Counting curves on irrational surfaces

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May 3, 1999

Abstract

In this paper we survey recent results and conjectures concerning enumeration problems on irrational surfaces.

Contents

*Supported by a grant from the Ford Foundation and NSF grant DMS-9802612
†Supported by NSF grant DMS-9803616

1 Introduction

Problems in enumerative algebraic geometry associated to counting curves on projective varieties have been heavily influenced in recent years by the introduction of stable maps, Gromov-Witten invariants, and quantum cohomology. Using these new ideas as well as classical methods, much progress has been made for many enumeration problems on rational surfaces.

In this survey paper, we focus on the situation for irrational surfaces. The first breakthrough in this direction was the work of Yau and Zaslow [91] in 1995 which gave a formula for the number of rational curves on a $K3$ surface. Their formula expresses a generating function for the number of curves as a modular form. Their method is motivated from mirror symmetry considerations and it is strikingly different both the "classical" algebro-geometric approach and the methods using quantum cohomology or Gromov-Witten invariants.

Methods from quantum cohomology and Gromov-Witten theory are not very useful in solving enumeration problems on most irrational surfaces. For instance, a generic $K3$ surface has no curves at all and so its ordinary Gromov-Witten invariants are zero. Nevertheless, some modifications of Gromov-Witten theory can be made and have been recently applied with success in the case of $K3$ and Abelian surfaces.

In [34], Göttsche gave an intriguing generalization of the Yau-Zaslow formula which conjecturally applies to any surface and any genus. The Göttsche-Yau-Zaslow formula has been verified to order eight by the work of Vainsencher [82] and Kleiman-Piene [51] and it has also been proved in the case of $K3$ and Abelian surfaces [15][17]. The conjecture is also consistant with various recursive computations on rational surfaces [18][83]. The Göttsche-Yau-Zaslow formula conjecturally provides a very nice answer to a fairly general set of enumeration problems.

Our survey is organized as follows.

We begin in Section 2 by describing the method of counting rational curves due to Yau and Zaslow which led to their discovery of the presence of modular forms in enumeration problems on surfaces.

In Section 3 we formulate the various kinds of enumeration problems on surfaces and then we focus on the primary problem of interest: counting curves in a linear system passing through a fixed number of points. We describe in detail Göttsche's generalization of the Yau-Zaslow formula.

In Section 4 we start with a short general exposition of the Gromov-Witten invariants including a general enough version to include families of symplectic structures. We then discuss the problems with the Gromov-Witten invariants on surfaces whose geometric genus and/or irregularity is non-zero. We discuss Taubes' "Seiberg-Witten equals Gromov-Witten" theorem and describe its relation to enumerative geometry. Finally, we discuss when the Gromov-Witten invariants are "enumerative".

In Section 5 we give an expository account of our computation of modified Gromov-Witten invariants to prove the Göttsche-Yau-Zaslow formula for $K3$ and Abelian surfaces. We include a description of how to use a "matching technique" to compute the contribution of multiple covers of nodal rational curves to the invariants. We end the section with a brief description of recent work of Behrend and Fantechi who give a purely algebraic modification of the Gromov-Witten invariants that generalize the (non-algebraic) modifications used in the $K3$ and Abelian surface case.

2 Counting curves via the method of Yau and Zaslow

The first big breakthrough for counting curves on irrational surfaces came in the 1995 paper of Yau and Zaslow [91] who discovered the unexpected link between modular forms and enumerating curves. In [91], Yau and Zaslow describe a method to count rational curves on $K3$ surfaces (for an exposition see Beauville [8]). They found that the numbers are given by the coefficients of the series

$$\frac{q}{\Delta(q)} = \prod_{m=1}^{\infty} (1 - q^m)^{-24}. \tag{1}$$

They prove this formula under the assumption that all the rational curves are reduced, irreducible, and nodal. More generally, if the curves are all irreducible and reduced (but possibly having complicated singularities), their argument counts curves with multiplicities that are shown by Fantechi-Göttsche-van Straten [26] to be related to Gromov-Witten multiplicities (see also Chen [19]).

The formula was later generalized by Göttsche [34] to a conjectural formula that applies to all surfaces and any genus (see Section 3).

Although it is not clear if Yau and Zaslow's argument can be generalized to other situations, it is so beautiful and strikingly different that we feel it is worthwhile to describe it here.

Let $\mathbf{P}^r$ be an r-dimensional linear system on a surface X with a finite number of rational curves (for example $\mathbf{P}^r$ could be the sublinear system of a complete linear system $|C|$ obtained by imposing the appropriate number of point conditions). Consider the compactified family of Jacobians:

$$\pi : \mathcal{J} \to \mathbf{P}^r$$

so that $\pi^{-1}(p)$ is $\mathrm{Jac}(C)$ if p is a point representing a smooth divisor C and if C is singular then $\pi^{-1}(p)$ is $\overline{\mathrm{Jac}}(C)$, a compactification of the Jacobian of C (such a family exists by [4][3], c.f. [1][2]). Yau and Zaslow show that if we assume all the rational curves in the linear system are nodal, then

$$\#\text{ of rational curves in the linear system } \mathbf{P}^r = \text{Euler characteristic}(\mathcal{J}).$$

The crucial observation is that the fibers $\overline{\mathrm{Jac}}(C)$ have Euler characteristic zero unless C is rational[1]. We denote the Euler characteristic of a space M by $e(M)$. Recall that the Euler characteristic of a fiber bundle is the product of the Euler characteristics of the fiber and the base and if $X = U \cup U^c$ is a disjoint union of an algebraic space X into a Zariski open set U and its compliment, then $e(X) = e(U) + e(U^c)$. The linear system $\mathbf{P}^r$ has a natural stratification given by the geometric genus of the corresponding divisor. The map $\pi : \mathcal{J} \to \mathbf{P}^r$ is a fiber bundle restricted to each strata so that $e(\mathcal{J})$ is given as the sum over all the strata of the product of the Euler characteristic of the fiber times the Euler characteristic of the corresponding strata.

[1]This is because $\overline{\mathrm{Jac}}(C)$ fibers over $\mathrm{Jac}(\tilde{C})$, the Jacobian of the normalization and $e(\mathrm{Jac}(\tilde{C})) = 0$ if $g(\tilde{C}) \neq 0$, *i.e.* C is not rational (see section 2 in Beauville [8]).

We see that only the strata corresponding to rational curves contribute to the Euler characteristic. Consequently, we get

$$e(\mathcal{J}) = \sum_{\text{rational curves } C \text{ in } \mathbf{P}^r} e(\overline{\operatorname{Jac}}(C)).$$

If C is a nodal rational curve, then $e(\overline{\operatorname{Jac}}(C)) = 1$ and so $e(\mathcal{J})$ is exactly the number of rational curves. In general, $e(\mathcal{J})$ counts the rational curves with multiplicities given by $e(\overline{\operatorname{Jac}}(C))$. Fantechi-Göttsche-van Straten [26] show that if C is irreducible, then $e(\overline{\operatorname{Jac}}(C))$ coincides with the length of the (zero dimensional) moduli space of genus 0 stable maps with image C and so the Yau-Zaslow multiplicities agree with the multiplicities arising in Gromov-Witten theory (in the language of Subsection 4.4 we can thus say $e(\mathcal{J})$ is *weakly enumerative*).

In general, there is no easy way to compute $e(\mathcal{J})$, but for $K3$ surfaces we can utilize some very special properties of $\mathcal{J}$. A r-dimensional complete linear system $|C|$ on a $K3$ surface X has a finite number of rational curves. If we assume that the linear system consists of only reduced and irreducible curves (for example, if $\mathcal{O}(C)$ generates $\operatorname{Pic}(X)$), then the associated family of compactified Jacobians $\mathcal{J} \to |C|$ is a *smooth hyperkähler* manifold (see Mukai [68] ex. 0.5). Furthermore, it is birational[2] to $X^{[r]}$, The Hilbert scheme of r points on X. $X^{[r]}$ is also a *smooth hyperkähler* manifold and by a result of Batyrev [7], the Betti numbers (and hence the Euler characteristics) of two birationally equivalent, smooth hyperkähler manifolds agree.

The Euler characteristic of the Hilbert scheme $X^{[r]}$ was determined in [35] by Göttsche using Deligne's proof of the Weil conjectures. Equation 1 then follows from his calculation.

3 Problems and conjectures

In this section we describe the general set up for enumeration problems on surfaces and we explain the Göttsche-Yau-Zaslow formula which conjecturally gives the answer to a very general set of enumerative problems. The

[2]The birational morphism between $\mathcal{J}$ and $X^{[r]}$ can be seen as follows. The generic point of $\mathcal{J}$ corresponds to a smooth genus r curve $C \subset X$ and an element of $\operatorname{Jac}(C)$. Since $\operatorname{Jac}(C)$ is generically isomorphic to $\operatorname{Sym}^r(C)$ the generic point in $\mathcal{J}$ gives us r (unordered) points on C and hence r points in X. Conversely, r generic points in X are contained in a unique smooth genus r curve C in $|C|$; the points on C then further determine an element of $\operatorname{Jac}(C)$ via the map $\operatorname{Sym}^r(C) \to \operatorname{Jac}(C)$.

discussion of this section for the most part applies equally well to rational surfaces as well as irrational.

