INSTANTONS ON S^4 AND $\overline{\mathbf{CP}}^2$, RANK STABILIZATION, AND BOTT PERIODICITY.

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Abstract. We study the large n limit of the moduli spaces of G_n-instantons on S^4 and $\overline{\mathbf{CP}}^2$ where G_n is $SU(n)$, $Sp(n/2)$, or $SO(n)$. We show that in the direct limit topology, the moduli space is homotopic to a classifying space. For example, the moduli space of $Sp(\infty)$ or $SO(\infty)$ instantons on $\overline{\mathbf{CP}}^2$ has the homotopy type of $BU(k)$ where k is the charge of the instantons. We use our results along with Taubes' result concerning the $k \to \infty$ limit to obtain a novel proof of the homotopy equivalences in the eight-fold Bott periodicity spectrum. We work with the algebro-geometric realization of the instanton spaces as moduli spaces of framed holomorphic bundles on $\mathbf{CP}^2$ and $\mathbf{CP}^2$ blown-up at a point. We give explicit constructions for these moduli spaces (see Table 1).

1. Introduction

Let $\mathcal{M}_k^{G_n}(X)$ denote the space of (based) G_n-instantons on X where G_n is $SU(n)$, $SO(n)$, or $Sp(n/2)$. In 1989, Taubes [19] showed that there is a "gluing" map $\mathcal{M}_k^{G_n}(X) \hookrightarrow \mathcal{M}_{k'}^{G_n}(X)$ when $k' > k$. He proved that in the direct limit topology, the instantons capture all the topology of connections modulo gauge equivalence. In other words, there is a homotopy equivalence:

$$\lim_{k\to\infty} \mathcal{M}_k^{G_n}(X) \sim \mathrm{Map}_0(X, BG_n).$$

There is also an inclusion $\mathcal{M}_k^{G_n}(X) \hookrightarrow \mathcal{M}_k^{G_{n'}}(X)$ where $n' > n$ induced by the inclusion $G_n \hookrightarrow G_{n'}$. Not much is known about the homotopy type of $\mathcal{M}_k^{G}(X) = \lim_{n\to\infty} \mathcal{M}_k^{G_n}(X)$ for general X. In this paper we determine the homotopy type of $\mathcal{M}_k^{G}(X)$ when X is S^4 or $\overline{\mathbf{CP}}^2$ with their standard metrics. The results are

$$\mathcal{M}_k^G(S^4) \sim \begin{cases} BU(k) & \text{if } G = SU, \\ BO(k) & \text{if } G = Sp, \\ BSp(k/2) & \text{if } G = SO; \end{cases} \tag{1}$$

$$\mathcal{M}_k^G(\overline{\mathbf{CP}}^2) \sim \begin{cases} BU(k)\times BU(k) & \text{if } G = SU, \\ BU(k) & \text{if } G \text{ is } Sp \text{ or } SO. \end{cases}$$

The results for the S^4 case were first proved in [18], [16], and [14] (c.f. [20]) and we proved the $\mathcal{M}_k^{SU}(\overline{\mathbf{CP}}^2)$ result in [4]. In this paper we are able to provide a unified approach to these moduli spaces and stabilization results.

By employing Taubes' theorem and by utilizing the conformal map $f : \overline{\mathbf{CP}}^2 \to S^4$ to compare $\mathcal{M}_k^{G_n}(S^4)$ and $\mathcal{M}_k^{G_n}(\overline{\mathbf{CP}}^2)$, and we are able to give a novel proof

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of the homotopy equivalences in the real and unitary Bott periodicity spectrums. Work in this direction has been done by Tian using instantons on S^4 (see [18],[20]) where one can prove some of the 4-fold equivalences. By using the comparison with instantons on $\overline{\mathbf{CP}}^2$ we are able to recover the finer 2-fold equivalences in the periodicity spectrum.

The moduli spaces $\mathcal{M}_k^{G_n}(S^4)$ and $\mathcal{M}_k^{G_n}(\overline{\mathbf{CP}}^2)$ are known to be isomorphic to moduli spaces of certain holomorphic bundles and have been constructed in various guises ([5], [13], [8]). Using work of Donaldson and King we construct the spaces from a unified viewpoint (see Table 1). We describe the relevant moduli spaces of holomorphic bundles as follows:

Let $H \subset \mathbf{CP}^2$ be a fixed hyperplane and let $\widetilde{\mathbf{CP}}^2$ be the blow-up of $\mathbf{CP}^2$ at a point away from H. Donaldson showed [8] that $\mathcal{M}_k^{SU(n)}(S^4)$ is isomorphic to the moduli space of pairs $(\mathcal{E},\tau)$ where $\mathcal{E} \to \mathbf{CP}^2$ is a rank n holomorphic bundle with $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = k$ and $\tau : \mathcal{E}|_H \to \mathbf{C}^n \otimes \mathcal{O}_H$ is a trivialization of $\mathcal{E}$ on H. In [13], King extended this result to $\overline{\mathbf{CP}}^2$ by showing that $\mathcal{M}_k^{SU(n)}(\overline{\mathbf{CP}}^2)$ is isomorphic to the moduli space of pairs $(\mathcal{E},\tau)$ where $\mathcal{E} \to \widetilde{\mathbf{CP}}^2$ is a rank n holomorphic bundle with $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = k$ and $\tau : \mathcal{E}|_H \to \mathbf{C}^n \otimes \mathcal{O}_H$ is a trivialization of $\mathcal{E}$ on H. They also construct the moduli spaces in terms of "linear algebra data."

One can extend their results to $Sp(n/2)$ and $SO(n)$. Let X denote S^4 or $\overline{\mathbf{CP}}^2$. The moduli space of Sp-instantons (respectively SO-instantons) is isomorphic to the moduli space of triples $(\mathcal{E},\tau,\phi)$ where ϕ is a symplectic (resp. real) structure:

$$\begin{aligned}
\mathcal{M}_k^{Sp(n/2)}(X) &\cong \{(\mathcal{E},\tau,\phi) : (\mathcal{E},\tau) \in \mathcal{M}_k^{SU(n)}(X), \phi : \mathcal{E} \xrightarrow{\cong} \mathcal{E}^*, \phi^* = -\phi\},\\
\mathcal{M}_k^{SO(n)}(X) &\cong \{(\mathcal{E},\tau,\phi) : (\mathcal{E},\tau) \in \mathcal{M}_k^{SU(n)}(X), \phi : \mathcal{E} \xrightarrow{\cong} \mathcal{E}^*, \phi^* = \phi\}.
\end{aligned}$$

Our construction realizes these moduli spaces as quotients of affine varieties $A_k^{G_n}(X)$ (the "linear algebra data") by free actions. The key to proving our stability theorem is to show that in the large n limit, the space of "linear algebra data" becomes contractible.

