetric functions $\phi \in H_0^1(B_R)$ with $\int_{B_R}\phi Z_0 = 0$, we have*

$$c_0\lambda_R\int_{B_R}|\phi|^2 \le Q(\phi,\phi),$$

where λ_R is given in (7.7) and $Q(\phi,\phi) := \int_{B_R}(|\nabla\phi|^2 - pU^{p-1}|\phi|^2)$.

Note that in [5, Lemma 7.2], there is above coercive estimate only for higher dimensions $n \ge 5$. The proof in lower dimensions $n = 3,\ 4$ is in fact similar and by slight modifications.

Lemma 7.4. *Consider*

$$\begin{cases}\partial_\tau\phi = \Delta\phi + pU^{p-1}(1-\chi_M)\phi + f_1\phi + f_2 y\cdot\nabla\phi + h \ \text{ in } \mathcal{D}_R\\ \phi = 0 \ \text{ on } \partial\mathcal{D}_R, \quad \phi(\cdot,\tau_0) = 0 \ \text{ in } B_{R(\tau_0)}\end{cases}$$

where $\chi_M(y) = \eta(\frac{y}{M})$, $M > 0$ is a large constant, $R^2\ln R \ll \tau^{\min\{1,d\}}$, $\|h\|_{v,a} < \infty$, $a \ge 0$. Then when τ_0 sufficiently large, for $\Theta_{Ra}^0(|y|)$ given in (7.7), the unique solution $\phi_*[h]$ has the following estimate:

$$|\phi_*[h]| \lesssim C(M,a,n)v\Theta_{Ra}^0(|y|)\|h\|_{v,a}.$$

Proof. Set $\bar a = \min\{a, n-\}$, $r = |y|$, $L_M\phi = \Delta\phi + pU^{p-1}(y)(1-\chi_M)\phi$. Set a barrier function as $\bar\phi(r,\tau) = Cvg(r,R)$, where

$$L_M g(r,R) = -\langle r\rangle^{-\bar a}, \quad g(r,R) = g_2(r)\int_r^R \frac{d\rho}{g_2^2(\rho)\rho^{n-1}}\int_0^\rho g_2(s)s^{n-1}\langle s\rangle^{-\bar a}ds$$

and $g_2(r) > 0$ is the positive kernel of L_M and $g_2(r) \sim 1$ for $r \in (0,\infty)$. By direct calculation, one has

$$\langle r\rangle^{\bar a}g(r,R) \lesssim \langle r\rangle^{\bar a}\Theta_{R\bar a}^0(|y|) \lesssim R^2\ln R.$$

By scaling argument, one has $\langle r\rangle^{\bar a}|r\partial_r g(r,R)| \lesssim R^2\ln R$. Then

$$\begin{aligned} P(\bar\phi) &:= L_M(Cvg(r,R)) + h(y,\tau) + Cv(f_1 g(r,R) + f_2 r\partial_r g(r,R)) - \partial_\tau(Cvg(r,R))\\ &= -Cv\langle r\rangle^{-\bar a} + h(y,\tau) + Cv(f_1 g(r,R) + f_2 r\partial_r g(r,R)) - Cv'g(r,R) - \frac{Cvg_2(r)R'}{g_2^2(R)R^{n-1}}\int_0^R g_2(s)s^{n-1}\langle s\rangle^{-\bar a}ds\\ &\le Cv\langle r\rangle^{-\bar a}\Bigg[-1 + \langle r\rangle^{\bar a}(f_1 g(r,R) + f_2 r\partial_r g(r,R)) - v'v^{-1}g(r,R)\langle r\rangle^{\bar a}\\ &\quad - \frac{\langle r\rangle^{\bar a}g_2(r)R'}{g_2^2(R)R^{n-1}}\int_0^R g_2(s)s^{n-1}\langle s\rangle^{-\bar a}ds\Bigg] + v\langle r\rangle^{-a}\|h\|_{v,a} \le -\frac{3}{4}Cv\langle r\rangle^{-\bar a} + v\langle r\rangle^{-\bar a}\|h\|_{v,a}\end{aligned}$$

where we have used

$$|\langle r\rangle^{\bar a}(f_1 g(r,R) + f_2 r\partial_r g(r,R))| \lesssim (|f_1| + |f_2|)R^2\ln R \lesssim C_f\tau^{-d}R^2\ln R \ll 1,$$

$$|v'v^{-1}g(r,R)\langle r\rangle^{\bar a}| \lesssim |v'|v^{-1}R^2\ln R \lesssim \tau^{-1}R^2\ln R \ll 1,$$

$$\frac{\langle r\rangle^{\bar a}g_2(r)|R'|}{g_2^2(R)R^{n-1}}\int_0^R g_2(s)s^{n-1}\langle s\rangle^{-\bar a}ds \sim \frac{\langle r\rangle^{\bar a}|R'|}{R^{n-1}}\int_0^R s^{n-1}\langle s\rangle^{-\bar a}ds \lesssim R|R'| \lesssim \tau^{-1}R^2 \ll 1.$$

Set $C = 2\|h\|_{v,a}$, then $P(\bar\phi) \le 0$.

□

7.1. **Mode 0 without orthogonality.**

Lemma 7.5. *Consider*

$$\begin{cases}\partial_\tau \phi^0 = \Delta \phi^0 + pU^{p-1}\phi^0 + f_1\phi^0 + f_2 y\cdot\nabla\phi^0 + h^0 & \text{in } \mathcal{D}_R,\\ \phi(\cdot,\tau_0) = e_0 Z_0(y) & \text{in } B_{R(\tau_0)}\end{cases} \tag{7.13}$$

where $\|h^0\|_{v,a} < \infty$, $a \ge 0$. *Assume* $\lambda_R \tau^d \gg 1$, $R^2 \ln R \ll \tau^{\min\{1,d\}}$. *Then for* τ_0 *sufficiently large, there exists a linear mapping* $(\phi^0, e_0) = (\mathcal{T}_{1i}[h^0], \mathcal{T}_{1e}[h^0])$ *solving* (7.13) *with the following estimates*

$$\langle y\rangle |\nabla \phi^0| + |\phi^0| \lesssim v\left(\min\{\tau^{\frac12}, \lambda_R^{-\frac12}\}\lambda_R^{-\frac12}\theta^0_{Ra}\langle y\rangle^{2-n} + \Theta^0_{Ra}(|y|)\right)\|h^0\|_{v,a},$$

$$|e_0| \lesssim v(\tau_0)\theta^0_{R(\tau_0)a}\|h^0\|_{v,a}.$$

Proof. First, we decompose ϕ^0 into two parts

$$\phi^0 = \phi_*[h^0] + \tilde\phi,$$

where $\phi_*[h^0]$ is the solution derived from Lemma 7.4 with the following estimate

$$|\phi_*[h^0]| \lesssim v\Theta^0_{Ra}(|y|)\|h^0\|_{v,a}. \tag{7.14}$$

Then

$$\begin{aligned}\partial_\tau \phi_*[h^0] + \partial_\tau\tilde\phi = {}& \Delta\phi_*[h^0] + \Delta\tilde\phi + pU^{p-1}(y)\chi_M\phi_*[h^0] + pU^{p-1}(y)(1-\chi_M)\phi_*[h^0]\\ &+ pU^{p-1}(y)\tilde\phi + f_1(\phi_*[h^0]+\tilde\phi) + f_2 y\cdot\nabla(\phi_*[h^0]+\tilde\phi) + h^0 \ \text{ in } \mathcal{D}_R,\end{aligned}$$

which implies that

$$\partial_\tau\tilde\phi = \Delta\tilde\phi + pU^{p-1}(y)\tilde\phi + f_1\tilde\phi + f_2 y\cdot\nabla\tilde\phi + pU^{p-1}(y)\chi_M\phi_*[h^0] \ \text{ in } \mathcal{D}_R.$$

We will construct a linear mapping $\tilde\phi = \tilde\phi[h^0]$. Take $\tilde\phi = \tilde\phi_1 + e(\tau)Z_0(y)$ and consider the following equation

$$\begin{cases}\partial_\tau\tilde\phi_1 = \Delta\tilde\phi_1 + pU^{p-1}\tilde\phi_1 + f_1\tilde\phi_1 + f_2 y\cdot\nabla\tilde\phi_1 - \partial_\tau e(\tau)Z_0 + \gamma_0 e(\tau)Z_0\\ \qquad + pU^{p-1}\chi_M\phi_*[h^0] + e(\tau)(f_1 Z_0(y) + f_2 y\cdot\nabla Z_0(y)) \ \text{ in } \mathcal{D}_R,\\ \tilde\phi_1 = 0 \ \text{ on } \partial\mathcal{D}_R, \quad \tilde\phi_1(\cdot,\tau_0) = 0 \ \text{ in } B_{R(\tau_0)}, \quad \int_{B_{R(\tau)}}\tilde\phi_1(y,\tau)Z_0(y)dy = 0 \quad \forall\tau>\tau_0.\end{cases} \tag{7.15}$$

Here $e(\tau)$ will be chosen to make $\int_{B_{R(\tau)}}\tilde\phi_1(y,\tau)Z_0(y)dy = 0$ for all $\tau > \tau_0$. Indeed, multiplying (7.15) by Z_0 and integrating by parts, one has

$$\begin{aligned}\partial_\tau\int_{B_{R(\tau)}}\tilde\phi_1 Z_0(y)dy = {}& \int_{B_{R(\tau)}}\partial_\tau\tilde\phi_1 Z_0(y)dy = \gamma_0\int_{B_{R(\tau)}}\tilde\phi_1 Z_0(y)dy + \int_{\partial B_{R(\tau)}} Z_0(y)\partial_n\tilde\phi_1 dy\\ &+ \int_{B_{R(\tau)}}(f_1\tilde\phi_1 + f_2 y\cdot\nabla\tilde\phi_1)Z_0(y)dy - (\partial_\tau e(\tau) - \gamma_0 e(\tau))\int_{B_{R(\tau)}} Z_0^2(y)dy\\ &+ \int_{B_{R(\tau)}} pU^{p-1}\chi_M\phi_*[h^0]Z_0(y)dy + e(\tau)\int_{B_{R(\tau)}}(f_1 Z_0(y) + f_2 y\cdot\nabla Z_0(y))Z_0(y)dy.\end{aligned}$$

By $\tilde\phi_1(\cdot,\tau_0) = 0$, the orthogonality $\int_{B_{R(\tau)}}\tilde\phi_1(y,\tau)Z_0(y)dy = 0$ holds for all $\tau > \tau_0$ if and only if

$$\begin{aligned}\partial_\tau e(\tau) - \tilde\gamma_0(\tau)e(\tau) = {}& \Big(\int_{B_{R(\tau)}} Z_0^2(y)dy\Big)^{-1}\Bigg[\int_{\partial B_{R(\tau)}} Z_0(y)\partial_n\tilde\phi_1 dy + \int_{B_{R(\tau)}}(f_1\tilde\phi_1 + f_2 y\cdot\nabla\tilde\phi_1)Z_0(y)dy\\ &+ \int_{B_{R(\tau)}} pU^{p-1}\chi_M\phi_*[h^0]Z_0(y)dy\Bigg],\end{aligned}$$

where $\tilde\gamma_0(\tau) = \gamma_0 + (\int_{B_{R(\tau)}} Z_0^2(y)dy)^{-1}\int_{B_{R(\tau)}}(f_1 Z_0(y) + f_2 y\cdot\nabla Z_0(y))Z_0(y)dy$. By (7.2), $\lim\limits_{\tau\to\infty}\tilde\gamma_0(\tau) = \gamma_0$ as $\tau\to\infty$. We take $e(\tau)$ as

$$\begin{aligned}e(\tau) = {}& -e^{\int^\tau\tilde\gamma_0(u)du}\int_\tau^\infty e^{-\int^s\tilde\gamma_0(u)du}\Big(\int_{B_{R(s)}} Z_0^2(y)dy\Big)^{-1}\Bigg[\int_{\partial B_{R(s)}} Z_0(y)\partial_n\tilde\phi_1(y,s)dy\\ &+ \int_{B_{R(s)}}(f_1(y,s)\tilde\phi_1(y,s) + f_2(y,s)y\cdot\nabla\tilde\phi_1(y,s))Z_0(y)dy + \int_{B_{R(s)}} pU^{p-1}(y)\chi_M(y)\phi_*[h^0](y,s)Z_0(y)dy\Bigg]ds.\end{aligned}$$

Set

$$\|\tilde\phi_1\|_w = \sup_{\tau>\tau_0}\left(\min\{\tau^{\frac12}, \lambda_R^{-\frac12}\}\lambda_R^{-\frac12}v\theta^0_{Ra}\right)^{-1}\left(\|\tilde\phi_1(\cdot,\tau)\|_{L^\infty(B_{R(\tau)})} + \|\langle\cdot\rangle\nabla\tilde\phi_1(\cdot,\tau)\|_{L^\infty(B_{R(\tau)})}\right).$$

By (7.14), it is straightforward to get

$$
\begin{aligned}
|e(\tau)| &\lesssim e^{\int^{\tau}\tilde{\gamma}_0(u)du}\int_{\tau}^{\infty} e^{-\int^{s}\tilde{\gamma}_0(u)du}\Big(e^{-cR(s)}\|\nabla\tilde{\phi}_1\|_{L^\infty(B_{R(s)})}+\||f_1\tilde{\phi}_1|+|f_2\nabla\tilde{\phi}_1|\|_{L^\infty(B_{R(s)})}+v(s)\theta^0_{Ra}(s)\|h^0\|_{v,a}\Big)ds\\
&\lesssim e^{\int^{\tau}\tilde{\gamma}_0(u)du}\int_{\tau}^{\infty} e^{-\int^{s}\tilde{\gamma}_0(u)du}\Big[(s^{-d}+e^{-cR(s)})\min\{s^{\frac12},\lambda_R^{-\frac12}(s)\}\lambda_R^{-\frac12}(s)v(s)\theta^0_{Ra}(s)\|\tilde{\phi}_1\|_w+v(s)\theta^0_{Ra}(s)\|h^0\|_{v,a}\Big]ds\\
&\lesssim (\tau_0^{-d}+e^{-cR(\tau_0)})\min\{\tau^{\frac12},\lambda_R^{-\frac12}\}\lambda_R^{-\frac12}v\theta^0_{Ra}\|\tilde{\phi}_1\|_w+v\theta^0_{Ra}\|h^0\|_{v,a}
\end{aligned}
\tag{7.16}
$$

for some constant $c>0$, and θ^0_{Ra} is given in (7.6). It follows that

$$
|\partial_\tau e(\tau)|\lesssim (\tau_0^{-d}+e^{-cR(\tau_0)})\min\{\tau^{\frac12},\lambda_R^{-\frac12}\}\lambda_R^{-\frac12}v\theta^0_{Ra}\|\tilde{\phi}_1\|_w+v\theta^0_{Ra}\|h^0\|_{v,a}.
$$

With the above choice of $e(\tau)$, the global existence of (7.15) can be deduced by the local existence.