3.1 Formulation of the problem

Since the only interesting subvarieties of a fixed algebraic surface X are curves, the general enumeration problem for X is to count the number of curves on X satisfying some set of prescribed properties. It is natural to begin by fixing the geometric genus g of the curves to be counted and to fix the homology class $[C] \in H_2(X, \mathbf{Z})$ of their image[3]. The set of curves with fixed geometric genus and homology class will in general form a positive dimensional family and so to get a well defined counting problem one imposes additional conditions. Some typical conditions on a curve C are given below:

Point: Require C to pass through a prescribed set of fixed points. The condition that a curve pass through a single fixed point is a codimension one condition on the family of all curves.

FLS: Require C to lie in a fixed linear system (FLS), by considering C as a divisor. Equivalently, fix the holomorphic structure on the line bundle $\mathcal{O}(C)$. For irregular surfaces ($H^1(X, \mathcal{O}) \neq 0$), this imposes a non-trivial constraint of codimension equal to $\dim H^1(X, \mathcal{O})$.

Loop: Require C to pass through fixed loops in X representing non-trivial elements in $H^1(X; \mathbf{Z})$ (this also is a non-trivial constraint only on irregular surfaces). Each loop imposes a *real* codimension one condition on the family of curves. In Section 4 we will show that these loop constraints are directly related to the FLS constraint (see Theorem 4.1).

Multi-point: Require C to pass through a fixed set of points with a prescribed set of multiplicities. This can be reformulated as a homological condition on the blown-up manifold; *i.e.* curves on X in the class $[C]$ passing through fixed points $x_1, \ldots, x_l$ with prescribed multiplicities α_i are in one-to-one correspondence with curves on $Bl_{\{x_1,\ldots,x_l\}}X$ in the class

[3]Notation: if C is a curve in X, then we denote its homology class in $H_2(X, \mathbf{Z})$ by $[C]$, its Poincaré dual in $H^2(X, \mathbf{Z})$ by $[C]^\vee$, the corresponding line bundle $\mathcal{O}(C)$, and the linear system $|C| := \mathbf{P}(H^0(X, \mathcal{O}(C)))$. We will also use the shorthand $C^2 := [C] \cdot [C]$ and $KC := c_1(TX)([C])$. In subscripts for moduli spaces and invariants we will just write C for the homology class, *e.g.* $\mathcal{M}_{g,C}(X) := \mathcal{M}_{g,[C]}(X)$.

$[C] - \sum_{i=1}^{l} \alpha_i [E_i]$ where $Bl_{\{x_1,\dots,x_l\}}(X)$ is the blow-up of X at the points $x_1, \dots, x_l$ and $E_1, \dots, E_l$ are the the exceptional divisors.

Tangency: Fix an auxiliary smooth curve D and require that C meet D with a prescribed degree of tangency. This can be reformulated in terms of curves on a certain family of blow-ups (see [16]). See also [24][25] where the problem was considered on $\mathbf{P}^2$ and attacked using certain stable maps to the incidence variety.

CX structure: One can also impose conditions on the complex structure of C itself. For example, one can count only curves with a fixed complex structure; for $g \geq 2$, this imposes $3g - 3$ constraints (for example see [42],[69]). For another example, one can count only those curves that are hyper-elliptic. The expected number of constraints this imposes is $g - 2$ since the codimension of the hyper-elliptic locus in the moduli space of curves is $g - 2$ (for example see [37]).

For the most part, we will focus on the most straight forward problem: counting genus g curves in a fixed linear system passing through a fixed set of points (with multiplicity one), although we will comment on the other kinds of constraints as they come up. We formalize the problem below:

Let C be a curve on a surface X and let $|C|$ denote the corresponding complete linear system and let K denote the canonical class. We assume that $\mathcal{O}(C - K)$ is ample so that the dimension of $|C|$ is determined by the Riemann-Roch formula.

Define the Severi variety $V_g(C)$ to be the closure of the set of curves in $|C|$ with geometric genus g. The condition that a divisor passes through a fixed point imposes a linear condition on $|C|$. We can thus interpret the degree of $V_g(C) \subset |C|$ as the number of genus g curves in $|C|$ passing through r points where r is the expected dimension of $V_g(C)$. The expected dimension of $V_g(C)$ is given by

$$\begin{aligned} \dim V_g(C) &= r \\ &= -KC + g - 1 + p_g - q \\ &= -KC + g - 2 + \chi(\mathcal{O}_X) \end{aligned}$$

where $p_g = \dim H^2(X, \mathcal{O})$, $q = \dim H^1(X, \mathcal{O})$, and $\chi(\mathcal{O}_X) = 1 - q + p_g$ is the holomorphic Euler characteristic of $\mathcal{O}_X$.

Main Enumeration Problem: The main problem we consider is computing $N_g(X,C)$ which is defined to be the number (when finite) of genus g curves in the linear system $|C|$ passing through $r = -KC + g - 1 + p_g - q$ generic points. Equivalently, $N_g(X,C)$ is the degree of $V_g(C)$ (when $V_g(C)$ is of the expected dimension).

3.2 The Göttsche-Yau-Zaslow formula

A priori, $N_g(X,C)$ depends heavily on the complex structure of X and the choice of the linear system $|C|$. Under sufficient ampleness conditions on C it is conjectured (essentially in Vainsencher [82]) that $N_g(X,C)$ only depends on the numbers g, C^2, CK, K^2, and $c_2(X)$, where $c_2(X)$ is the Euler characteristic of X.

Göttsche has found a remarkable generalization of the Yau-Zaslow formula. The formula is a generating function for the numbers $N_g(X,C)$ in terms of five universal power series and the numbers g, C^2, CK, K^2, and $c_2(X)$. Three of the five universal power series are describes explicitly as (quasi-)modular forms while the remaining two series have coefficients that can be determined recursively.

Conjecture 1 ([34]) *Let C be a sufficiently ample*[4] *divisor on X. Then $N_g(X,C)$, the number of genus g curves in $|C|$ passing through*

$$r = -KC + g - 2 + \chi(\mathcal{O}_X)$$

points, is given as the coefficient of $q^{\frac{1}{2}C(C-K)}$ in the following power series in q:

$$B_1^{K^2} B_2^{CK} (DG_2)^r \frac{D^2 G_2}{(\Delta(D^2G_2))^{\chi(\mathcal{O}_X)/2}} \tag{2}$$

where $D = q\frac{d}{dq}$, G_2 is the Eisenstein series:

$$G_2(q) = -\frac{1}{24} + \sum_{k>0}(\sum_{d|k} d)q^k,$$

[4]For the precise formulation of the ampleness condition see [34]. Depending on X, the conjecture is expected to hold under weaker assumptions on C.

Δ *is the discriminant:*

$$\Delta(q) = q\prod_{k>0}(1-q^k)^{24},$$

and $B_i(q)$ *are universal power series whose first terms are*

$$\begin{aligned} B_1(q) &= 1 - q - 5q^2 + 39q^3 - 345q^4 + \cdots \\ B_2(q) &= 1 + 5q + 2q^2 + 35q^3 - 140q^4 + \cdots \end{aligned}$$

(see [34] for the coefficients of B_i *to order 20).*

Note that when K is numerically trivial (X is a $K3$, Abelian, Enriques, or hyper-elliptic surface), then the power series (2) does not depend on B_1 or B_2 and so is given by an explicit (quasi-)modular form. In the case when X is a $K3$ or Abelian surface, the conjecture was proved in [15] and [17] to hold for *all* C representing a primitive homology class (using a slightly modified definition of $N_g(X,C)$, see section 5).

The impetus for Göttsche's generalization stemmed largely from two sources: the work of Vainsencher [82] (and its subsequent generalization by Kleiman and Piene [51]), and the formula of Yau and Zaslow [91].

In [82], Vainsencher gives universal polynomials that count the number of curves with 6 or fewer nodes, passing through the appropriate number of points, in a sufficiently ample linear system on *any* surface. In other words, he computes $N_g(X,C)$ for any g, X, and C provided that C is sufficiently ample and

$$\frac{1}{2}C(C+K) - g \leq 5.$$

Kleiman and Piene have refine the methods of Vainsencher to extend his results up to eight nodes. They also provide explicit bounds for the power of the ample class required to guarantee that the formulas hold. The methods used to obtain these results are classical in the sense that they do not use physics, quantum cohomology, or even stable maps. They also provide precise enumerative information as oppose to the "virtual" or "weakly enumerative" count sometimes determined by the Gromov-Witten invariants (c.f. subsection 4.4), in particular, they show that when the ampleness conditions are satisfied, all the curves passing through a generic choice of points are nodal. We refer the reader to [82] and [51] for more details.

