**Bessel Functions**

If we separate variable in

\[ u_t = D \left( u_{rr} + \frac{1}{r} u_r \right) \quad 0 \leq r < a, \quad t > 0 \]

\[ u(r, 0) = f(r), \quad u(a, t) = 0 \]

we obtain \[ u(r, t) = R(r) T(t) \quad \frac{T'}{T} = \frac{R'' + \frac{1}{r} R'}{R} = -\lambda \]

This leads to the singular Sturm-Liouville problem

\[ (x) \begin{cases} \phi'' + \frac{1}{x} \phi' + \lambda \phi = 0, \quad 0 \leq x \leq a, \\ \phi(a) = 0, \quad \phi(0) \text{ finite} \end{cases} \]

This can be written as \( (\frac{1}{x} \phi')' + \lambda \phi = 0 \) so that in Sturm-Liouville form \( p(x) = x \) and \( w(x) = x \) is the weight. It is a singular Sturm-Liouville problem since \( p(0) = 0 \).

The solutions to \((x)\) are denoted by \( J_0(\sqrt{\lambda} x) \) and \( Y_0(\sqrt{\lambda} x) \) (Bessel functions of the first kind of order zero) and so

\[ (+) \quad \phi = A J_0(\sqrt{\lambda} x) + B Y_0(\sqrt{\lambda} x) \]

where \( J_0(x) \) and \( Y_0(x) \) are the two linearly independent solutions of \( x^2 \phi'' + x \phi' + x^2 \phi = 0 \) in \( x > 0 \). Notice that \( x = 0 \) is a regular singular point. If we put \( \gamma = \Gamma(\Gamma-1) + \Gamma > 0 \) so

\[ J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots \]

be bavior \[ Y_0(x) = \frac{2}{\pi} \left[ \log \left( \frac{x}{2} \right) + \gamma \right] J_0(x) + \frac{2}{\pi} \left( \frac{x^2}{2^2} + \cdots \right) \]

Notice that \( J_0(x) \sim 1 \) as \( x \to 0^+ \), \( Y_0(x) \sim \frac{2}{\pi} \log x \) as \( x \to 0^+ \).
REMARK IF WE LET $x = \sqrt{\lambda} \Gamma$

THEN $y(x) = \phi(x/\sqrt{\lambda})$ transforms $\phi'' + \frac{1}{x} \phi' + \phi = 0$

INTO $y'' + \frac{1}{x} y' + y = 0$.

PROOF $\phi' = y'/\sqrt{\lambda}$, $\phi'' = \lambda y''$

SO $\phi'' + \frac{1}{x} \phi' + \lambda \phi = \lambda y'' + \frac{\lambda}{x} y' + \lambda y = \lambda (y'' + \frac{1}{x} y' + y) = 0$

THUS $y'' + \frac{1}{x} y' + y = 0$.

NOW RETURNING TO (+) WE SET $\phi(0)$ FINITE TO GET $B = 0$.

THEN $\phi(\alpha) = 0$ IMPLIES $J_0 (\sqrt{\lambda} \alpha) = 0$.

\begin{align*}
J_0(x_1) & \quad \text{and} \quad Y_0(x) \\
X_1 = 2.405 & \quad \text{and} \quad X
\end{align*}

THUS THE EIGENVALUES ARE $\sqrt{\lambda} \alpha = X_k$ OR $\lambda_k = X_k^2 / \alpha^2$, $k = 1, 2, ..$

WHERE $X_k$ FOR $k = 1, 2, ..$ ARE THE ROOTS OF $J_0 (x) = 0$.

THUS $\phi_k (\Gamma) = A J_0 (\sqrt{\lambda_k} \Gamma)$.

SINCE THE WEIGHT FUNCTION IS $W(\Gamma) = \Gamma$ WE HAVE THE ORTHOGONALITY PROPERTY

$$\int_0^a \Gamma J_0 (\sqrt{\lambda_k} \Gamma) J_0 (\sqrt{\lambda_n} \Gamma) \, d\Gamma = 0 \quad k \neq n.$$ 

IN ADDITION, SOME FURTHER WORK (NOT GIVEN) SHOWS THAT

$$\int_0^a \Gamma (J_0 (\sqrt{\lambda_k} \Gamma))^2 \, d\Gamma = \frac{a}{2} \left[ J_0 (\sqrt{\lambda_k} \alpha) \right]^2.$$ 

THIS DERIVATION IS GIVEN IN APPENDIX A PAGE (16) BELOW.
OSCILLATIONS: LARGE X BEHAVIOR OF $J_0(x)$, $Y_0(x)$.

$x^2 y'' + x y' + x^2 y = 0$

We let $y = p \psi$. Then

$x^2 (p \psi'' + p' \psi' + p'' \psi) + x (p \psi' + p' \psi) + x^2 p \psi = 0$

Choose $p$ to eliminate the middle term:

$p' = -\frac{1}{2} \frac{1}{x}$, so $\ln p = -\frac{1}{2} \ln x + C \rightarrow p = x^{-1/2}$.