Multiplying equation (7.15) by $\tilde{\phi}_1$ and integrating by parts, one has

$$
\begin{aligned}
&\frac12\partial_\tau\int_{B_{R(\tau)}}(\tilde{\phi}_1)^2dy+\int_{B_{R(\tau)}}(|\nabla\tilde{\phi}_1|^2-pU^{p-1}(\tilde{\phi}_1)^2)dy\\
=&\int_{B_{R(\tau)}}(f_1\tilde{\phi}_1+f_2y\cdot\nabla\tilde{\phi}_1)\tilde{\phi}_1dy+\int_{B_{R(\tau)}}pU^{p-1}\chi_M\phi_*[h^0]\tilde{\phi}_1dy+e(\tau)\int_{B_{R(\tau)}}(f_1Z_0(y)+f_2y\cdot\nabla Z_0(y))\tilde{\phi}_1dy.
\end{aligned}
$$

Then by Lemma 7.3 and (7.2), we get

$$
\begin{aligned}
&\frac12\partial_\tau\int_{B_{R(\tau)}}(\tilde{\phi}_1)^2dy+c\lambda_R\int_{B_{R(\tau)}}(\tilde{\phi}_1)^2dy\le C\tau^{-d}\int_{B_{R(\tau)}}(\tilde{\phi}_1)^2dy+\int_{B_{R(\tau)}}\frac{4}{c\lambda_R}(pU^{p-1}\chi_M\phi_*[h^0])^2dy\\
&+\int_{B_{R(\tau)}}\frac{c\lambda_R}{4}(\tilde{\phi}_1)^2dy+\int_{B_{R(\tau)}}\frac{4}{c\lambda_R}e^2(\tau)(f_1Z_0(y)+f_2y\cdot\nabla Z_0(y))^2dy+\int_{B_{R(\tau)}}\frac{c\lambda_R}{4}(\tilde{\phi}_1)^2dy
\end{aligned}
$$

for some constant $c>0$. By (7.2), (7.14), (7.16) and the assumption $\lambda_R\tau^d\gg 1$, we get

$$
\begin{aligned}
\frac12\partial_\tau\int_{B_{R(\tau)}}(\tilde{\phi}_1)^2dy+\frac{c\lambda_R}{4}\int_{B_{R(\tau)}}(\tilde{\phi}_1)^2dy&\lesssim\lambda_R^{-1}[(v\theta^0_{Ra}\|h^0\|_{v,a})^2+\tau^{-2d}e^2(\tau)]\\
&\lesssim\lambda_R^{-1}(v\theta^0_{Ra})^2\left[\|h^0\|_{v,a}+(\tau_0^{-d}+e^{-cR(\tau_0)})\|\tilde{\phi}_1\|_w\right]^2.
\end{aligned}
$$

Since $\tilde{\phi}_1(\cdot,\tau_0)=0$, one has

$$
\begin{aligned}
\int_{B_{R(\tau)}}(\tilde{\phi}_1)^2dy&\lesssim e^{-\int^{\tau}\frac{c\lambda_R(u)}{2}du}\int_{\tau_0}^{\tau}e^{\int^{s}\frac{c\lambda_R(u)}{2}du}\lambda_R^{-1}(s)(v(s)\theta^0_{Ra}(s))^2[\|h^0\|_{v,a}+(\tau_0^{-d}+e^{-cR(\tau_0)})\|\tilde{\phi}_1\|_w]^2ds\\
&\lesssim\min\{\tau,\lambda_R^{-1}\}\lambda_R^{-1}(v\theta^0_{Ra})^2\left[\|h^0\|_{v,a}+(\tau_0^{-d}+e^{-cR(\tau_0)})\|\tilde{\phi}_1\|_w\right]^2.
\end{aligned}
$$

Applying parabolic estimate to (7.15), one has

$$
\begin{aligned}
&\|\tilde{\phi}_1(\cdot,\tau)\|_{L^\infty(B_{R(\tau)})}\lesssim\min\{\tau^{\frac12},\lambda_R^{-\frac12}\}\lambda_R^{-\frac12}v\theta^0_{Ra}[\|h^0\|_{v,a}+(\tau_0^{-d}+e^{-cR(\tau_0)})\|\tilde{\phi}_1\|_w]\\
&+\tau^{-d}(\|\tilde{\phi}_1(\cdot,\tau)\|_{L^\infty(B_{R(\tau)})}+\|y\cdot\nabla\tilde{\phi}_1\|_{L^\infty(B_{R(\tau)})})+|\partial_\tau e(\tau)|+|e(\tau)|+v\theta^0_{Ra}\|h^0\|_{v,a}\\
\lesssim&\min\{\tau^{\frac12},\lambda_R^{-\frac12}\}\lambda_R^{-\frac12}v\theta^0_{Ra}[\|h^0\|_{v,a}+(\tau_0^{-d}+e^{-cR(\tau_0)})\|\tilde{\phi}_1\|_w].
\end{aligned}
$$

By comparison principle, the spatial decay of $\tilde{\phi}_1$ can be improved and scaling argument will give the spatial decay about $\nabla\tilde{\phi}_1$. Then one has

$$
\langle y\rangle|\nabla\tilde{\phi}_1|+|\tilde{\phi}_1|\lesssim\min\{\tau^{\frac12},\lambda_R^{-\frac12}\}\lambda_R^{-\frac12}v\theta^0_{Ra}[\|h^0\|_{v,a}+(\tau_0^{-d}+e^{-cR(\tau_0)})\|\tilde{\phi}_1\|_w]\langle y\rangle^{2-n},
$$

which implies

$$
\langle y\rangle|\nabla\tilde{\phi}_1|+|\tilde{\phi}_1|\lesssim\min\{\tau^{\frac12},\lambda_R^{-\frac12}\}\lambda_R^{-\frac12}v\theta^0_{Ra}\langle y\rangle^{2-n}\|h^0\|_{v,a}.
$$

Reviewing the computations in (7.16) and using $\lambda_R\tau^d\gg 1$, one has $|e(\tau)|\lesssim v\theta^0_{Ra}\|h^0\|_{v,a}$ and then

$$
|\tilde{\phi}|=|\tilde{\phi}_1+e(\tau)Z_0(y)|\lesssim\min\{\tau^{\frac12},\lambda_R^{-\frac12}\}\lambda_R^{-\frac12}v\theta^0_{Ra}\langle y\rangle^{2-n}\|h^0\|_{v,a}
\tag{7.17}
$$

Finally, we take $e_0=e(\tau_0)$. Combining (7.14) and (7.17), we complete the proof of this Lemma. □

7.2. **Modes** 1 **to** n **without orthogonality.**

Lemma 7.6. *Consider*

$$\begin{cases}\partial_\tau \phi^1 = \Delta\phi^1 + pU^{p-1}\phi^1 + f_1\phi^1 + f_2 y\cdot\nabla\phi^1 + h^1(y,\tau) \ \text{ in } \mathcal{D}_R \\ \phi^1 = 0 \ \text{ on } \partial\mathcal{D}_R \quad \phi^1(\cdot,\tau_0) = 0 \ \text{ in } B_{R(\tau_0)}\end{cases} \tag{7.18}$$

where $h^1(y,\tau) = \sum_{j=1}^{n} h_j(|y|,\tau)\Upsilon_j$*. Assume* $R^n\theta^1_{Ra} \ll \tau^{\min\{1,d\}}$*, where* θ^1_{Ra} *is given in (7.6). Then for* τ_0 *sufficiently large, there exists a unique linear mapping* $\phi^1 = \phi^1[h^1]$ *solving (7.18) of the form* $\phi^1 = \sum_{j=1}^{n}\phi_j(|y|,\tau)\Upsilon_j$ *with the following estimate*

$$\langle y\rangle|\nabla\phi^1| + |\phi^1| \lesssim v\theta^1_{Ra}R^n\langle y\rangle^{1-n}\|h^1\|_{v,a}.$$

Proof. Set $r = |y|$. Notice $y\cdot\nabla(\phi_j(r,\tau)\Upsilon_j) = r\partial_r\phi_j(r,\tau)\Upsilon_j$. It is equivalent to considering

$$\begin{cases}\partial_\tau\phi_j = \mathcal{L}_1[\phi_j] + f_1\phi_j(r,\tau) + f_2 r\partial_r\phi_j(r,\tau) + h_j(r,\tau) \ \text{ for } \ r\in(0,R(\tau)),\ \tau\in(\tau_0,\infty) \\ \partial_r\phi_j(0,\tau) = 0 = \phi_j(R(\tau),\tau) \ \text{ for } \ \tau\in(\tau_0,\infty), \quad \phi_j(r,\tau_0) = 0 \ \text{ for } \ r\in(0,R(\tau_0))\end{cases} \tag{7.19}$$

where $\mathcal{L}_1[\phi_j] := \partial_{rr}\phi_j + \frac{n-1}{r}\partial_r\phi_j - \frac{n-1}{r^2}\phi_j + pU(r)^{p-1}\phi_j$, $|h_j| \le v\langle y\rangle^{-a}\|h_j\|_{v,a}$, $\|h_j\|_{v,a} \lesssim \|h^1\|_{v,a}$.

One positive kernel of $\mathcal{L}_1$ is given by $Z(r) := -U_r = (n(n-2))^{\frac{n-2}{4}}(n-2)r(1+r^2)^{-\frac{n}{2}}$. Set a barrier function of (7.19) as $\phi_s = Cv\bar\phi(r,R)$, where

$$\mathcal{L}_1[\bar\phi] = -\langle r\rangle^{-\bar a}, \quad \bar a = \min\{a, n-1\}$$

with $\bar\phi$ given by the variation of parameter formula

$$\bar\phi(r,R) = Z(r)\int_r^R \frac{1}{\rho^{n-1}Z^2(\rho)}\int_0^\rho \langle s\rangle^{-\bar a}Z(s)s^{n-1}dsd\rho.$$

Then

$$\bar\phi \lesssim R^n\theta^1_{R\bar a}r\langle r\rangle^{-n}, \quad |\partial_R\bar\phi| = \left|\frac{Z(r)}{R^{n-1}Z^2(R)}\int_0^R\langle s\rangle^{-\bar a}Z(s)s^{n-1}ds\right| \lesssim R^{n-1}\theta^1_{R\bar a}r\langle r\rangle^{-n}$$

for all $r > 0$. This estimate holds for all $n > 2$, and $a \le 0$ is also allowed here. Next, we compute

$$\begin{aligned}P(\phi_s) &:= \mathcal{L}_1\phi_s + f_1\phi_s + f_2 r\partial_r\phi_s - \partial_\tau\phi_s + h_j = -Cv\langle r\rangle^{-\bar a} + Cv(f_1\bar\phi + f_2 r\partial_r\bar\phi) - Cv'\bar\phi - Cv\partial_R\bar\phi R' + h_j \\ &\le Cv\langle r\rangle^{-\bar a}\left[-1 + \langle r\rangle^{\bar a}(f_1\bar\phi + f_2 r\partial_r\bar\phi) - v^{-1}v'\langle r\rangle^{\bar a}\bar\phi - \langle r\rangle^{\bar a}\partial_R\bar\phi R' + C^{-1}\langle r\rangle^{\bar a - a}\|h_j\|_{v,a}\right] \\ &\le Cv\langle r\rangle^{-\bar a}(-\frac{3}{4} + C^{-1}\|h_j\|_{v,a})\end{aligned}$$

where we have used

$$\begin{aligned}|\langle r\rangle^{\bar a}(f_1\bar\phi + f_2 r\partial_r\bar\phi)| &\lesssim \tau^{-d}R^n\theta^1_{R\bar a}\langle r\rangle^{\bar a+1-n} \lesssim \tau^{-d}R^n\theta^1_{Ra} \ll 1, \\ |v^{-1}v'\langle r\rangle^{\bar a}\bar\phi| &\lesssim \tau^{-1}R^n\theta^1_{R\bar a}\langle r\rangle^{\bar a+1-n} \lesssim \tau^{-1}R^n\theta^1_{Ra} \ll 1, \\ |\langle r\rangle^{\bar a}\partial_R\bar\phi R'| &\lesssim \langle r\rangle^{\bar a}R^{n-1}\theta^1_{R\bar a}r\langle r\rangle^{-n}|R'| \lesssim \tau^{-1}R^n\theta^1_{Ra} \ll 1\end{aligned}$$

by (7.2), $\theta^1_{R\bar a} = \theta^1_{Ra}$ and the assumption $R^n\theta^1_{Ra} \ll \tau^{\min\{1,d\}}$. Taking $C = 4\|h_j\|_{v,a}$, one has $P(\phi_s) < 0$. ∎

7.3. **Higher modes.**

Lemma 7.7. *Consider*

$$\begin{cases}\partial_\tau\phi^\perp = \Delta\phi^\perp + pU^{p-1}\phi^\perp + f_1\phi^\perp + f_2 y\cdot\nabla\phi^\perp + h^\perp \ \text{ in } \mathcal{D}_R \\ \phi^\perp = 0 \ \text{ on } \partial\mathcal{D}_R, \quad \phi^\perp(\cdot,\tau_0) = 0 \ \text{ in } B_{R(\tau_0)}\end{cases}$$

where $\|h^\perp\|_{v,a} < \infty$*,* $a \ge 0$*. Assume* $R^2\ln R \ll \tau^{\min\{1,d\}}$*. Then there exists a unique linear mapping* $\phi^\perp = \phi^\perp[h^\perp]$ *of the form*

$$\phi^\perp = \sum_{j=n+1}^{\infty}\phi_j^\perp(|y|,\tau)\Upsilon_j \tag{7.20}$$

with the following estimate

$$\langle y\rangle|\nabla\phi^\perp| + |\phi^\perp| \lesssim v\left(\Theta^0_{Ra}(|y|) + \theta^0_{Ra}R\langle y\rangle^{2-n}\right)\|h^\perp\|_{v,a}.$$

First we give the following technical lemma.

Lemma 7.8. *For $f \in C^2(B_R)\cap C_0(B_R)$, by the expansion of spherical harmonic functions, $f = \sum_{j=0}^{\infty} f_j(r)\Upsilon_j$, where $r = |y|$, $f_j(r) = \int_{S^{n-1}} f(r\theta)\Upsilon_j(\theta)d\theta \in C^2[0,R]$. Then*

$$Q(f,f) = \int_{B_R} (|\nabla f|^2 - pU^{p-1}f^2)dy = |S^{n-1}|\sum_{j=0}^{\infty} Q_j(f_j,f_j),$$

where $|S^{n-1}|$ is the volume of the unit $(n-1)$-sphere and

$$Q_j(f_j,f_j) = \int_0^R \left(f_j'^2 + \frac{\iota_j}{r^2}f_j^2 - pU^{p-1}f_j^2\right) r^{n-1}dr.$$

Specially, if $f_j = 0$ for $j = 0,1,\ldots,n$, it holds that

$$Q(f,f) \geq (n+1)\int_{B_R} \frac{|f|^2}{|y|^2}dy. \tag{7.21}$$

Proof. Since $\Delta_{S^{n-1}}\Upsilon_i = -\iota_i\Upsilon_i$, $\iota_i = i(n-2+i)$ for a nonnegative integer i, we have

$$\Delta(f_i\Upsilon_i) = (f_i'' + \frac{n-1}{r}f_i' - \frac{\iota_i}{r^2}f_i)\Upsilon_i.$$

$f|_{\partial B_R} = 0$ implies $f_j(R) = 0$, $j = 0,1,\ldots$. Then

$$\begin{aligned} Q(f,f) &= \int_{B_R} |\nabla f|^2 - pU^{p-1}f^2 dy = -\int_{B_R} (f\Delta f + pU^{p-1}f^2)dy \\ = &-|S^{n-1}|\int_0^R \left[\sum_{i=0}^{\infty} f_i(f_i'' + \frac{n-1}{r}f_i' - \frac{\iota_i}{r^2}f_i) + pU^{p-1}\sum_{i=0}^{\infty} f_i^2\right] r^{n-1}dr = |S^{n-1}|\sum_{i=0}^{\infty} Q_i(f_i,f_i). \end{aligned}$$

For $i \geq n+1$, $\iota_i \geq 2n$, we have

$$Q_i(f_i,f_i) \geq Q_1(f_i,f_i) + (n+1)\int_0^R \frac{f_i^2}{r^2}r^{n-1}dr \geq (n+1)\int_0^R \frac{f_i^2}{r^2}r^{n-1}dr$$

since $\Delta u - \frac{n-1}{r^2}u + pU^{p-1}(y)u = 0$ has a positive kernel $-U_r$, and by [41, Lemma 4.2], one has $Q_1(f_i,f_i) \geq 0$. Specially, if $f_j = 0$ for $j = 0,1,\ldots,n$, we have (7.21). □

Proof of Lemma 7.7. The existence and uniqueness of the linear mapping $\phi^\perp = \phi^\perp[h^\perp]$ are guaranteed by the classical parabolic theory. The form (7.20) is derived from the existence of every component ϕ_j with

$$\begin{cases} \partial_\tau\phi_j = \partial_{rr}\phi_j + \frac{n-1}{r}\partial_r\phi_j - \frac{\iota_j}{r^2}\phi_j + pU^{p-1}\phi_j + f_1\phi_j + f_2 r\partial_r\phi_j + h_j \ \text{ for } \ r \in (0,R(\tau)), \tau \in (\tau_0,\infty) \\ \partial_r\phi_j(0,\tau) = \phi_j(R(\tau),\tau) = 0 \ \text{ for } \ \tau \in (\tau_0,\infty), \quad \phi_j(r,\tau_0) = 0 \ \text{ for } \ r \in (0,R(\tau_0)). \end{cases}$$

By similar operation in mode 0, we set $\phi^\perp = \phi_*[h^\perp] + \tilde\phi[h^\perp]$, where $\phi_*[h^\perp]$ satisfies

$$\begin{cases} \partial_\tau\phi_* = \Delta\phi_* + pU^{p-1}(1-\chi_M)\phi_* + f_1\phi_* + f_2 y\cdot\nabla\phi_* + h^\perp \ \text{ in } \ \mathcal{D}_R, \\ \phi_* = 0 \ \text{ on } \ \partial\mathcal{D}_R, \quad \phi_*(\cdot,\tau_0) = 0 \ \text{ in } \ B_{R(\tau_0)}, \end{cases}$$

and $\tilde\phi[h^\perp]$ satisfies

$$\begin{cases} \partial_\tau\tilde\phi = \Delta\tilde\phi + pU^{p-1}\tilde\phi + f_1\tilde\phi + f_2 y\cdot\nabla\tilde\phi + pU^{p-1}\chi_M\phi_*[h^\perp] \ \text{ in } \ \mathcal{D}_R, \\ \tilde\phi = 0 \ \text{ on } \ \partial\mathcal{D}_R, \quad \tilde\phi(\cdot,\tau_0) = 0 \ \text{ in } \ B_{R(\tau_0)}. \end{cases} \tag{7.22}$$

Under the assumption $R^2\ln R \ll \tau^{\min\{1,d\}}$, by Lemma 7.4, we have

$$|\phi_*[h^\perp]| \lesssim v\Theta^0_{Ra}(|y|)\|h^\perp\|_{v,a}. \tag{7.23}$$

$\phi_*[h^\perp]$ has the form $\phi_*[h^\perp] = \sum_{j=n+1}^{\infty} \phi_{*j}(r,\tau)\Upsilon_j$ by the same reason as (7.20). By the same argument, there exists a linear mapping $\tilde\phi = \tilde\phi[h^\perp]$ and $\tilde\phi$ has the same form as (7.20). Thus we are able to apply (7.21) to $\tilde\phi$.