Göttsche's crucial observation is that if there exist universal polynomial formulas for $N_g(X,C)$ that apply to any surface, then they must satisfy very strong multiplicative properties. The reason is that they should also apply to disconnected surfaces, and Göttsche shows that the obvious relationship

$$N_g(X_1 \coprod X_2, C_1 \coprod C_2) = \sum_{g_1+g_2=g} N_{g_1}(X_1, C_1) N_{g_2}(X_2, C_2)$$

forces the formulas to take on a very special form. In particular, the polynomials for $N_g(X, C)$ must be determined by five universal power series.

On the other hand, it was the work of Yau and Zaslow that suggested that the universal power series may be related to modular forms. Using ideas from physics, Yau and Zaslow gave predictions for the number of rational curves with n nodes on a $K3$ surface (see Section 2). The numbers appear as the coefficients of the Fourier expansion of a well known modular form, the discriminant. This work is in a sense complimentary to the Vainsencher work; while Vainsencher's formulas apply to any surface but for only a small number of nodes, the Yau-Zaslow formula applies to only genus 0 and a $K3$ surface, but to any number of nodes. Thus, while Vainsencher's formulas determine the first few coefficients of each of the five power series, the Yau-Zaslow formula provides a closed form for a certain product of three of the power series.

By building on this knowledge along with other known results (particularly the recursive formulas of Caporaso-Harris [18] and Vakil [83]) and using some remarkable pattern recognition, Göttsche arrived at Conjecture 1 and verified it to a fairly high degree of redundancy. Closed formulas for the series B_1 and B_2 are unknown, but a recursive scheme for the coefficients can be derived from the Caporaso-Harris or Vakil formulas.

One intriguing aspect of the conjecture is the appearance of modular forms. The underlying "reason" for the modularity is currently a mystery.

4 Gromov-Witten invariants

In this section we begin with a short general exposition of the Gromov-Witten invariants including a general enough version to include families of symplectic structures. We then discuss the problems with the Gromov-Witten invariants on surfaces whose geometric genus and/or irregularity is non-zero. We discuss Taubes' "Seiberg-Witten equals Gromov-Witten" theorem and describe its relation to enumerative geometry. Finally, we discuss when the Gromov-Witten invariants are "enumerative".

Many papers have been written on Gromov-Witten theory, for the reader's convenience we give an extensive list in the bibliography: [63] [61] [62] [43] [12] [6] [73] [85] [69] [50] [52] [87] [67] [40] [30] [11] [65] [27] [89] [49] [21] [9] [33] [66] [39] [86] [14] [57] [48] [46] [70] [54] [32] [29] [74] [23] [76] [58] [56] [38] [31] [75] [20] [13] [47] [45] [41] [53] [64] [72] [71] [55] .

4.1 Gromov-Witten invariants

Gromov-Witten invariants have their origins in symplectic geometry and conformal field theory but have been recently defined purely algebro-geometrically [11][62]. The basic object of study is the moduli space $\mathcal{M}_{g,n,C}(X)$ of *stable maps* of n-marked, genus g curves to X in the class C. In general, Gromov-Witten invariants are certain intersection numbers of cycles on $\mathcal{M}_{g,n,C}(X)$ which are shown to be invariant under deformations of the (almost) Kähler structure of X.

We briefly outline the framework of the Gromov-Witten invariants in order to fix notations and we refer the reader to (for example) [5] or [28] for complete accounts. We include here a straight forward generalization of the usual framework to include families of (almost) Kähler structures (see [15][17]).

Let (X,ω) be any compact (almost) Kähler manifold. Recall that an n-marked, genus g *stable map* of degree $[C] \in H_2(X,\mathbf{Z})$ is a (pseudo)-holomorphic map $f:\Sigma\to X$ from an n-marked nodal curve $(\Sigma,x_1,\dots,x_n)$ of geometric genus g to X with $f_*([\Sigma])=[C]$ that has no infinitesimal automorphisms. Two stable maps $f:\Sigma\to X$ and $f':\Sigma'\to X$ are equivalent if there is a biholomorphism $h:\Sigma\to\Sigma'$ such that $f=f'\circ h$. We write $\mathcal{M}_{g,n,C}(X,\omega)$ for the moduli space of equivalence classes of genus g, n-marked, stable maps of degree $[C]$ to X. We will often drop the ω or X from the notation if they are understood and we sometimes will drop the n from the notation when it is 0. If B is a family of almost Kähler structures, we denote parameterized version of the moduli space:

$$\mathcal{M}_{g,n,C}(X,B)=\coprod_{t\in B}\mathcal{M}_{g,n,C}(X,\omega_t).$$

If B is a compact, connected, oriented manifold then $\mathcal{M}_{g,n,C}(X,B)$ has a fiduciary cycle $[\mathcal{M}_{g,n,C}(X,B)]^{vir}$ called the virtual fundamental cycle (see [15] and the fundamental papers of Li and Tian [62][59][60], or alternatively

Behrend-Fantechi and Siebert [11][77]). The dimension of the cycle is

$$\dim_{\mathbf{R}}[\mathcal{M}_{g,n,C}(X,B)]^{vir} = -2KC + (6 - \dim_{\mathbf{R}} X)(g-1) + 2n + \dim_{\mathbf{R}} B.$$

The invariants are defined by evaluating cohomology classes of $\mathcal{M}_{g,n,C}$ on the virtual fundamental cycle. The cohomology classes are defined via incidence relations of the maps with cycles in X. The framework is as follows. There are maps

$$\begin{array}{ccc} \mathcal{M}_{g,1,C} & \xrightarrow{ev} & X \\ \downarrow{\scriptstyle ft} & & \\ \mathcal{M}_{g,C} & & \end{array}$$

called the *evaluation* and *forgetful* maps defined by $ev(\{f:(\Sigma,x_1)\to X\}) = f(x_1)$ and $ft(\{f:(\Sigma,x_1)\to X\}) = \{f:\Sigma\to X\}$.[5] The diagram should be regarded as the universal map over $\mathcal{M}_{g,C}$.

Given geometric cycles $\alpha_1,\dots,\alpha_l$ in X representing classes $[\alpha_1],\dots,[\alpha_l]\in H_*(X,\mathbf{Z})$ with Poincaré duals $[\alpha_1]^\vee,\dots,[\alpha_l]^\vee$, we can define the Gromov-Witten invariant

$$\Phi^{(X,B)}_{g,C}(\alpha_1,\dots,\alpha_l) = \int_{[\mathcal{M}_{g,C}(X,B)]^{vir}} ft_*ev^*([\alpha_1]^\vee)\cup\dots\cup ft_*ev^*([\alpha_l]^\vee). \quad (3)$$

$\Phi^{(X,B)}_{g,C}(\alpha_1,\dots,\alpha_l)$ counts the number of genus g, degree $[C]$ maps which are pseudo-holomorphic with respect to some almost Kähler structure in B and such that the image of the map intersects each of the cycles $\alpha_1,\dots,\alpha_l$.[6] The Gromov-Witten invariants are multi-linear in the α's and they are symmetric for α's of even degree and skew symmetric for α's of odd degree. If $p_1,\dots,p_k$ are points in a path-connected X, we will use the shorthand

$$\Phi^{(X,B)}_{g,C}(\mathrm{pt.}^k,\alpha_{k+1},\dots,\alpha_l) := \Phi^{(X,B)}_{g,C}(p_1,\dots,p_k,\alpha_{k+1},\dots,\alpha_l).$$

Now suppose that X is a Kähler surface. The only non-trivial constraints arising from intersecting with cycles are those coming from zero and one dimensional cycles, *i.e.* points and loops. The constraints imposed by intersecting two dimensional cycles (divisors) are determined purely homologically (the so-called "divisor equation") and cycles of dimension three and four impose no constraints.

[5] There is some subtlety to making this definition rigorous since forgetting the point may make a stable map unstable, but it can be done.

[6] The integral is defined to be 0 if the integrand is not a form of the correct degree.

The **loop** constraints (C passes through a fixed set of loops) are related to the **FLS** constraint (fixing the linear system of C) by the following:

Theorem 4.1 (Thm. 2.1 of [17]) *Let $\gamma_1, \dots, \gamma_{b_1}$ be loops representing an oriented basis of $H_1(X, \mathbf{Z})$. Then the invariant:*

$$\Phi_{g,C}(\gamma_1, \dots, \gamma_{b_1}, pt.^l)$$

counts the number of genus g maps whose image lies in the fixed linear system $|C|$ and passes through l points.