The constructions also allow us to identify the universal bundles over $\mathcal{M}_k^G(X)$ inducing the homotopy equivalences of Equation 1. In the holomorphic setting they can be described as certain higher direct image bundles and in the connection setting they can be described as the index bundles of a certain family of coupled Dirac operators.

In section 2 we fix notation, state the theorems, and prove Bott periodicity. In the subsequent sections we construct the moduli spaces and prove the theorems. We conclude with a short appendix discussing a more differentio-geometric construction of the universal bundles. The authors would like to thank John Jones and Ralph Cohen for suggesting that the homotopy equivalences of Equation 1 should exist.

2. The main results and Bott periodicity

2.1. **Statement of the theorems.** Let $G_n \hookrightarrow P \to X$ be a principal bundle on a Riemannian 4-manifold X with structure group $G_n = SU(n)$, $SO(n)$, or $Sp(n/2)$.

Using the defining representations for $SU(n)$ or $Sp(n/2)$ and the complexified standard representation for $SO(n)$, we associate to P a rank n complex vector bundle E and we define the *charge* k to be $c_2(E)[X]$. [1]

A bundle isomorphism $\phi : E \to E^*$ is called a *real structure* if $\phi^* = \phi$ and it is called a *symplectic structure* if $\phi^* = -\phi$. We can regard a $SO(n)$ or a $Sp(n/2)$ bundle as a $SU(n)$ bundle E along with ϕ, a real or symplectic structure respectively. Obviously, n must be even for E to have a symplectic structure and it is also not hard to see that if E has a real structure, then our k must be even.

Let $\mathcal{A}(E)$ denote the space of connections on E that are compatible with ϕ and let F_E^+ be the self-dual part of the curvature of a connection $A \in \mathcal{A}(E)$. Let $\mathcal{G}_E^0$ be the group of gauge transformations of E commuting with ϕ and preserving a fixed isomorphism $E_{x_0} \cong \mathbf{C}^n$ of the fiber over a base point $x_0 \in X$. We define the (based) instanton moduli spaces to be (c.f. [9]):

$$\mathcal{M}_k^{G_n}(X) = \{A \in \mathcal{A}(E) : F_A^+ = 0\}/\mathcal{G}_E^0.$$

From here on let X denote S^4 or $\overline{\mathbf{CP}}^2$ with their standard metrics. We describe how the moduli spaces $\mathcal{M}_k^{G_n}(X)$ can be constructed from configurations of linear algebra data satisfying certain "integrability" conditions, modulo natural automorphisms. The configurations are laid out by Table 1 where we have adopted the following notations: Our vector spaces are always complex and our maps are always complex linear. We regard a map $f : U \to W$ as an element $f \in U^* \otimes W$. An isomorphism $\phi : W \to W^*$ is called a *symplectic structure* if $\phi \in \Lambda^2 W^*$ and a *real structure* if $\phi \in S^2 W^*$. $Gl(W)$ denotes the group of isomorphism of W and if ϕ is a symplectic (respectively real) structure on W, the let $Sp(W)$ (resp. $O(W)$) denote the group of isomorphisms of W compatible with ϕ (*i.e.* $f^*\phi f = \phi$). When n is even, let J denote the standard symplectic structure on $\mathbf{C}^n$. Unless otherwise noted, the vector spaces in Table 1 are k-dimensional.

If $f : V_1 \to V_2$, then $Gl(V_1) \times Gl(V_2)$ acts on f by $f \mapsto g_2 f g_1^{-1}$ and thus on f^* by $(g_1^{-1})^* f^* g_2^*$. So the action of the automorphism group on $\overline{\mathbf{CP}}^2$ configurations is given by [2]

$$\begin{aligned}
(g,h)\cdot(a_1,a_2,x,b,c) &= (ga_1h^{-1}, ga_2h^{-1}, hxg^{-1}, gb, ch^{-1}) \\
g\cdot(\alpha_1,\alpha_2,\xi,\gamma) &= (g\alpha_1 g^*, g\alpha_2 g^*, (g^*)^{-1}\xi g^{-1}, \gamma g^*),
\end{aligned}$$

and on S^4 configurations by

$$\begin{aligned}
g\cdot(a_1,a_2,b,c) &= (ga_1g^{-1}, ga_2g^{-1}, gb, cg^{-1}) \\
g\cdot(\alpha_1,\alpha_2,\gamma) &= (g\alpha_1g^{-1}, g\alpha_2g^{-1}, \gamma g^{-1}).
\end{aligned}$$

The three main theorems of this paper are the following:

Theorem 2.1 (Moduli Construction). *Let $\overline{A}_k^{G_n}(X)$ denote the space of integrable configurations as given by Table 1. There is an open dense set $A_k^{G_n}(X) \subset \overline{A}_k^{G_n}(X)$ (the "non-degenerate" configurations) such that the instanton moduli space $\mathcal{M}_k^{G_n}(X)$ is isomorphic to the quotient of $A_k^{G_n}(X)$ by the automorphism group. Furthermore, the action of the automorphism group on $A_k^{G_n}(X)$ is free and the vector spaces W*

[1] Our definition of k in the $SO(n)$ case differs from some of the literature by a factor of 2.

[2] Note that if we fix bases for the vector spaces, f^* is the transpose matrix and should not be confused with conjugate transpose.

G_n	X	Configurations	Integrability Condition	Automorphism Group	$\dim_{\mathbf{C}} \mathcal{M}$
$SU(n)$	$\overline{\mathbf{CP}}^2$	(a_1, a_2, x, b, c) $a_i \in U^* \otimes W$ $x \in W^* \otimes U$ $b \in \mathbf{C}^n \otimes W$ $c \in U^* \otimes \mathbf{C}^n$	$a_1 x a_2 - a_2 x a_1 + bc = 0$	$Gl(W) \times Gl(U)$	$4nk$
$Sp(n/2)$	$\overline{\mathbf{CP}}^2$	$(\alpha_1, \alpha_2, \xi, \gamma)$ $\alpha_i \in S^2 W$ $\xi \in S^2 W^*$ $\gamma \in W \otimes \mathbf{C}^n$	$\alpha_1 \xi \alpha_2 - \alpha_2 \xi \alpha_1 - \gamma^* J \gamma = 0$	$Gl(W)$	$(n+2)k$
$SO(n)$	$\overline{\mathbf{CP}}^2$	$(\alpha_1, \alpha_2, \xi, \gamma)$ $\alpha_i \in \Lambda^2 W$ $\xi \in \Lambda^2 W^*$ $\gamma \in W \otimes \mathbf{C}^n$	$\alpha_1 \xi \alpha_2 - \alpha_2 \xi \alpha_1 + \gamma^* \gamma = 0$	$Gl(W)$	$(n-2)k$
$SU(n)$	S^4	(a_1, a_2, b, c) $a_i \in W^* \otimes W$ $b \in \mathbf{C}^n \otimes W$ $c \in W^* \otimes \mathbf{C}^n$	$[a_1, a_2] + bc = 0$	$Gl(W)$	$4nk$
$Sp(n/2)$	S^4	$(\alpha_1, \alpha_2, \gamma)$ Real str. Φ $\Phi \alpha_i \in S^2 W^*$ $\gamma \in W^* \otimes \mathbf{C}^n$	$\Phi[\alpha_1, \alpha_2] - \gamma^* J \gamma = 0$	$O(W)$	$(n+2)k$
$SO(n)$	S^4	$(\alpha_1, \alpha_2, \gamma)$ Sympl. str. Φ $\Phi \alpha_i \in \Lambda^2 W^*$ $\gamma \in W^* \otimes \mathbf{C}^n$	$\Phi[\alpha_1, \alpha_2] + \gamma^* \gamma = 0$	$Sp(W)$	$(n-2)k$

TABLE 1. Configurations that construct $\mathcal{M}_k^{G_n}(X)$.

and U of Table 1 are canonically isomorphic to $H^1(\mathcal{E}(-H))$ and $H^1(\mathcal{E}(-H+E))$ respectively.