Multiplying (7.22) by $\tilde\phi$ and integrating both sides, we have

$$\frac{1}{2}\partial_\tau\int_{B_R} \tilde\phi^2 dy + Q(\tilde\phi,\tilde\phi) = \int_{B_R} (f_1\tilde\phi + f_2 y\cdot\nabla\tilde\phi)\tilde\phi dy + \int_{B_R} pU^{p-1}(y)\chi_M\phi_*[h^\perp]\tilde\phi dy.$$

By (7.21), (7.2) and Hölder inequality, one has

$$\frac{1}{2}\partial_\tau \int_{B_R} \tilde{\phi}^2 dy + (n+\frac{1}{2}) \int_{B_R} \frac{\tilde{\phi}^2}{|y|^2} dy \le C\tau^{-d} \int_{B_R} \tilde{\phi}^2 dy + \frac{1}{2} \int_{B_R} \left(pU^{p-1}(y)\chi_M \phi_*[h^\perp]|y|\right)^2 dy.$$

Then, by (7.23) and the assumption $R^2 \ln R \ll \tau^{\min\{1,d\}}$, we have

$$\partial_\tau \int_{B_R} \tilde{\phi}^2 dy + R^{-2} \int_{B_R} |\tilde{\phi}|^2 dy \lesssim (v\theta^0_{Ra})^2 \|h^\perp\|^2_{v,a}.$$

Since $\tilde{\phi}(\cdot,\tau_0) = 0$ and the assumption $R^2 \ln R \ll \tau^{\min\{1,d\}}$, we have

$$\int_{B_R} \tilde{\phi}^2 dy \lesssim e^{-\int^\tau R^{-2}(u)du} \int_{\tau_0}^\tau e^{\int^s R^{-2}(u)du} (v(s)\theta^0_{Ra}(s))^2 ds \|h^\perp\|^2_{v,a} \lesssim (v\theta^0_{Ra} R\|h^\perp\|_{v,a})^2.$$

Using the same argument in Lemma 7.5, one has

$$|\tilde{\phi}(y,\tau)| \lesssim v\theta^0_{Ra} R\langle y\rangle^{2-n} \|h^\perp\|_{v,a}. \tag{7.24}$$

Combining (7.23), (7.24) and scaling argument, we get

$$\langle y\rangle|\nabla\phi^\perp| + |\phi^\perp| \lesssim v\left(\Theta^0_{Ra}(|y|) + \theta^0_{Ra} R\langle y\rangle^{2-n}\right) \|h^\perp\|_{v,a}.$$

□

Proof of Proposition 7.2. The case for higher modes has been given in Lemma 7.7. Since the fast spatial decay of the right hand side h cannot be recovered in non-orthogonal case in lower modes i, $0 \le i \le n$, we transform the fast decay right hand side into slower decay function by solving the corresponding elliptic equation.

7.4. **Mode 0 with orthogonality.** Consider

$$\Delta H^0 + pU^{p-1}H^0 = \tilde{h}^0(r,\tau) \ \text{ in } \mathbb{R}^n$$

where $\tilde{h}^0$ is the extension of h^0 as zero outside $\mathcal{D}_R$. The orthogonal condition is reformulated as

$$\int_0^R h^0(r,\tau) Z_{n+1}(r) r^{n-1} dr = 0 \ \text{ for all } \ \tau > \tau_0.$$

Take $H^0(r,\tau)$ as in the following form

$$H^0(r,\tau) = \tilde{Z}_{n+1}(r) \int_0^r \tilde{h}^0(s,\tau) Z_{n+1}(s) s^{n-1} ds - Z_{n+1}(r) \int_0^r \tilde{h}^0(s,\tau)\tilde{Z}_{n+1}(s) s^{n-1} ds, \ \text{ if } \ a \le n-2,$$

$$H^0(r,\tau) = \tilde{Z}_{n+1}(r) \int_0^r \tilde{h}^0(s,\tau) Z_{n+1}(s) s^{n-1} ds + Z_{n+1}(r) \int_r^\infty \tilde{h}^0(s,\tau)\tilde{Z}_{n+1}(s) s^{n-1} ds, \ \text{ if } \ a > n-2,$$

where $\tilde{Z}_{n+1}(r)$ is the other linearly independent kernel of the homogeneous equation, which satisfies that the Wronskian $W[Z_{n+1}, \tilde{Z}_{n+1}] = r^{1-n}$, $\tilde{Z}_{n+1}(r) \sim r^{2-n}$ if $r \to 0$ and $\tilde{Z}_{n+1}(r) \sim 1$ if $r \to \infty$. It is straightforward to check

$$\|H^0\|_{v,\hat{a}_0} \lesssim \|h^0\|_{v,2+a},$$

where $\hat{a}_0$ is given in (7.5) and $a > 0$ is used to ensure that the spatial decay of $\tilde{h}^0(s,\tau)Z_{n+1}(s)s^{n-1}$ is faster than s^{-1} for $s \ge 1$. Next, consider

$$\begin{cases} \partial_\tau \Phi^0 = \Delta\Phi^0 + pU^{p-1}\Phi^0 + H^0 & \text{in } \mathcal{D}_{2R}, \\ \Phi^0(\cdot,\tau_0) = \bar{e}_0 Z_0 & \text{in } B_{2R(\tau_0)}, \end{cases} \tag{7.25}$$

where $(\Phi^0, \bar{e}_0)$ is given by Lemma 7.5 under the condition $f_1 = f_2 = 0$. By scaling argument, one has

$$\langle y\rangle^2|\nabla^2\Phi^0| + \langle y\rangle|\nabla\Phi^0| + |\Phi^0| \lesssim v\left(\min\{\tau^{\frac{1}{2}}, \lambda_R^{-\frac{1}{2}}\}\lambda_R^{-\frac{1}{2}}\theta^0_{R\hat{a}_0}\langle y\rangle^{2-n} + \Theta^0_{R\hat{a}_0}(|y|)\right) \|h^0\|_{v,2+a},$$

$$|\bar{e}_0| \lesssim v(\tau_0)\theta^0_{R(\tau_0)\hat{a}_0} \|h^0\|_{v,2+a}.$$

Acting the operator $L := \Delta + pU^{p-1}$ on both sides of (7.25) and denoting $\phi^0_1 = L\Phi^0$, we obtain

$$\begin{cases} \partial_\tau \phi^0_1 = \Delta\phi^0_1 + pU^{p-1}\phi^0_1 + h^0 & \text{in } \mathcal{D}_R \\ \phi^0_1(\cdot,\tau_0) = \gamma_0\bar{e}_0 Z_0 & \text{in } B_R \end{cases}$$

with the following estimate

$$\langle y\rangle|\nabla\phi^0_1| + |\phi^0_1| \lesssim v\left(\min\{\tau^{\frac{1}{2}}, \lambda_R^{-\frac{1}{2}}\}\lambda_R^{-\frac{1}{2}}\theta^0_{R\hat{a}_0}\langle y\rangle^{-n} + \Theta^0_{R\hat{a}_0}(|y|)\langle y\rangle^{-2}\right) \|h^0\|_{v,2+a}.$$

Taking into account $f_1\phi^0 + f_2 y\cdot\nabla\phi^0$, we consider

$$\begin{cases} \partial_\tau \phi_2^0 = \Delta\phi_2^0 + pU^{p-1}\phi_2^0 + f_1\phi_2^0 + f_2 y\cdot\nabla\phi_2^0 + f_1\phi_1^0 + f_2 y\cdot\nabla\phi_1^0 & \text{in } \mathcal{D}_R \\ \phi_2^0(\cdot,\tau_0) = e_{02} Z_0 & \text{in } B_R \end{cases}$$

where

$$\left|f_1\phi_1^0 + f_2 y\cdot\nabla\phi_1^0\right| \lesssim C_f \tau^{-d} \min\{\tau^{\frac12}, \lambda_R^{-\frac12}\}\lambda_R^{-\frac12} v\theta^0_{R\hat{a}_0}\langle y\rangle^{-2}\|h^0\|_{v,2+a}.$$

Using Lemma 7.5 again, one can find a solution (ϕ_2^0, e_{02}) with the following estimates

$$|\phi_2^0| \lesssim C_f\tau^{-d}\min\{\tau^{\frac12},\lambda_R^{-\frac12}\}\lambda_R^{-\frac12} v\theta^0_{R\hat{a}_0}\left(\min\{\tau^{\frac12},\lambda_R^{-\frac12}\}\lambda_R^{-\frac12}\ln R\langle y\rangle^{2-n} + \ln R\right)\|h^0\|_{v,2+a},$$

$$|e_{02}| \lesssim C_f\tau_0^{-d}\min\{\tau_0^{\frac12},\lambda_{R(\tau_0)}^{-\frac12}\}\lambda_{R(\tau_0)}^{-\frac12} v(\tau_0)\theta^0_{R(\tau_0)\hat{a}_0}\ln R(\tau_0)\|h^0\|_{v,2+a}.$$

Finally, we take $(\phi^0, e_0) = (\phi_1^0 + \phi_2^0, \gamma_0\bar{e}_0 + e_{02})$ and conclude the result for mode 0:

$$|\phi^0| \lesssim v\min\{\tau^{\frac12},\lambda_R^{-\frac12}\}\lambda_R^{-\frac12}\theta^0_{R\hat{a}_0}\left(\langle y\rangle^{-n} + C_f\tau^{-d}\min\{\tau^{\frac12},\lambda_R^{-\frac12}\}\lambda_R^{-\frac12}\ln R\langle y\rangle^{2-n}\right)\|h^0\|_{v,2+a}$$
$$+ v\left(\Theta^0_{R\hat{a}_0}(|y|)\langle y\rangle^{-2} + C_f\tau^{-d}\min\{\tau^{\frac12},\lambda_R^{-\frac12}\}\lambda_R^{-\frac12}\theta^0_{R\hat{a}_0}\ln R\right)\|h^0\|_{v,2+a},$$
$$|e_0| \lesssim v(\tau_0)\theta^0_{R(\tau_0)\hat{a}_0}\left(1 + C_f\tau_0^{-d}\min\{\tau_0^{\frac12},\lambda_{R(\tau_0)}^{-\frac12}\}\lambda_{R(\tau_0)}^{-\frac12}\ln R(\tau_0)\right)\|h^0\|_{v,2+a}.$$

7.5. **Modes 1 to n with orthogonality.** Set $r=|y|$. Consider $h^1(y,\tau) = \sum_{j=1}^n h_j(r,\tau)\Upsilon_j$ satisfying $\int_{B_{2R}} h^1 Z_j = 0$ for all $j=1,\dots,n$, $\tau\in(\tau_0,\infty)$. Then

$$\int_0^{2R} h_j(r,\tau)U_r(r)r^{n-1}dr = 0 \ \text{ for all } \ \tau\in(\tau_0,\infty) \tag{7.26}$$

where $U_r(r) = (n(n-2))^{\frac{n-2}{4}}(2-n)r(1+r^2)^{-\frac n2}$. Let $H = H_j(r,\tau)\Upsilon_j$ satisfying $\mathcal{L}_1 H_j + \tilde{h}_j = 0$ in $\mathbb{R}^n$, where $\tilde{h}_j$ is the extension of h_j as zero outside $\mathcal{D}_R$. H_j is given by

$$H_j(r,\tau) = U_r(r)\int_0^r \frac{1}{\rho^{n-1}U_r(\rho)^2}\int_\rho^\infty \tilde{h}_j(s,\tau)U_r(s)s^{n-1}dsd\rho \ \text{ for } \ -1<a\le n-1,$$
$$H_j(r,\tau) = -U_r(r)\int_r^\infty \frac{1}{\rho^{n-1}U_r(\rho)^2}\int_\rho^\infty \tilde{h}_j(s,\tau)U_r(s)s^{n-1}dsd\rho \ \text{ for } \ a>n-1$$

where $a>-1$ is used to guarantee that the spatial decay of $\tilde{h}_j(s,\tau)U_r(s)s^{n-1}$ is faster than s^{-1}. Using (7.26), one has the following estimate

$$\|H_j\|_{v,\hat{a}_1} \lesssim \|h_j\|_{v,2+a},$$

where $\hat{a}_1$ is given in (7.5). Next, consider

$$\begin{cases} \partial_\tau\Phi = \Delta\Phi + pU^{p-1}\Phi + H_j(r,\tau)\Upsilon_j \ \text{ in } \ D_{2R}, \\ \Phi = 0 \ \text{ on } \ \partial D_{2R} \quad \Phi(\cdot,\tau_0) = 0 \ \text{ in } \ B_{2R(\tau_0)}. \end{cases}$$

By Lemma 7.6, we find a solution Φ_j with the estimate

$$|\Phi_j| \lesssim v\theta^1_{R\hat{a}_1}R^n\langle y\rangle^{1-n}\|h^1\|_{v,2+a}.$$

It follows that

$$\phi_{j1} = L\Phi_j \ \text{ with } \ |\phi_{j1}| \lesssim v\theta^1_{R\hat{a}_1}R^n\langle y\rangle^{-1-n}\|h^1\|_{v,2+a}.$$

Consider

$$\begin{cases} \partial_\tau\phi_{j2} = \mathcal{L}_1\phi_{j2} + f_1\phi_{j2} + f_2 r\partial_r\phi_{j2} + f_1\phi_{j1} + f_2 r\partial_r\phi_{j1} \ \text{ for } \ r\in(0,R(\tau)),\ \tau\in(\tau_0,\infty) \\ \partial_r\phi_{j2}(0,\tau) = 0 = \phi_{j2}(R(\tau),\tau) \ \text{ for } \ \tau\in(\tau_0,\infty), \quad \phi_{j2}(r,\tau_0) = 0 \ \text{ for } \ r\in(0,R(\tau_0)) \end{cases}$$

where

$$|f_1\phi_{j1} + f_2 r\partial_r\phi_{j1}| \lesssim C_f\tau^{-d}v\theta^1_{R\hat{a}_1}R^n\langle y\rangle^{-1-n}\|h^1\|_{v,2+a}.$$

Using Lemma 7.6 again, we get ϕ_{j2} with the following estimate

$$|\phi_{j2}| \lesssim C_f\tau^{-d}v\theta^1_{R\hat{a}_1}R^{2n}\langle y\rangle^{1-n}\|h^1\|_{v,2+a},$$

Set $\phi_j[h_j] = \phi_{j1} + \phi_{j2}$. Then $\phi^1[h^1] = \sum_{j=1}^n \phi_j[h_j]\Upsilon_j$ with the following estimate

$$\langle y\rangle|\nabla\phi^1| + |\phi^1| \lesssim v\theta^1_{R\hat{a}_1}R^n\left(\langle y\rangle^{-1-n} + C_f\tau^{-d}R^n\langle y\rangle^{1-n}\right)\|h^1\|_{v,2+a}$$

as desired. ∎

APPENDIX A. ESTIMATES FOR HEAT EQUATIONS

Recalling $\mathcal{T}_n^{out}$ defined in (2.3), we only require $t_0 \ge 0$ in Lemma A.1 and Lemma A.2.

A.1. Heat equation with right hand side $v(t)|x|^{-b}\mathbf{1}_{\{l_1(t)\le |x|\le l_2(t)\}}$.

