# THE CLOSED TOPOLOGICAL VERTEX VIA THE CREMONA TRANSFORM

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ABSTRACT. We compute the local Gromov-Witten invariants of the "closed vertex", that is, a configuration of three $\mathbb{P}^1$'s meeting in a single triple point in a Calabi-Yau threefold. The method is to express the local invariants of the vertex in terms of ordinary Gromov-Witten invariants of a certain blowup of $\mathbb{P}^3$ and then to compute those invariants via the geometry of the Cremona transformation.

## 1. INTRODUCTION

Let $C \subset Y$ be a curve in a Calabi-Yau threefold consisting of three $\mathbb{P}^1$'s meeting in a single point $p$. With some minor assumptions on the formal neighborhood of $C \subset Y$ (see Assumption 1), the contribution to the genus $g$ Gromov-Witten invariants of $Y$ by maps to $C$ is well defined and denoted by $N^g_{d_1,d_2,d_3}(C)$, where $d_i$ is the degree of the map to $i$th component. We call $C$ the closed topological vertex and we say that $N^g_{d_1,d_2,d_3}(C)$ are the local invariants of $C$.

**Theorem 1.** *The local invariants of the closed topological vertex are given as follows.*

$$N^g_{d_1,d_2,d_3}(C) = 0$$

*if $\{d_1, d_2, d_3\}$ contains two distinct non-zero values, otherwise*

$$N^g_{d,d,d}(C) = N^g_{d,d,0}(C) = N^g_{d,0,0}(C).$$

Note that $N^g_{d,0,0}(C)$ is just the local invariant of a smooth $\mathbb{P}^1$ in a Calabi-Yau threefold embedded with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. These were computed by Faber and Pandharipande [7] and are given by:

$$N^g_{d,0,0}(C) = \frac{|B_{2g}(2g-1)|}{(2g)!} d^{2g-3}$$

where $B_{2g}$ is the $2g$th Bernoulli number.

If one of the $d_i$'s is zero, then Theorem 1 is a special case of the computation of local invariants of ADE configurations done by Bryan, Katz, and Leung in [2].

The local invariants $N^g_{d_1,d_2,d_3}(C)$ are related to the "topological vertex" of Aganagic, Klemm, Marino, and Vafa [1]. Using the conjectural Chern-Simons/string theory duality, they compute the open string amplitudes of 3 rational curves meeting in a triple point in a Calabi-Yau threefold. Open string amplitudes correspond to a version of Gromov-Witten theory using Riemann surfaces with boundary. The correct mathematical formulation of open string Gromov-Witten theory is not currently known. Our local invariants $N^g_{d_1,d_2,d_3}(C)$ correspond to closed string amplitudes, which is why we call our configuration the closed topological vertex.

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In the course of the proof of Theorem 1, we prove a certain symmetry of the Gromov-Witten invariants of $\mathbb{P}^3$ blown up at points. This symmetry arises from the Cremona transformation and may be of independent interest.

Let $X$ be the blowup of $\mathbb{P}^3$ at $n$ (distinct) points $\{x_1, \ldots, x_n\}$ where $n > 4$. Let $h$ be the pullback of the class of a line in $\mathbb{P}^3$ and let $e_1, \ldots, e_n$ be the classes of the lines in the corresponding exceptional divisors.

**Theorem 2.** *Let $\beta = dh - \sum_{i=1}^{n} a_i e_i$ with $2d = \sum_{i=1}^{n} a_i$ and assume that $a_i \neq 0$ for some $i > 4$. Then we have the following equality of Gromov-Witten invariants:*

$$\langle\ \rangle^X_{g,\beta} = \langle\ \rangle^X_{g,\beta'}$$

*where $\beta' = d'h - \sum_{i=1}^{n} a'_i e_i$ has coefficients given by*

$$\begin{aligned}
d' &= 3d - 2(a_1 + a_2 + a_3 + a_4) \\
a'_1 &= \ d - \ (a_2 + a_3 + a_4) \\
a'_2 &= \ d - \ (a_1 + a_3 + a_4) \\
a'_3 &= \ d - \ (a_1 + a_2 + a_4) \\
a'_4 &= \ d - \ (a_1 + a_2 + a_3) \\
a'_5 &= \ a_5 \\
&\ \ \vdots \\
a'_n &= \ a_n.
\end{aligned}$$

**Remark 3.** The condition that $a_i \neq 0$ for some $i > 4$ is necessary; for example, the theorem fails for the class $\beta = h - e_1 - e_2$.