Theorem 2.2 (Lifts of instanton maps to configurations)**.** *There are commuting inclusions of configurations (defined in section 4)*

$$\begin{aligned} i &: A_k^{G_n}(X) \hookrightarrow A_k^{G_{n'}}(X) \\ j &: A_k^{G_n}(S^4) \hookrightarrow A_k^{G_n}(\overline{\mathbf{CP}}^2) \end{aligned}$$

for $n < n'$ and $k < k'$. These maps intertwine the automorphisms and consequently descend to maps on the instanton moduli spaces. The map i descends to the map induced by the inclusion $G_n \hookrightarrow G_{n'}$ and the map j descends to the map induced by pulling back connections via $f : \overline{\mathbf{CP}}^2 \to S^4$.

Theorem 2.3 (Rank Stabilization)**.** *Let $A_k^G(X)$ be the direct limit space*

$$\lim_{n\to\infty} A_k^{G_n}(X)$$

defined by the inclusions i. Then $A_k^G(X)$ is contractible and consequently

$$\mathcal{M}_k^G = \lim_{n\to\infty} \mathcal{M}_k^{G_n}$$

is homotopic to the classifying space for the associated automorphism group. This theorem implies the homotopy equivalences in Equation 1.

Remark 2.1. A naïve dimension count for $\mathcal{M}_k^{G_n}(X)$ is obtained by subtracting the dimension of the automorphism group and the number of conditions imposed by integrability from the dimension of the configurations. This agrees with the dimension predicted by the Atiyah-Singer index formula and appears in the far right column of the table.

We prove Theorems 2.1, 2.2, and 2.3 in Sections 3, 4 , and 5 respectively.

2.2. **Bott Periodicity.** In this subsection we show how Theorems 2.3, 2.2, and Taubes' stabilization leads to an alternative, relatively quick proof of the following homotopy equivalences in the periodicity spectrum:

Theorem 2.4 (Bott). *Let SU, U, SO, and Sp denote the direct limit groups of $SU(n)$, $U(n)$, $SO(n)$, and $Sp(n)$ as $n \to \infty$. Let $\Omega^j X$ denote the j-fold loop space of X. The following are homotopy equivalences:*

$$\begin{aligned} \Omega^2 SU &\sim U, \\ \Omega^2 Sp &\sim U/O, \\ \Omega^2 SO &\sim U/Sp, \\ \Omega^4 SO &\sim Sp, \\ \Omega^4 Sp &\sim O. \end{aligned}$$

Remark 2.2. The first equivalence is Bott periodicity for the unitary group and the next four appear in the real periodicity spectrum. The only missing homotopy equivalences:

$$\Omega^2(Sp/U) \sim BO \times \mathbf{Z} \text{ and } \Omega^2(SO/U) \sim BSp \times \mathbf{Z}$$

are related to monopoles (see Cohen and Jones [7] and the thesis of Ernesto Lupercio [15]).

Proof. Let i', j' and t' denote the maps on the moduli spaces induced by rank inclusion, pull-back from S^4 to $\overline{\mathbf{CP}}^2$, and Taubes' gluing respectively (i' and j' are the descent of the maps i and j in Theorem 2.2). We will argue that i', j', and t' commute up to homotopy. The maps i' and j' commute (on the nose) from Theorem 2.2; and we can see that t' commutes up to homotopy with i' and j' from some general properties of t': The Taubes' map for any semi-simple compact Lie group G is obtained from the Taubes map for $SU(2)$ via any homomorphism $SU(2) \to G$ generating $\pi_3(G)$. Since the inclusions $G_n \hookrightarrow G_{n'}$, $n' > n$ ($n > 4$ if $G_n = SO(n)$) induce isomorphisms on π_3, t' automatically commutes with i'. To see that t' commutes up to homotopy with j' we use almost instantons: connections with Yang-Mills energy smaller than a small constant ϵ. The space of almost instantons $\mathcal{M}_{k,\epsilon}^{G_n}(X)$ has a strong deformation retract onto the space of instantons and there is a map $t'_\epsilon : \mathcal{M}_{k,\epsilon}^{G_n}(X) \to \mathcal{M}_{k+1,\epsilon}^{G_n}(X)$ homotopic to t'. It is local in the sense that $t'_\epsilon(A)$ agrees with A up to gauge in the complement of a ball about the gluing point. On the other hand, the natural map $\overline{\mathbf{CP}}^2 \to S^4$ is a conformal isometry on the complement of the hyperplane that gets mapped to a point. Connections pulled back by this map have the same Yang-Mills energy and we get a map j'_ϵ on almost instantons. Thus as long as we choose our gluing point away from the hyperplane, t'_ϵ and j'_ϵ commute and so t' and j' commute up to homotopy.

The maps t', j', and j then induce commuting maps on the corresponding direct limit moduli and configurations spaces when $n \to \infty$. We will assume that we have passed to that limit throughout the rest of this section. From Theorem 2.3, we have that $A_k^{G_\infty}(X)$ is contractible. We can thus identify the homotopy fibers of the j' maps to get the following fibrations: [3]

$$\begin{array}{ccccc} U(k)\times U(k)/U(k) & \longrightarrow & \mathcal{M}_k^{SU}(S^4) & \xrightarrow{j'} & \mathcal{M}_k^{SU}(\overline{\mathbf{CP}}^2), \\ U(k)/O(k) & \longrightarrow & \mathcal{M}_k^{Sp}(S^4) & \xrightarrow{j'} & \mathcal{M}_k^{Sp}(\overline{\mathbf{CP}}^2), \\ U(k)/Sp(k/2) & \longrightarrow & \mathcal{M}_k^{SO}(S^4) & \xrightarrow{j'} & \mathcal{M}_k^{SO}(\overline{\mathbf{CP}}^2) \end{array} \tag{2}$$

where $U(k)$ is included into $U(k)\times U(k)$ via the diagonal. Here we are using the fact that Theorem 2.2 gives us j, the lift of j' to the principle bundles $A_k^{G_n}(X)$ that intertwines the actions.