Lemma A.1. *Assume $n > 2$, $v(t) \ge 0$, $b \in \mathbb{R}$, $0 \le l_1(t) \le l_2(t) \le C_* t^{\frac12}$, $C_l^{-1} l_i(t) \le l_i(s) \le C_l l_i(t)$, $i = 1, 2$, for all $\frac{t}{2} \le s \le t$, $t \ge t_0 \ge 0$, where $C_* > 0, C_l \ge 1$. Then*

$$\mathcal{T}_n^{out}\left[v(t)|x|^{-b}\mathbf{1}_{\{l_1(t)\le|x|\le l_2(t)\}}\right] \lesssim t^{-\frac{n}{2}} e^{-\frac{|x|^2}{16t}} \int_{\frac{t_0}{2}}^{\frac{t}{2}} v(s) \begin{cases} l_2^{n-b}(s) & \text{if } b<n \\ \ln(\frac{l_2(s)}{l_1(s)}) & \text{if } b=n \\ l_1^{n-b}(s) & \text{if } b>n \end{cases} ds$$

$$+ \sup_{t_1\in[t/2,t]} v(t_1) \begin{cases} \begin{cases} l_2^{2-b}(t) & \text{if } b<2 \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } b=2 \\ l_1^{2-b}(t) & \text{if } b>2 \end{cases} & \text{for } |x|\le l_1(t) \\ \begin{cases} l_2^{2-b}(t) & \text{if } b<2 \\ \langle \ln(\frac{l_2(t)}{|x|})\rangle & \text{if } b=2 \\ |x|^{2-b} & \text{if } 2<b<n \\ |x|^{2-n}\langle \ln(\frac{|x|}{l_1(t)})\rangle & \text{if } b=n \\ |x|^{2-n} l_1^{n-b}(t) & \text{if } b>n \end{cases} & \text{for } l_1(t)<|x|\le l_2(t) \\ |x|^{2-n} e^{-\frac{|x|^2}{16t}} \begin{cases} l_2^{n-b}(t) & \text{if } b<n \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } b=n \\ l_1^{n-b}(t) & \text{if } b>n \end{cases} & \text{for } |x|>l_2(t) \end{cases}.$$

Proof.

$$\mathcal{T}_n^{out}\left[v(t)|x|^{-b}\mathbf{1}_{\{l_1(t)\le|x|\le l_2(t)\}}\right] \lesssim t^{-\frac{n}{2}} \int_{\frac{t_0}{2}}^{\frac{t}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} v(s)|y|^{-b}\mathbf{1}_{\{l_1(s)\le|y|\le l_2(s)\}} dyds$$

$$+ \sup_{t_1\in[t/2,t]} v(t_1) \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n} (t-s)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} |y|^{-b}\mathbf{1}_{\{C_l^{-1}l_1(t)\le|y|\le C_l l_2(t)\}} dyds := u_1 + \sup_{t_1\in[t/2,t]} v(t_1) u_2.$$

For u_1, notice $|y| \le C_* t^{\frac12}$. For $|x| \le 2C_* t^{\frac12}$, we have

$$u_1 \lesssim t^{-\frac{n}{2}} \int_{\frac{t_0}{2}}^{\frac{t}{2}} \int_{\mathbb{R}^n} v(s)|y|^{-b}\mathbf{1}_{\{l_1(s)\le|y|\le l_2(s)\}} dyds \lesssim t^{-\frac{n}{2}} \int_{\frac{t_0}{2}}^{\frac{t}{2}} v(s) \begin{cases} l_2^{n-b}(s) & \text{if } b<n \\ \ln(\frac{l_2(s)}{l_1(s)}) & \text{if } b=n \\ l_1^{n-b}(s) & \text{if } b>n \end{cases} ds.$$

For $|x| > 2C_* t^{\frac12}$, one has $|x-y| \ge \frac{|x|}{2}$. Then

$$u_1 \lesssim t^{-\frac{n}{2}} e^{-\frac{|x|^2}{16t}} \int_{\frac{t_0}{2}}^{\frac{t}{2}} v(s) \begin{cases} l_2^{n-b}(s) & \text{if } b<n \\ \ln(\frac{l_2(s)}{l_1(s)}) & \text{if } b=n \\ l_1^{n-b}(s) & \text{if } b>n \end{cases} ds.$$

Let us estimate u_2 in different regions.

For $|x| \le (2C_l)^{-1} l_1(t)$, since $\frac{|y|}{2} \le |x-y| \le 2|y|$, then

$$u_2 \le \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n} (t-s)^{-\frac{n}{2}} e^{-\frac{|y|^2}{16(t-s)}} |y|^{-b}\mathbf{1}_{\{C_l^{-1}l_1(t)\le|y|\le C_l l_2(t)\}} dyds \sim \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{b}{2}} \int_{\frac{l_1^2(t)}{16C_l^2(t-s)}}^{\frac{C_l^2 l_2^2(t)}{16(t-s)}} e^{-z} z^{\frac{n-b}{2}-1} dzds$$

$$= \left(\int_{\frac{t}{2}}^{t-l_2^2(t)} + \int_{t-l_2^2(t)}^{t-l_1^2(t)} + \int_{t-l_1^2(t)}^{t} \cdots \right) := u_{21} + u_{22} + u_{23} \lesssim \begin{cases} l_2^{2-b}(t) & \text{if } b<2 \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } b=2 \\ l_1^{2-b}(t) & \text{if } b>2. \end{cases}$$

In order to get the last inequality above, we need the following estimates. For u_{21}, since $n>2$, we have

$$u_{21} \sim \int_{\frac{t}{2}}^{t-l_2^2(t)} (t-s)^{-\frac{b}{2}} \int_{\frac{l_1^2(t)}{16C_l^2(t-s)}}^{\frac{C_l^2 l_2^2(t)}{16(t-s)}} z^{\frac{n-b}{2}-1} dzds \lesssim \begin{cases} l_2^{2-b}(t) & \text{if } b<n \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle l_2^{2-n}(t) & \text{if } b=n \\ l_1^{n-b}(t) l_2^{2-n}(t) & \text{if } b>n. \end{cases}$$

For u_{22}, since $l_1^2(t) \lesssim t-s \lesssim l_2^2(t)$, then

$$u_{22} \lesssim \int_{t-l_2^2(t)}^{t-l_1^2(t)} (t-s)^{-\frac{b}{2}} \begin{cases} 1 & \text{if } b<n \\ \langle \ln(\frac{t-s}{l_1^2(t)})\rangle & \text{if } b=n \\ (\frac{l_1^2(t)}{t-s})^{\frac{n-b}{2}} & \text{if } b>n \end{cases} ds \lesssim \begin{cases} l_2^{2-b}(t) & \text{if } b<2 \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } b=2 \\ l_1^{2-b}(t) & \text{if } b>2. \end{cases}$$

For u_{23}, we have

$$u_{23} \lesssim \int_{t-l_1^2(t)}^{t} (t-s)^{-\frac{b}{2}} \int_{\frac{l_1^2(t)}{16C_l^2(t-s)}}^{\frac{C_l^2 l_2^2(t)}{16(t-s)}} e^{-\frac{z}{2}} dzds \lesssim \int_{t-l_1^2(t)}^{t} (t-s)^{-\frac{b}{2}} e^{-\frac{l_1^2(t)}{32C_l^2(t-s)}} ds \sim l_1^{2-b}(t).$$

For $(2C_l)^{-1} l_1(t) \le |x| \le 2C_l l_2(t)$, then

$$\begin{aligned} u_2 &\le \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n} (t-s)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} |y|^{-b} \left(\mathbf{1}_{\{(4C_l)^{-1}l_1(t) \le |y| \le \frac{|x|}{2}\}} + \mathbf{1}_{\{\frac{|x|}{2} \le |y| \le 2|x|\}} + \mathbf{1}_{\{2|x| \le |y| \le 4C_l l_2(t)\}} \right) dyds \\ &:= u_{21}+u_{22}+u_{23} \lesssim \begin{cases} l_2^{2-b}(t) & \text{if } b<2 \\ \langle \ln(\frac{l_2(t)}{|x|})\rangle & \text{if } b=2 \\ |x|^{2-b} & \text{if } 2<b<n \\ |x|^{2-n}\langle \ln(\frac{|x|}{l_1(t)})\rangle & \text{if } b=n \\ |x|^{2-n} l_1^{n-b}(t) & \text{if } b>n. \end{cases} \end{aligned}$$

For the last inequality above, we need to estimate u_{21}, u_{22} and u_{23}. For u_{21}, since $n>2$, one has

$$u_{21} \le \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n} (t-s)^{-\frac{n}{2}} e^{-\frac{|x|^2}{16(t-s)}} |y|^{-b} \mathbf{1}_{\{(4C_l)^{-1}l_1(t) \le |y| \le \frac{|x|}{2}\}} dyds \lesssim \begin{cases} |x|^{2-b} & \text{if } b<n \\ |x|^{2-n}\langle \ln(\frac{|x|}{l_1(t)})\rangle & \text{if } b=n \\ |x|^{2-n} l_1^{n-b}(t) & \text{if } b>n. \end{cases}$$

For u_{22}, we have

$$u_{22} \lesssim |x|^{-b} \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n} (t-s)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \mathbf{1}_{\{|x-y| \le 3|x|\}} dyds \sim |x|^{-b} \int_{\frac{t}{2}}^{t} \int_0^{\frac{9|x|^2}{4(t-s)}} e^{-z} z^{\frac{n}{2}-1} dzds \sim |x|^{2-b}.$$

For u_{23}, we have

$$\begin{aligned} u_{23} &\le \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n} (t-s)^{-\frac{n}{2}} e^{-\frac{|y|^2}{16(t-s)}} |y|^{-b} \mathbf{1}_{\{2|x| \le |y| \le 4C_l l_2(t)\}} dyds \sim \int_{\frac{t}{2}}^{t} \int_{\frac{|x|^2}{4(t-s)}}^{\frac{C_l^2 l_2^2(t)}{t-s}} (t-s)^{-\frac{b}{2}} e^{-z} z^{\frac{n-b}{2}-1} dzds \\ &= \left(\int_{\frac{t}{2}}^{t-\frac{l_2^2(t)}{2C_*^2}} + \int_{t-\frac{l_2^2(t)}{2C_*^2}}^{t-\frac{|x|^2}{8C_l^2 C_*^2}} + \int_{t-\frac{|x|^2}{8C_l^2 C_*^2}}^{t} \cdots \right) := u_{231}+u_{232}+u_{233} \lesssim \begin{cases} l_2^{2-b}(t) & \text{if } b<2 \\ \langle \ln(\frac{l_2(t)}{|x|})\rangle & \text{if } b=2 \\ |x|^{2-b} & \text{if } b>2. \end{cases} \end{aligned}$$

In order to get the last inequality above, we need the following estimates. For u_{231}, we have

$$u_{231} \sim \int_{\frac{t}{2}}^{t-\frac{l_2^2(t)}{2C_*^2}} \int_{\frac{|x|^2}{4(t-s)}}^{\frac{C_l^2 l_2^2(t)}{t-s}} (t-s)^{-\frac{b}{2}} z^{\frac{n-b}{2}-1} dzds \lesssim \begin{cases} l_2^{2-b}(t) & \text{if } b<n \\ l_2^{2-n}(t)\langle \ln(\frac{l_2(t)}{|x|})\rangle & \text{if } b=n \\ |x|^{n-b} l_2^{2-n}(t) & \text{if } b>n. \end{cases}$$

For u_{232}, since $n>2$, we estimate

$$u_{232} \lesssim \int_{t-\frac{l_2^2(t)}{2C_*^2}}^{t-\frac{|x|^2}{8C_l^2 C_*^2}} (t-s)^{-\frac{b}{2}} \begin{cases} 1 & \text{if } b<n \\ \langle \ln(\frac{|x|^2}{t-s})\rangle & \text{if } b=n \\ (\frac{|x|^2}{t-s})^{\frac{n-b}{2}} & \text{if } b>n \end{cases} ds \lesssim \begin{cases} l_2^{2-b}(t) & \text{if } b<2 \\ \langle \ln(\frac{l_2(t)}{|x|})\rangle & \text{if } b=2 \\ |x|^{2-b} & \text{if } b>2. \end{cases}$$

For u_{233}, one has

$$u_{233} \lesssim \int_{t-\frac{|x|^2}{8C_l^2C_*^2}}^{t} (t-s)^{-\frac{b}{2}} e^{-\frac{|x|^2}{8(t-s)}} ds \sim |x|^{2-b} \int_{C_l^2C_*^2}^{\infty} e^{-z} z^{\frac{b}{2}-2} dz \sim |x|^{2-b}.$$

For $|x| \geq 2C_l l_2(t)$, since $\frac{|x|}{2} \leq |x-y| \leq 2|x|$, then for $n > 2$, it follows that

$$u_2 \lesssim \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{n}{2}} e^{-\frac{|x|^2}{16(t-s)}} ds \begin{cases} l_2^{n-b}(t) & \text{if } b < n \\ \langle \ln(\frac{l_2(t)}{l_1(t)}) \rangle & \text{if } b = n \\ l_1^{n-b}(t) & \text{if } b > n \end{cases} \lesssim |x|^{2-n} e^{-\frac{|x|^2}{16t}} \begin{cases} l_2^{n-b}(t) & \text{if } b < n \\ \langle \ln(\frac{l_2(t)}{l_1(t)}) \rangle & \text{if } b = n \\ l_1^{n-b}(t) & \text{if } b > n. \end{cases}$$

□

A.2. **Heat equation with right hand side** $v(t)|x|^{-b}\mathbf{1}_{\{|x|\geq t^{1/2}\}}$**.**

Lemma A.2. *Assume $n > 0$, $v(t) \geq 0$, $b \in \mathbb{R}$, $t_0 \geq 0$, then*

$$\mathcal{T}_n^{out}\left[v(t)|x|^{-b}\mathbf{1}_{\{|x|\geq t^{\frac{1}{2}}\}}\right]$$
$$\lesssim \begin{cases} t^{-\frac{n}{2}} \int_{t_0/2}^{t/2} v(s) \begin{cases} t^{\frac{n-b}{2}} & \text{if } b < n \\ \langle \ln(ts^{-1}) \rangle & \text{if } b = n \\ s^{\frac{n-b}{2}} & \text{if } b > n \end{cases} ds + t^{1-\frac{b}{2}} \sup_{t_1\in[t/2,t]} v(t_1) & \text{if } |x| \leq t^{\frac{1}{2}} \\ |x|^{-b}\left(t \sup_{t_1\in[t/2,t]} v(t_1) + \int_{t_0/2}^{t/2} v(s) ds\right) + t^{-\frac{n}{2}} e^{-\frac{|x|^2}{16t}} \int_{t_0/2}^{t/2} v(s) \begin{cases} 0 & \text{if } b < n \\ \langle \ln(|x| s^{-\frac{1}{2}}) \rangle & \text{if } b = n \\ s^{\frac{n-b}{2}} & \text{if } b > n \end{cases} ds & \text{if } |x| > t^{\frac{1}{2}} \end{cases}.$$

Proof. By definition, we write

$$\mathcal{T}_n^{out}\left[v(t)|x|^{-b}\mathbf{1}_{\{|x|\geq t^{\frac{1}{2}}\}}\right] \lesssim t^{-\frac{n}{2}} \int_{t_0/2}^{t/2} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} v(s)|y|^{-b}\mathbf{1}_{\{|y|\geq s^{\frac{1}{2}}\}} dyds$$
$$+ \sup_{t_1\in[t/2,t]} v(t_1) \int_{t/2}^{t}\int_{\mathbb{R}^n} (t-s)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} |y|^{-b}\mathbf{1}_{\{|y|\geq 2^{-\frac{1}{2}} t^{\frac{1}{2}}\}} dyds := t^{-\frac{n}{2}} u_1 + \sup_{t_1\in[t/2,t]} v(t_1) u_2.$$

For u_1, when $|x| \leq 2t^{\frac{1}{2}}$, we have

$$u_1 \lesssim \int_{t_0/2}^{t/2}\int_{\mathbb{R}^n} v(s)|y|^{-b}\mathbf{1}_{\{s^{\frac{1}{2}}\leq|y|\leq 4t^{\frac{1}{2}}\}} dyds + \int_{t_0/2}^{t/2}\int_{\mathbb{R}^n} e^{-\frac{|y|^2}{16t}} v(s)|y|^{-b}\mathbf{1}_{\{4t^{\frac{1}{2}}\leq|y|\}} dyds$$
$$\lesssim \int_{t_0/2}^{t/2} v(s) \begin{cases} t^{\frac{n-b}{2}} & \text{if } b < n \\ \langle \ln(ts^{-1}) \rangle & \text{if } b = n \\ s^{\frac{n-b}{2}} & \text{if } b > n \end{cases} ds + t^{\frac{n-b}{2}} \int_{t_0/2}^{t/2} v(s) ds \sim \int_{t_0/2}^{t/2} v(s) \begin{cases} t^{\frac{n-b}{2}} & \text{if } b < n \\ \langle \ln(ts^{-1}) \rangle & \text{if } b = n \\ s^{\frac{n-b}{2}} & \text{if } b > n \end{cases} ds.$$

For u_1, when $|x| > 2t^{\frac{1}{2}}$, we have

$$u_1 = \int_{t_0/2}^{t/2}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} v(s)|y|^{-b}\left(\mathbf{1}_{\{s^{\frac{1}{2}}\leq|y|\leq\frac{|x|}{2}\}} + \mathbf{1}_{\{\frac{|x|}{2}\leq|y|\leq 2|x|\}} + \mathbf{1}_{\{2|x|\leq|y|\}}\right) dyds$$
$$\lesssim \int_{t_0/2}^{t/2} v(s) \int_{\mathbb{R}^n}\left(e^{-\frac{|x|^2}{16t}}|y|^{-b}\mathbf{1}_{\{s^{\frac{1}{2}}\leq|y|\leq\frac{|x|}{2}\}} + |x|^{-b} e^{-\frac{|x-y|^2}{4t}}\mathbf{1}_{\{|x-y|\leq 3|x|\}} + e^{-\frac{|y|^2}{16t}}|y|^{-b}\mathbf{1}_{\{2|x|\leq|y|\}}\right) dyds$$
$$\lesssim e^{-\frac{|x|^2}{16t}} \int_{t_0/2}^{t/2} v(s) \begin{cases} |x|^{n-b} & \text{if } b < n \\ \langle \ln(|x|s^{-\frac{1}{2}}) \rangle & \text{if } b = n \\ s^{\frac{n-b}{2}} & \text{if } b > n \end{cases} ds + t^{\frac{n}{2}}|x|^{-b}\int_{t_0/2}^{t/2} v(s) ds + t^{\frac{n-b}{2}} e^{-\frac{|x|^2}{8t}} \int_{t_0/2}^{t/2} v(s) ds$$
$$\lesssim e^{-\frac{|x|^2}{16t}} \int_{t_0/2}^{t/2} v(s) \begin{cases} 0 & \text{if } b < n \\ \langle \ln(|x|s^{-\frac{1}{2}}) \rangle & \text{if } b = n \\ s^{\frac{n-b}{2}} & \text{if } b > n \end{cases} ds + t^{\frac{n}{2}}|x|^{-b}\int_{t_0/2}^{t/2} v(s) ds.$$