Our basic strategy of first identifying the local invariant with an invariant of a blowup of projective space and then utilizing the Cremona transformation was first employed in [3]. In that paper, the technique was used to compute the local contributions of nodal fibers in an elliptically fibered $K3$ surface (see section 5.3 of [3])

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## 2. LOCAL INVARIANTS

Let $i : Z \subset Y$ be a closed subvariety of a smooth projective Calabi-Yau threefold. Let $\beta$ be a curve class in $Z$; we define the "local Gromov-Witten invariants of $Z \subset Y$", denoted $N^g_\beta(Z \subset Y)$, whenever the substack of $\overline{M}_g(Y, i_*\beta)$ consisting of maps whose image lies in $Z$ is a union of path connected componets. This substack then inherits a degree 0 virtual class and $N^g_\beta(Z \subset Y)$ is defined to be its degree. In general, $N^g_\beta(Z \subset Y)$ depends on a (formal or analytic) neighborhood of $Z \subset Y$, but in many cases, it depends only on the normal bundle. We write $N^g_\beta(Z)$ when the structure of the neighborhood is understood.

Local invariants have been studied extensively in the literature. Examples include the local invariants of surfaces such as $\mathbb{P}^2$ [9, 12], $K3$ [15], and rational elliptic surfaces [10]. Local invariants of curves have been studied in [2, 4, 5, 14]. Recently, Aganagic, Klemm, Marino, and Vafa ([1], c.f. [6]) have developed a technique, based on a conjectural physical duality, which computes the local invariants of any toric curve or surface. The basic building block in their algorithm is the open string amplitude of the vertex configuration $C$. The correct general mathematical definition of the open-string invariants is not currently known. However, a mathematical version of the open vertex has been announced by Li-Liu-Liu-Zhou [13].

In order to define $N^g_{d_1,d_2,d_3}(C)$ we must specify the geometry of the formal neighborhood of $C$ in $Y$.

**Assumption 1.** *We assume that the local geometry of $C \subset Y$ is as follows. The curve $C$ consists of three components $C = C_1 \cup C_2 \cup C_3$, with $C_i \cong \mathbb{P}^1$, and meeting in a single triple point $p$. $C$ is embedded in $Y$ such that the normal bundle of each component of $C$ is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Additionally, for the case of $d_i > 0$, we assume the formal neighborhood of the triple point has the geometry of the coordinate axes in $\mathbb{C}^3$ with respect to the local coordinates defined by the normal bundles. For the case where one of the $d_i$'s is zero, say $d_3$, we assume that the curve $C_1 \cup C_2$ is contractable (c.f.* [2], *section 2).*

We remark that the above assumption leads to two different formal neighborhoods in the two cases $d_i > 0$ and $d_3 = 0$. In the former case, the curve $C_1 \cup C_2$ admits a deformation in $Y$ that smooths the node and hence it cannot be contractable.

## 3. The closed vertex and invariants of a blowup of $\mathbb{P}^3$

In this section we prove that the local invariants of the closed topological vertex are equal to certain ordinary Gromov-Witten invariants:

**Proposition 4.** *Let $X \to \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ at six points $\{x_1, x_2, x_3, x_1', x_2', x_3'\}$. Let $h$ be the pullback of the class of a line in $\mathbb{P}^3$ and let $\{e_1, e_2, e_3, e_1', e_2', e_3'\}$ be the classes of the lines in the corresponding exceptional divisors. Assume that $d_1, d_2, d_3 > 0$ and let*

$$\beta = \sum_{i=1}^{3} d_i(h - e_i - e_i').$$

*Then the local invariants of the vertex are given by the ordinary Gromov-Witten invariants of $X$ in the class of $\beta$:*

$$N^g_{d_1,d_2,d_3}(C) = \langle\ \rangle^X_{g,\beta}\,.$$

PROOF: By the deformation invariance property of Gromov-Witten invariants, we may replace $X$ with any threefold that is deformation equivalent to the blowup of $\mathbb{P}^3$ at six points. We will construct such an $X$ that is compatible with the toric structure of $\mathbb{P}^3$. Let $x_0, x_1, x_2, x_3$ be the fixed points of the standard $T = (\mathbb{C}^\times)^3$ action on $\mathbb{P}^3$. Let $X'$ be the blowup of $\mathbb{P}^3$ at the three points $\{x_1, x_2, x_3\}$, let $C_i'' \subset \mathbb{P}^3$ be the line between $x_0$ and $x_i$, and let $C_i' \subset X'$ be the proper transform of $C_i''$. Let $x_i'$ be the intersection of $C_i'$ with the exceptional divisor of $X' \to \mathbb{P}^3$. Define $X$ to be the blowup of $X'$ at the three points $\{x_1', x_2', x_3'\}$ and let $C_i$ be the proper transform of $C_i'$. Clearly, $X$ is deformation equivalent to a blowup of $\mathbb{P}^3$ at six distinct points.