Since j' commutes with t, the above fibrations are valid for the direct limit spaces when $k\to\infty$.

We now use Taubes' theorem to compare the above fibrations with the fibration on the space of connections induced by the cofibration $S^2 \hookrightarrow \overline{\mathbf{CP}}^2 \to S^4$. Let $\mathcal{B}_k^{G_n}(X)$ denote the space of all G_n-connections of charge k modulo based gauge equivalence. $\mathcal{B}_k^{G_n}(X)$ is homotopy equivalent to the mapping space $Map_k(X, BG_n)$ and the cofibration $S^2 \hookrightarrow \overline{\mathbf{CP}}^2 \to S^4$ gives rise to a fibration

$$\Omega_k^4 BG_n \xrightarrow{j'} Map_k(\overline{\mathbf{CP}}^2, BG_n) \to \Omega^2 BG_n.$$

Up to homotopy, the map j' in the above sequence is induced by pulling back connections via $\overline{\mathbf{CP}}^2 \to S^4$ (thus justifying the notation). These maps also commute with the group inclusions i and so give a fibration in the $n\to\infty$ limit. Also, $\mathcal{B}_k^{G_n}(X)$ and $\mathcal{B}_{k+1}^{G_n}(X)$ are naturally homotopy equivalent and so we implicitly identify them and drop the notational dependence; we have:

$$\mathcal{B}^G(S^4) \xrightarrow{j'} \mathcal{B}^G(\overline{\mathbf{CP}}^2) \to \Omega G. \tag{3}$$

Taubes' stabilization theorem states that the inclusions $\mathcal{M}_k^{G_n}(X) \hookrightarrow \mathcal{B}^{G_n}(X)$ induce a homotopy equivalence in the limit $k\to\infty$. Since the inclusion of the moduli spaces into $\mathcal{B}$ commutes with both j' and i' we can pass to the $k\to\infty$ and $n\to\infty$ limits and use Equation 2 to identify the homotopy fiber of $j' : \mathcal{B}^G(S^4) \to \mathcal{B}^G(\overline{\mathbf{CP}}^2)$. This fiber is in turn homotopy equivalent to $\Omega^2 G$ by the sequence 3.

[3] As we will see in Table 1, the structure groups of the various principle bundles $A_k^{G_n}(X)$ are the complex forms of the groups $U(k)$, $Sp(k/2)$, and $O(k)$ (*i.e.* $Gl(k,\mathbf{C})$, $Sp(k/2,\mathbf{C})$, and $O(k,\mathbf{C})$). Since the complex forms of the groups are homotopic to their compact form, their classifying spaces are the same (up to homotopy). It is traditional to use the compact form when refering to classifying spaces so we will use the notation of the compact group for the rest of this section.

Thus for $G = SU$, Sp, and SO respectively, we get

$$\begin{aligned}\Omega^2 SU &\sim U \times U/U \sim U,\\ \Omega^2 Sp &\sim U/O, \text{ and}\\ \Omega^2 SO &\sim U/Sp.\end{aligned}$$

The final two homotopy equivalences are arrived at by applying Theorem 2.3 and Taubes' stabilization directly to $\mathcal{M}_k^{Sp}(S^4)$ and $\mathcal{M}_k^{SO}(S^4)$. $\square$

2.3. **The algebro-geometric moduli spaces and $A_k^{G_n}$.** From now on, we will use the notation X and Y to denote S^4 and $\mathbf{CP}^2$ or $\overline{\mathbf{CP}}^2$ and $\widetilde{\mathbf{CP}}^2$ (Recall from the introduction that $\widetilde{\mathbf{CP}}^2$ is the blown-up projective plane). Consider the moduli space $\mathcal{M}_{alg}^{n,k}(Y)$ consisting of pairs $(\mathcal{E}, \tau)$ where $\mathcal{E}$ is a rank n holomorphic bundle on Y and $\tau : \mathcal{E}|_H \xrightarrow{\cong} \mathbf{C}^n \otimes \mathcal{O}_H$ is an isomorphism.

Let $p : Y \to X$ be the smooth map that sends $H \mapsto x_0$ and is one-to-one elsewhere. The map p is compatible with the natural orientations (we think of p as a "anti-holomorphic blowdown"). We can construct a natural map

$$\Xi : \mathcal{M}_k^{SU(n)}(X) \to \mathcal{M}_{alg}^{n,k}(Y)$$

by defining the holomorphic structure on $p^*(E)$ corresponding to $\Xi([A])$ to be $(d_{p^*A})^{0,1}$ and τ is induced by the fixed isomorphism of E_{x_0} (c.f. [3]).

Theorem 2.5 (Donaldson [8], King [13]). *The map Ξ is an isomorphism of moduli spaces.*

Consider the moduli space $\mathcal{M}_{alg,\pm}^{n,k}(Y)$ of triples $(\mathcal{E}, \tau, \phi)$ where $(\mathcal{E}, \tau) \in \mathcal{M}_{alg}^{n,k}(Y)$ and $\phi : \mathcal{E} \to \mathcal{E}^*$ is a isomorphism such that $\phi^* = \pm\phi$. We can construct maps $\Xi_\pm$ in the same fashion as Ξ. A consequence of Theorem 2.5 is

Corollary 2.6. *The maps*

$$\begin{aligned}\Xi_+ &: \mathcal{M}_k^{SO(n)}(X) \to \mathcal{M}_{alg,+}^{n,k}(Y) \text{ and}\\ \Xi_- &: \mathcal{M}_k^{Sp(n)}(X) \to \mathcal{M}_{alg,-}^{n,k}(Y)\end{aligned}$$

are moduli space isomorphisms ((X, Y) is $(S^4, \mathbf{CP}^2)$ or $(\overline{\mathbf{CP}}^2, \widetilde{\mathbf{CP}}^2)$).

Proof. Recall that we consider $SO(n)$ or $Sp(n/2)$ connections to be $SU(n)$ connections that are compatible with a real or symplectic structure ϕ, *i.e.*

$$\nabla_{A^*}(\phi s) = \phi \nabla_A s$$

where $A \in \mathcal{A}(\mathcal{E})$ and A^* is the induced connection in $\mathcal{A}(E^*)$. Compatibility implies that ϕ will be a holomorphic map with respect to the holomorphic structures defined by $(d_{\pi^*A})^{0,1}$ and $(d_{\pi^*(A^*)})^{0,1}$. Conversely, let $(\mathcal{E}, \tau, \phi)$ be in $\mathcal{M}_k^{Sp(n/2)}(X)$ or $\mathcal{M}_k^{SO(n)}(X)$. Choose a hermitian structure on $\mathcal{E}$ compatible with ϕ and τ. By Theorem 2.5, the unique hermitian connection is the pullback of an anti-self-dual $SU(n)$ connection on E which is, by construction, compatible with ϕ. $\square$

Henceforth we will drop the $\mathcal{M}_{alg}$ notation and use $\mathcal{M}_k^{G_n}(X)$ to refer to either moduli space.