We can formulate a more precise version of this. In order to count curves in a fixed linear system $|C|$ one would like to restrict the integral of Equation 3 to the cycle defined by $\Psi_{\Sigma_0}^{-1}(0)$ where Ψ_{Σ_0} is the map

$$\Psi_{\Sigma_0} : \mathcal{M}_{g,C}(X, \omega) \to \mathrm{Pic}^0(X)$$

given by

$$f \mapsto \mathcal{O}(\mathrm{Im}(f) - \Sigma_0)$$

where $\Sigma_0 \in |C|$ is a fixed divisor. Dually, one can add the pullback by Ψ_{Σ_0} of the volume form on $\mathrm{Pic}^0(X)$ to the integrand defining the invariant:

$$\int_{[\mathcal{M}_{g,C}(X)]^{vir}} \Psi_{\Sigma_0}^*([\mathrm{pt.}]^\vee) \cup ft_* ev^*([\alpha_1]^\vee) \cup \cdots \cup ft_* ev^*([\alpha_l]^\vee).$$

The class $\Psi_{\Sigma_0}^*([\mathrm{pt.}]^\vee)$ can be expressed in terms of classes arising from the constraints imposed by loops:

Theorem 4.2 *Let X be a Kähler surface and let $[\gamma] \in H_1(X, \mathbf{Z})$ and let $\tilde{\gamma}$ be the corresponding class in $H^1(\mathrm{Pic}^0(X), \mathbf{Z})$ induced by the identification $\mathrm{Pic}^0(X) \cong H^1(X, \mathbf{R})/H^1(X, \mathbf{Z})$. Then*

$$\Psi_{\Sigma_0}^*(\tilde{\gamma}) = ft_* ev^*([\gamma]^\vee).$$

Corollary 4.3 $\Psi_{\Sigma_0}^*([pt.]^\vee) = ft_* ev^*([\gamma_1]^\vee) \cup \cdots \cup ft_* ev^*([\gamma_{b_1}]^\vee).$

Proof: Theorem 4.2 is proved in the appendix of [17]. Roughly the idea is this: We view classes in $H^1(\mathcal{M}_{g,C}(X); \mathbf{Z})$ as homotopy classes of circle valued functions on $\mathcal{M}_{g,C}(X)$. The values of a certain circle valued function on $\mathcal{M}_{g,C}(X)$ representing the class $ft_* ev^*([\gamma]^\vee)$ are given as integrals of a form representing $[\gamma]^\vee$ over 3-cycles that are obtained by sweeping out a path of curves in X. On the other hand, it is shown that the values of a circle valued function representing $\Phi^*_{\Sigma_0}(\tilde{\gamma})$ are given by an integral over X of a certain 1-form wedged with $[\gamma]^\vee$. The equality of the two circle valued functions is then established with essentially a residue calculation.

The corollary follows immediately and is essentially a restatement of Theorem 4.1.

4.2 Difficulties with ordinary Gromov-Witten invariants

Gromov-Witten invariants have been remarkably effective in answering many questions in enumerative geometry for rational surfaces (see for example [84][83][36][56][24][37] and many others). Rational surfaces all have $p_g = q = 0$; however, the ordinary Gromov-Witten invariants are not very effective for counting curves when p_g or q are not zero[7]. One basic reason is that the moduli space of stable maps fails to be a good model for a linear system (and the corresponding Severi varieties) for dimensional reasons: For an effective divisor C such that $C - K$ is ample, the dimension of the Severi variety $V_g(C)$ (the closure of the set of geometric genus g curves in the complete linear system $|C|$) is

$$\dim_{\mathbf{C}} V_g(C) = -KC + g - 1 + p_g - q.$$

On the other hand, the virtual dimension of the moduli space $\mathcal{M}_{g,C}(X)$ of stable maps of genus g in the class $[C]$ is

$$\operatorname{virdim}_{\mathbf{C}} \mathcal{M}_{g,C}(X) = -KC + g - 1.$$

If the virtual dimension of the stable maps doesn't match the number of constraints required for $N_g(X, C)$, the corresponding Gromov-Witten invariant *must be zero.*

The discrepancy $p_g - q$ arises from two sources. Since the image of maps in $\mathcal{M}_{g,C}(X)$ are divisors not only in $|C|$ but also potentially in every linear

[7]Not much is known about the enumerative geometry of irrational surfaces with $p_g = q = 0$ even with quantum cohomology and the usual Gromov-Witten invariants at our disposal. However, it may be possible to adapt the methods of Subsections 5.1 and 5.2 to the case of the Enriques surface (which is irrational and has $p_g = q = 0$).

system in $\mathrm{Pic}^{[C]}(X)$, one would expect $\dim \mathcal{M}_{g,C}(X)$ to exceed $\dim V_g(C)$ by $q = \dim \mathrm{Pic}^{[C]}(X)$.[8] As we discussed in the previous section, this discrepancy can be accounted for within the framework of the usual Gromov-Witten invariants using loop constraints (see Theorem 4.1).

However, even if we consider $\mathcal{M}_{g,C}$ as a model for the parameterized Severi varieties

$$V_g([C]) \equiv \coprod_{\mathcal{O}(C')\in \mathrm{Pic}^{[C]}(X)} V_g(C'),$$

there is still a p_g dimensional discrepancy (Donaldson also discusses this in detail [22]).

The reason is the following. The virtual dimension of $\mathcal{M}_{g,C}(X)$ is the dimension of the space of curves that persist as *pseudo-holomorphic* curves when we perturb the Kähler structure to a generic almost Kähler structure. The difference of p_g in the dimensions of $\mathcal{M}_{g,C}$ and $V_g([C])$ means that only a codimension p_g subspace of $V_g([C])$ persists as pseudo-holomorphic curves when we perturb the Kähler structure. Algebraically, this arises as the obstruction to those infinitesimal deformations of the map that deform the image of the map in the $H^{0,2}(X)$ direction (see Subsection 5.3). One way to rectify this situation is to find a compact p_g-dimensional[9] *family* of almost Kähler structures that has the property that the only almost Kähler structure in the family that supports pseudo-holomorphic curves in the class $[C]$ is the original Kähler structure. If T is such a family, then the moduli space $\mathcal{M}_{g,C}(X,T)$ of stable maps for the family T is a better model for the space $V_g([C])$ in the sense that its dimension is stable under generic perturbations of the family $T \mapsto T'$.

Given the existence of a p_g-dimensional family as described above, these invariants can be used to answer enumerative geometry questions for the corresponding surface and linear system. In general, it is not clear when such a family will exist; however, if X has a hyperkähler metric g (*i.e.* X is an Abelian or $K3$ surface), then there is a natural candidate for T, namely the hyperkähler family of Kähler structures. We call this family the *twistor family* associated to the metric g and we denote it T_g. It is parameterized by a 2-sphere and so $\dim_{\mathbf{R}} T_g = 2 = 2p_g$ as it should. Furthermore, the property

[8] we use $\mathrm{Pic}^{[C]}(X)$ to denote the component of $\mathrm{Pic}(X)$ corresponding to line bundles with first Chern class $[C]^\vee$.

[9] By this we really mean a real $2p_g$ dimensional family; the parameter space for the family need not have a complex structure or even an almost complex structure.

that all the curves in $\mathcal{M}_{g,C}(X,T_g)$ are holomorphic for the original complex structure can be proved with Hodge theory (of course this need no longer be the case for a perturbation of T_g to a generic family of almost Kähler structure.

We will discuss this case in further detail in Section 5 and in Subsection 5.3 we discuss recent work of Behrend and Fantechi that in many cases fixes the "p_g discrepancy" purely algebraically.

4.3 Relationship between the Seiberg-Witten and Gromov-Witten invariants

In a series of papers [78][79][80][81] in 1994 and 1995, Taubes proved that the Seiberg-Witten invariants of a symplectic 4-manifold are given by a certain set of Gromov-Witten invariants. This work has had a profound impact on symplectic and smooth topology and has found numerous applications; however, from the point of view of enumerative algebraic geometry, it gives us no information. The Seiberg-Witten invariants are topological invariants only when $b_2^+(X) > 1$; for a Kähler surface this is equivalent to $p_g(X) > 0$. As we explained in the last section, the Gromov-Witten invariants are *zero* in any class $[C]$ with $\mathcal{O}(C-K)$ ample, so there is only a small range of possible classes with non-zero invariants. In fact, in the Kähler case Taubes' theorem reduced to a fact already explained in Witten's original paper [88]: the Seiberg-Witten invariants only "count" connected components of the canonical divisor. That is, the only spin^c structures with non-zero solutions to the Seiberg-Witten equations correspond to the classes in $H_2(X,\mathbf{Z})$ that are the various sums of the components of the canonical divisor[10]. In particular, on a minimal surface of general type, the Seiberg-Witten and Gromov-Witten invariants are non-trivial only for the canonical class (and the zero class).

Taubes shows in general that the Seiberg-Witten invariants count pseudo-holomorphic curves that are always *smoothly embedded* (though possibly dis-

[10] On a symplectic manifold there are two different ways to identify spin^c structures with elements of $H_2(X,\mathbf{Z})$. One can always take the dual to the first Chern class of the bundle of positive spinors, *i.e.* $c_1(W^+)^\vee \in H_2(X,\mathbf{Z})$; alternatively, on a symplectic 4-manifold every spin^c structure can be obtained by twisting the canonical spin^c structure W_0^+ by a line bundle L and so we can consider the class $c_1(L)^\vee \in H_2(X,\mathbf{Z})$. The latter correspondence is more natural from the point of view of Gromov-Witten theory since on a Kähler surface, a smooth curve C corresponds to a solution of the Seiberg-Witten equations for the spin^c structure $\mathcal{O}(C)\otimes W_0^+$.

connected). From the point of view of enumerative geometry, smoothly embedded curves are uninteresting (the corresponding Severi variety is a linear system and so $N_g(X, C) = 1$, *i.e.* there is exactly one smooth curve in $|C|$ passing through the appropriate number of points).