We choose the basis $\{H, E_1, E_2, E_3, E_1', E_2', E_3'\}$ for $H_4(X;\mathbb{Z})$ where $H$ is the pullback of a hyperplane in $\mathbb{P}^3$, $E_i'$ are the exceptional divisors of $X \to X'$, and $E_i$ are the total transforms of the exceptional divisors of $X' \to \mathbb{P}^3$. We choose the basis $\{h, e_1, e_2, e_3, e_1', e_2', e_3'\}$ for $H_2(X;\mathbb{Z})$ where $h$ is a line in $H$, $e_i'$ is a line in $E_i'$, and $e_i$ is a line in $E_i$ which is disjoint

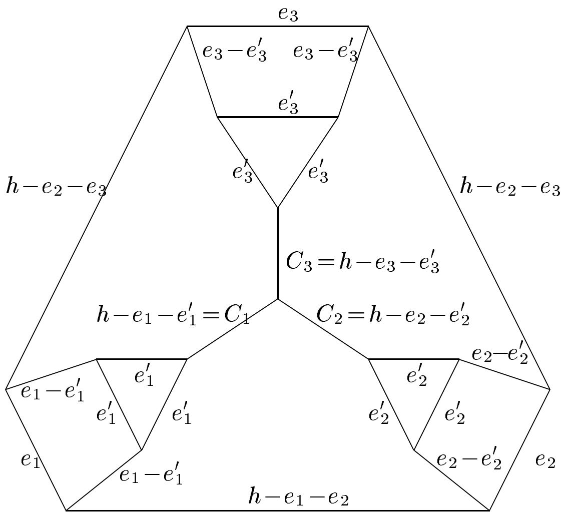


FIGURE 1. The curve $F$, which is the union of the $T$ invariant curves of $X$

from $x_i'$. In this basis, we have $H \cdot H = h$, $E_i \cdot E_i = -e_i$, and all other intersections between divisors are zero. With respect to this basis, the class of the curve $C_i$ is $h - e_i - e_i'$.

The $T$ action on $\mathbb{P}^3$ lifts to a $T$ action on $X'$ and $X$ since the centers of each blowup are invariant. Let $F$ be the union of all the $T$ invariant curves in $X$. The $T$ invariant curves and their classes are configured as in Figure 1. The fixed points of $T$ correspond to the vertices in the diagram.

The normal bundle of each $C_i$ is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. This can be seen as follows. Let $P'' \subset \mathbb{P}^3$ be a plane containing the line $C_i''$ and let $P' \subset X'$ and $P \subset X$ denote the successive proper transforms. Then $C_i \subset P$ so $N_{C_i/P}$ defines a sub line bundle in $N_{C_i/X}$ of degree $C_i \cdot C_i$ where the intersection product is in $P$. By functoriality of blowups, $P' \to P''$ is the blowup along $x_i$ and $P \to P'$ is the blowup along $x_i'$. We then compute the intersection product in $P$:

$$C_i \cdot C_i = (h - e_i - e_i') \cdot (h - e_i - e_i') = -1.$$

Since we can apply this argument to any plane $P$ containing $C_i$, and they span the normal bundle, we conclude that $N_{C_i/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Moreover, this argument shows that the configuration $C = C_1 \cup C_2 \cup C_3$ satisfies (the $d_i > 0$ case of) Assumption 1 since we can take the two spanning planes of the normal bundle of (say) $C_1$ to contain $C_2$ and $C_3$ respectively and so the local geometry of the triple point is that of the coordinate axes.

We've shown that $C \subset X$ has the local geometry of the closed vertex; then to prove the proposition, we need to show that the only contributions to $\langle\ \rangle^X_\beta$ are given by maps to the configuration $C$. This is accomplished with the following:

**Lemma 5.** *Let $X$ and $C = C_1 \cup C_2 \cup C_3$ be as above and let*

$$\beta = \sum_{i=1}^{3} d_i (h - e_i - e_i')$$

*where we assume that $d_i > 0$. Then every stable map $[f] \in \overline{M}_g(X, \beta)$ has image $C$.*

PROOF: Since the torus $T$ acts on $X$ it acts on $\overline{M}_g(X, \beta)$. Let

$$[f : D \to X] \in \overline{M}_g(X, \beta)$$

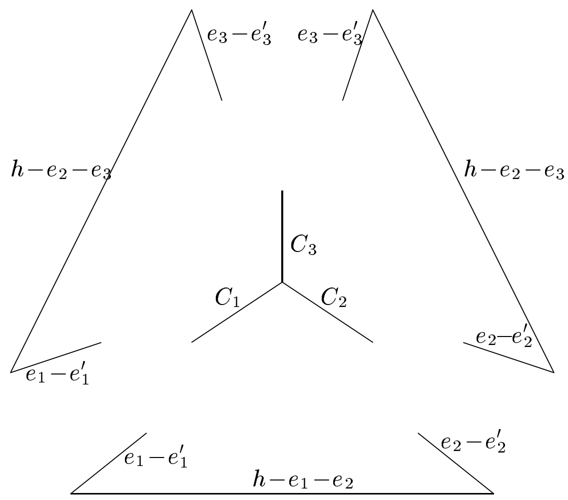


FIGURE 2. Possible curves in $\mathrm{Im}(f'')$

be any stable map. Assume that $\mathrm{Im}(f)$ contains a point $x$ not contained in $C$.