For u_2, when $|x| \leq 2^{-\frac{3}{2}} t^{\frac{1}{2}}$, we have $|y| \geq 2|x|$. Then

$$u_2 \leq \int_{t/2}^{t}\int_{\mathbb{R}^n} (t-s)^{-\frac{n}{2}} e^{-\frac{|y|^2}{16(t-s)}} |y|^{-b}\mathbf{1}_{\{|y|\geq 2^{-\frac{1}{2}} t^{\frac{1}{2}}\}} dyds \lesssim \int_{t/2}^{t} (t-s)^{-\frac{b}{2}} e^{-\frac{t}{64(t-s)}} ds \sim t^{1-\frac{b}{2}}.$$

For u_2, when $|x| \geq 2^{-\frac{3}{2}} t^{\frac{1}{2}}$, one has

$$u_2 \leq \int_{t/2}^{t} \int_{\mathbb{R}^n} (t-s)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} |y|^{-b} \left(\mathbf{1}_{\{9^{-1}t^{\frac{1}{2}} \leq |y| \leq \frac{|x|}{2}\}} + \mathbf{1}_{\{\frac{|x|}{2} \leq |y| \leq 4|x|\}} + \mathbf{1}_{\{4|x| \leq |y|\}} \right) dyds$$

$$\lesssim \int_{t/2}^{t} \int_{\mathbb{R}^n} (t-s)^{-\frac{n}{2}} \left(e^{-\frac{|x|^2}{16(t-s)}} |y|^{-b} \mathbf{1}_{\{9^{-1}t^{\frac{1}{2}} \leq |y| \leq \frac{|x|}{2}\}} + |x|^{-b} e^{-\frac{|x-y|^2}{4(t-s)}} \mathbf{1}_{\{|x-y| \leq 5|x|\}} + e^{-\frac{|y|^2}{16(t-s)}} |y|^{-b} \mathbf{1}_{\{4|x| \leq |y|\}} \right) dyds$$

$$\lesssim |x|^{2-n} e^{-\frac{|x|^2}{16t}} \begin{cases} |x|^{n-b} & \text{if } b < n \\ \langle \ln(|x| t^{-\frac{1}{2}}) \rangle & \text{if } b = n + t|x|^{-b} + |x|^{2-b} e^{-\frac{|x|^2}{2t}} \sim t|x|^{-b}. \\ t^{\frac{n-b}{2}} & \text{if } b > n \end{cases}$$

□

A.3. **Cauchy problem with initial value $\langle y \rangle^{-b}$.**

Lemma A.3. *For $n \geq 1$, $b \in \mathbb{R}$ and $t > 0$, it holds that*

$$(4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \langle y \rangle^{-b} dy$$

$$\lesssim \begin{cases} \langle t \rangle^{-\frac{b}{2}} \mathbf{1}_{\{|x| \leq \langle t \rangle^{\frac{1}{2}}\}} + |x|^{-b} \mathbf{1}_{\{|x| > \langle t \rangle^{\frac{1}{2}}\}} & \text{if } b < n \\ \langle t \rangle^{-\frac{n}{2}} \ln(t+2) \mathbf{1}_{\{|x| \leq \langle t \rangle^{\frac{1}{2}}\}} + \left(|x|^{-n} + t^{-\frac{n}{2}} e^{-\frac{|x|^2}{16t}} \ln(|x|+2) \right) \mathbf{1}_{\{|x| > \langle t \rangle^{\frac{1}{2}}\}} & \text{if } b = n \\ \langle t \rangle^{-\frac{n}{2}} \mathbf{1}_{\{|x| \leq \langle t \rangle^{\frac{1}{2}}\}} + \left(|x|^{-b} + t^{-\frac{n}{2}} e^{-\frac{|x|^2}{16t}} \right) \mathbf{1}_{\{|x| > \langle t \rangle^{\frac{1}{2}}\}} & \text{if } b > n \end{cases}.$$

Proof. Set

$$u(x,t) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \langle y \rangle^{-b} dy \sim t^{-\frac{n}{2}} \left(\int_{|y| \leq \frac{|x|}{2}} + \int_{\frac{|x|}{2} \leq |y| \leq 2|x|} + \int_{2|x| \leq |y|} \right) e^{-\frac{|x-y|^2}{4t}} \langle y \rangle^{-b} dy.$$

We estimate term by term:

$$\int_{|y| \leq \frac{|x|}{2}} e^{-\frac{|x-y|^2}{4t}} \langle y \rangle^{-b} dy \lesssim e^{-\frac{|x|^2}{16t}} \int_{|y| \leq \frac{|x|}{2}} \langle y \rangle^{-b} dy \sim \begin{cases} e^{-\frac{|x|^2}{16t}} |x|^n & \text{if } |x| \leq 1 \\ \begin{cases} e^{-\frac{|x|^2}{16t}} |x|^{n-b} & \text{if } b < n \\ e^{-\frac{|x|^2}{16t}} \ln(|x|+2) & \text{if } b = n \\ e^{-\frac{|x|^2}{16t}} & \text{if } b > n \end{cases} & \text{if } |x| > 1 \end{cases},$$

$$\int_{\frac{|x|}{2} \leq |y| \leq 2|x|} e^{-\frac{|x-y|^2}{4t}} \langle y \rangle^{-b} dy \lesssim \langle x \rangle^{-b} \int_{|x-y| \leq 3|x|} e^{-\frac{|x-y|^2}{4t}} dy \sim \begin{cases} \langle x \rangle^{-b} |x|^n & \text{if } |x| \leq t^{\frac{1}{2}} \\ \langle x \rangle^{-b} t^{\frac{n}{2}} & \text{if } |x| > t^{\frac{1}{2}} \end{cases},$$

and

$$\int_{2|x| \leq |y|} e^{-\frac{|x-y|^2}{4t}} \langle y \rangle^{-b} dy \leq \int_{2|x| \leq |y|} e^{-\frac{|y|^2}{16t}} \langle y \rangle^{-b} dy.$$

For $|x| \geq 1$, we have

$$\int_{2|x| \leq |y|} e^{-\frac{|y|^2}{16t}} \langle y \rangle^{-b} dy \sim t^{\frac{n-b}{2}} \int_{\frac{|x|^2}{4t}}^{\infty} e^{-z} z^{\frac{n-b}{2}-1} dz \lesssim \begin{cases} t^{\frac{n-b}{2}} & \text{if } |x| \leq t^{\frac{1}{2}}, b < n \\ \ln(\frac{t}{|x|^2}) + 1 & \text{if } |x| \leq t^{\frac{1}{2}}, b = n \\ |x|^{n-b} & \text{if } |x| \leq t^{\frac{1}{2}}, b > n \\ t^{\frac{n-b}{2}} e^{-\frac{|x|^2}{8t}} & \text{if } |x| > t^{\frac{1}{2}} \end{cases}.$$

For $|x| < 1$, we have

$$\int_{2|x| \leq |y|} e^{-\frac{|y|^2}{16t}} \langle y \rangle^{-b} dy \sim \int_{2|x|}^{2} e^{-\frac{r^2}{16t}} r^{n-1} dr + \int_{2}^{\infty} e^{-\frac{r^2}{16t}} r^{n-1-b} dr$$

$$\lesssim \begin{cases} t^{\frac{n}{2}} e^{-\frac{|x|^2}{8t}} & \text{if } t \leq |x|^2 \\ t^{\frac{n}{2}} & \text{if } |x|^2 < t \leq 1 \\ 1 & \text{if } t > 1 \end{cases} + \begin{cases} t^{\frac{n-b}{2}} e^{-\frac{1}{8t}} & \text{if } t < 1 \\ t^{\frac{n-b}{2}} & \text{if } t \geq 1, b < n \\ \ln(t+2) & \text{if } t \geq 1, b = n \\ 1 & \text{if } t \geq 1, b > n \end{cases} \lesssim \begin{cases} t^{\frac{n}{2}} e^{-\frac{|x|^2}{16t}} & \text{if } t \leq |x|^2 \\ t^{\frac{n}{2}} & \text{if } |x|^2 < t \leq 1 \\ t^{\frac{n-b}{2}} & \text{if } t \geq 1, b < n \\ \ln(t+2) & \text{if } t \geq 1, b = n \\ 1 & \text{if } t \geq 1, b > n \end{cases}.$$

Combining above estimates, one has

$$u(x,t) \lesssim \begin{cases} \begin{cases} 1 & \text{if } \ t \le 1 \\ t^{-\frac{b}{2}} & \text{if } \ t \ge 1, b < n \\ t^{-\frac{n}{2}} \ln(t+2) & \text{if } \ t \ge 1, b = n \\ t^{-\frac{n}{2}} & \text{if } \ t \ge 1, b > n \end{cases} & \text{if } \ |x| < 1 \\ \begin{cases} t^{-\frac{b}{2}} & \text{if } \ |x| \le t^{\frac{1}{2}}, b < n \\ t^{-\frac{n}{2}} \ln(t+2) & \text{if } \ |x| \le t^{\frac{1}{2}}, b = n \\ t^{-\frac{n}{2}} & \text{if } \ |x| \le t^{\frac{1}{2}}, b > n \\ \langle x \rangle^{-b} & \text{if } \ |x| > t^{\frac{1}{2}}, b < n \\ \langle x \rangle^{-n} + t^{-\frac{n}{2}} e^{-\frac{|x|^2}{16t}} \ln(|x|+2) & \text{if } \ |x| > t^{\frac{1}{2}}, b = n \\ \langle x \rangle^{-b} + t^{-\frac{n}{2}} e^{-\frac{|x|^2}{16t}} & \text{if } \ |x| > t^{\frac{1}{2}}, b > n \end{cases} & \text{if } \ |x| \ge 1 \end{cases}$$

$$\lesssim \begin{cases} \langle t \rangle^{-\frac{b}{2}} & \text{if } \ |x| \le \max\{1, t^{\frac{1}{2}}\}, b < n \\ \langle t \rangle^{-\frac{n}{2}} \ln(t+2) & \text{if } \ |x| \le \max\{1, t^{\frac{1}{2}}\}, b = n \\ \langle t \rangle^{-\frac{n}{2}} & \text{if } \ |x| \le \max\{1, t^{\frac{1}{2}}\}, b > n \\ \langle x \rangle^{-b} & \text{if } \ |x| > \max\{1, t^{\frac{1}{2}}\}, b < n \\ \langle x \rangle^{-n} + t^{-\frac{n}{2}} e^{-\frac{|x|^2}{16t}} \ln(|x|+2) & \text{if } \ |x| > \max\{1, t^{\frac{1}{2}}\}, b = n \\ \langle x \rangle^{-b} + t^{-\frac{n}{2}} e^{-\frac{|x|^2}{16t}} & \text{if } \ |x| > \max\{1, t^{\frac{1}{2}}\}, b > n. \end{cases}$$

This completes the proof of Lemma A.3. □

APPENDIX B. PROOF OF PROPOSITION 3.1: SOLVING THE OUTER PROBLEM

Proof. It suffices to find a fixed point for $\psi = \mathcal{T}_4^{out}[\mathcal{G}[\psi, \phi, \mu_1, \xi]]$. Set

$$w_o(x,t) = \ln t (t(\ln t)^2)^{5\delta-\kappa} R^{-a} \left(\mathbf{1}_{\{|x| \le t^{\frac{1}{2}}\}} + t|x|^{-2} \mathbf{1}_{\{|x| > t^{\frac{1}{2}}\}} \right),$$

$$\|g\|_o = \sup_{(x,t) \in \mathbb{R}^4 \times (t_0, \infty)} w_o^{-1}(x,t) |g(x,t)|, \quad B_o = \{g(x,t) : \ \|g\|_o \le D_o\},$$

where $D_o \ge 1$ will be determined later. For any $\psi_1 \in B_o$, let us estimate $\mathcal{G}[\psi_1, \phi, \mu_1, \xi]$ term by term. In this proof, we will apply Lemma A.1 and Lemma A.2 multiple times to estimate convolution $\mathcal{T}_4^{out}$ and will not state them repetitively.

By the definitions of the norms (3.8), (3.9), (3.10), one has

$$|\phi(y,t)| + \langle y \rangle |\nabla \phi(y,t)| \lesssim (t(\ln t)^2)^{5\delta-\kappa} \langle y \rangle^{-a} \|\phi\|_{i,\kappa-5\delta,a},$$

$$|\mu_1| + t|\mu_{1t}| \lesssim t \ln t (t(\ln t)^2)^{5\delta-\kappa} R^{-a} \|\mu_1\|_{*1}, \quad |\xi| + t|\xi_t| \lesssim t(\ln t)^2 (t(\ln t)^2)^{5\delta-\kappa} R^{-a} \|\xi\|_{*2}.$$

Then

$$\left| \Delta_x \eta_R \mu^{-1} \phi(\frac{x-\xi}{\mu}, t) \right| \lesssim (\mu_0 R)^{-2} \mathbf{1}_{\{\mu_0 R \le |x-\xi| \le 2\mu_0 R\}} \mu^{-1} (t(\ln t)^2)^{5\delta-\kappa} \langle y \rangle^{-a} \|\phi\|_{i,\kappa-5\delta,a}$$
$$\sim \Lambda_1 (\mu_0 R)^{-2} \ln t (t(\ln t)^2)^{5\delta-\kappa} R^{-a} \mathbf{1}_{\{\mu_0 R \le |x-\xi| \le 2\mu_0 R\}},$$

$$\left| 2\nabla_x \eta_R \cdot \mu^{-2} \nabla_y \phi(\frac{x-\xi}{\mu}, t) \right| \lesssim (\mu_0 R)^{-1} \mathbf{1}_{\{\mu_0 R \le |x-\xi| \le 2\mu_0 R\}} \mu^{-2} (t(\ln t)^2)^{5\delta-\kappa} \langle y \rangle^{-1-a} \|\phi\|_{i,\kappa-5\delta,a}$$
$$\lesssim \Lambda_1 (\mu_0 R)^{-2} \ln t (t(\ln t)^2)^{5\delta-\kappa} R^{-a} \mathbf{1}_{\{\mu_0 R \le |x-\xi| \le 2\mu_0 R\}},$$

$$\left| \partial_t \eta_R \mu^{-1} \phi(\frac{x-\xi}{\mu}, t) \right| = \left| \nabla \eta(\frac{x-\xi}{\mu_0 R}) \cdot (\frac{-\xi_t}{\mu_0 R} - \frac{x-\xi}{\mu_0 R} \frac{(\mu_0 R)_t}{\mu_0 R}) \mu^{-1} \phi(\frac{x-\xi}{\mu}, t) \right|$$
$$\lesssim \Lambda_1^2 (\mu_0 R)^{-2} \ln t (t(\ln t)^2)^{5\delta-\kappa} R^{-a} \mathbf{1}_{\{\mu_0 R \le |x-\xi| \le 2\mu_0 R\}},$$

where we have used $\gamma < \frac{1}{2}$ and $5\delta - \kappa < -1$ in the last inequality. Then one has

$$\mathcal{T}_4^{out} \left[(\mu_0 R)^{-2} \ln t (t(\ln t)^2)^{5\delta-\kappa} R^{-a} \mathbf{1}_{\{\mu_0 R \le |x-\xi| \le 2\mu_0 R\}} \right] \lesssim \mathcal{T}_4^{out} \left[(\mu_0 R)^{-2} \ln t (t(\ln t)^2)^{5\delta-\kappa} R^{-a} \mathbf{1}_{\{\mu_0 R/2 \le |x| \le 4\mu_0 R\}} \right]$$
$$\lesssim t^{-2} e^{-\frac{|x|^2}{16t}} \int_{\frac{t_0}{2}}^{\frac{t}{2}} (\ln s)^{-1} (s(\ln s)^2)^{5\delta-\kappa} R^{2-a}(s) ds + \begin{cases} \ln t (t(\ln t)^2)^{5\delta-\kappa} R^{-a} & \text{if } \ |x| \le \mu_0 R \\ \ln t (t(\ln t)^2)^{5\delta-\kappa} R^{-a} (\mu_0 R)^2 |x|^{-2} e^{-\frac{|x|^2}{16t}} & \text{if } \ |x| > \mu_0 R \end{cases}$$
$$\lesssim w_o$$