It is worth noting here that although the enumerative information in the Seiberg-Witten/Gromov-Witten invariants is trivial, the actual Seiberg-Witten/Gromov-Witten multiplicities can be somewhat subtle. Non-trivial multiplicities for curves with a smooth image occur when the curve is a multiple of a square-zero curve of genus one. Multiplicities for the ordinary Gromov-Witten invariants arise because of the possibility of many different maps multiply covering the same image. Taubes' defines his own version of Gromov-Witten invariants which count embedded curves (rather than maps) and allow for the possibility of many components. His multiplicities are defined using the spectral flow of a certain operator that arises naturally in the Seiberg-Witten context. The exact relationship between the two definitions of Gromov-Witten invariants was clarified by Ionel and Parker [43] who showed that the invariants contain equivalent information.

4.4 When are Gromov-Witten invariants "enumerative"?

By counting maps to X rather than subvarieties of X, the Gromov-Witten invariants acquire many advantageous properties such as deformation invariance and numerous relations. The disadvantage to enumerative applications is that the maps may contract or multiply-cover some of their components. Thus Gromov-Witten invariants may count genus g maps whose image does not have geometric genus g. Furthermore, a given curve in X may be the image of many different maps, possibly even a family of maps. Thus the Gromov-Witten invariants may count a given isolated curve with a non-trivial multiplicity that may be negative and/or non-integral.

For these reasons, the Gromov-Witten invariants are said to give a *virtual* count of curves. When the count defined by a Gromov-Witten invariant coincides with the actual number of curves, the invariant is said to be *enumerative*. Here we introduce an intermediate notion:

Definition 4.4 *A Gromov-Witten invariant is said to be* weakly enumerative *if it counts only curves with geometric genus g each with positive, integral multiplicity that is one for curves with (at worst) nodal singularities.*

This notion is particularly applicable to surfaces where one can often rule out maps that collapse or multiply-cover components by dimensional arguments. The basic reason for this is the dependence of the dimension of the space of stable maps on $\dim X$ and g. When X is complex dimension 2 or less, the dimension of the space of stable maps grows linearly with g.

For surfaces, a Gromov-Witten invariant will be weakly invariant if all the maps counted by the invariant are birational isomorphisms onto their image. For example, Göttsche and Pandharipande [36] show that all the genus 0 Gromov-Witten invariants of $\mathbf{P}^2$ blown-up at n generic points are enumerative for $n \leq 10$ but their arguments also imply that the invariants are weakly enumerative for *all* n. If the location of the blow-up points are not generic, then the invariants may fail to even be weakly enumerative (c.f. Subsection 5.2). Other examples include the modified Gromov-Witten invariants for $K3$ and Abelian surfaces discussed in Section 5 which are weakly enumerative for generic choices of the $K3$ or Abelian surface.

If a genus 0 Gromov-Witten invariant on a surface is weakly enumerative, then the work of Fantechi-Göttsche-Van Staten [26] shows that the multiplicities of the irreducible curves are determined solely by the type and number of singularities (see also Section 2).

5 Modified invariants and the case of $K3$ and Abelian surfaces

In this section we give an expository account of our use of modified Gromov-Witten invariants to prove the Göttsche-Yau-Zaslow formula for $K3$ and Abelian surfaces. We include a description of how to use a "matching technique" to compute the contribution of multiple covers of nodal rational curves to the invariants. We end the section with a brief description of recent work of Behrend and Fantechi who give a purely algebraic modification of the Gromov-Witten invariants that generalizes the (non-algebraic) modifications used in the case of $K3$ and Abelian surfaces.

5.1 The $K3$ and Abelian surface case

In this section we explain the proof of the Göttsche-Yau-Zaslow formula for primitive classes in $K3$ and Abelian surfaces [15][17]. Let X be a $K3$ or

Abelian surface and let $C \subset X$ be a curve representing a primitive homology class. To verify the conjecture we need to show first that the numbers $N_g(X, C)$ only depend on g and $[C]^2$ (and whether X is a $K3$ or Abelian surface) and then show the numbers are given as the coefficients of the predicted modular forms.

Since $p_g(X) = 1$, there are difficulties with the ordinary Gromov-Witten invariants (see Subsection 4.2). In fact, it is easy to see the following:

Lemma 5.1 *All the (ordinary) Gromov-Witten invariants of X are zero.*

PROOF: Gromov-Witten invariants are invariant under deformations of the (almost) Kähler structure. Any $K3$ or Abelian surface can be deformed to a (non-algebraic) Kähler surface that has *no* holomorphic curves at all. *q.e.d.*

However, this problem can be rectified and a proof of the Göttsche-Yau-Zaslow formula can be obtained. The following two theorems are the main theorems:

Theorem 5.2 *There exists a family Gromov-Witten invariant that computes $N_g(X, C)$. Furthermore, the invariant only depends on C^2 and g (and possibly the divisibility of $[C]$). The invariant is weakly enumerative for generic X.*[11]

The following theorem computes the invariant of Theorem 5.2 thus verifying the Göttsche-Yau-Zaslow formula:

Theorem 5.3 *The Göttsche-Yau-Zaslow formula holds for all $C \subset X$ representing a primitive homology class. That is, if X is a $K3$ surface then $N_g(X, C)$ is the coefficient of $q^{\frac{1}{2}C^2}$ in the series*

$$\begin{aligned} & q^{-1} \prod_{m=1}^{\infty} (1 - q^m)^{-24} \left(\sum_{k=1}^{\infty} k\sigma(k) q^k \right)^g \\ = \;& \Delta^{-1} (DG_2)^g \end{aligned}$$

[11]By generic in this setting we mean that X is generic among those $K3$ surfaces that admit a curve in the class $[C]$. See subsection 4.4 for the definition of weakly enumerative.

and if X is an Abelian surface then $N_g(X,C)$ is the coefficient of $q^{\frac{1}{2}C^2}$ in the series

$$\begin{aligned} &\left(\sum_{k=1}^{\infty} k^2\sigma(k)q^k\right)\left(\sum_{k=1}^{\infty} k\sigma(k)q^k\right)^{g-2} \\ = \;& D^2G_2(DG_2)^{g-2} \end{aligned}$$

where $\sigma(k)=\sum_{d|k} d$ is the sum of the divisors of k.

Remark 5.4 There is also a formula for the number of genus g curves on an Abelian surface passing through g points (without imposing the **FLS** condition). The numbers are the coefficients of the series $g(DG_2)^{g-1}$ (see [17]).

We first outline the proof of Theorem 5.2. The family of Kähler structures used in Theorem 5.2 is provided by the existence of hyperkähler metrics on X. Since $c_1(TX)=0$, Yau's proof of the Calabi conjecture [90] provides a Ricci flat Kähler Einstein metric, which for surfaces is a hyperkähler metric.

A hyperkähler metric is characterized by a 2-sphere's worth of Kähler structures $a\omega_I + b\omega_J + c\omega_K$ where

$$S^2 = \{(a,b,c)\in \mathbf{R}^3 : a^2+b^2+c^2=1\}$$

and the associated complex structures I, J, and K satisfy the algebra of the imaginary quaternions. We call this family the twistor family associated to a hyperkähler metric. The following theorem lists the key properties of the twistor family which allow us to use it with the family version of Gromov-Witten invariants to compute $N_g(X,C)$.

Theorem 5.5 *Denote the twistor family associated to a hyperkähler metric h by T_h. Then,*

1. *For any two hyperkähler metrics h and h', T_h is deformation equivalent to $T_{h'}$. We denote the deformation equivalence class by simply T.*

2. *For any orientation preserving diffeomorphism f the family $f^*(T)$ is deformation equivalent to T.*

3. *For any class $[C]\in H_2(X;\mathbf{Z})$ with $[C]^2\geq -2$, there is exactly one member of the family T_h which admits holomorphic curves in the class $[C]$.*

If $C \subset X$ is a curve on a $K3$ surface, choose a hyperkähler metric h on X and consider the invariant $\Phi_{g,C}^{(X,T_h)}(\mathrm{pt.}^g)$. Property 1 of the previous theorem shows that this invariant is independent of the chosen hyperkähler structure on X. The orientation preserving diffeomorphisms act transitively on elements of $H_2(X;\mathbf{Z})$ with the same square and divisibility and so by property 2, the invariant $\Phi_{g,C}^{(X,T)}(\mathrm{pt.}^g)$ is a universal number that *only depends on g, C^2, and the divisibility of* $[C]$. Finally, property 3 shows that $\Phi_{g,C}^{(X,T)}(\mathrm{pt.}^g)$ only counts curves that are holomorphic with respect to a single complex structure on X.