The moduli space $\mathcal{M}_k^{G_n}(X)$ has a universal bundle (see Lemma 3.2 of [4])

$$\begin{array}{c}\mathbb{E}\\ \downarrow\\ \mathcal{M}_k^{G_n}(X)\times Y\end{array}$$

so that $\mathbb{E}|_{\{\mathcal{E}\}\times Y}\cong\mathcal{E}$.

Consider the cohomology groups $H^i(\mathcal{E}(-H))$. The fact that $\mathcal{E}$ is trivial on H (and thus on nearby lines) implies that $H^i(\mathcal{E}(-H))=0$ for $i=0$ or 2 (see [4]). The Riemann-Roch theorem then gives $\dim H^1(\mathcal{E}(-H))=k$. We will see from the construction of section 3 that the vector space W of Table 1 can be canonically identified with $H^1(\mathcal{E}(-H))$. In the case of $X=\overline{\mathbf{CP}}^2$ and $G_n=SU(n)$, a similar argument shows $\dim H^1(\mathcal{E}(-2H+E))=k$ and U is canonically identified with $H^1(\mathcal{E}(-2H+E))$.

Let $\pi:\mathcal{M}_k^{G_n}(X)\times Y\to\mathcal{M}_k^{G_n}(X)$ be projection. One consequence of the above discussion is that the higher direct image sheaf $R^1\pi_*(\mathbb{E}(-H))$ is a rank k bundle on $\mathcal{M}_k^{G_n}(X)$. Consequently, we have the following geometric interpretation of the configuration spaces $A_k^{G_n}(X)$ (c.f. Appendix A):

Theorem 2.7. *The space of configurations $A_k^{G_n}(X)$ (see Table 1) is homeomorphic to the total space of the frame bundle of $R^1\pi_*(\mathbb{E}(-H))$ except for the case $A_k^{SU(n)}(\overline{\mathbf{CP}}^2)$ which is homeomorphic to the frame bundle of*

$$R^1\pi_*(\mathbb{E}(-H))\oplus R^1\pi_*(\mathbb{E}(-2H+E)).$$

Proof. The fiber of $A_k^{G_n}(X)\to\mathcal{M}_k^{G_n}(X)$ over a point $\mathcal{E}$ is the orbit of a representative configuration by the automorphism group. This can be identified with the space $\mathrm{Iso}(W,\mathbf{C}^k)$ (or $\mathrm{Iso}(W,\mathbf{C}^k)\times\mathrm{Iso}(U,\mathbf{C}^k)$ in the $X=\overline{\mathbf{CP}}^2$, $G_n=SU(n)$ case) where we also understand $\mathrm{Iso}(W,\mathbf{C}^k)$ to be isomorphisms of symplectic or real vector spaces in the $X=S^4$, $G_n=SO(n)$ or $Sp(n/2)$ cases.

3. Construction of the moduli spaces

3.1. **Preliminaries.** To fill in Table 1, we rely heavily on the constructions of Donaldson and King; we will recall what we need from their constructions in subsection 3.2. Let us first begin by introducing some general notation. For an n-dimensional projective manifold M and a coherent sheaf $\mathcal{E}$ on M let $SD_{p,\mathcal{E}}$ denote the Serre duality isomorphism

$$SD_{p,\mathcal{E}}:H^p(\mathcal{E})\to H^{n-p}(\mathcal{E}^*(K))^*.$$

Let $H^i(\phi):H^i(\mathcal{E}\otimes\mathcal{G})\to H^i(\mathcal{F}\otimes\mathcal{G})$ denote the map in cohomology induced by a sheaf map $\phi:\mathcal{E}\to\mathcal{F}$. If $s\in H^0(\mathcal{O}(D))$ is a section vanishing on D, we let $\delta_s:H^0(\mathcal{E}|_D)\to H^1(\mathcal{E}(-D))$ denote the coboundary map arising in the long exact sequence associated to

$$0\to\mathcal{E}(-D)\xrightarrow{s}\mathcal{E}\xrightarrow{r}\mathcal{E}_D\to 0.$$

We will use the following elementary properties of Serre duality:

1. $SD_{p,\mathcal{E}}=(-1)^{p(n-p)}(SD_{n-p,\mathcal{E}^*(K)})^*$,

2. $SD_{p,\mathcal{E}}$ is natural in the sense that

$$\begin{array}{ccc} H^p(\mathcal{E}) & \xrightarrow{SD_{p,\mathcal{E}}} & H^{n-p}(\mathcal{E}^*(K))^* \\ \downarrow{\scriptstyle H^p(\phi)} & & \downarrow{\scriptstyle H^{n-p}(\phi^*)^*} \\ H^p(\mathcal{F}) & \xrightarrow{SD_{p,\mathcal{F}}} & H^{n-p}(\mathcal{F}^*(K))^* \end{array}$$

commutes.

The sign in the first property arises from commuting the cup product.

A *monad* is a three term complex of vector bundles over a complex manifold

$$\mathcal{U} \xrightarrow{A} \mathcal{V} \xrightarrow{B} \mathcal{W}$$

such that A is injective, B is surjective, and $B \circ A$ is 0. The monad determines its *cohomology bundle* $\mathcal{E} = \operatorname{Ker}(B)/\operatorname{Im}(A)$.

The point is that one can build complicated holomorphic bundles from relatively simple bundles using monads. By fixing the bundles $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ and allowing the maps A and B to vary, one parameterizes a family of bundles. We say that $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ *effectively parameterizes bundles* if the morphisms of $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-monads are in one-to-one correspondence with morphisms of the associated cohomology bundles. This will be the case under favorable cohomological conditions on $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ (for details see [13], [10]).

The Chern character of the cohomology bundle can be computed by the formula

$$ch(\mathcal{E}) = ch(\mathcal{V}) - ch(\mathcal{U}) - ch(\mathcal{W}).$$

Note that the cohomology bundle associated to the dual monad

$$\mathcal{U}^* \xrightarrow{B^*} \mathcal{V}^* \xrightarrow{A^*} \mathcal{W}^*$$

is the dual bundle E^*. We call a monad *self-dual* (or *anti-self-dual*) if it is of the form

$$\mathcal{U} \xrightarrow{A} \mathcal{V} \xrightarrow{A^*\beta^*} \mathcal{U}^*$$

where $\beta : \mathcal{V} \to \mathcal{V}^*$ is a real (or symplectic) structure; *i.e.* $\beta^* = \beta$ (or $\beta^* = -\beta$).