Then $p' = -\frac{1}{2} x^{-3/2}$, $p'' = \frac{3}{4} x^{-5/2}$

So $\frac{p''}{p} = \frac{3}{4 x^2}$, $\frac{p'}{xp} = \frac{-1/2 x^{-3/2}}{x^{1/2}} = -\frac{1}{2 x^2}$.

This yields that $\psi'' + (1 + \frac{1}{4 x^2}) \psi = 0$.

For $x >> 1$ we have $\psi'' + \psi \approx 0$ so $\psi \approx \cos x$.

It turns out that

$J_0(x) \sim \left(\frac{2}{\pi x}\right)^{1/4} \cos \left(x - \frac{\pi}{4}\right)$ for $x >> 1$

$Y_0(x) \sim \left(\frac{2}{\pi x}\right)^{1/4} \sin \left(x - \frac{\pi}{4}\right)$

Decaying oscillations for large $x$.
Finally, returning to \( u_t = D \left( u_{rr} + \frac{1}{r} u_r \right) \)

we obtain

\[
U(r, t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} J_0 \left( \sqrt{\lambda_k} r \right) C_k
\]

then

\[
U(r, 0) = f(r) = \sum_{k=1}^{\infty} C_k J_0 \left( \sqrt{\lambda_k} r \right).
\]

By orthogonality, we obtain

\[
C_k = \frac{\int_0^a f(r) J_0 \left( \sqrt{\lambda_k} r \right) r \, dr}{\int_0^a \left( J_0 \left( \sqrt{\lambda_k} r \right) \right)^2 r \, dr}.
\]

Example: Find an eigenfunction expansion solution for

\[
u_t = D \left( u_{rr} + \frac{1}{r} u_r \right), \quad 0 < r < a, \quad t > 0
\]

\[
U(r, 0) = f(r), \quad U(a, t) = e^{-t}, \quad U \text{ bounded as } r \to 0.
\]

We let \( U(r, t) = e^{-t} + V(r, t) \) to obtain homogeneous boundary conditions so that

\[
V_t = D \left( V_{rr} + \frac{1}{r} V_r \right) + e^{-t}
\]

\[
V(a, t) = 0, \quad V \text{ bounded as } r \to 0
\]

\[
V(r, 0) = f(r) - 1
\]

We separate variables to get \( \frac{T'}{D} \frac{\phi''}{\phi} + \frac{1}{r} \frac{\phi'}{\phi} = -\lambda \)

for the homogeneous problem.

This gives \( \phi'' + \frac{1}{r} \phi' + \lambda \phi = 0 \) \( 0 < r < a, \phi(a) = 0, \phi(0) \text{ finite} \)

so \( \phi_k = J_0 \left( \sqrt{\lambda_k} r \right) \) where \( \lambda_k = \frac{z_k^2}{a^2}, \quad J_0 (z_k) = 0, \quad \lambda_k = 1, 2, \ldots \).
Then we write

$$V(r, t) = \sum_{n=1}^{\infty} b_n(t) \phi_n(r), \quad \phi_n(r) = J_0(\sqrt{\lambda_n} r)$$

Substituting into the PDE we obtain

$$\sum_{n=1}^{\infty} b_n'(t) \phi_n(r) = \sum_{n=1}^{\infty} D \lambda_n b_n(t) (\phi_n'' + \frac{1}{r} \phi_n') + e^{-t}$$

Then use $\phi_n'' + \frac{1}{r} \phi_n' = -\lambda_n \phi_n$ and expand $I = \sum_{n=1}^{\infty} \chi_n J_0(\sqrt{\lambda_n} r)$. This yields that

$$\chi_n = \int_0^a r J_0(\sqrt{\lambda_n} r) \, dr / \int_0^a r J_0^2(\sqrt{\lambda_n} r) \, dr.$$ 

We therefore obtain

$$\sum_{n=1}^{\infty} b_n' \phi_n = -\sum_{n=1}^{\infty} D \lambda_n b_n \phi_n + \sum_{n=1}^{\infty} e^{-t} \chi_n \phi_n$$

This yields

$$\sum_{n=1}^{\infty} \left( b_n' + D \lambda_n b_n - e^{-t} \chi_n \right) \phi_n = 0.$$ 

By orthogonality of eigenfunctions we obtain

$$\begin{cases} b_n' = -D \lambda_n b_n + e^{-t} \chi_n \\ b_n(0) \text{ given} \end{cases}$$

Notice that

$$V(r, 0) = F(r) - 1 = \sum_{n=1}^{\infty} b_n(0) \phi_n(r).$$

This implies that

$$b_n(0) = \int_0^a (F(r) - 1) r \phi_n(r) \, dr / \int_0^a r (\phi_n(r)^2) \, dr.$$ 

We solve for $b_n$:

$$b_n' + D \lambda_n b_n = e^{-t} \chi_n.$$ 

Hence

$$\left( b_n e^{D \lambda_n t} \right)' = e^{-t} e^{D \lambda_n t} \chi_n \to b_n e^{D \lambda_n t} = b_n(0) + \int_0^t \chi_n e^{-\tau} (e^{D \lambda_n \tau} - 1) \, d\tau.$$ 