With a little further work, one can show that $\Phi_{g,C}^{(X,T)}(\mathrm{pt.}^g)$ is weakly enumerative for generic X. That is, for a generic $K3$ surface X,

$$N_g(X,C) = \Phi_{g,C}^{(X,T)}(\mathrm{pt.}^g)$$

as long as we understand that curves should be counted with certain positive integral multiplicities if the singularities are other than nodes.

A similar discussion applies to an Abelian surface X with the addition complication due to non-trivial $q = \dim H^1(X,\mathcal{O}_X)$. Since we wish to count curves in a fixed linear system passing through $g-2$ points, we combine Theorem 4.1 and Theorem 5.2 to show that for a generic X and loops $\gamma_1,\ldots,\gamma_4$ giving an oriented basis of $H_1(X,\mathbf{Z})$,

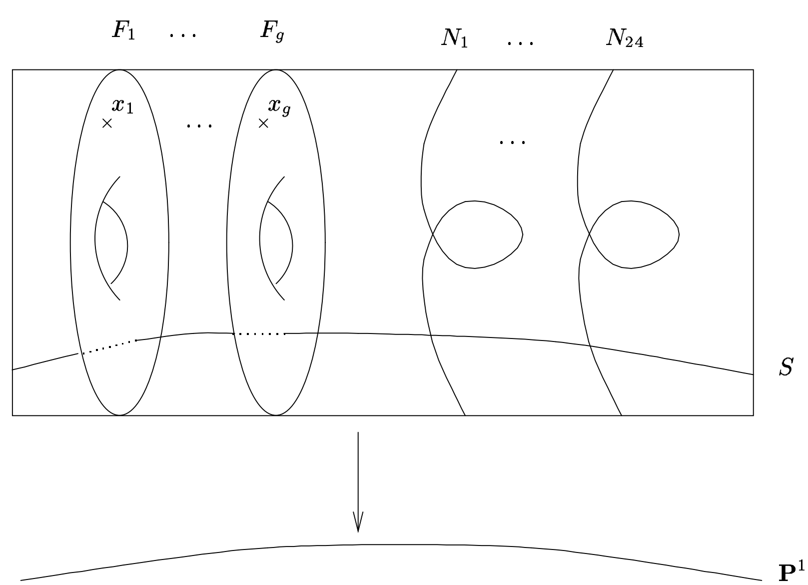
$$N_g(X,C) = \Phi_{g,C}^{(X,C)}(\gamma_1,\ldots,\gamma_4,\mathrm{pt.}^{g-2}) \tag{4}$$

(again, in the weakly enumerative sense).

This explains the proof of Theorem 5.2. Then with the invariants in hand, it suffices to compute the invariant for some particular choice of X and C to prove Theorem 5.3. We choose X to be elliptically fibered with a section and we take the class $[C]$ to be the section class plus a multiple of the fiber class. This is a primitive homology class and so our computation will give us a proof of the Göttsche-Yau-Zaslow formula for all primitive classes (Theorem 5.3). We do not know how to make the analogous computation for multiples of this class (or any other non-primitive class), but we make a brief remark about this case below:

Remark 5.6 To even formulate the enumeration problem for non-primitive classes on a $K3$ or Abelian surface, one must always deal with non-reduced curves. On these surfaces, there is a curve in a class $d[C]$ if and only if there is a curve in the class $[C]$. So for example, when counting rational curves in the class $d[C]$ one must decide how to count those rational curves in the class $[C]$ with their non-reduced structure. Our definition of $N_g(X, dC)$ using the family Gromov-Witten invariant applies in the non-primitive case as well, but its enumerative significance is less clear. When we show that our Gromov-Witten invariant is weakly enumerative for generic X and primitive $[C]$ we use the fact that all the curves will be irreducible and reduced. *A priori*, the family Gromov-Witten invariant may assign a negative and/or non-integral multiplicity to non-reduced curves.

We illustrate the computation of Theorem 5.3 with the $K3$ case and afterwards we discuss the new issues involved for the Abelian surface calculation. A elliptically fibered $K3$ surface with a section generically has 24 singular fibers consisting of nodal rational curves $N_1, \ldots, N_{24}$ and a section S which is a rational curve of square -2. We choose our g points $x_1, \ldots, x_g$ away from the section and lying on g distinct, generic, smooth fibers $F_1, \ldots, F_g$:



Fix $n \geq 0$ and let $[C_n] = [S] + (n + g)[F]$. Note that $[C_n]^2 = 2g - 2 + 2n$

and so to verify the Göttsche-Yau-Zaslow formula we need to show:

$$\sum_{n=0}^{\infty} \Phi_{g,C_n}^{(X,T)}(\text{pt.}^g) q^{g-1+n} = q^{-1} \prod_{m=1}^{\infty} (1-q^m)^{-24} \left(\sum_{k=1}^{\infty} k\sigma(k) q^k \right)^g .$$

The advantage of our choice of X and C_n is that we can really *see* all the curves in the linear system $|C_n|$. They are all reducible and their components consists of the section S along with $n+g$ fiber curves (possibly non-reduced). In order for a curve in $|C_n|$ to pass through the points $x_1, \ldots, x_g$ *and* have geometric genus no more than g (as it must in order to be the image of a genus g map) it must consist of the section, the g fibers $F_1, \ldots, F_g$ (possibly with multiplicities), and some number of the rational fibers $N_1, \ldots, N_{24}$ (again possibly with multiplicities). In other words, the curve must be

$$S + \sum_{i=1}^{g} b_i F_i + \sum_{j=1}^{24} a_j N_j$$

where the g-tuple $\mathbf{b} = (b_1, \ldots, b_g)$ and the 24-tuple $\mathbf{a} = (a_1, \ldots, a_{24})$ satisfy $b_i \geq 1$, $a_j \geq 0$, and $\sum_{i=1}^{g} b_i + \sum_{j=1}^{24} a_j = n + g$.

To compute the invariant $\Phi_{g,C_n}^{(X,T)}(\text{pt.}^g)$ we must compute the number of *maps* with the various images determined by $\mathbf{a}$ and $\mathbf{b}$. What we mean by this, strictly speaking, is that we must compute the virtual fundamental class of the moduli space of stable maps with image given by $\mathbf{a}$ and $\mathbf{b}$. The virtual dimension of the moduli space is zero and so the virtual fundamental class is a number, but the moduli space may actually be higher dimensional. In the case at hand, we show that the moduli space splits as a product of other moduli spaces and the virtual fundamental class splits into a product of virtual classes coming from each of the factors. The factors in this product can be identified with the moduli spaces of maps whose images multiple cover a single fiber F_i or N_j and the corresponding factor of the virtual class is (essentially[12]) the usual virtual class of each factor.

In this way we show that the contribution to the invariant from maps with image corresponding to $\mathbf{a}$ and $\mathbf{b}$ is a product of "local" contributions

[12] The only difference is the role that the section plays. It turns out that the obstruction to deforming the section is one dimensional and exactly cancels the obstruction to deforming the complex structure on X in the direction of the twistor family. The end result is the same as if we pretend the section is a square -1 curve and there is no family.

from multiple covers of the fibers $F_1, \dots, F_g$ and $N_1, \dots, N_{24}$. The invariant is thus of the form

$$\Phi^{(X,T)}_{g,C_n}(\mathrm{pt.}^g) = \sum_{\substack{(\mathbf{a},\mathbf{b}):\\ \sum_j a_j + \sum_i b_i = n+g}} \prod_{i=1}^{g} r(b_i) \prod_{j=1}^{24} p(a_j)$$

where $r(b)$ and $p(a)$ are the "local contributions" of b-fold covers of a smooth fiber and a-fold covers of a nodal rational fiber respectively. Multiplying both sides of the equation by q^{g-1+n} we get

$$\Phi^{(X,T)}_{g,C_n}(\mathrm{pt.}^g) q^{g-1+n} = q^{-1} \sum_{\substack{(\mathbf{a},\mathbf{b}):\\ \sum_j a_j + \sum_i b_i = n+g}} \prod_{i=1}^{g} r(b_i) q^{b_i} \prod_{j=1}^{24} p(a_j) q^{a_j}$$

and then summing over n:

$$\begin{aligned} \sum_{n=0}^{\infty} \Phi^{(X,T)}_{g,C_n}(\mathrm{pt.}^g) q^{g-1+n} &= q^{-1} \sum_{(\mathbf{a},\mathbf{b})} \prod_{i=1}^{g} r(b_i) q^{b_i} \prod_{j=1}^{24} p(a_j) q^{a_j} \\ &= q^{-1} \left(\sum_{b=1}^{\infty} r(b) q^b \right)^g \left(\sum_{a=0}^{\infty} p(a) q^a \right)^{24}. \end{aligned}$$

To prove the theorem then, it remains to be shown (which we will do in the next subsection) that

1. the local contribution $r(b)$ of b-fold covers of a smooth fiber is given by $b\sigma(b)$, and

2. the local contribution $p(a)$ of a-fold covers of a nodal fiber is given by the number of partitions of a since the generating function for the number of partitions is given by

$$\prod_{m=1}^{\infty} (1 - q^m)^{-1}.$$

The computation for an Abelian surface follows by similar methods by again assuming that the Abelian surface is elliptically fibered, *i.e.* it is a product of two elliptic curves $E_1 \times E_2$. The only new complication is the use of loops as geometric constraints (see Equation 4). Using a careful choice of loops, all the possible images of the genus g maps satisfying the geometric constraints are easily identified. The invariant is then again computed by calculating the contribution of the various multiple covers of a given image.