A self-dual monad is isomorphic to its dual by the isomorphism $(\mathbb{1}, \beta, \mathbb{1})$ and an anti-self-dual monad is isomorphic to its dual by $(\mathbb{1}, \beta, -\mathbb{1})$. Thus if $(\mathcal{U}, \mathcal{V}, \mathcal{U}^*)$ effectively parameterizes bundles, then $(\mathbb{1}, \beta, \pm\mathbb{1})$ induces a real (or symplectic) structure $\phi : \mathcal{E} \to \mathcal{E}^*$ on the cohomology bundle.

3.2. **The $SU(n)$ constructions.** We wish to show that the $SU(n)$ configurations of Table 1 give rise to bundles in $\mathcal{M}_k^{SU(n)}(X)$.

For $(a_1, a_2, b, c) \in A_k^{SU(n)}(S^4)$ consider the sequence of bundles on $Y = \mathbf{CP}^2$:

$$W \otimes \mathcal{O}(-H) \xrightarrow{A} (W \oplus W \oplus \mathbf{C}^n) \otimes \mathcal{O} \xrightarrow{B} W \otimes \mathcal{O}(H) \tag{4}$$

where

$$\begin{aligned} A &= \begin{pmatrix} x_1 - a_1 x_3 \\ x_2 - a_2 x_3 \\ c x_3 \end{pmatrix}, \\ B &= \begin{pmatrix} -x_2 + a_2 x_3 & x_1 - a_1 x_3 & b x_3 \end{pmatrix} \end{aligned}$$

and $\langle x_1, x_2, x_3 \rangle$ generates $H^1(\mathcal{O}(H))$ and H is the zero set of x_3.

The integrability condition is equivalent to $B \circ A = 0$. We define $A_k^{SU(n)}(S^4)$ to be the open dense set of the integrable configurations such that A and B are pointwise injective and surjective respectively. Thus for configurations in $A_k^{SU(n)}(S^4)$, Sequence 4 is a monad. By computing Chern classes and restricting Sequence 4 to H, one can see that the cohomology bundle $\mathcal{E}$ lies in $\mathcal{M}_k^{SU(n)}(S^4)$. In fact the converse is true:

Theorem 3.1 (Donaldson). *Every $\mathcal{E} \in \mathcal{M}_k^{SU(n)}(S^4)$ is given by a monad of the form in Sequence 4 and the correspondence is unique up to the natural action of $Gl(W)$. Furthermore, W is canonically identified with $H^1(\mathcal{E}(-H))$ (Okonek, et. al.* [17] *pg. 275).*

To finish the proof of Theorem 2.1 for $X = S^4$ and $G_n = SU(n)$ we only need to show that the automorphism group acts freely on $A_k^{SU(n)}(S^4)$. This follows from the identification of $A_k^{SU(n)}(S^4)$ with the frame bundle of $R^1\pi_*(\mathbb{E}(-H))$ (see Theorem 2.7).

For $(a_1, a_2, x, b, c) \in A_k^{SU(n)}(\overline{\mathbf{CP}}^2)$ consider the sequence of bundles on $Y = \widetilde{\overline{\mathbf{CP}}}^2$:

$$\begin{array}{c} U \otimes \mathcal{O}(-H) \\ \oplus \\ W \otimes \mathcal{O}(-H+E) \end{array} \xrightarrow{A} V \otimes \mathcal{O} \xrightarrow{B} \begin{array}{c} W \otimes \mathcal{O}(H) \\ \oplus \\ U \otimes \mathcal{O}(H-E) \end{array} \tag{5}$$

where $V = W \oplus U \oplus W \oplus U \oplus \mathbf{C}^n$ and

$$A = \begin{pmatrix} a_1x_3 & -y_2 \\ x_1 - xa_1x_3 & 0 \\ a_2x_3 & y_1 \\ x_2 - xa_2x_3 & 0 \\ cx_3 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} x_2 & a_2x_3 & -x_1 & -a_1x_3 & bx_3 \\ xy_1 & y_1 & xy_2 & y_2 & 0 \end{pmatrix}.$$

We have chosen sections $\langle x_1, x_2, x_3 \rangle$ spanning $H^0(\mathcal{O}(H))$ and $\langle y_1, y_2 \rangle$ spanning $H^0(\mathcal{O}(H-E))$ so that x_3 vanishes on H and $x_1y_1 + x_2y_2$ spans the kernel of $H^0(\mathcal{O}(H)) \otimes H^0(\mathcal{O}(H-E)) \to H^0(2H-E)$.

The integrability condition is equivalent to $B \circ A = 0$. We define $A_k^{SU(n)}(\overline{\mathbf{CP}}^2)$ to be the open dense set of the integrable configurations that are such that A and B are pointwise injective and surjective respectively. Thus for configurations in $A_k^{SU(n)}(\overline{\mathbf{CP}}^2)$, Sequence 5 is a monad. By computing Chern classes and restricting Sequence 5 to H, one can see that the cohomology bundle $\mathcal{E}$ lies in $\mathcal{M}_k^{SU(n)}(\overline{\mathbf{CP}}^2)$. Once again the converse is true:

Theorem 3.2 (King). *Every $\mathcal{E} \in \mathcal{M}_k^{SU(n)}(\overline{\mathbf{CP}}^2)$ is given by a monad of the form in Sequence 5 and the correspondence is unique up to the natural action of $Gl(W) \times Gl(U)$. Furthermore, W and U are canonically identified with $H^1(\mathcal{E}(-H))$ and $H^1(\mathcal{E}(-H+E))$ respectively.* [4]

[4] Historically, the correspondence between holomorphic bundles and instantons on S^4 or $\overline{\mathbf{CP}}^2$ was proved by constructing the bundle moduli spaces as in this section and showing that the construction is equivalent to the "twistor" construction of instantons. Now there is a direct analytic proof of the correspondence due to Buchdahl [6] that also applies to $\overline{\mathbf{CP}}^2 \# \cdots \# \overline{\mathbf{CP}}^2$.

Once again we see that automorphism group acts freely on $A_k^{SU(n)}(\overline{\mathbf{CP}}^2)$ from the identification of $A_k^{SU(n)}(\overline{\mathbf{CP}}^2)$ with the frame bundle of

$$R^1\pi_*(\mathbb{E}(-H)) \oplus R^1\pi_*(\mathbb{E}(-H+E)).$$

Remark 3.1. If $\mathcal{E} \in \mathcal{M}_k^{SU(n)}(X)$ then $\mathcal{E}^* \in \mathcal{M}_k^{SU(n)}(X)$ and is given by the cohomology of the dual monad. To find the "dual configuration", we need to use a monad automorphism to put the dual monads into the form determined by a configuration. One can then see that the correspondence $\mathcal{E} \mapsto \mathcal{E}^*$ is realized on the level of configurations by

$$(a_1, a_2, b, c) \mapsto (a_1^*, a_2^*, -c^*, b^*)$$

in the S^4 case and

$$(a_1, a_2, x, b, c) \mapsto (a_1^*, a_2^*, x^*, -c^*, b^*)$$

in the $\overline{\mathbf{CP}}^2$ case.