This gives

$$b_n(t) = b_n(0) e^{-D \lambda_n t} + e^{-D \lambda_n t} \int_0^t \chi_n e^{-\tau} (e^{D \lambda_n \tau} - 1) \, d\tau.$$ 

With

$$u(r, t) = e^{-t} + \sum_{n=1}^{\infty} b_n(t) J_0(\sqrt{\lambda_n} r).$$
EXAMPLE FIND THE SOLUTION TO

\[ u_t = D \left( u_{rr} + \frac{1}{r} u_r + u_{zz} \right) \]

IN \( 0 < z < h \)
\( 0 < r < a \)

WITH \( u(r, 0, t) = u(r, h, t) = 0 \)
\( u(z, 0, t) = 0, \) \( u \) FINE AS \( r \to 0 \)
\( u(r, z, 0) = F(r, z). \)

WE SEPARATE VARIABLES TO OBTAIN \( u(r, z, t) = R(r) T(t) Z(z). \)

WE CALCULATE

\[ \frac{T'}{T} = \frac{R'' + \frac{1}{r} R'}{R} + \frac{Z''}{Z} = -\lambda. \]

WE SET \( R'' + \frac{1}{r} R' + \mu R = 0 \)
SO THAT

\( R'(a) = 0, \) \( R(0) \) FINE
\( R(a) = 0, \) \( R(0) \) FINE
WHERE \( \sqrt{\mu} a = X_0, \) \( J_0 (X_0) = 0 \)

THUS \( \mu_n = \frac{X_0^2}{a^2} \)
WHERE \( J_0 (X_0) = 0. \)

THEN \( -\mu_n + \frac{Z''}{Z} = -\lambda \)
SO THAT

\( Z'' + (\lambda - \mu_n) Z = 0 \)
\( Z(0) = Z(h) = 0 \)

THUS Yields \( Z_0 = \sin \left( \sqrt{\lambda - \mu_n} z \right) \)
\( \sqrt{\lambda - \mu_n} h = m\pi \)
\( m = 1, 2, \ldots \)

THUS GIVE

\[ \lambda = \mu_n + \frac{m^2 \pi^2 \alpha^2}{H^2} \]
\( \mu_n = \frac{X_0^2}{a^2} \)
AND \( J_0 (X_0) = 0. \)

WE OBTAIN \( Z_m(z) = \sin \left( \frac{m\pi z}{H} \right), \)
\( m = 1, 2, \ldots \)

\( R_n(r) = J_0 \left( \frac{X_0 r}{a} \right), \)
\( n = 1, 2, \ldots \)
Then \[
\frac{T'}{DT} = -\lambda_{mn} \quad \rightarrow \quad T = e^{-D(m_n + \frac{m^2 \pi^2}{H^2})t}
\]

This yields the eigenfunction expansion

\[
\psi(r, z, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-D(m_n + \frac{m^2 \pi^2}{H^2})t} J_0(\sqrt{m_n} r) \sin \left( \frac{m\pi z}{H} \right)
\]

Finally to satisfy the initial condition we obtain an equation for \(A_{mn}\):

\[
f(r, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} J_0(\sqrt{m_n} r) \sin \left( \frac{m\pi z}{H} \right)
\]

By orthogonality,

\[
\int_0^a \left[ \int_0^b f(r, z) r J_0(\sqrt{m_n} r) \sin \left( \frac{m\pi z}{H} \right) \, dr \right] \, dz = \int_0^a \left[ \int_0^b f(r, z) \sin \left( \frac{m\pi z}{H} \right) \, dz \right] \, dr
\]

\[
= A_{mn} \left[ \int_0^a \left[ \int_0^b r J_0^2(\sqrt{m_n} r) \, dr \right] \sin^2 \left( \frac{m\pi z}{H} \right) \, dz \right]
\]

\[
= A_{mn} \frac{H}{2} \left[ \int_0^a J_0^2(\sqrt{m_n} r) \, dr \right] = A_{mn} \frac{H}{2} \left( \frac{\alpha^2}{2} \left( J_0'(\sqrt{m_n} \alpha) \right)^2 \right)
\]

\[
= \frac{H \alpha^2}{4} A_{mn} \left( J_0'(\sqrt{m_n} \alpha) \right)^2.
\]

This yields

\[
A_{mn} = \frac{4}{H \alpha^2 \left[ J_0'(\sqrt{m_n} \alpha) \right]^2} \left[ \int_0^a f(r, z) J_0(\sqrt{m_n} r) \sin \left( \frac{m\pi z}{H} \right) \, dr \right] \, dz
\]

which yields the coefficient in (7).
Example

Solve the wave equation

\[ u_{tt} = c^2 \left( u_{rr} + \frac{1}{r} u_r \right), \quad 0 < r < a, \quad t > 0 \]

\[ u(r, 0) = 0, \quad u_t(r, 0) = 0, \quad u(a, t) = 1, \quad u \text{ bounded as } r \to 0. \]

This corresponds to deflecting a membrane along the rim to generate waves.