provided $5\delta-\kappa-a\gamma>-2$. Also,

$$\left|\eta_R\mu^{-2}\xi_t\cdot\nabla_y\phi(\frac{x-\xi}{\mu},t)\right|\lesssim\Lambda_1^2\mathbf{1}_{\{|x|\le 3\mu_0R\}}(\ln t)^2(\ln t)^2(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(t(\ln t)^2)^{5\delta-\kappa}\langle y\rangle^{-1-a}$$
$$\sim\Lambda_1^2(\ln t)^4(t(\ln t)^2)^{10\delta-2\kappa}R^{-a}\left(\mathbf{1}_{\{|x|\le\mu_0\}}+(\ln t)^{-1-a}|x|^{-1-a}\mathbf{1}_{\{\mu_0<|x|\le 3\mu_0R\}}\right),$$

and

$$\begin{aligned}&\mathcal{T}_4^{out}\left[(\ln t)^4(t(\ln t)^2)^{10\delta-2\kappa}R^{-a}(\ln t)^{-1-a}|x|^{-1-a}\mathbf{1}_{\{\mu_0<|x|\le 3\mu_0R\}}\right]\\ &\le\mathcal{T}_4^{out}\left[(\ln t)^3(t(\ln t)^2)^{10\delta-2\kappa}R^{-a}|x|^{-1}\mathbf{1}_{\{\mu_0<|x|\le 3\mu_0R\}}\right]\\ &\lesssim t^{-2}e^{-\frac{|x|^2}{16t}}\int_{\frac{t_0}{2}}^{\frac{t}{2}}(\ln s)^3(s(\ln s)^2)^{10\delta-2\kappa}R^{-a}(s)(\mu_0R)^3(s)ds\\ &\quad+\begin{cases}\mu_0R(\ln t)^3(t(\ln t)^2)^{10\delta-2\kappa}R^{-a} & \text{if } |x|\le\mu_0R\\ (\ln t)^3(t(\ln t)^2)^{10\delta-2\kappa}R^{-a}|x|^{-2}e^{-\frac{|x|^2}{16t}}(\mu_0R)^3 & \text{if } |x|>\mu_0R\end{cases}\lesssim t_0^{-\epsilon}w_o,\\ &\mathcal{T}_4^{out}[(\ln t)^4(t(\ln t)^2)^{10\delta-2\kappa}R^{-a}\mathbf{1}_{\{|x|\le\mu_0\}}]\lesssim t_0^{-\epsilon}w_o\end{aligned}$$

provided $5\delta-\kappa-a\gamma>-2$ and $\epsilon>0$ is sufficiently small.

Using (2.30), one has

$$\begin{aligned}&\left|(1-\eta_R)S\Big[u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})\Big]\right|\\ &\lesssim\Bigg[t^{-2}(\ln t)^{-1}|x|^{-2}+t^{-1}(\ln t)^{-2}|\mu_1||x|^{-4}\\ &\quad+(\ln t)^{-2}\left(|\tilde g[\bar\mu_0,\mu_1]|+|\bar\mu_{0t}|\ln t\sup_{t_1\in[t/2,t]}(\frac{|\mu_1(t_1)|}{\bar\mu_0(t)}+\frac{|\mu_{1t}(t_1)|}{|\bar\mu_{0t}(t)|})\right)|x|^{-4}+|\xi_t|(\ln t)^{-1}|x|^{-3}\Bigg]\mathbf{1}_{\{\frac{\mu_0R}{2}\le|x|\le 9t^{\frac12}\}}\\ &\quad+|\xi_t|t^{\frac32}(\ln t)^{-1}|x|^{-6}\mathbf{1}_{\{|x|>2t^{\frac12}\}}+(t^2(\ln t)^{-1}|x|^{-6})^3\mathbf{1}_{\{|x|>2t^{\frac12}\}}\\ &\lesssim\left(t^{-2}(\ln t)^{-1}|x|^{-2}+\Lambda_1^2\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}|x|^{-3}\right)\mathbf{1}_{\{\frac{\mu_0R}{2}\le|x|\le 9t^{\frac12}\}}\\ &\quad+\Lambda_1t^{\frac32}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}|x|^{-6}\mathbf{1}_{\{|x|>2t^{\frac12}\}}+(t^2(\ln t)^{-1}|x|^{-6})^3\mathbf{1}_{\{|x|>2t^{\frac12}\}}\end{aligned}$$

since

$$|\bar\mu_{0t}|\ln t\sup_{t_1\in[t/2,t]}\left(\frac{|\mu_1(t_1)|}{\bar\mu_0(t)}+\frac{|\mu_{1t}(t_1)|}{|\bar\mu_{0t}(t)|}\right)\lesssim\Lambda_1(\ln t)^2(t(\ln t)^2)^{5\delta-\kappa}R^{-a},$$

$$\begin{aligned}\tilde g[\bar\mu_0,\mu_1]&\lesssim\Lambda_1^2t^{-2}\int_{t_0/2}^{t}(\ln s(s(\ln s)^2)^{5\delta-\kappa}R^{-a}(s)(\ln s)^{-2}+s\ln s(s(\ln s)^2)^{5\delta-\kappa}R^{-a}(s))ds\\ &\quad+\Lambda_1^2(t\ln t)^{-1}[t(\ln t)^3(t(\ln t)^2)^{5\delta-\kappa}R^{-a}]^2\lesssim\Lambda_1^2\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}\end{aligned}\tag{B.1}$$

when $5\delta-\kappa-a\gamma>-2$.

Then we estimate

$$\begin{aligned}&\mathcal{T}_4^{out}\left[t^{-2}(\ln t)^{-1}|x|^{-2}\right]\mathbf{1}_{\{\frac{\mu_0R}{2}\le|x|\le 9t^{\frac12}\}}\\ &\lesssim t^{-2}\ln\ln te^{-\frac{|x|^2}{16t}}+\begin{cases}t^{-2} & \text{if } |x|\le\mu_0R\\ t^{-2}(\ln t)^{-1}(\ln(|x|^{-1}t^{\frac12})+1) & \text{if } \mu_0R<|x|\le t^{\frac12}\\ (t\ln t)^{-1}|x|^{-2}e^{-\frac{|x|^2}{16t}} & \text{if } |x|>t^{\frac12}\end{cases}\lesssim t_0^{-\epsilon_0}w_o\end{aligned}$$

since $5\delta-\kappa-a\gamma>-2$.

$$\begin{aligned}&\mathcal{T}_4^{out}\left[\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}|x|^{-3}\mathbf{1}_{\{\frac{\mu_0R}{2}\le|x|\le 9t^{\frac12}\}}\right]\\ &\lesssim t^{-2}\int_{\frac{t_0}{2}}^{\frac{t}{2}}\ln s(s(\ln s)^2)^{5\delta-\kappa}R^{-a}(s)s^{\frac12}ds+\begin{cases}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(\mu_0R)^{-1} & \text{if } |x|\le\mu_0R\\ \ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}|x|^{-1} & \text{if } \mu_0R<|x|\le t^{\frac12}\\ \ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}t^{\frac12}|x|^{-2}e^{-\frac{|x|^2}{16t}} & \text{if } |x|>t^{\frac12}\end{cases}\\ &\lesssim t_0^{-\epsilon}w_o\end{aligned}$$

since $5\delta-\kappa-a\gamma>-2$.

$$
\begin{aligned}
&\mathcal{T}_4^{out}\left[t^{\frac32}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}|x|^{-6}\mathbf{1}_{\{|x|>2t^{\frac12}\}}\right]\\
&\lesssim\begin{cases} t^{-2}\int_{\frac{t_0}{2}}^{t}s^{\frac12}\ln s(s(\ln s)^2)^{5\delta-\kappa}R^{-a}(s)ds & \text{if } |x|\le t^{\frac12}\\ |x|^{-6}\int_{\frac{t_0}{2}}^{t}s^{\frac32}\ln s(s(\ln s)^2)^{5\delta-\kappa}R^{-a}(s)ds+t^{-2}e^{-\frac{|x|^2}{16t}}\int_{\frac{t_0}{2}}^{\frac t2}s^{\frac12}\ln s(s(\ln s)^2)^{5\delta-\kappa}R^{-a}(s)ds & \text{if } |x|>t^{\frac12}\end{cases}\\
&\lesssim t_0^{-\epsilon}w_o.
\end{aligned}
$$

$$
\mathcal{T}_4^{out}\left[(t^2(\ln t)^{-1}|x|^{-6})^3\mathbf{1}_{\{|x|>2t^{\frac12}\}}\right]\lesssim t^{-2}\mathbf{1}_{\{|x|\le t^{\frac12}\}}+\left(t^7(\ln t)^{-3}|x|^{-18}+t^{-2}e^{-\frac{|x|^2}{16t}}\right)\mathbf{1}_{\{|x|>t^{\frac12}\}}\lesssim t_0^{-\epsilon}w_o
$$

when $5\delta-\kappa-a\gamma>-2$.

$$
\begin{aligned}
&\left(u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})+\psi_1+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right)^3\\
&-\left(u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})\right)^3-3\left(\mu^{-1}w(\frac{x-\xi}{\mu})\right)^2\left(\psi_1+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right)\\
&-\Bigg[3\left(u_1+\varphi[\mu]-\mu^{-1}w(\frac{x-\xi}{\mu})\right)\left(u_1+\varphi[\mu]+\mu^{-1}w(\frac{x-\xi}{\mu})\right)\\
&+6(u_1+\varphi[\mu])\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})\Bigg]\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\\
=\,&3\left(u_1+\varphi[\mu]-\mu^{-1}w(\frac{x-\xi}{\mu})\right)\left(u_1+\varphi[\mu]+\mu^{-1}w(\frac{x-\xi}{\mu})\right)\psi_1\\
&+6(u_1+\varphi[\mu])\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})\psi_1\\
&+3\left(\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})\right)^2\left(\psi_1+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right)\\
&+3\left(u_1+\varphi[\mu]+\bar\mu_0^{-1}\Phi_0(\frac{x-\xi}{\bar\mu_0},t)\eta(\frac{4(x-\xi)}{\sqrt t})\right)\left(\psi_1+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right)^2+\left(\psi_1+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right)^3\\
\lesssim\,&\left((t\ln t)^{-1}\mathbf{1}_{\{|\bar x|\le 2t^{\frac12}\}}+t^2(\ln t)^{-1}|\bar x|^{-6}\mathbf{1}_{\{|\bar x|>2t^{\frac12}\}}+\ln t\langle y\rangle^{-2}\mathbf{1}_{\{|\bar x|\ge t^{\frac12}\}}\right)\\
&\times\left((t\ln t)^{-1}\mathbf{1}_{\{|\bar x|\le 2t^{\frac12}\}}+t^2(\ln t)^{-1}|\bar x|^{-6}\mathbf{1}_{\{|\bar x|>2t^{\frac12}\}}+\ln t\langle y\rangle^{-2}\right)|\psi_1|\\
&+(\ln t\langle y\rangle^{-2}\mathbf{1}_{\{|\bar x|\le 2t^{\frac12}\}}+(t\ln t)^{-1}\mathbf{1}_{\{|\bar x|\le 2t^{\frac12}\}}+t^2(\ln t)^{-1}|\bar x|^{-6}\mathbf{1}_{\{|\bar x|>2t^{\frac12}\}})(t\ln t)^{-1}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)\mathbf{1}_{\{|\bar x|\le 8t^{\frac12}\}}|\psi_1|\\
&+(t\ln t)^{-2}\langle\bar y\rangle^{-4}\ln^2(2+|\bar y|)\mathbf{1}_{\{|\bar x|\le 8t^{\frac12}\}}\left(|\psi_1|+|\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)|\right)\\
&+\Bigg(\ln t\langle y\rangle^{-2}\mathbf{1}_{\{|\bar x|\le 2t^{\frac12}\}}+(t\ln t)^{-1}\mathbf{1}_{\{|\bar x|\le 2t^{\frac12}\}}+O(t^2(\ln t)^{-1}|\bar x|^{-6})\mathbf{1}_{\{|\bar x|>2t^{\frac12}\}}\\
&+(t\ln t)^{-1}\langle\bar y\rangle^{-2}\ln(2+|\bar y|)\mathbf{1}_{\{|\bar x|\le 8t^{\frac12}\}}\Bigg)\left|\psi_1+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right|^2+\left|\psi_1+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right|^3\\
\lesssim\,&\left(t^{-1}\langle y\rangle^{-2}\mathbf{1}_{\{|x|\le t^{\frac12}\}}+(\ln t)^{-2}|x|^{-4}\mathbf{1}_{\{|x|>t^{\frac12}\}}\right)|\psi_1|\\
&+(t\ln t)^{-2}\langle\bar y\rangle^{-4}\ln^2(2+|\bar y|)\mathbf{1}_{\{|\bar x|\le 8t^{\frac12}\}}\left|\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right|\\
&+\left(\ln t\langle y\rangle^{-2}\mathbf{1}_{\{|x|\le t^{\frac12}\}}+t^2(\ln t)^{-1}|x|^{-6}\mathbf{1}_{\{|x|>t^{\frac12}\}}\right)\left|\psi_1+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right|^2+\left|\psi_1+\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right|^3,
\end{aligned}
$$

where we have used Corollary 2.3 and (2.28).

Consider the terms involving ϕ:

$$\begin{aligned}
&\mathcal{T}_4^{out}\left[(t\ln t)^{-2}\langle\bar{y}\rangle^{-4}\ln^2(2+|\bar{y}|)\mathbf{1}_{\{|\bar{x}|\le 8t^{\frac12}\}}\left|\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right|\right]\\
&\lesssim \Lambda_1\mathcal{T}_4^{out}[t^{-2}\langle\bar{y}\rangle^{-4}\mathbf{1}_{\{|\bar{x}|\le 2\mu_0 R\}}\ln t(t(\ln t)^2)^{5\delta-\kappa}\langle y\rangle^{-a}]\\
&\sim \Lambda_1\mathcal{T}_4^{out}[t^{-2}\ln t(t(\ln t)^2)^{5\delta-\kappa}\langle y\rangle^{-4-a}\mathbf{1}_{\{|\bar{x}|\le 2\mu_0 R\}}]\lesssim \Lambda_1 t^{-2}e^{-\frac{|x|^2}{16t}}
\end{aligned}$$

since

$$\begin{aligned}
&\mathcal{T}_4^{out}[t^{-2}\ln t(t(\ln t)^2)^{5\delta-\kappa}\langle y\rangle^{-4-a}\mathbf{1}_{\{\mu_0\le|\bar{x}|\le 2\mu_0 R\}}]\\
&\lesssim \mathcal{T}_4^{out}[t^{-2}(\ln t)^{-3-a}(t(\ln t)^2)^{5\delta-\kappa}|x|^{-4-a}\mathbf{1}_{\{\frac{\mu_0}{2}\le|x|\le 4\mu_0 R\}}]\\
&\lesssim t^{-2}e^{-\frac{|x|^2}{16t}}+\begin{cases} t^{-2}(\ln t)^{-1}(t(\ln t)^2)^{5\delta-\kappa} & \text{if } |x|\le\mu_0\\ t^{-2}(\ln t)^{-3}(t(\ln t)^2)^{5\delta-\kappa}|x|^{-2}e^{-\frac{|x|^2}{16t}} & \text{if } |x|>\mu_0\end{cases}\lesssim t^{-2}e^{-\frac{|x|^2}{16t}}\lesssim t_0^{-\epsilon}w_o,\\
&\mathcal{T}_4^{out}[t^{-2}\ln t(t(\ln t)^2)^{5\delta-\kappa}\langle y\rangle^{-4-a}\mathbf{1}_{\{|\bar{x}|<\mu_0\}}]\lesssim \mathcal{T}_4^{out}[t^{-2}\ln t(t(\ln t)^2)^{5\delta-\kappa}\mathbf{1}_{\{|x|<2\mu_0\}}]\\
&\lesssim t^{-2}e^{-\frac{|x|^2}{16t}}+\begin{cases} t^{-2}(\ln t)^{-1}(t(\ln t)^2)^{5\delta-\kappa} & \text{if } |x|\le\mu_0\\ t^{-2}(\ln t)^{-3}(t(\ln t)^2)^{5\delta-\kappa}|x|^{-2}e^{-\frac{|x|^2}{16t}} & \text{if } |x|>\mu_0\end{cases}\lesssim t^{-2}e^{-\frac{|x|^2}{16t}}.
\end{aligned}$$