We study the limits of $[f]$ under the $T$ action on $\overline{M}_g(X, \beta)$. For any one parameter subgroup $\mathbb{C}^\times \to T$, the limit of $x$, under the action of $\lambda \in \mathbb{C}^\times$ as $\lambda \to 0$, is in the fixed point set of the $\mathbb{C}^\times$ action. Moreover, since $x \notin C$, we can find a subgroup $\mathbb{C}^\times \to T$ such that the limit of $x$ is $v_0$, a fixed point of the $T$ action but *not* contained in $C$. It follows that the limit of the same $\mathbb{C}^\times$ subgroup acting on $[f] \in \overline{M}_g(X, \beta)$ is a stable map $f'$ such that $v_0 \in \mathrm{Im}(f')$. Consequently, $v_0$ is in the image of all maps in the closure of the $T$ orbit of $[f']$. In particular, there exists a stable map $[f''] \in \overline{M}_g(X, \beta)$ fixed by $T$ and such that $v_0 \in \mathrm{Im}(f'')$. Therefore, we have constructed a stable map $f'' : D \to X$ whose image is contained in $F$, the union of the $T$ invariants curves, but is not contained in $C$.

We now show that this leads to a contradiction. The class

$$f''_*[D] = \beta = \sum_{i=1}^{3} d_i(h - e_i - e_{i'})$$

has the property that the total multiplicity of the $e$ terms is $-2(d_1 + d_2 + d_3)$ while the multiplicity of $h$ is $(d_1 + d_2 + d_3)$. Observe that every component of $F$ whose class has an $h$ also has two negative $e$ terms. It follows that the sum of all the non-$h$ components in $f''_*[D]$ must have the same number of positive $e$ terms as negative $e$ terms. Since the classes of the non-$h$ components are all either $e_i$, $e_i'$, or $e_i - e_i'$, we can conclude that $\mathrm{Im}(f'') \subset F$ *does not contain any of the components of class* $e_i$ *or* $e_i'$.

Removing those components from the diagram of $F$, we find that $\mathrm{Im}(f'')$ must lie in the components pictured in Figure 2. Since the domain of $f''$ is connected, the image is connected and must lie in one of the components of the graph shown in Figure 2. Since $\mathrm{Im}(f'') \not\subset C$, it must be one of the outer components. However, this contradicts the assumption that $d_i > 0$ for all $i$ since none of the outer components contain all three classes $\{e_1, e_2, e_3\}$.

This proves that our initial assumption is false and so $\mathrm{Im}(f) \subset C$ for all $[f] \in \overline{M}_g(X, \beta)$ and the lemma is proved. $\square$

## 4. COMPUTING THE LOCAL INVARIANTS $N^g_{d_1,d_2,d_3}(C)$

We need the following vanishing lemma (c.f. [8] Proposition 3.1):

**Lemma 6.** *Let $X$ and $\beta$ be as in Theorem 2 with $n = 6$, then if $a_i < 0$ for some $i$, $\langle\,\rangle^X_{g,\beta} = 0$.*

PROOF: Without loss of generality we may assume that $i = 1$ so that $a_1 < 0$. Let $[f] \in \overline{M}_g(X, \beta)$ be any stable map. Since $\beta \cdot E_1 < 0$, $\mathrm{Im}(f)$ must have a component $C''$ contained in $E_1$. The class of $C''$ is therefore $me_1$ for some $m > 0$. Writing $\mathrm{Im}(f) = C' \cup C''$ we have

$$0 = -K_X \cdot \beta = -K_X \cdot C' - K_X \cdot C'' = -K_X \cdot C' + 2m$$

and so $-K_X \cdot C'$ is negative. However, this contradicts Lemma 10 which states that $-K_X$ is nef. Therefore $\overline{M}_g(X, \beta) = \emptyset$ and the Lemma follows. □

We now prove Theorem 1 assuming Theorem 2 which we will prove in the subsequent sections.

We can assume that $d_1, d_2, d_3 > 0$ since the other cases are covered by [2] and [7]. We order the $d_i$'s so that $d_1 \geq d_2 \geq d_3$.