We also will need some finer information about these constructions. Namely, there is cohomological interpretations for the maps occurring in the configurations. For $(a_1, a_2, b, c) \in A_k^{SU(n)}(S^4)$ the maps are given by the following compositions:

$$\begin{aligned}
a_i &: H^1(\mathcal{E}(-H)) \xrightarrow{H^1(x_3)^{-1}} H^1(\mathcal{E}(-2H)) \xrightarrow{H^1(x_i)} H^1(\mathcal{E}(-H)),\\
b &: H^0(\mathcal{E}|_H) \xrightarrow{\delta_{x_3}} H^1(\mathcal{E}(-H)),\\
c^* &: H^0(\mathcal{E}^*|_H) \xrightarrow{\delta_{x_3}} H^1(\mathcal{E}^*(-H)) \xrightarrow{H^1(x_3)^{-1}} H^1(\mathcal{E}^*(-2H)) \xrightarrow{SD} H^1(\mathcal{E}(-H))^*.
\end{aligned}$$

With our definition of $\langle x_1, x_2, x_3\rangle$ and $\langle y_1, y_2\rangle$ we get a well defined section $s = x_2/y_1 = -x_1/y_2$ of $H^0(\mathcal{O}(E))$. For $(a_1, a_2, x, b, c) \in A_k^{SU(n)}(\overline{\mathbf{CP}}^2)$ the maps are given by the following compositions:

$$\begin{aligned}
a_1 &: H^1(\mathcal{E}(-H)) \xrightarrow{H^1(x_3)^{-1}} H^1(\mathcal{E}(-2H)) \xrightarrow{H^1(-y_2)} H^1(\mathcal{E}(-H)),\\
a_2 &: H^1(\mathcal{E}(-H)) \xrightarrow{H^1(x_3)^{-1}} H^1(\mathcal{E}(-2H)) \xrightarrow{H^1(y_1)} H^1(\mathcal{E}(-H)),\\
x &: H^1(\mathcal{E}(-H+E)) \xrightarrow{H^1(s)} H^1(\mathcal{E}(-H)),\\
b &: H^0(\mathcal{E}|_H) \xrightarrow{\delta_{x_3}} H^1(\mathcal{E}(-H)),\\
c^* &: H^0(\mathcal{E}^*|_H) \xrightarrow{\delta_{x_3}} H^1(\mathcal{E}^*(-H)) \xrightarrow{H^1(x_3)^{-1}} H^1(\mathcal{E}^*(-2H)) \xrightarrow{SD} H^1(\mathcal{E}(-H+E))^*.
\end{aligned}$$

King gives a detailed discussion of this description. In the S^4 case, one can also ferret these maps out of the Beilinson spectral sequence derivation of the monads on $\mathbf{CP}^2$ ([17] pg. 249-251,275) using the triviality of $\mathcal{E}$ on H.

Remark 3.2. In general, if z is the defining section of a divisor D and D is geometrically a rational curve, then it is easy to see from the long exact sequence that

$$H^1(z) : H^1(\mathcal{E}(-2D)) \to H^1(\mathcal{E}(-D))$$

is an isomorphism if and only if $\mathcal{E}|_D$ is trivial. Thus, in the above cohomological interpretations of configurations, $H^1(x_3)$ is always an isomorphism. We see then, for example, that the map x is singular if and only if $\mathcal{E}$ has the exceptional curve E as a "jumping line". Likewise we can interpret a_1 and a_2: The complement of H in Y is either a complex plane or a complex plane blown-up at the origin. In either case lines through the origin are given by the zeros of $\mu_1 x_1 + \mu_2 x_2$. Thus $\mathcal{E}$ will have jumping lines at those lines parameterized by (μ_1, μ_2) for which $\mu_1 a_1 + \mu_2 a_2$ is singular. This circle of ideas has been utilized heavily by Hurtubise, Milgram, *et. al.* who use jumping lines to give a filtration of the moduli spaces ([11], [12], [1]).

3.3. **Construction of $\mathcal{M}_k^{Sp(n/2)}(X)$ and $\mathcal{M}_k^{SO(n)}(X)$.** We now use the constructions of the previous subsection to construct the moduli spaces $\mathcal{M}_k^{Sp(n/2)}(X)$ and $\mathcal{M}_k^{SO(n)}(X)$. We first show that given $Sp(n/2)$ or $SO(n)$ configurations from Table 1, one produces an appropriate self-dual or anti-self-dual monad determining an element of the corresponding moduli space. We then show the converse, *i.e.* given an element $(\mathcal{E}, \tau, \phi)$ of $\mathcal{M}_k^{Sp(n/2)}(X)$ or $\mathcal{M}_k^{SO(n)}(X)$ we can get an equivalence class of the corresponding configurations from Table 1.

For each of the four cases with $G_n = Sp(n/2)$ or $SO(n)$, we will use configurations to define a sequence

$$\mathcal{U} \xrightarrow{A} \mathcal{V} \xrightarrow{A^*\beta^*} \mathcal{U}^*.$$

For integrable configurations (those in $\overline{A}_k^{G_n}(X)$), the sequence will satisfy

$$A^*\beta^*A = 0$$

and for each of the cases we define $A_k^{G_n}(X) \subset \overline{A}_k^{G_n}(X)$ to be the open dense subset such that the corresponding map A is pointwise injective. The sequence will then be a monad.

For $(\alpha_1, \alpha_2, \gamma) \in A_k^{Sp(n/2)}(S^4)$ we define an anti-self-dual monad by

$$W \otimes \mathcal{O}(-H) \xrightarrow{A} (W \oplus W \oplus \mathbf{C}^n) \otimes \mathcal{O} \xrightarrow{A^*\beta^*} W^* \otimes \mathcal{O}(H) \tag{6}$$

where

$$A = \begin{pmatrix} x_1 - \alpha_1 x_3 \\ x_2 - \alpha_2 x_3 \\ \gamma x_3 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & \Phi & 0 \\ -\Phi & 0 & 0 \\ 0 & 0 & J \end{pmatrix}.$$

For $(\alpha_1, \alpha_2, \gamma) \in A_k^{SO(n)}(S^4)$ we define an self-dual monad by

$$W \otimes \mathcal{O}(-H) \xrightarrow{A} (W \oplus W \oplus \mathbf{C}^n) \otimes \mathcal{O} \xrightarrow{A^*\beta^*} W^* \otimes \mathcal{O}(H) \tag{7}$$

where

$$A = \begin{pmatrix} x_1 - \alpha_1 x_3 \\ x_2 - \alpha_2 x_3 \\ \gamma x_3 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & \Phi & 0 \\ -\Phi & 0 & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix}.$$

For $(\alpha_1, \alpha_2, \xi, \gamma) \in A_k^{Sp(n/2)}(\overline{\mathbf{CP}}^2)$ we define an anti-self-dual monad by

$$\begin{matrix} W^* \otimes \mathcal{O}(-H) \\ \oplus \\ W \otimes \mathcal{O}(-H+E) \end{matrix} \xrightarrow{A} V' \otimes \mathcal{O} \xrightarrow{A^*\beta^*} \begin{matrix} W \otimes \mathcal{O}(H) \\ \oplus \\ W^* \otimes \mathcal{O}(H-E) \end{matrix} \tag{8}$$

where

$$
\begin{aligned}
A &= \begin{pmatrix} \alpha_1 x_3 & -y_2 \\ x_1 - \xi\alpha_1 x_3 & 0 \\ \alpha_2 x_3 & y_1 \\ x_2 - \xi\alpha_2 x_3 & 0 \\ \gamma x_3 & 0 \end{pmatrix}, \\
\beta &= \begin{pmatrix} 0 & 0 & \xi & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\xi & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & J \end{pmatrix}.
\end{aligned}
$$

and $V' = W \oplus W^* \oplus W \oplus W^* \oplus \mathbf{C}^n$.