Next, we have

$$\begin{aligned}
&\mathcal{T}_4^{out}\left[\left(\ln t\langle y\rangle^{-2}\mathbf{1}_{\{|x|\le t^{\frac12}\}}+t^2(\ln t)^{-1}|x|^{-6}\mathbf{1}_{\{|x|>t^{\frac12}\}}\right)\left|\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right|^2\right]\\
&\lesssim \Lambda_1^2\mathcal{T}_4^{out}[(\ln t\mathbf{1}_{\{|x|\le\mu_0\}}+(\ln t)^{-1}|x|^{-2}\mathbf{1}_{\{\mu_0<|x|\le 4\mu_0 R\}})(\ln t)^2(t(\ln t)^2)^{10\delta-2\kappa}\langle y\rangle^{-2a}]\\
&\sim \Lambda_1^2\mathcal{T}_4^{out}\left[(\ln t)^3(t(\ln t)^2)^{10\delta-2\kappa}\mathbf{1}_{\{|x|\le\mu_0\}}+(t(\ln t)^2)^{10\delta-2\kappa}(\ln t)^{1-2a}|x|^{-2-2a}\mathbf{1}_{\{\mu_0<|x|\le 4\mu_0 R\}}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{T}_4^{out}\left[(t(\ln t)^2)^{10\delta-2\kappa}(\ln t)^{1-2a}|x|^{-2-2a}\mathbf{1}_{\{\mu_0<|x|\le 4\mu_0 R\}}\right]\lesssim \mathcal{T}_4^{out}\left[(t(\ln t)^2)^{10\delta-2\kappa}\ln t|x|^{-2}\mathbf{1}_{\{\mu_0<|x|\le 4\mu_0 R\}}\right]\\
&\lesssim t^{-2}e^{-\frac{|x|^2}{16t}}\int_{\frac{t_0}{2}}^{\frac{t}{2}}(s(\ln s)^2)^{10\delta-2\kappa}\ln s(\mu_0 R)^2(s)ds\\
&\quad+\begin{cases}(t(\ln t)^2)^{10\delta-2\kappa}(\ln t)^2 & \text{if } |x|\le\mu_0\\ (t(\ln t)^2)^{10\delta-2\kappa}\ln t(\ln(\frac{\mu_0 R}{|x|})+1) & \text{if } \mu_0<|x|\le\mu_0 R\\ (t(\ln t)^2)^{10\delta-2\kappa}\ln t(\mu_0 R)^2|x|^{-2}e^{-\frac{|x|^2}{16t}} & \text{if } |x|>\mu_0 R\end{cases}\lesssim t_0^{-\epsilon}w_o
\end{aligned}$$

when $5\delta-\kappa<-1$ and $5\delta-\kappa-a\gamma>-2$. Also,

$$\mathcal{T}_4^{out}\left[(\ln t)^3(t(\ln t)^2)^{10\delta-2\kappa}\mathbf{1}_{\{|x|\le\mu_0\}}\right]\lesssim t_0^{-\epsilon}w_o.$$

When $5\delta-\kappa+(2-a)\gamma<0$, one has

$$\left|\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right|^3\lesssim \Lambda_1\ln t(t(\ln t)^2)^{5\delta-\kappa}\langle y\rangle^{-a}\left|\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right|^2\lesssim \ln t\langle y\rangle^{-2}\left|\eta_R\mu^{-1}\phi(\frac{x-\xi}{\mu},t)\right|^2.$$

Let us now estimate terms involving ψ_1.

$$\begin{aligned}
&\left|\mu^{-2}w^2(\frac{x-\xi}{\mu})\psi_1(1-\eta_R)\right|\lesssim(\ln t)^{-2}|x|^{-4}|\psi_1|\mathbf{1}_{\{|x|\ge\frac{\mu_0 R}{2}\}}\\
&\lesssim D_o(\ln t)^{-2}|x|^{-4}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}\left(\mathbf{1}_{\{\frac{\mu_0 R}{2}\le|x|\le t^{\frac12}\}}+t|x|^{-2}\mathbf{1}_{\{|x|>t^{\frac12}\}}\right)\\
&=D_o(\ln t)^{-1}(t(\ln t)^2)^{5\delta-\kappa}R^{-a}|x|^{-4}\mathbf{1}_{\{\frac{\mu_0 R}{2}\le|x|\le t^{\frac12}\}}+D_o t(\ln t)^{-1}(t(\ln t)^2)^{5\delta-\kappa}R^{-a}|x|^{-6}\mathbf{1}_{\{|x|>t^{\frac12}\}}.
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
&\mathcal{T}_4^{out}\left[(\ln t)^{-1}(t(\ln t)^2)^{5\delta-\kappa}R^{-a}|x|^{-4}\mathbf{1}_{\{\frac{\mu_0 R}{2}\le|x|\le t^{\frac12}\}}\right]\\
&\lesssim t^{-2}e^{-\frac{|x|^2}{16t}}+\begin{cases}(\ln t)^{-1}(t(\ln t)^2)^{5\delta-\kappa}R^{-a}(\mu_0 R)^{-2} & \text{if } |x|\le\mu_0 R\\ (\ln t)^{-1}(t(\ln t)^2)^{5\delta-\kappa}R^{-a}|x|^{-2}(\ln(\frac{|x|}{\mu_0 R})+1) & \text{if } \mu_0 R<|x|\le t^{\frac12}\\ (t(\ln t)^2)^{5\delta-\kappa}R^{-a}|x|^{-2}e^{-\frac{|x|^2}{16t}} & \text{if } |x|>t^{\frac12}\end{cases}\lesssim t_0^{-\epsilon}w_o
\end{aligned}$$

when $5\delta - \kappa - a\gamma > -2$. For the second term, one has

$$\mathcal{T}_4^{out}\left[t(\ln t)^{-1}(t(\ln t)^2)^{5\delta-\kappa}R^{-a}|x|^{-6}\mathbf{1}_{\{|x|>t^{\frac{1}{2}}\}}\right] \lesssim t^{-2}\mathbf{1}_{\{|x|\le t^{\frac{1}{2}}\}} + \left(|x|^{-6} + t^{-2}e^{-\frac{|x|^2}{16t}}\right)\mathbf{1}_{\{|x|>t^{\frac{1}{2}}\}} \lesssim t_0^{-\epsilon}w_o.$$

Thus

$$\left|\mathcal{T}_4^{out}\left[\mu^{-2}w^2(\frac{x-\xi}{\mu})\psi_1(1-\eta_R)\right]\right| \lesssim t_0^{-\epsilon}w_o.$$

Notice

$$\left(\ln t\langle y\rangle^{-2}\mathbf{1}_{\{|x|\le t^{\frac{1}{2}}\}} + t^2(\ln t)^{-1}|x|^{-6}\mathbf{1}_{\{|x|>t^{\frac{1}{2}}\}}\right)|\psi_1|^2 + |\psi_1|^3 \lesssim D_o^2\left(t^{-1}\langle y\rangle^{-2}\mathbf{1}_{\{|x|\le t^{\frac{1}{2}}\}} + (\ln t)^{-2}|x|^{-4}\mathbf{1}_{\{|x|>t^{\frac{1}{2}}\}}\right)|\psi_1|$$

when $5\delta - \kappa - a\gamma < -1$. And

$$(\ln t)^{-2}|x|^{-4}\mathbf{1}_{\{|x|>t^{\frac{1}{2}}\}}|\psi_1| \lesssim (\ln t)^{-2}|x|^{-4}|\psi_1|\mathbf{1}_{\{|x|\ge\frac{\mu_0 R}{2}\}},$$

where the last term has been estimated above. So we only need to estimate the following term

$$\begin{aligned} t^{-1}\langle y\rangle^{-2}\mathbf{1}_{\{|x|\le t^{\frac{1}{2}}\}}|\psi_1| &\lesssim D_o\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}t^{-1}\langle y\rangle^{-2}\mathbf{1}_{\{|x|\le t^{\frac{1}{2}}\}} \\ &\lesssim D_o\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}t^{-1}\left(\mathbf{1}_{\{|x|\le\mu_0\}} + (\ln t)^{-2}|x|^{-2}\mathbf{1}_{\{\mu_0<|x|\le t^{\frac{1}{2}}\}}\right), \end{aligned}$$

and

$$\begin{aligned} &\mathcal{T}_4^{out}\left[(\ln t)^{-1}(t(\ln t)^2)^{5\delta-\kappa}R^{-a}t^{-1}|x|^{-2}\mathbf{1}_{\{\mu_0<|x|\le t^{\frac{1}{2}}\}}\right] \\ \lesssim\ & t^{-2}e^{-\frac{|x|^2}{16t}} + \begin{cases} (t(\ln t)^2)^{5\delta-\kappa}R^{-a}t^{-1} & \text{if } |x|\le\mu_0 \\ (\ln t)^{-1}(t(\ln t)^2)^{5\delta-\kappa}R^{-a}t^{-1}(\ln(|x|^{-1}t^{\frac{1}{2}})+1) & \text{if } \mu_0<|x|\le t^{\frac{1}{2}} \\ (\ln t)^{-1}(t(\ln t)^2)^{5\delta-\kappa}R^{-a}|x|^{-2}e^{-\frac{|x|^2}{16t}} & \text{if } |x|>t^{\frac{1}{2}} \end{cases} \lesssim t_0^{-\epsilon}w_o, \end{aligned}$$

$$\mathcal{T}_4^{out}[\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}t^{-1}\mathbf{1}_{\{|x|\le\mu_0\}}] \lesssim t_0^{-\epsilon}w_o.$$

These imply

$$\mathcal{T}_4^{out}\left[t^{-1}\langle y\rangle^{-2}\mathbf{1}_{\{|x|\le t^{\frac{1}{2}}\}}|\psi_1|\right] \lesssim t_0^{-\epsilon}w_o.$$

Taking $D_o = D_o(\Lambda_1)$ large depending on Λ_1 and then choosing t_0 large enough, we have

$$\mathcal{T}_4^{out}[\mathcal{G}[\psi_1,\phi,\mu_1,\xi]] \in B_o.$$

The contraction property is given by the similar method which is used in dealing with terms including ψ_1. Then the unique solution ψ is found in B_o for (3.11) by the contraction mapping theorem.

From now on, we also regard $D_o(\Lambda_1)$ as a constant depending on Λ_1. Reviewing the estimates above and utilizing $\mathbf{1}_{\{|x|\ge\frac{\mu_0 R}{2}\}}$ to transform the spatial decay to time decay, one has

$$|\mathcal{G}[\psi,\phi,\mu_1,\xi]| \lesssim C(\Lambda_1)[(\mu_0 R)^{-2}\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a} + (\ln t)^3(t(\ln t)^2)^{10\delta-2\kappa}]$$

where $C(\Lambda_1)$ is a constant depending on Λ_1 which changes from line to line.

By gradient estimate, we have

$$|\nabla\psi| \lesssim C(\Lambda_1)\ln t(t(\ln t)^2)^{5\delta-\kappa}R^{-a}.$$

Next, we will use scaling argument to deduce the Hölder estimate of $\psi(x,t)$ in time variable t. For $x_1\in\mathbb{R}^4$, $t_1>4t_0$, set

$$\tilde{\psi}(z,s) = \psi(x_1+\lambda(t_1)z, t_1+\lambda^2(t_1)s)$$

where $0<\lambda(t_1)\le t_1^{\frac{1}{2}}$. Then

$$\partial_s\tilde{\psi} = \Delta_z\tilde{\psi} + \tilde{\mathcal{G}}(z,s)$$

where $\tilde{\mathcal{G}}(z,s) = \lambda^2(t_1)\mathcal{G}[\psi,\phi,\mu_1,\xi](x_1+\lambda(t_1)z, t_1+\lambda^2(t_1)s)$, and standard parabolic regularity theory implies

$$\|\tilde{\psi}\|_{C^{2\alpha,\alpha}(B(0,\frac{1}{16})\times(-\frac{1}{4},0))} \le C(\alpha)\left(\|\tilde{\psi}\|_{L^\infty(B(0,\frac{1}{4})\times(-\frac{1}{2},0))} + \|\tilde{\mathcal{G}}\|_{L^\infty(B(0,\frac{1}{4})\times(-\frac{1}{2},0))}\right)$$

where α can be chosen as any constant in $(0,1)$ and $C(\alpha)$ is a constant depending on α. Moreover, one has

$$\begin{aligned} &\|\tilde{\psi}\|_{L^\infty(B(0,\frac{1}{4})\times(-\frac{1}{2},0))} \lesssim C(\Lambda_1)\ln t_1(t_1(\ln t_1)^2)^{5\delta-\kappa}R^{-a}(t_1), \\ &\|\tilde{\mathcal{G}}\|_{L^\infty(B(0,\frac{1}{4})\times(-\frac{1}{2},0))} \lesssim C(\Lambda_1)\lambda^2(t_1)\left[(\mu_0 R)^{-2}(t_1)\ln t_1(t_1(\ln t_1)^2)^{5\delta-\kappa}R^{-a}(t_1) + (\ln t_1)^3(t_1(\ln t_1)^2)^{10\delta-2\kappa}\right], \end{aligned}$$

and

$$\|\tilde{\psi}\|_{C^{2\alpha,\alpha}(B(0,\frac{1}{16})\times(-\frac{1}{4},0))} \geq \sup_{s_1,s_2\in(-1/4,0)} \frac{|\psi(x_1,t_1+\lambda(t_1)^2 s_1)-\psi(x_1,t_1+\lambda(t_1)^2 s_2)|}{|s_1-s_2|^\alpha}$$

$$= \lambda^{2\alpha}(t_1) \sup_{s_1,s_2\in(-1/4,0)} \frac{|\psi(x_1,t_1+\lambda(t_1)^2 s_1)-\psi(x_1,t_1+\lambda(t_1)^2 s_2)|}{|(t_1+\lambda(t_1)^2 s_1)-(t_1+\lambda(t_1)^2 s_2)|^\alpha}$$

$$= \lambda^{2\alpha}(t_1) \sup_{s_1,s_2\in(t_1-\frac{\lambda(t_1)^2}{4},t_1)} \frac{|\psi(x_1,s_1)-\psi(x_1,s_2)|}{|s_1-s_2|^\alpha}.$$

Thus

$$\sup_{s_1,s_2\in(t_1-\frac{\lambda(t_1)^2}{4},t_1)} \frac{|\psi(x_1,s_1)-\psi(x_1,s_2)|}{|s_1-s_2|^\alpha} \lesssim C(\Lambda_1,\alpha)\{\lambda^{-2\alpha}(t_1)\ln t_1(t_1(\ln t_1)^2)^{5\delta-\kappa}R^{-a}(t_1)$$
$$+\lambda^{2-2\alpha}(t_1)[(\mu_0 R)^{-2}(t_1)\ln t_1(t_1(\ln t_1)^2)^{5\delta-\kappa}R^{-a}(t_1)+(\ln t_1)^3(t_1(\ln t_1)^2)^{10\delta-2\kappa}]\}.$$

□

Appendix C. Estimates for $\nabla_{\bar{x}}\varphi[\bar{\mu}_0]$ and $\partial_t\varphi[\bar{\mu}_0]$

In this section, we will revisit the calculations in Section 2.2 and derive the following estimates

$$|\partial_t\varphi[\bar{\mu}_0]| \lesssim t^{-2}(\ln t)^{-1}, \quad |\nabla_{\bar{x}}\varphi[\bar{\mu}_0]| \lesssim \begin{cases} (t\ln t)^{-1} & \text{if } |\bar{x}|\leq \mu_0 \\ t^{-1}(\ln t)^{-2}|\bar{x}|^{-1}+t^{-\frac{3}{2}}(\ln t)^{-1} & \text{if } \mu_0<|\bar{x}|\leq t^{\frac{1}{2}} \\ t^{-\frac{3}{2}}(\ln t)^{-1}e^{-\frac{|\bar{x}|^2}{16t}}+t(\ln t)^{-2}|\bar{x}|^{-5} & \text{if } |\bar{x}|>t^{\frac{1}{2}} \end{cases}. \tag{C.1}$$

Proof. Notice $\bar{\mu}_0\sim(\ln t)^{-1}$ and $|\bar{\mu}_{0t}|\sim t^{-1}(\ln t)^{-2}$. By (2.7), we have

$$|\nabla_{\bar{x}}\tilde{\varphi}_1[\bar{\mu}_0]| \lesssim |\bar{x}|t^{-2}(\ln t)^{-1}\mathbf{1}_{\{|\bar{x}|\leq 2t^{\frac{1}{2}}\}}+|\bar{x}|^{-1}(t\ln t)^{-1}e^{-\frac{|\bar{x}|^2}{4t}}\mathbf{1}_{\{|\bar{x}|>2t^{\frac{1}{2}}\}}.$$

For $\nabla_{\bar{x}}\tilde{\varphi}_{1b}[\bar{\mu}_0]$, we abbreviate $\nabla_{\bar{x}}\tilde{\varphi}_{1b}[\bar{\mu}_0]$ as $\nabla_{\bar{x}}\tilde{\varphi}_{1b}$. By (2.8), then

$$\nabla_{\bar{x}}\tilde{\varphi}_{1b}(\bar{x},t)=\mathcal{T}_4^{out}[\nabla_{\bar{x}}(-\bar{\mu}_{0t}\hat{\varphi}_1+(E-\tilde{E})[\bar{\mu}_0])](\bar{x},t).$$