We let \( u(r, t) = 1 + v(r, t) \) so that

\[ v_{tt} = c^2 \left( v_{rr} + \frac{1}{r} v_r \right) \]

\[ v(r, 0) = -1, \quad v_t(r, 0) = 0 \]

\[ v(a, t) = 0, \quad v \text{ bounded as } r \to 0. \]

We separate variables \( v = R(r) T(t) \) to obtain

\[ \frac{T''}{c^2 T} = \frac{R'' + \frac{1}{r} R'}{R} = -\lambda \]

\[ T'' + c^2 \lambda T = 0 \]

This leads to the eigenvalue problem

\[ \phi'' + \frac{1}{r} \phi' + \lambda \phi = 0 \]

\[ \phi(a) = 0, \quad \phi(0) \text{ finite} \]

The solution is

\[ \phi(r) = J_0(\sqrt{\lambda} r) \]

where \( \sqrt{\lambda} a = z_K \) and \( J_0(z_K) = 0 \) for \( K = 1, 2, 3, \ldots \)

Hence, \( \lambda_K = z_K^2 / a^2 \).

This yields that

\[ T_K'' + w_K^2 T_K = 0 \]

\[ w_K = c \sqrt{\lambda_K} = cz_K / a. \]

The solution is

\[ T_K = A_K \cos(w_K t) + B_K \sin(w_K t) \]

This yields that

\[ v(r, t) = \sum_{K=1}^{\infty} \left( A_K \cos(w_K t) + B_K \sin(w_K t) \right) J_0(\sqrt{\lambda_K} r) \]
Now we satisfy $V(r, 0) = -1$, $V_t (r, 0) = 0$ to obtain

\[ -1 = \sum_{k=1}^{\infty} A_k J_0 (\sqrt{A_k} r) \quad A_k = \frac{\int_0^a r J_0 (\sqrt{A_k} r) \, dr}{\int_0^a r J_0^2 (\sqrt{A_k} r) \, dr} \]

\[ 0 = \sum_{k=1}^{\infty} B_k w_k J_0 (\sqrt{A_k} r) \quad \rightarrow \quad B_k = 0 \]

Then \( W(r, t) = 1 + \sum_{k=1}^{\infty} A_k \cos (w_k t) J_0 (\sqrt{A_k} r) \)

where \( A_k \) and \( w \) are given above.

**Example** Find an eigenfunction representation for the solution to

\[ U_t = D \left( U_{rr} + \frac{1}{r} U_r \right) + F \quad 0 < r < a, \quad t > 0 \]

\[ -D U_r = h (U - T_1) \text{ on } r = a; \quad U \text{ finite at } r = 0 \]

\[ U(r, 0) = T_2. \]

Here \( D, h, T_1, T_2 \) are constants.

We first calculate the steady-state solution which satisfies

\[ D \left( U_s'' + \frac{1}{r} U_s' \right) + F = 0 \]

\[ -D U_s' = h (U_s - T_1) \text{ on } r = a \]

We let \( U_s = \frac{F}{4D} r^2 + A_0 \), which solves \( U_s'' + \frac{1}{r} U_s' = -\frac{F}{D} \).

Then to find \( A_0 \) we have

\[ -\left( \frac{F}{4D} \right) 2 a D = h \left( \frac{F}{4D} a^2 + A_0 - T_1 \right) \]

This yields

\[ A_0 = T_1 - \frac{Fa_2}{2h} - \frac{Fa^2}{4D} \].
This yields the steady-state solution:

\[ U_3(\Gamma) = \frac{F}{4\theta}(\Gamma^2 - a^2) + T_x - \frac{Fa}{2h}. \]

We then write \( V(\Gamma, t) = V_0(\Gamma) + U_3(\Gamma). \)

This leads to

\[ V_t = D \left( \nu \frac{\Gamma}{D} \right), \]

- \( D V_0 = \nu V_0 \) on \( \Gamma = a, \) \( \nu \) finite as \( \Gamma \to 0 \)

\[ V(\Gamma, 0) = T_2 - U_3(\Gamma). \]

Separating variables, we obtain \( \frac{T'}{DT} = \frac{R^2 + \Gamma R'}{R} = -\lambda. \)

This gives the eigenvalue problem

\[ \phi'' + \frac{\Gamma}{D} \phi' + \lambda \phi = 0, \quad 0 < \Gamma < a. \]

- \( D \phi'(a) = h \phi(a), \) \( \phi(0) \) finite.