By Proposition 4 we have

$$N^g_{d_1,d_2,d_3}(C) = \langle\,\rangle^X_{g,\beta}$$

where

$$\beta = (d_1 + d_2 + d_3)h - d_1(e_1 + e_2) - d_2(e_3 + e_4) - d_3(e_5 + e_6).$$

Then applying Theorem 2 we get

$$N^g_{d_1,d_2,d_3}(C) = \langle\,\rangle^X_{g,\beta'}$$

where

$$\beta' = (3d_3 - d_1 - d_2)h - (d_3 - d_2)e_1 - (d_3 - d_2)e_2 - (d_3 - d_1)e_3 - (d_3 - d_1)e_4 - d_3e_5 - d_3e_6.$$

Lemma 6 then implies that $N^g_{d_1,d_2,d_3}(C) = 0$ unless $d_3 = d_2 = d_1 = d$, in which case

$$\beta' = d(h - e_5 - e_6)$$

and so $N^g_{d,d,d}(C) = N^g_{0,0,d}(C)$ and Theorem 1 is proved. □

## 5. THE GEOMETRY OF THE CREMONA TRANSFORMATION

In this section, we prove Theorem 2 by studying the geometry of $X$, the blowup of $\mathbb{P}^3$ at $n$ points, and $\hat{X}$, the blowup of $X$ along a certain configuration of six lines.

Let $X$ be the blowup of $\mathbb{P}^3$ at $n$ distinct points $x_1, \ldots, x_n$ where $n > 4$. We take the first four points to be the fixed points of the standard torus action on $\mathbb{P}^3$ and we take the remaining points to be any fixed points of the Cremona transformation:

$$\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$$
$$(z_0 : z_1 : z_2 : z_3) \mapsto (\frac{1}{z_0} : \frac{1}{z_1} : \frac{1}{z_2} : \frac{1}{z_3}).$$

**Remark 7.** The case where $n-4$ is greater than the number of fixed points is easily handled by including in the blowup locus pairs of points exchanged by the Cremona transformation. However, for notational convenience we will assume that the points $x_1, \ldots, x_n$ are fixed.

Let $l_{jk}$, $1 \leq j < k \leq 4$ be the proper transform of the line through $x_j$ and $x_k$. Let

$$\pi : \hat{X} \to X$$

be the blowup of $X$ along the six (disjoint) lines $l_{jk}$.

$\hat{X}$ admits an involution $\tau : \hat{X} \to \hat{X}$ which resolves the Cremona transformation. The map $\tau$ is discussed in more detail by Gathmann in [8], although note that our $\hat{X}$ has the additional blowups at $x_5, \ldots, x_n$ whose corresponding exceptional divisors are simply fixed by $\tau$, or possibly exchanged if the points $x_5 \ldots x_n$ include non-trivial orbits (c.f. Remark 7).

We briefly describe the divisors and the curves on $X$ and $\hat{X}$ and their intersections. Generally, we denote divisor classes with upper case letters and curve classes with lower case letters. Classes on $\hat{X}$ will have a hat, and classes on $X$ will not.

The homology groups $H_4(X;\mathbb{Z})$ and $H_2(X;\mathbb{Z})$ are spanned by the divisor and curve classes respectively:

$$H_4(X;\mathbb{Z}) = \langle H, E_1, \ldots, E_n \rangle, \quad H_2(X;\mathbb{Z}) = \langle h, e_1, \ldots, e_n \rangle.$$

Here $H$ is the pullback of the hyperplane in $\mathbb{P}^3$, $h$ is the class of the line in $H$, $E_i$ is the exceptional divisor over $x_i$, and $e_i$ is the class of a line in $E_i$.

The intersection pairing on $X$ is given by:

$$\begin{aligned} H \cdot H &= h, & E_i \cdot E_i &= -e_i, \\ H \cdot h &= p, & E_i \cdot e_i &= -p \end{aligned}$$

where $p \in H_0(X;\mathbb{Z})$ is the class of the point and all other pairings are zero.

The homology groups $H_4(\hat{X};\mathbb{Z})$ and $H_2(\hat{X};\mathbb{Z})$ are also spanned by divisor and curve classes:

$$H_4(\hat{X};\mathbb{Z}) = \left\langle \hat{H}, \hat{E}_i, \hat{F}_{jk} \right\rangle, \quad H_2(\hat{X};\mathbb{Z}) = \left\langle \hat{h}, \hat{e}_i, \hat{f}_{jk} \right\rangle,$$

where $1 \leq i \leq n$ and $1 \leq j < k \leq 4$. Here $\hat{H}$ is the proper transform of $H$ and $\hat{h}$ is the generic line in $\hat{H}$. $\hat{E}_i$ is the proper transform of $E_i$ and $\hat{e}_i$ is the class of the generic line in $\hat{E}_i$. $\hat{F}_{jk}$ is the component of the exceptional divisor of $\hat{X} \to X$ lying over $l_{jk}$, and $\hat{f}_{jk}$ is the fiber class of $\pi : \hat{F}_{jk} \to l_{jk}$.