Notice by (2.7), we have

$$|\bar{\mu}_{0t}\nabla_{\bar{x}}\hat{\varphi}_1(\bar{x},t)| \lesssim |\bar{x}|t^{-3}(\ln t)^{-2}\mathbf{1}_{\{|\bar{x}|\leq t^{\frac{1}{2}}\}}+|\bar{x}|^{-1}t^{-2}(\ln t)^{-2}e^{-\frac{|\bar{x}|^2}{4t}}\mathbf{1}_{\{|\bar{x}|>t^{\frac{1}{2}}\}},$$

$$|\nabla_{\bar{x}}(E-\tilde{E})[\bar{\mu}_0]| \lesssim t^{-\frac{7}{2}}(\ln t)^{-3}\mathbf{1}_{\{t^{\frac{1}{2}}\leq|\bar{x}|\leq 2t^{\frac{1}{2}}\}},$$

and

$$\mathcal{T}_4^{out}\left[t^{-\frac{7}{2}}(\ln t)^{-3}\mathbf{1}_{\{t^{\frac{1}{2}}\leq|\bar{x}|\leq 2t^{\frac{1}{2}}\}}+|\bar{x}|t^{-3}(\ln t)^{-2}\mathbf{1}_{\{|\bar{x}|\leq t^{\frac{1}{2}}\}}\right] \lesssim \mathcal{T}_4^{out}\left[t^{-\frac{5}{2}}(\ln t)^{-2}\mathbf{1}_{\{|\bar{x}|\leq 2t^{\frac{1}{2}}\}}\right] \lesssim t^{-\frac{3}{2}}(\ln t)^{-2}e^{-\frac{|\bar{x}|^2}{16t}},$$

$$\mathcal{T}_4^{out}\left[|\bar{x}|^{-1}t^{-2}(\ln t)^{-2}e^{-\frac{|\bar{x}|^2}{4t}}\mathbf{1}_{\{|\bar{x}|>t^{\frac{1}{2}}\}}\right] \lesssim \mathcal{T}_4^{out}\left[(\ln t)^{-2}|\bar{x}|^{-5}\mathbf{1}_{\{|\bar{x}|>t^{\frac{1}{2}}\}}\right] \lesssim \begin{cases} t^{-\frac{3}{2}}(\ln t)^{-2} & \text{if } |\bar{x}|\leq t^{\frac{1}{2}} \\ t(\ln t)^{-2}|\bar{x}|^{-5} & \text{if } |\bar{x}|>t^{\frac{1}{2}} \end{cases}.$$

Thus

$$|\nabla_{\bar{x}}\tilde{\varphi}_{1b}| \lesssim t^{-\frac{3}{2}}(\ln t)^{-2}\mathbf{1}_{\{|\bar{x}|\leq t^{\frac{1}{2}}\}}+t(\ln t)^{-2}|\bar{x}|^{-5}\mathbf{1}_{\{|\bar{x}|>t^{\frac{1}{2}}\}}.$$

Next, let us consider $\nabla_{\bar{x}}\varphi_2[\bar{\mu}_0]$. Recall the definition of φ_2 in Lemma 2.2, then

$$\nabla_{\bar{x}}\varphi_2=\mathcal{T}_4^{out}\left[\bar{\mu}_0^{-2}\bar{\mu}_{0t}\nabla_{\bar{x}}\left(Z_5(\frac{\bar{x}}{\bar{\mu}_0})\eta(\frac{\bar{x}}{\sqrt{t}})\right)\right].$$

Notice

$$\left|\bar{\mu}_0^{-2}\bar{\mu}_{0t}\nabla_{\bar{x}}(Z_5(\frac{\bar{x}}{\bar{\mu}_0})\eta(\frac{\bar{x}}{\sqrt{t}}))\right|=\left|\bar{\mu}_0^{-2}\bar{\mu}_{0t}(\bar{\mu}_0^{-1}\nabla Z_5(\frac{\bar{x}}{\bar{\mu}_0})\eta(\frac{\bar{x}}{\sqrt{t}})+t^{-\frac{1}{2}}Z_5(\frac{\bar{x}}{\bar{\mu}_0})\nabla\eta(\frac{\bar{x}}{\sqrt{t}}))\right|$$
$$\lesssim t^{-1}\left[\ln t(1+|\frac{\bar{x}}{\bar{\mu}_0}|)^{-3}\mathbf{1}_{\{|\bar{x}|\leq 2t^{\frac{1}{2}}\}}+t^{-\frac{1}{2}}(1+|\frac{\bar{x}}{\bar{\mu}_0}|)^{-2}\mathbf{1}_{\{t^{\frac{1}{2}}\leq|\bar{x}|\leq 2t^{\frac{1}{2}}\}}\right] \sim t^{-1}\ln t(1+|\frac{\bar{x}}{\bar{\mu}_0}|)^{-3}\mathbf{1}_{\{|\bar{x}|\leq 2t^{\frac{1}{2}}\}}$$
$$\sim t^{-1}\ln t\mathbf{1}_{\{|\bar{x}|\leq\mu_0\}}+t^{-1}(\ln t)^{-2}|\bar{x}|^{-3}\mathbf{1}_{\{\mu_0<|\bar{x}|\leq 2t^{\frac{1}{2}}\}}.$$

Therefore, we obtain

$$|\nabla_{\bar{x}}\varphi_2| \lesssim \mathcal{T}_4^{out}\left[t^{-1}\ln t\mathbf{1}_{\{|\bar{x}|\le\mu_0\}} + t^{-1}(\ln t)^{-2}|\bar{x}|^{-3}\mathbf{1}_{\{\mu_0<|\bar{x}|\le 2t^{\frac12}\}}\right] \lesssim \begin{cases}(t\ln t)^{-1} & \text{if } |\bar{x}|\le\mu_0\\ t^{-1}(\ln t)^{-2}|\bar{x}|^{-1} & \text{if } \mu_0<|\bar{x}|\le t^{\frac12}\\ t^{-\frac32}(\ln t)^{-2}e^{-\frac{|\bar{x}|^2}{16t}} & \text{if } |\bar{x}|>t^{\frac12}\end{cases}.$$

Since $\varphi = \tilde{\varphi}_1 + \tilde{\varphi}_{1b} + \varphi_2$, one concludes the upper bound of $|\nabla_{\bar{x}}\varphi[\bar{\mu}_0]|$ in (C.1).

The left part is devoted to estimating $\partial_t\varphi[\bar{\mu}_0]$. For $\partial_t\tilde{\varphi}_1[\bar{\mu}_0]$, by (2.6),

$$|\partial_t\tilde{\varphi}_1[\bar{\mu}_0]| = \left|2^{\frac32}\bar{\mu}_{0t}|\bar{x}|^{-2}\left(e^{-\frac{|\bar{x}|^2}{4t}} - \eta(\frac{|\bar{x}|}{\sqrt{t}})\right) + 2^{\frac32}\bar{\mu}_0|\bar{x}|^{-2}\left(\frac{|\bar{x}|^2}{4t^2}e^{-\frac{|\bar{x}|^2}{4t}} + \frac{|\bar{x}|}{2t^{\frac32}}\eta'(\frac{|\bar{x}|}{\sqrt{t}})\right)\right| \lesssim t^{-2}(\ln t)^{-1}e^{-\frac{|\bar{x}|^2}{4t}}.$$

Next, we estimate $\partial_t\tilde{\varphi}_{1b}[\bar{\mu}_0]$. For any integer $n\ge 1$ and $f(x,t)\in C^1(\mathbb{R}^n\times(t_0,\infty))$,

$$\begin{aligned}
&\partial_t\left(\int_{t_0}^t\int_{\mathbb{R}^n}[4\pi(t-s)]^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}f(y,s)dyds\right)\\
&=\partial_t\left(\int_{t_0}^{\frac{t}{2}}\int_{\mathbb{R}^n}[4\pi(t-s)]^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}f(y,s)dyds + \int_0^{\frac{t}{2}}\int_{\mathbb{R}^n}(4\pi a)^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4a}}f(y,t-a)dyda\right)\\
&=\frac12\int_{\mathbb{R}^n}(2\pi t)^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{2t}}f(y,\frac{t}{2})dy + \int_{t_0}^{\frac{t}{2}}\int_{\mathbb{R}^n}\partial_t\left\{[4\pi(t-s)]^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}\right\}f(y,s)dyds\\
&\quad+\int_{\frac{t}{2}}^t\int_{\mathbb{R}^n}[4\pi(t-s)]^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}(\partial_t f)(y,s)dyds.
\end{aligned}\tag{C.2}$$

As a consequence of (2.8) and (C.2), we have

$$\begin{aligned}
\partial_t\tilde{\varphi}_{1b} =& \frac12\int_{\mathbb{R}^4}(2\pi t)^{-2}e^{-\frac{|x-y|^2}{2t}}\left(-\bar{\mu}_{0t}\hat{\varphi}_1 + (E-\tilde{E})[\bar{\mu}_0]\right)(y,\frac{t}{2})dy\\
&+\int_{t_0}^{\frac{t}{2}}\int_{\mathbb{R}^4}\partial_t\left\{[4\pi(t-s)]^{-2}e^{-\frac{|x-y|^2}{4(t-s)}}\right\}\left(-\bar{\mu}_{0t}\hat{\varphi}_1 + (E-\tilde{E})[\bar{\mu}_0]\right)(y,s)dyds\\
&+\int_{\frac{t}{2}}^t\int_{\mathbb{R}^4}[4\pi(t-s)]^{-2}e^{-\frac{|x-y|^2}{4(t-s)}}\left[\partial_t\left(-\bar{\mu}_{0t}\hat{\varphi}_1 + (E-\tilde{E})[\bar{\mu}_0]\right)\right](y,s)dyds.
\end{aligned}$$

where by (2.9) and (2.10), it follows that

$$\begin{aligned}
&\left|\left(-\bar{\mu}_{0t}\hat{\varphi}_1 + (E-\tilde{E})[\bar{\mu}_0]\right)(\bar{x},\frac{t}{2})\right| + \left|\left(-\bar{\mu}_{0t}\hat{\varphi}_1 + (E-\tilde{E})[\bar{\mu}_0]\right)(\bar{x},t)\right|\\
&\lesssim (t\ln t)^{-2}\mathbf{1}_{\{|\bar{x}|\le t^{\frac12}\}} + t^{-1}(\ln t)^{-2}|\bar{x}|^{-2}e^{-\frac{|\bar{x}|^2}{4t}}\mathbf{1}_{\{|\bar{x}|>t^{\frac12}\}} + (t\ln t)^{-3}\mathbf{1}_{\{t^{\frac12}\le|\bar{x}|\le 2t^{\frac12}\}} \lesssim (t\ln t)^{-2}e^{-\frac{|\bar{x}|^2}{4t}},
\end{aligned}$$

and

$$\begin{aligned}
&|\partial_t(\bar{\mu}_{0t}\hat{\varphi}_1)| = |\bar{\mu}_{0tt}\hat{\varphi}_1 + \bar{\mu}_{0t}\partial_t\hat{\varphi}_1|\\
\lesssim& \left|(t\ln t)^{-2}\left(t^{-1}\mathbf{1}_{\{|\bar{x}|\le 2t^{\frac12}\}} + |\bar{x}|^{-2}e^{-\frac{|\bar{x}|^2}{4t}}\mathbf{1}_{\{|\bar{x}|>2t^{\frac12}\}}\right) + t^{-3}(\ln t)^{-2}e^{-\frac{|\bar{x}|^2}{4t}}\right| \sim t^{-3}(\ln t)^{-2}e^{-\frac{|\bar{x}|^2}{4t}},
\end{aligned}$$

and

$$|\partial_t(E-\tilde{E})[\bar{\mu}_0]| \lesssim t^{-4}(\ln t)^{-3}\mathbf{1}_{\{\sqrt{t}\le|\bar{x}|\le 2\sqrt{t}\}}.$$

Thus by Lemma A.3 and same calculation for deducing (2.12), we have

$$|\partial_t\tilde{\varphi}_{1b}| \lesssim (t\ln t)^{-2}\int_{\mathbb{R}^4}t^{-2}e^{-\frac{|x-y|^2}{2t}}dy + t^{-1}\int_{t_0}^t\int_{\mathbb{R}^4}(t-s)^{-2}e^{-\frac{|x-y|^2}{8(t-s)}}(s\ln s)^{-2}e^{-\frac{|y|^2}{4s}}dyds \lesssim (t\ln t)^{-2}.$$

Finally, we consider $\partial_t\varphi_2[\bar{\mu}_0]$. By (C.2) and the definition of φ_2 in Lemma 2.2, we have

$$\begin{aligned}
\partial_t\varphi_2 =& \frac12\int_{\mathbb{R}^4}(2\pi t)^{-2}e^{-\frac{|x-y|^2}{2t}}\bar{\mu}_0^{-2}(\frac{t}{2})\bar{\mu}_{0t}(\frac{t}{2})Z_5(\frac{y}{\bar{\mu}_0(\frac{t}{2})})\eta(\frac{\sqrt{2}y}{\sqrt{t}})dy\\
&+\int_{t_0}^{\frac{t}{2}}\int_{\mathbb{R}^4}\partial_t\left\{[4\pi(t-s)]^{-2}e^{-\frac{|x-y|^2}{4(t-s)}}\right\}\bar{\mu}_0^{-2}(s)\bar{\mu}_{0t}(s)Z_5(\frac{y}{\bar{\mu}_0(s)})\eta(\frac{y}{\sqrt{s}})dyds\\
&+\int_{\frac{t}{2}}^t\int_{\mathbb{R}^4}[4\pi(t-s)]^{-2}e^{-\frac{|x-y|^2}{4(t-s)}}\left[\partial_s\left(\bar{\mu}_0^{-2}(s)\bar{\mu}_{0t}(s)Z_5(\frac{y}{\bar{\mu}_0(s)})\eta(\frac{y}{\sqrt{s}})\right)\right]dyds,
\end{aligned}$$

and by (2.16),

$$\left|\bar\mu_0^{-2}(\frac{t}{2})\bar\mu_{0t}(\frac{t}{2})Z_5(\frac{\bar x}{\bar\mu_0(\frac{t}{2})})\eta(\frac{\sqrt{2}\bar x}{\sqrt t})\right| + \left|\bar\mu_0^{-2}(t)\bar\mu_{0t}(t)Z_5(\frac{\bar x}{\bar\mu_0(t)})\eta(\frac{\bar x}{\sqrt t})\right|$$
$$\lesssim t^{-1}\mathbf{1}_{\{|\bar x|\le (\ln t)^{-1}\}} + t^{-1}(\ln t)^{-2}|\bar x|^{-2}\mathbf{1}_{\{(\ln t)^{-1}<|\bar x|\le 2t^{\frac12}\}},$$

and

$$\left|\partial_t(\bar\mu_0^{-2}\bar\mu_{0t}Z_5(\frac{\bar x}{\bar\mu_0})\eta(\frac{\bar x}{\sqrt t}))\right|$$
$$= \left|\partial_t(\bar\mu_0^{-2}\bar\mu_{0t})Z_5(\frac{\bar x}{\bar\mu_0})\eta(\frac{\bar x}{\sqrt t}) - \bar\mu_0^{-2}\bar\mu_{0t}\frac{\bar x}{\bar\mu_0}\cdot\nabla Z_5(\frac{\bar x}{\bar\mu_0})\frac{\bar\mu_{0t}}{\bar\mu_0}\eta(\frac{\bar x}{\sqrt t}) - \bar\mu_0^{-2}\bar\mu_{0t}Z_5(\frac{\bar x}{\bar\mu_0})\frac{\bar x}{2t^{\frac32}}\cdot\nabla\eta(\frac{\bar x}{\sqrt t})\right|$$
$$\lesssim t^{-2}\langle\frac{\bar x}{\bar\mu_0}\rangle^{-2}\mathbf{1}_{\{|\bar x|\le 2t^{\frac12}\}}.$$

Thus, by similar calculation for Lemma A.3 and the upper bound of φ_2 in Lemma 2.2, we have

$$|\partial_t\varphi_2| \lesssim \int_{\mathbb{R}^4} t^{-2}e^{-\frac{|x-y|^2}{2t}}\left(t^{-1}\mathbf{1}_{\{|y|\le(\ln t)^{-1}\}} + t^{-1}(\ln t)^{-2}|y|^{-2}\mathbf{1}_{\{(\ln t)^{-1}<|y|\le 2t^{\frac12}\}}\right)dy$$
$$+ t^{-1}\int_{t_0}^{t}\int_{\mathbb{R}^4}(t-s)^{-2}e^{-\frac{|x-y|^2}{8(t-s)}}\left(s^{-1}\mathbf{1}_{\{|y|\le(\ln s)^{-1}\}} + s^{-1}(\ln s)^{-2}|y|^{-2}\mathbf{1}_{\{(\ln s)^{-1}<|y|\le 2s^{\frac12}\}}\right)dyds \lesssim t^{-2}(\ln t)^{-1}.$$

Collecting above estimates, we obtain $|\partial_t\varphi[\bar\mu_0]| \lesssim t^{-2}(\ln t)^{-1}$.

□

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