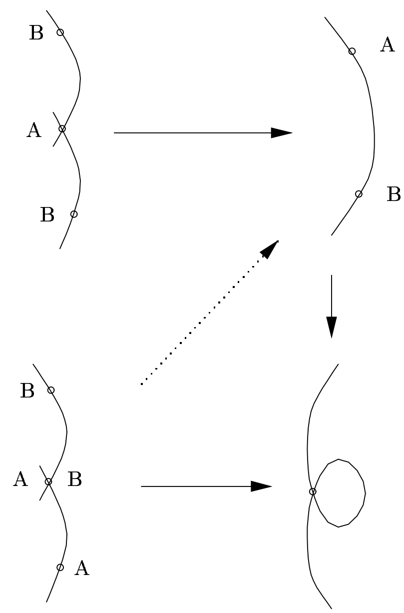
5.2 Local contributions

In this subsection we compute the local contributions $r(b)$ and $p(a)$ of multiple covers of smooth and nodal fibers.

The contribution $r(b)$ is easy to understand. It is the virtual fundamental class of the moduli space of genus 1, 2-marked, degree b maps to a fixed smooth genus 1, 2-marked curve that send the marked points to the marked points. The two marked points correspond to the intersection with the section and the point x_i. By declaring one of the marked points to be the origin, we give the domain and range the structure of elliptic curves and then the map must be a homomorphism. Thus $r(b)$ is the number of elliptic curves with a non-zero marked point that admit a degree b homomorphism onto a fixed elliptic curve with a non-zero marked point mapping the marked point to the marked point. The number of elliptic curves admitting a degree b homomorphism to a fixed elliptic curve is the number of index b sublattices of $\mathbf{Z}\oplus\mathbf{Z}$ which is classically known to be $\sigma(b)$. The number of choices for the location of the marked point in the domain is then just b and so the moduli space of b-fold covers of a smooth fiber is a discrete space consisting of $b\sigma(b)$ points.

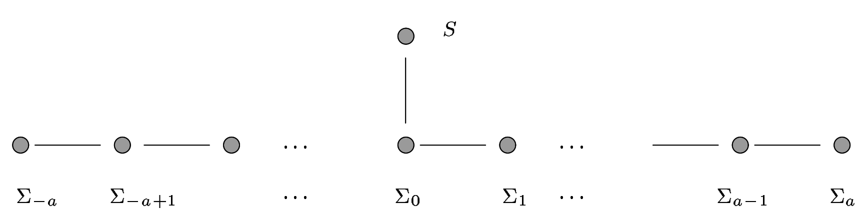
It is more difficult to directly see the contribution $p(a)$ of a-fold covers of a nodal fiber. This moduli space does not have the expected dimension zero so we need to compute its virtual fundamental class.

Define $\mathcal{M}(S+aN)$ to be the moduli space of genus 0 maps to X in the class $[S]+a[N]$ with image $S+aN_j$ for some fixed N_j (it doesn't matter which one—their neighborhoods are all biholomorphic). Denote its virtual fundamental class by $[\mathcal{M}(S+aN)]^{vir}$. The moduli space $\mathcal{M}(S+aN)$ has a number of different path components arising from the possible "jumping" behavior of the map at the node. The basic phenomenon is illustrated below for degree two maps. The moduli space $\mathcal{M}(S+2N)$ has different components depending on whether the map factors through the normalization or not. In the figure below, points labeled by A and B are mapped to corresponding points labeled A and B and the normalization map identifies A and B to the nodal point. Notice that the bottom map cannot be factored through the normalization (the dashed map doesn't exist) even though the map to the nodal curve is well defined.



In fact, the space $\mathcal{M}(S+2N)$ has three connected components: one for maps that factor through the normalization and the two maps that consist of the bottom map in the figure along with the section attached (it can attach to either of the components).

For higher degree, maps that factor through the bottom map of the figure will lie in a different component from maps that factor through the normalization. In general, the components of $\mathcal{M}(S+aN)$ are labeled by sequences $\dots, s_{-2}, s_{-1}, s_0, s_1, \dots$ of non-negative integers with $\sum s_i = a$. This can be seen as follows. Consider the nodal rational curve Σ whose dual graph is:



The curve Σ maps to $S+N$ by sending the component S isomorphically to S and each component Σ_i by a degree one map to N in such a way that the nodes connecting Σ_i to Σ_{i+1} are mapped by a local isomorphism onto the node of N. It can be shown that every map in $\mathcal{M}(S+aN)$ factors uniquely through a map to Σ. The components of $\mathcal{M}(S+aN)$ are determined by the various ways the degree of the map can distribute among the Σ_i components of Σ. We denote by $\mathcal{M}(S+\sum_i s_i\Sigma_i)$ the moduli space of genus 0 maps to Σ in the class $[S]+\sum_i s_i[\Sigma_i]$.

The preceding discussion leads to

$$\begin{aligned} p(a) &= [\mathcal{M}(S+aN)]^{vir} \\ &= \sum_{\{s_i\}:\sum_i s_i=a} [\mathcal{M}(S+\sum_i s_i\Sigma_i)]^{vir} \end{aligned}$$

where by $[\mathcal{M}(S+\sum_i s_i\Sigma_i)]^{vir}$ we mean the virtual fundamental class pulled back from $\mathcal{M}(S+aN)$. This fundamental class is induced by assigning normal bundles $\mathcal{O}(-2)$ to the Σ_i components and $\mathcal{O}(-1)$ to the S component.

We can now compute $[\mathcal{M}(S+\sum_i s_i\Sigma_i)]^{vir}$ indirectly by identifying it with another moduli-obstruction problem arising from certain blow-ups of $\mathbf{P}^2$. It will turn out that $[\mathcal{M}(S+\sum_i s_i\Sigma_i)]^{vir}$ is always zero or one depending on whether the sequence $\{s_i\}$ satisfies a certain property or not.

To identify the moduli-obstruction problem with one coming from a blow-up of $\mathbf{P}^2$, we must blow-up $\mathbf{P}^2$ in such a way that it contains a configuration of rational curves isomorphic to Σ with normal bundles $\mathcal{O}(-2)$ on the Σ_i curves and $\mathcal{O}(-1)$ on the S curve. To see that this can be done, begin by blowing up a line at three points. Its proper transform is a -2 curve which we identify with Σ_0; it meets three -1 curves, one of which we identify with S. The remaining two -1 curves can be made into -2 curves by blowing up a point on each of them. We identify their proper transforms with Σ_{-1} and Σ_1 and we repeat this process with the new -1 curves, continuing until we have the configuration Σ.

We then consider the Gromov-Witten invariants of this blow-up of $\mathbf{P}^2$ in the class corresponding to $S+\sum_i s_i\Sigma_i$. Assuming that all the rational curves in the blown up $\mathbf{P}^2$ in the class $[S]+\sum_i s_i[\Sigma_i]$ actually lie in the configuration Σ,[13] the number $[\mathcal{M}(S+\sum_i s_i\Sigma_i)]^{vir}$ is given by the corresponding invariant on the blown up $\mathbf{P}^2$.

[13]Some extra work is required to justify this assumption. One can perform the blow-ups in such a way that the resulting surface has an action of $\mathbf{C}^*$ preserving the configuration Σ. This action provides the tool needed to show that there are no curves outside Σ in the relevant homology class. One assumes that such a curve exists and studies its limits under the $\mathbf{C}^*$ action; the desired contradiction is then arrived at by a homological argument.

Genus 0 Gromov-Witten invariants of blow ups of $\mathbf{P}^2$ have been thoroughly studied in general by Göttsche and Pandharipande [36] but the particular invariants arising in this moduli-obstruction problem can be computed from elementary properties. The key property we use is the invariance of the Gromov-Witten invariants under Cremona transformations. By successive applications of the Cremona transformation, one shows that the invariant corresponding to $[\mathcal{M}(S+\sum_i s_i\Sigma_i)]^{vir}$ is either zero or equivalent to the number of lines in the plane through two points, *i.e.* one. The latter case occurs if and only if the sequence $\{s_i\}$ is such that for each $i \geq 0$, s_{i+1} is either s_i or $s_i - 1$ and for each $i \leq 0$, s_{i-1} is either s_i or $s_i - 1$. Following [15] we call a sequence with this property 1-admissible.