Note that $\hat{F}_{jk} \to l_{jk}$ is the trivial fibration and the class of the section $\hat{s}_{jk}$ is given by

$$\hat{s}_{jk} = \hat{h} - \hat{e}_j - \hat{e}_k + \hat{f}_{jk}.$$

The intersections are given as follows:

$$\begin{aligned} \hat{H} \cdot \hat{H} &= \hat{h}, & \hat{E}_i \cdot \hat{E}_i &= -\hat{e}_i, & \hat{F}_{jk} \cdot \hat{F}_{jk} &= -\hat{s}_{jk} - \hat{f}_{jk}, \\ \hat{H} \cdot \hat{F}_{jk} &= \hat{f}_{jk}, & \hat{E}_j \cdot \hat{F}_{jk} &= \hat{f}_{jk}, & & \\ \hat{H} \cdot \hat{h} &= \hat{p}, & \hat{E}_i \cdot \hat{e}_i &= -\hat{p}, & \hat{F}_{jk} \cdot \hat{f}_{jk} &= -\hat{p}, \end{aligned}$$

where $\hat{p} \in H_0(\hat{X};\mathbb{Z})$ is the class of the point and all other intersections are zero.

The action of $\tau$ on divisors is described by Gathmann [8] in section 6. The action of $\tau$ on the curve classes of $\hat{X}$ is then easily obtained using Poincaré duality and is given as follows:

$$\begin{aligned}
\tau_*\hat{h} &= 3\hat{h} - (\hat{e}_1 + \hat{e}_2 + \hat{e}_3 + \hat{e}_4),\\
\tau_*\hat{e}_1 &= 2\hat{h} - (\hat{e}_2 + \hat{e}_3 + \hat{e}_4),\\
\tau_*\hat{e}_2 &= 2\hat{h} - (\hat{e}_1 + \hat{e}_3 + \hat{e}_4),\\
\tau_*\hat{e}_3 &= 2\hat{h} - (\hat{e}_1 + \hat{e}_2 + \hat{e}_4),\\
\tau_*\hat{e}_4 &= 2\hat{h} - (\hat{e}_1 + \hat{e}_2 + \hat{e}_3),\\
\tau_*\hat{e}_5 &= \hat{e}_5,\\
&\vdots\\
\tau_*\hat{e}_n &= \hat{e}_n,\\
\tau_*\hat{f}_{jk} &= \hat{s}_{j'k'},
\end{aligned}$$

where $\{j',k'\}$ is defined by the condition $\{j,k\} \cup \{j',k'\} = \{1,2,3,4\}$.

For a class $\hat{\beta} = d\hat{h} - \sum_{i=1}^n a_i\hat{e}_i$ with $2d = \sum_{i=1}^n a_i$, we have $-K_{\hat{X}} \cdot \hat{\beta} = 0$ and so the degree $\hat{\beta}$ Gromov-Witten invariants have no insertions. Since $\tau$ is an isomorphism, it preserves the Gromov-Witten invariants of $\hat{X}$ so in particular,

$$\langle\,\rangle^{\hat{X}}_{g,\hat{\beta}} = \langle\,\rangle^{\hat{X}}_{g,\tau_*\hat{\beta}}$$

where

$$\tau_*\hat{\beta} = d'\hat{h} - \sum_{i=1}^n a'_i\hat{e}_i$$

has coefficients $d', a'_1, \ldots, a'_n$ given by the equations of Theorem 2.

To prove Theorem 2 then, it suffices to prove the following

**Lemma 8.** *Let $d, a_1, \ldots, a_n$ be such that $2d = \sum_{i=1}^n a_i$ and $a_i \neq 0$ for some $i > 4$. Then*

$$\langle\,\rangle^{X}_{g,\beta} = \langle\,\rangle^{\hat{X}}_{g,\hat{\beta}}$$

*where $\beta = dh - \sum_{i=1}^n a_ie_i$ and $\hat{\beta} = d\hat{h} - \sum_{i=1}^n a_i\hat{e}_i$.*

**Remark 9.** The condition that $a_i \neq 0$ for some $i > 4$ is necessary. For example,

$$1 = \langle\,\rangle^{X}_{0,h-e_1-e_2} \neq \langle\,\rangle^{\hat{X}}_{0,\hat{h}-\hat{e}_1-\hat{e}_2} = 0.$$

PROOF: The lemma follows from the general results of Hu [11]. We warn the reader that the theorems in [11] are incorrect as stated; the above example provides a counterexample. However, the author has informed us that a crucial hypothesis is missing in the main theorems of [11]. Namely, in Hu's notation, he must additionally assume that the class $p_!(A)$ is not exceptional.

The paper [11] uses the machinery of relative Gromov-Witten invariants and gluing. To make our paper self-contained, we provide below an independent proof of Lemma 8 in the case of $n = 6$ which is what is needed for Theorem 1.