We obtain \( \phi_n(\Gamma) = J_0(\sqrt{\lambda_n} \Gamma) \) where \( \sqrt{\lambda_n} \) is found from the root of the transcendental relation:

\[ -D J_0'(\sqrt{\lambda_n} \Gamma) = h J_0(\sqrt{\lambda_n} \Gamma). \]

The solution is then

\[ V(\Gamma, t) = \sum_{n=1}^{\infty} A_n J_0(\sqrt{\lambda_n} \Gamma) e^{-\lambda_n D t}. \]

Satisfying \( V(\Gamma, 0) = T_2 - U_3(\Gamma) = \sum_{n=1}^{\infty} A_n J_0(\sqrt{\lambda_n} \Gamma). \)

We obtain that

\[ A_n = \frac{\int_0^a \Gamma \left[ T_2 - U_3(\Gamma) \right] J_0(\sqrt{\lambda_n} \Gamma) d\Gamma}{\int_0^a \Gamma (J_0(\sqrt{\lambda_n} \Gamma))^2 d\Gamma}. \]
WE BEGIN WITH

\[ U_t = D \left( \frac{1}{r} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \varphi^2} \right) \right) \text{ in } 0 < \varphi < 2\pi, \ 0 < r < a \]

\[ U(\varphi, a, t) = 0, \quad U \text{ finite as } r \to 0 \]

\[ U(r, \varphi, 0) = f(r, \varphi) \]

WE SEPARATE VARIABLES

\[ U(r, \varphi, t) = T(t) R(r) \Phi(\varphi) \]

THEN

\[ T' R \Phi = D T \left( \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} \right) \Phi + \frac{1}{r^2} R \Phi'' = \text{constant} \]

THEN

\[ \frac{T'}{DT} = \frac{R'' + \frac{1}{r} R'}{R} + \frac{1}{r^2} \frac{\Phi''}{\Phi} = \text{constant} = -\lambda \]

WE HAVE

\[ \frac{r^2 (R'' + \frac{1}{r} R')}{R} + \frac{\Phi''}{\Phi} = r^2 (\text{constant}) \]

HENCE

\[ \frac{\Phi''}{\Phi} = -\lambda \]

WITH \( \Phi \) 2\pi PERIODIC.

WE HAVE

\[ \Phi'' + \lambda \Phi = 0, \quad 0 < \varphi < 2\pi \]

\[ \Phi(0) = \Phi(2\pi), \quad \Phi'(0) = \Phi'(2\pi) \]

\[ \Phi(\varphi) = A \cos \lambda \varphi + B \sin \lambda \varphi \]

THEN WE GET

\[ \frac{R'' + \frac{1}{r} R'}{R} - \frac{\lambda}{r^2} = \lambda \]

THIS YIELDS THAT

\[ R'' + \frac{1}{r} R' + \left( \lambda - \frac{\lambda}{r^2} \right) R = 0 \]

THEN WE OBTAIN

\[ r^2 R'' + r R' + (\lambda r^2 - \lambda) R = 0 \quad 0 < r < a \]

\[ R(a) = 0, \quad R \text{ finite as } a \to 0. \]

WITH \( \lambda = \pi^2 \) THIS LEADS TO THE EIGENVALUE PROBLEM
\[ r^2 \phi'' + r \phi' + \left( \frac{\lambda}{r^2} - n^2 \right) \phi = 0 \]
\[ \phi(a) = 0, \quad \phi \text{ finite as } r \to 0. \]

If we let \( x = \sqrt{\lambda} r \) and replace \( \phi(r) = y(\sqrt{\lambda} r) \) to obtain
\[ x^2 y'' + x y' + \left( x^2 - n^2 \right) y = 0 \]
\[ y(a) = 0, \quad y(0) \text{ finite} \]

If we substitute \( y = x^d \) we get \( \alpha(\lambda - 1) + \lambda - n^2 = 0 \)

and so \( d = \pm n \). This implies \( y_1 \sim c_1 x^n \) and \( y_2 \sim c_2 x^{-n} \) as \( x \to 0 \).

The two solutions are \( J_0, Y_0 \) Bessel functions of the first kind of order \( n \).

\[ y = A J_0(x) + B Y_0(x) \]

\[ J_0(x) \sim c x^n \text{ as } x \to 0, \quad Y_0(x) \sim c / x^n \text{ as } x \to 0. \]

\( Y_0(0) \) unbounded, \( J_0(0) = 0 \quad n > 0. \)

Hence we have \( \phi(r) = A J_0(\sqrt{\lambda} r) \quad B = 0 \text{ for boundedness} \)

then with \( \phi(a) = 0 \quad \to J_0(\sqrt{\lambda} a) = 0 \)

so \( \sqrt{\lambda} a = z_m \) where \( J_0(z_m) = 0 \) for \( m = 1, 2, 3, \ldots \) and each \( n \).

\[ \lambda_m = z_m^2 / a^2 \]

This leads to \( u(\theta, \phi, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{-\lambda_m t} J_0(\sqrt{\lambda_m} r) \left[ A_{mn} \cos n \phi + B_{mn} \sin n \phi \right] \)

Now with \( u(\theta, \phi, 0) = f(\theta, \phi) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} J_0(\sqrt{\lambda_m} r) \left[ A_{mn} \cos n \phi + B_{mn} \sin n \phi \right] \)

Then
\[ A_{mn} = \frac{\int_0^{2\pi} \int_0^{2\pi} r f(\theta, \phi) \cos (n \phi) J_0(\sqrt{\lambda_m} r) \, dr \, d\theta}{\int_0^{2\pi} \int_0^{2\pi} \cos^2 (n \phi) \, r J_0^2(\sqrt{\lambda_m} r) \, dr \, d\theta}, \quad \text{similar for } B_{mn} \]
BY ORTHOGONALITY WE MUST HAVE

\[
\int_0^\varkappa \mathcal{J}_n \left( \sqrt{\Lambda_{mn}} \ v \right) \mathcal{J}_n \left( \sqrt{\Lambda_{nj}} \ v \right) \, dv = 0 \quad \text{FOR} \ m \neq j.
\]