By the preceding arguments we conclude that

$$p(a) = \#\text{ of 1-admissable sequences } \{s_i\} \text{ with } \sum_i s_i = a$$

and we simply need to see that 1-admissible sequences of total sum a are in one-to-one correspondence with partitions of a. This is achieved by exhibiting a bijection between 1-admissible sequences of total sum a and Young diagrams of size a (which are well known to correspond bijectively to partitions). Given a Young diagram define a 1-admissible sequence $\{s_i\}$ by setting s_0 equal to the number of blocks on the diagonal, s_1 equal to the number of blocks on the first lower diagonal, s_2 equal to the number of blocks on the second lower diagonal, and so on, doing the same for $s_{-1}, s_{-2}, \ldots$ with the upper diagonals. It is easily seen that this defines a bijection and thus concludes the proof of Theorem 5.3.

5.3 Other modifications via algebraic means

Although the results of the previous section are purely algebraic, the use of the twistor family to modify the usual Gromov-Witten invariants is a non-algebraic tool. Behrend and Fantechi have recently announced [10] a purely algebraic modification of the Gromov-Witten invariants that apply to any smooth algebraic variety with $H^{0,2}(X) > 0$, and in particular to surfaces with $p_g > 0$. For $K3$ and Abelian surfaces, their modification is equivalent to the twistor family invariant. In this section we outline their modification for surfaces in general.

In the usual algebraic definition of Gromov-Witten invariants, one defines a virtual fundamental class on the space of stable maps that is invariant under deformations. The ingredients in Behrend and Fantechi's approach to this [11] are the intrinsic normal cone and the obstruction complex of the moduli space of stable maps $\mathcal{M}_{g,n,C}(X)$. Behrend and Fantechi modify the usual obstruction complex so that the resulting virtual fundamental class has its dimension larger than the usual dimension by p_g and is invariant under deformations of X preserving the $(1,1)$ type of $[C]$.

The tangent-obstruction complex for $\mathcal{M}_{g,n,C}(X)$ is built from the tangent-obstruction complex of the Deligne-Mumford moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ and the relative tangent-obstruction complex of

$$\mathcal{M}_{g,n,C}(X) \to \overline{\mathcal{M}}_{g,n}.$$

The relative tangent and obstruction spaces at the map $\{f : C \to X\}$ can be identified with $H^0(C, f^*TX)$ and $H^1(C, f^*TX)$ respectively. An infinitesimal deformation of X together with f is automatically obstructed unless the class $[\operatorname{Im} f]$ remains type $(1,1)$. For such deformations, the obstructions always lie in the kernel of

$$H^1(C, f^*TX) \to H^2(X, \mathcal{O}).$$

We can define this map via its dual map

$$H^0(X, \Omega_X^2) \to H^0(C, f^*\Omega_X^1 \otimes \omega_C)$$

which is induced by the composition of $H^0(X, \Omega_X^2) \to H^0(C, f^*\Omega_X^2)$ and the map induced by

$$f^*\Omega_X^2 \to f^*\Omega_X^1 \otimes f^*\Omega_X^1 \to f^*\Omega_X^1 \otimes \Omega_C^1 \to f^*\Omega_X^1 \otimes \omega_C.$$

Behrend and Fantechi modify the usual tangent-obstruction complex by replacing the relative obstruction space by the kernel of $H^1(C, f^*TX) \to H^2(X, \mathcal{O})$. In order for their machinery to work, the obstruction complex must be "perfect", that is, equivalent in the derived category to a two term complex of vector bundles. Thus, if $H^1(C, f^*TX) \to H^2(X, \mathcal{O})$ is surjective (or of constant rank) for all f, then the modification leads to an invariant. The theorem is as follows:

Theorem 5.7 *Let X be a surface and $[C]$ a class of type $(1,1)$. There is a (modified) virtual class $[\mathcal{M}_{g,C}]^{vir}_{mod}$ of dimension $-KC + g - 1 + p_g$. If the map*

$$H^1(C, f^*TX) \to H^2(X, \mathcal{O})$$

is surjective for every $\{f : C \to X\}$ *in* $\mathcal{M}_{g,C}$*, then* $[\mathcal{M}_{g,C}]^{vir}_{mod}$ *defines modified Gromov-Witten invariants that are invariant under deformations of* X *preserving the* $(1,1)$ *type of* $[C]$.

The hypotheses of the theorem can be shown to hold for $K3$ and Abelian surfaces and also for more general surfaces with appropriate ampleness conditions on C.

Remark 5.8 Using these invariants and Theorem 4.2, one can define $N_g(X, C)$ using Gromov-Witten theory and perhaps use them to devise a proof of the Göttsche-Yau-Zaslow formula.

Remark 5.9 As with the families version of Gromov-Witten invariants, it is not clear what the analogues of the composition law and quantum cohomology should be.

Remark 5.10 Except for the cases where X admits a hyperkähler structure, it is not clear if there is a symplectic version of Behrend and Fantechi's modified invariants.

The computations for $K3$ and Abelian surfaces described in section 5 easily extend to compute the modified invariants of any elliptic surface with a section in the class $[C_n] = [S] + n[F]$ where S is the section and $[F]$ is the class of the fiber. For example, let $E(m)$ be a generic elliptic surface over $\mathbf{P}^1$ with a section and Euler characteristic $12m$.[14] With this convention, $E(1)$ is a rational elliptic surface, $E(2)$ is a $K3$ surface, and etcetera so that $p_g(E(m)) = m - 1$. Let $\tilde{\Phi}$ be the modified invariants of Behrend-Fantechi. Then the methods of Subsections 5.1 and 5.2 can be used to compute $\tilde{\Phi}^{E(m)}_{g,C_n}(\mathrm{pt.}^g)$:

$$\sum_{n=0}^{\infty} \tilde{\Phi}^{E(m)}_{g,C_n}(\mathrm{pt.}^g) q^{g+n} = \left(\frac{q}{\Delta}\right)^{m/2} (DG_2)^g. \tag{5}$$

There are a few remarks worth making about this formula.

Remark 5.11 The formula *is different* from the Göttsche-Yau-Zaslow formula except in the case $m = 2$ (the $K3$ case). In general, the classes $[S]+n[F]$ do not satisfy Göttsche's ampleness conditions so there is no contradiction with the conjecture. It does show that the ampleness conditions cannot be removed.

[14] The classes $[C_n]$ are exactly characterized as those classes $[C]$ such that $[C]\cdot K = m-2$.

Remark 5.12 The invariants $\tilde{\Phi}^{E(m)}_{g,C_n}$ only have enumerative significance for $E(1)$ and $E(2)$. For $m > 2$, one cannot deform away the elliptic fibration and so $\tilde{\Phi}^{E(m)}_{g,C_n}$ always gives the virtual count done in the computation of Subsections 5.1 and 5.2. For $E(1)$ and $E(2)$ (*i.e.* $\mathbf{P}^2$ blown up at nine points and $K3$) a generic deformation will no longer be elliptically fibered and $\tilde{\Phi}^{E(m)}_{g,C_n}$ will be weakly enumerative. In fact, for $E(1)$, $\tilde{\Phi}^{E(1)}_{g,C_n}$ is just the ordinary Gromov-Witten invariants for $\mathbf{P}^2$ blown-up at nine points. For genus zero they are determined recursively from the quantum cohomology and are enumerative in the strong sense (for a generic choice of blow-up points). Furthermore, in [15] it is shown that the Gromov-Witten invariant of $E(1)$ for *any* class C with $CK = -1$ is equivalent to the invariant for some C_n.

The fact that there is a closed formula (different from the Göttsche-Yau-Zaslow formula!) for these invariants of $E(1)$ in terms of modular forms is somewhat of a surprise. Although the genus zero numbers are determined recursively, it is not clear how to obtain the closed formula from the recursion or how the modularity is reflected in the structure of the quantum cohomology. Ionel and Parker [44] have recently outlined a new proof of the genus zero formula for $E(1)$ that is perhaps the most transparent. They use a formula for the invariants of a fiber sum along with a topological recursion relation to show that the left hand side of Equation 5 satisfies a differential equation that is solved by the modular form on the right hand side.

6 Acknowledgments

We would like to thank the many people with whom we have had valuable conversations about Gromov-Witten theory and enumerative geometry. In particular we thank K. Behrend, A. Bertram, H. Clemens, O. DeBarre, R. Donagi, S. Donaldson, B. Fantechi, E. Getzler, A. Givental, L. Göttsche, T. Graber, E. Ionel, A. K. Liu, S. Katz, S. Kleiman, J. Li, T. Parker, R. Pandharipande, Y.-B. Ruan, B. Siebert, C. Taubes, M. Thaddeus, G. Tian, A. Todorov, C. Vafa, I. Vainsencher, R. Vakil, S.-T. Yau, and E. Zaslow.

We would also like to thank the organizers of the conferences that made it possible to talk to many of the above people. In particular, Y. Eliashberg and L. Traynor for organizing symplectic geometry program in IAS/Park City summer institute (1997); H. Clemens, E. Arbarello, and J. Harer for organizing the workshop on "Hodge theory, mirror symmetry, and quantum cohomology" in Pisa/Cortona, and A. Bertram and Y.-B. Ruan for organizing the AMS summer workshop on Quantum Cohomology.

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