Assume that $n = 6$. Without loss of generality we may assume that $a_5 \neq 0$. We will show that any $[\hat{f}] \in \overline{M}_g(\hat{X}, \hat{\beta})$ has an image which is disjoint from $\hat{F} = \cup_{j<k}\hat{F}_{jk}$, and any $[f] \in \overline{M}_g(X, \beta)$ has an image which is disjoint from $l = \cup_{j<k} l_{jk}$. It follows that the natural map $\overline{M}_g(\hat{X}, \hat{\beta}) \to \overline{M}_g(X, \beta)$ induced by $\pi$ is an isomorphism of the moduli spaces and their virtual fundamental classes. Indeed, if both $\mathrm{Im}(\hat{f}) \cap \hat{F} = \emptyset$ and $\mathrm{Im}(f) \cap l = \emptyset$ for all stable maps $[\hat{f}] \in \overline{M}_g(\hat{X}, \hat{\beta})$ and $[f] \in \overline{M}_g(X, \beta)$, then both $\overline{M}_g(\hat{X}, \hat{\beta})$ and $\overline{M}_g(X, \beta)$ are canonically identified with $\overline{M}_g(\hat{X} \backslash \hat{F}, \hat{\beta})$.

Let $[f : C \to X] \in \overline{M}_g(X, \beta)$ and suppose that $\mathrm{Im}(f) \cap l_{jk} \neq \emptyset$ for some $j$ and $k$. $\mathrm{Im}(f) \not\subset l_{jk}$ since $a_5 \neq 0$ and so

$$f_*(C) = C' + b l_{jk}$$

where $C'$ meets $l_{jk}$ in a finite set of points ($b$ can be zero here). Let $\hat{C}'$ be the proper transform of $C'$. Since $C' \cap l_{jk} \neq \emptyset$, we have $\hat{C}' \cdot \hat{F}_{jk} = m > 0$. Therefore we have

$$\hat{C}' = d\hat{h} - \sum_{i=1}^{6} a_i \hat{e}_i - b(\hat{h} - \hat{e}_j - \hat{e}_k) - m\hat{f}_{jk}.$$

Define $\{j', k'\}$ by the condition $\{j', k'\} \cup \{j, k\} = \{1, 2, 3, 4\}$ and let

$$\hat{D}_{jk} = 2\hat{H} - (\hat{E}_1 + \cdots + \hat{E}_6) - \hat{F}_{jk} - \hat{F}_{j'k'}.$$

Then

$$\hat{D}_{jk} \cdot \hat{C}' = -m < 0.$$

However, this contradicts Lemma 12 which states that $\hat{D}_{jk}$ is nef. Thus $\mathrm{Im}(f) \cap l = \emptyset$ for all $[f] \in \overline{M}_g(X, \beta)$.

We argue in a similar fashion for $\overline{M}_g(\hat{X}, \hat{\beta})$. Let $[\hat{f} : C \to \hat{X}] \in \overline{M}_g(\hat{X}, \hat{\beta})$ and suppose that $\mathrm{Im}(\hat{f}) \cap \hat{F}_{jk} \neq \emptyset$ for some $j$ and $k$. Since $\hat{\beta} \cdot \hat{F}_{jk} = 0$, $\hat{f}_*(C)$ must have a component $C''$ contained in $\hat{F}_{jk}$. We then have

$$\hat{\beta} = \hat{f}_*(C) = C' + C''$$

where $C'$ is non-empty since $\hat{\beta} \cdot \hat{E}_5 = a_5 > 0$.

Since $C'' \subset \hat{F}_{jk}$ is an effective class in $\hat{F}_{jk} \cong \mathbb{P}^1 \times \mathbb{P}^1$, it is of the form $a\hat{s}_{jk} + b\hat{f}_{jk}$ with $a, b \geq 0$ and $a + b > 0$.

Define $\hat{D}_{jk}$ as above. Then $\hat{D}_{jk} \cdot \hat{\beta} = 0$ and $\hat{D}_{jk} \cdot C'' = a + b > 0$ and so $\hat{D}_{jk} \cdot C' < 0$, contradicting the fact that $\hat{D}_{jk}$ is nef.

This proves that $\mathrm{Im}(\hat{f}) \cap \hat{F} = \emptyset$ for all $[\hat{f}] \in \overline{M}_g(\hat{X}, \hat{\beta})$ and Lemma 8 is proved.

This then completes the proof of Theorem 2.

## 6. Nef divisors on $X$ and $\hat{X}$

In this section we prove the results about nef divisors on $X$ and $\hat{X}$ that were used earlier. We assume that $n = 6$, i.e. that $X$ is the blowup of $\mathbb{P}^3$ at 6 points.