EXAMPLE SOLVE \( U_\varkappa = \mathcal{D} \left( \mathcal{H} g + \frac{1}{v} \mathcal{U} g + \frac{1}{v^2} \mathcal{W} g \right) \) IN \( 0 \leq \rho < a, \ 0 < \phi < \phi_a \)

WITH \( \mathcal{U}(a, \phi, t) = \mathcal{U}(\rho, 0, t) = \mathcal{U}(\rho, a, t) = 0, \ \mathcal{U}(\rho, \phi, 0) = \mathcal{F}(\rho, \phi). \)

WE SEPARATE VARIABLES TO OBTAIN \( \mathcal{U}(\rho, \phi, t) = \mathcal{T}(t) \mathcal{R}(\rho) \mathcal{\Phi}(\phi) \) TO GET

\[
\frac{\mathcal{T}'}{\mathcal{T}} = \frac{\mathcal{R}'' + \frac{\mathcal{L}}{\rho} \mathcal{R}' + \frac{1}{\rho^2} \mathcal{\Phi}''}{\mathcal{R}} = -\lambda.
\]

\[
\mathcal{\Phi}'' + \frac{\mathcal{L}}{\rho} \mathcal{\Phi} = 0, \ 0 < \phi < \phi_a \rightarrow \mathcal{\Phi} = \sin \left( \frac{mn\phi}{\phi_a} \right)
\]

\( \mathcal{\Phi}(0) = 0 \), \( \mathcal{\Phi}(a) = 0 \) \( \lambda_m = m^2 \alpha^2 / \phi_a^2 \)

THEN \( \mathcal{R}(\rho) = \mathcal{J}_{\lambda m / \phi_a} \left( \sqrt{\Lambda_{mn}} \ rho \right) \)

\( \mathcal{R}(a) = 0 \), \( \mathcal{R}(0) \) Finte

THEN \( \mathcal{J}_{mn / \phi_a} \left( \sqrt{\Lambda_{mn}} \ a \right) = 0 \) SO \( \sqrt{\Lambda_{mn}} \ a = \sigma_{mn} \) AND \( \mathcal{J}_{mn / \phi_a} (\sigma_{mn}) = 0 \)

THEN \( \mathcal{T}(t) = e^{-D \lambda_{mn} t} \)

THE SOLUTION IS \( \mathcal{U}(\rho, \phi, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} e^{-D \lambda_{mn} t} \sin \left( \frac{mn\phi}{\phi_a} \right) \mathcal{J}_{\lambda m / \phi_a} \left( \sqrt{\Lambda_{mn}} \ \rho \right) \)

THEN WITH \( \mathcal{U}(\rho, \phi, 0) = \mathcal{F}(\rho, \phi) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin \left( \frac{mn\phi}{\phi_a} \right) \mathcal{J}_{\lambda m / \phi_a} \left( \sqrt{\Lambda_{mn}} \ \rho \right) \)

BY ORTHOGONALITY:

\[
A_{mn} = \frac{\int_0^\phi \int_0^a \rho \mathcal{F}(\rho, \phi) \sin \left( \frac{mn\phi}{\phi_a} \right) \mathcal{J}_{\lambda m / \phi_a} \left( \sqrt{\Lambda_{mn}} \ \rho \right) d\rho \, d\phi}{\int_0^\phi \int_0^a \rho \left( \mathcal{J}_{\lambda m / \phi_a} \left( \sqrt{\Lambda_{mn}} \ \rho \right) \right)^2 \sin^2 \left( \frac{mn\phi}{\phi_a} \right) d\rho \, d\phi}
\]
**Example** Find the solution to the heat equation in a cylinder. We have that \( u(\rho, \psi, t) \) satisfies

\[
\frac{\partial u}{\partial t} = D \left( \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \psi^2} \right) \quad 0 < \rho < a, \quad 0 < \psi < 2\pi, \quad t > 0
\]

\[
\begin{align*}
\frac{\partial u}{\partial \rho}(a, \psi, t) &= 0 \quad \text{on} \quad \rho = a, \\
u(\rho, 0) &= \left(1 - \frac{\rho}{a}\right) \cos \psi. 
\end{align*}
\]

We let \( U(\rho, \psi, t) = \psi(\rho, t) \cos \psi \).

We substitute into the PDE to obtain:

\[
\frac{\partial \psi}{\partial t} = D \left( \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \psi^2} \right) \quad 0 < \rho < a, \quad t > 0
\]

\[
\psi(a, t) = 0, \quad \psi(\rho, 0) = \left(1 - \frac{\rho}{a}\right)
\]

We separate variables to obtain

\[
\frac{\partial \psi}{\partial \rho} = \frac{R''}{\rho} + \frac{2}{\rho} R' - \frac{1}{\rho^2} R = -\Lambda R,
\]

this leads to the eigenvalue problem

\[
\phi'' + \frac{2}{\rho} \phi' + \left(\lambda - \frac{1}{\rho^2}\right) \phi = 0 \quad 0 < \rho < a
\]

\[
\phi(a) = 0, \quad \text{\phi bounded as} \ \rho \to 0
\]

We write this as \( (\Gamma \phi')' + (\Delta \rho - \frac{1}{\rho^2}) \phi = 0 \) we height \( W = \Gamma \).