**Lemma 10.** *$-K_X$ is a nef divisor on $X$.*

PROOF[1]: The anti-canonical divisor on $X$ is given by

$$-K_X = 4H - 2(E_1 + \cdots + E_6)$$

and so $-K_X$ is nef if and only if

$$\begin{aligned} D &= D' + D'' \\ &= (H - E_1 - E_2 - E_3) + (H - E_4 - E_5 - E_6) \end{aligned}$$

is nef. $D'$ and $D''$ are the proper transforms of the planes through $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$ respectively. Thus to see that $D$ is nef, it suffices to check that $D \cdot C \geq 0$ for a curve $C \subset D'$.

The class of $C$ is given by

$$dh - a_1e_1 - a_2e_2 - a_3e_3$$

for some $d$, $a_1$, $a_2$, and $a_3$.

The first Chern class of the normal bundle of $D' \subset X$ is

$$D' \cdot D' = h - e_1 - e_2 - e_3$$

so we get that

$$\begin{aligned} D' \cdot C &= (h - e_1 - e_2 - e_3) \cdot (dh - a_1e_1 - a_2e_2 - a_3e_3) \\ &= d - a_1 - a_2 - a_3 \end{aligned}$$

where the intersection product on the right hand side is in $D'$ which is isomorphic to the blowup of $\mathbb{P}^2$ at three points. The class

$$2h - e_1 - e_2 - e_3$$

is nef in the surface $D'$ since it is represented by an irreducible, effective curve of non-negative square. Intersecting this class with $C$ in $D'$ we conclude that $2d \geq a_1 + a_2 + a_3$.

Computing intersections on $X$ we get:

$$\begin{aligned} D \cdot C &= D' \cdot C + D'' \cdot C \\ &= (d - a_1 - a_2 - a_3) + d \\ &\geq 0 \end{aligned}$$

which proves the lemma.

**Remark 11.** A similar argument can be used to show that the anticanonical divisor is nef for the blowup of $\mathbb{P}^3$ at up to eight points. This is the optimal result since $(-K_X)^3$ is negative for blowups of more than eight points.

**Lemma 12.** *Let $1 \leq j < k \leq 4$ and define $j', k'$ by the condition $\{j, k\} \cup \{j', k'\} = \{1, 2, 3, 4\}$. Then the divisor*

$$\hat{D}_{jk} = 2\hat{H} - (\hat{E}_1 + \cdots + \hat{E}_6) - \hat{F}_{jk} - \hat{F}_{j'k'}$$

*is nef in $\hat{X}$.*

PROOF: Let $\hat{D}'$ and $\hat{D}''$ be the proper transforms of the planes through $\{x_j, x_k, x_5\}$ and $\{x_{j'}, x_{k'}, x_6\}$ respectively. Then

$$\hat{D}' = \hat{H} - \hat{E}_j - \hat{E}_k - \hat{E}_5 - \hat{F}_{jk}$$

$$\hat{D}'' = \hat{H} - \hat{E}_{j'} - \hat{E}_{k'} - \hat{E}_6 - \hat{F}_{j'k'}$$

so $\hat{D}_{jk} = \hat{D}' + \hat{D}''$.

[1]The idea for the proof of this lemma was suggested to us by Sándor Kovács.

To see that $\hat{D}_{jk}$ is nef, it suffices to check that $\hat{D}_{jk} \cdot C \geq 0$ for any curve $C \subset \hat{D}'$.

$\hat{D}'$ is isomorphic to the blowup of $\mathbb{P}^2$ at three points. Under this identification, the classes of the line and the three exceptional divisors are

$$h' = \hat{h} - \hat{f}_{jk}, \quad e'_j = \hat{e}_j - \hat{f}_{jk}, \quad e'_k = \hat{e}_k - \hat{f}_{jk}, \quad e'_5 = \hat{e}_5.$$

The curve $C \subset \hat{D}'$ has class

$$dh' - a_j e'_j - a_k e'_k - a_5 e'_5$$

and since $h' - e'_5$ is a nef divisor in $\hat{D}'$, we have

$$d \geq a_5.$$

The first Chern class of the normal bundle of $\hat{D}' \subset \hat{X}$ is

$$(\hat{H} - \hat{E}_j - \hat{E}_k - \hat{E}_5 - \hat{F}_{jk})^2 = -\hat{e}_5 = -e'_5$$

and so

$$\hat{D}' \cdot C = -e'_5 \cdot (dh' - a_j e'_j - a_k e'_k - a_5 e'_5) = -a_5$$

where the intersection product on the right hand side is on $\hat{D}'$.

Therefore

$$\begin{aligned} \hat{D}_{jk} \cdot C &= \hat{D}' \cdot C + \hat{D}'' \cdot C \\ &= -a_5 + d \\ &\geq 0. \end{aligned}$$

□

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