The solution is \( \phi = AJ_1(\sqrt{\lambda} \rho) + BJ_1(\sqrt{\lambda} \rho) \).

We need \( B = 0 \) for boundedness and \( \phi(a) = 0 \) yields that \( J_1(\sqrt{\lambda} a) = 0 \) so \( \sqrt{\lambda} a = Z_k \rightarrow \lambda_k = Z_k^2/a^2 \)

where \( J_1(Z_k) = 0 \). We obtain then \( T_k(t) = e^{-\lambda_k \rho t} J_1(\sqrt{\lambda_k} \rho) \).

By superposition \( \psi(\rho, t) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k \rho t} J_1(\sqrt{\lambda_k} \rho) \)

with \( \psi(\rho, 0) = 1 - \frac{\rho}{a} \) we get \( c_k = \frac{\int_0^a \left(1 - \frac{\rho}{a}\right) J_1(\sqrt{\lambda_k} \rho) d\rho}{\int_0^a \left[J_0(\sqrt{\lambda_k} \rho)\right]^2 d\rho} \).

\[
J_1(z)
\]

And \( u(\rho, \psi, t) = \psi(\rho, t) \cos \psi \).
Bessel's Equation of Order $\nu$

We write
\[ x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \quad y = x^\alpha \rightarrow \alpha = \pm \nu \]

then
\[ y = A J_{\nu}(x) + B Y_{\nu}(x) \quad \nu > 0 \text{ wlog} \]

$J_{\nu}(x)$ singular at $x \to 0$; $Y_{\nu}(x)$ analytic at $x \to 0$.

Thus
\[ x^2 \phi'' + x \phi' + (x^2 - \nu^2) \phi = 0 \quad 0 < x < a \]
\[ \phi(0) \text{ finite } \phi(a) = 0 \]

The solution is
\[ \phi = J_{\nu}(\sqrt{a} x) \]

with
\[ J_{\nu}(\sqrt{a} a) = 0 \]
\[ \text{so } \sqrt{a} a = z_k, \quad k = 1, 2, \ldots \]

which gives
\[ a_k = z_k^2 / a^2, \quad k = 1, 2, \ldots \]

There is an identity of the form
\[ \int_0^a x (J_{\nu}(\sqrt{a} x))^2 \, dx = \frac{a^2}{2} \left[ J_{\nu}'(\sqrt{a} a) \right]^2 \]

when
\[ J_{\nu}(\sqrt{a} a) = 0 \text{. (See Appendix A.)} \]
Lemma. Suppose that \( \phi'' + \frac{1}{\gamma} \phi' + \Lambda \phi = 0 \) in \( 0 < r < a \).
\( \phi(a) = 0, \ \phi, \phi' \text{ bounded at } \gamma \to 0. \)

Prove that \( \int_0^a \gamma \phi'^2 \, d\gamma = \frac{a^2}{2\Lambda} (\phi'(a))^2. \)

Proof.
\( (\gamma \phi')' + \Lambda \gamma \phi = 0. \)
Multiply by \( \gamma \phi' \) \( (\gamma \phi')(\gamma \phi')' + \Lambda \gamma^2 \phi \phi' = 0. \)
Integrate to get \( \frac{1}{2} \left[ (\gamma \phi')^2 \right]_0^a + \Lambda \left[ \int_0^a \gamma^2 \phi^3 \, d\gamma \right] = 0. \)

Now integrate by parts:
\( \frac{1}{2} a^2 (\phi'(a))^2 + \Lambda \left[ \int \gamma^2 \phi^3 \, d\gamma \right]_0^a - \left[ \int_0^a \gamma \phi^3 \, d\gamma \right] = 0. \)
But \( \phi(a) = 0 \) so \( \int_0^a \gamma \phi^3 \, d\gamma = \frac{1}{2} \frac{a^2}{\Lambda} (\phi'(a))^2. \)

Now if \( \phi = J_0(\sqrt{\Lambda} \gamma) \) and \( J_0(\sqrt{\Lambda} a) = 0 \) determine \( \Lambda \), then the identity above gives
\( \int_0^a \gamma (J_0(\sqrt{\Lambda} \gamma))^2 \, d\gamma = \frac{1}{2} \frac{a^2}{\Lambda} (J_0'(\sqrt{\Lambda} a))^2. \)

Remark. The same calculation can be done for
\( \gamma^2 \phi'' + \gamma \phi' + (\Lambda \gamma^2 - V^2) \phi = 0 \) \( \phi(0) \text{ bounded, } \phi(a) = 0. \)
The solution is \( \phi = J_1(\sqrt{\Lambda} \gamma) \) with \( J_1(\sqrt{\Lambda} a) = 0. \)
We claim that \( \int_0^a \gamma (J_1(\sqrt{\Lambda} \gamma))^2 \, d\gamma = \frac{1}{2} \frac{a^2}{\Lambda} (J_1'(\sqrt{\Lambda} a))^2. \)

Proof.
\( (\gamma \phi')' + (\Lambda \gamma - V^2/\gamma) \phi = 0. \)
Multiply by \( \gamma \phi' \) and integrate \( \int_0^a. \)
\[
\frac{1}{2} \left[ (r \phi')^2 \right]_0^a + \Lambda \int_0^a r^2 \frac{1}{2} \frac{d}{dr} (\phi^2) \, dr - \nu^2 \phi^2 \bigg|_0^a = 0.
\]

But for \( \nu > 0 \), \( \phi(0) = 0 \) and for \( \nu = 0 \) the last term vanishes. Since \( \phi(0) = 0 \) we get

\[
\frac{1}{2} a^2 (\phi' \vert a \vert)^2 + \frac{\Lambda}{2} \int_0^a r^2 (\phi^2)' \, dr = 0.
\]

Integrate by parts as before and we find \( \psi \vert a \vert = 0 \) to get

\[
\int_0^a r \phi^2 \, dr = \frac{1}{2 \Lambda} a^2 (\phi' \vert a \vert)^2
\]

Since

\[
\phi(r) = \frac{\nu}{r \sqrt{\Lambda}} \text{ then } \phi'(r) = \sqrt{\Lambda} \frac{\nu}{r \sqrt{\Lambda}}.
\]

So

\[
\int_0^a r \left( \frac{\nu}{r \sqrt{\Lambda}} \right) \phi'(a \sqrt{\Lambda}) \, dr = \frac{\Lambda}{2} \left( \nu \sqrt{\Lambda} \phi'(a \sqrt{\Lambda}) \right)^2.
\]
1. (30pts) Put the following two problems in Sturm-Liouville form, identify the weight function \( w(x) \), and calculate the eigenvalues and eigenfunctions. Also what is the orthogonality relation for the eigenfunctions?

\[
x^2 \phi'' + 5x \phi' + \lambda \phi = 0, \quad 1 \leq x \leq 2; \quad \phi(1) = \phi(2) = 0
\]

Hint: try \( \phi(x) = x^r \)

\[
\phi'' - 2\phi + \lambda \phi = 0, \quad 0 \leq x \leq 1; \quad \phi(0) = \phi(1) = 0
\]

Hint: try \( \phi(x) = e^{rx} \)

2. (20pts) Use the method of separation variables to solve

\[
\begin{cases}
  u_{tt} = u_{xx} + e^t \sin(3x), & 0 < x < \pi \\
  u(x,0) = \sin(3x), u_t(x,0) = \sin(5x) & 0 < x < \pi \\
  u(0,t) = t, u(\pi,t) = 0
\end{cases}
\]

3. (30pts) (a) (20pts) Use the method of separation variables to solve the following PDE:

\[
u_{xx} + u_{yy} = 0 \quad \text{in} \quad D = (0, \pi) \times (0, \pi)
\]

\[
u_y(x,0) = u(x,\pi) = 0, \quad u(\pi,y) = 0
\]

\[
u(0,y) = \cos^2(y)
\]

(b) (10pts) Prove that the solution obtained in (a) is unique.

4. (20pts) Use the method of separation of variables to solve the following PDE:

\[
u_{xx} + u_{yy} = 1 \quad \text{in} \quad D = \{(x,y) | x^2 + y^2 < 4 \}
\]

\[
u(x,y) = x^2 - y^2 \quad \text{on} \quad \partial D = \{(x,y) | x^2 + y^2 = 4 \}
\]
Homework Assignment 6 (Due Date: April 8, 2014)

1. (30pts) Put the following two problems in Sturm-Liouville form, identify the weight function $w(x)$, and calculate the eigenvalues and eigenfunctions. Also what is the orthogonality relation for the eigenfunctions?

\[ x^2 \phi'' + 5x \phi' + \lambda \phi = 0, \ 1 \leq x \leq 2; \ \phi(1) = \phi(2) = 0 \]

Hint: try $\phi(x) = x^r$

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    u(0, t) = t, u(\pi, t) = 0
\end{cases}
\end{align*}
\]

3. (30pts) (a) (20pts) Use the method of separation variables to solve the following PDE:

\[ \begin{align*}
    u_{xx} + u_{yy} &= 0 \text{ in } D = (0, \pi) \times (0, \pi) \\
    u_t(x, 0) &= u_x(x, \pi) = 0, u(\pi, y) = 0 \\
    u(0, y) &= \cos^2(y)
\end{align*} \]

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\end{align*} \]
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\[ x^2 \phi_{xx} + 5x \phi_x + \lambda \phi = 0, \quad 1 \leq x \leq 2; \phi(1) = \phi(2) = 0 \]

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\[ u(x,y) = x^2 - y^2 \quad \text{on} \quad \partial D = \{(x,y) | x^2 + y^2 = 4 \} \]