For $(\alpha_1, \alpha_2, \xi, \gamma) \in A_k^{SO(n)}(\overline{\mathbf{CP}}^2)$ we define a self-dual monad by

$$
\begin{array}{c} W^* \otimes \mathcal{O}(-H) \\ \oplus \\ W \otimes \mathcal{O}(-H+E) \end{array} \xrightarrow{A} V' \otimes \mathcal{O} \xrightarrow{A^*\beta^*} \begin{array}{c} W \otimes \mathcal{O}(H) \\ \oplus \\ W^* \otimes \mathcal{O}(H-E) \end{array} \tag{9}
$$

where

$$
\begin{aligned}
A &= \begin{pmatrix} \alpha_1 x_3 & -y_2 \\ x_1 - \xi\alpha_1 x_3 & 0 \\ \alpha_2 x_3 & y_1 \\ x_2 - \xi\alpha_2 x_3 & 0 \\ \gamma x_3 & 0 \end{pmatrix}, \\
\beta &= \begin{pmatrix} 0 & 0 & \xi & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -\xi & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

and $V' = W \oplus W^* \oplus W \oplus W^* \oplus \mathbf{C}^n$.

By computing the Chern characters and restricting the monads to H one can see that the cohomology bundle of Monads 6 and 7 lie in $\mathcal{M}_k^{SU(n)}(S^4)$ and the cohomology bundle of Monads 8 and 9 lie in $\mathcal{M}_k^{SU(n)}(\overline{\mathbf{CP}}^2)$. Furthermore, since the Monads 6 and 8 are anti-self-dual they induce a symplectic structure $\phi : \mathcal{E} \to \mathcal{E}^*$ on the corresponding cohomology bundle which restricts to J on H. The Monads 6 and 8 thus define elements of $\mathcal{M}_k^{Sp(n/2)}(S^4)$ and $\mathcal{M}_k^{Sp(n/2)}(\overline{\mathbf{CP}}^2)$ respectively. Similarly, the Monads 7 and 9 define elements of $\mathcal{M}_k^{SO(n)}(S^4)$ and $\mathcal{M}_k^{SO(n)}(\overline{\mathbf{CP}}^2)$. Finally, the group of monad automorphisms that preserve the given form of the above monads is induced by the natural action of the configuration automorphism groups listed in Table 1.

Now suppose that $(\mathcal{E}, \tau, \phi)$ is an element of $\mathcal{M}_k^{Sp(n/2)}(X)$ or $\mathcal{M}_k^{SO(n)}(X)$. We wish to produce an equivalence class of the corresponding configurations. Let (a_1, a_2, b, c) or (a_1, a_2, x, b, c) be a representative configuration for $(\mathcal{E}, \tau) \in \mathcal{M}_k^{SU(n)}(X)$.

We begin by defining the map Φ by the following composition of isomorphisms:

$$
\Phi = H^1(\phi)^* \circ SD_{\mathcal{E}(-2H)} \circ H^1(x_3)^{-1}. \tag{10}
$$

For $X = \overline{\mathbf{CP}}^2$, Φ is a map from W to U^* and for $X = S^4$, Φ is a map from W to W^*.

Proposition 3.3. *When $X = S^4$ the map Φ satisfies the following relations:*

$$\begin{aligned}
\Phi^* &= \begin{cases} \Phi & \textit{when } G_n = Sp(n/2), \\ -\Phi & \textit{when } G_n = SO(n), \end{cases} \\
\Phi a_i &= a_i^* \Phi, \\
c^* &= \begin{cases} \Phi b J & \textit{if } G_n = Sp(n/2), \\ \Phi b & \textit{if } G_n = SO(n)\ . \end{cases}
\end{aligned}$$

When $X = \overline{\mathbf{CP}}^2$ the map Φ satisfies the following relations:

$$\begin{aligned}
\Phi a_i &= \begin{cases} -a_i^* \Phi^* & \textit{if } G_n = SO(n), \\ a_i^* \Phi^* & \textit{if } G_n = Sp(n/2), \end{cases} \\
x^* \Phi &= \begin{cases} -\Phi^* x & \textit{if } G_n = SO(n), \\ \Phi^* x & \textit{if } G_n = Sp(n/2), \end{cases} \\
c^* &= \begin{cases} \Phi b J & \textit{if } G_n = Sp(n/2), \\ \Phi b & \textit{if } G_n = SO(n)\ . \end{cases}
\end{aligned}$$

Proof. The proof is a straight forward application of the properties of Serre duality to the cohomological interpretation of the configuration maps and the definition of Φ. For example, if $X = S^4$ we have a commutative diagram:

$$\begin{tikzcd}
H^0(\mathcal{E}^*|_H) \arrow[r, "\delta_{x_3}"] & H^1(\mathcal{E}^*(-H)) & H^1(\mathcal{E}^*(-2H)) \arrow[l, "H^1(x_3)"'] \arrow[r, "SD"] & H^1(\mathcal{E}(-H))^* \\
H^0(\mathcal{E}|_H) \arrow[u, "H^0(\phi|_H)"'] \arrow[r, "\delta_{x_3}"] & H^1(\mathcal{E}(-H)) \arrow[u, "H^1(\phi)"'] & H^1(\mathcal{E}(-2H)) \arrow[l, "H^1(x_3)"'] \arrow[u, "H^1(\phi)"'] \arrow[r, "SD"] & H^1(\mathcal{E}^*(-H))^* \arrow[u, "H^1(\phi^*)^*"']
\end{tikzcd}$$

Now follow the diagram from the lower left corner to the upper right using both directions along the perimeter. Since $\phi|_H$ is $\mathbb{1}$ in the SO case, J in the Sp case, and $J^{-1} = -J$, we see that $c^* \circ \phi|_H = \pm \Phi \circ b$ where $\phi^* = \pm \phi$. The result for c^* follows and a similar diagram shows the $\overline{\mathbf{CP}}^2$ case.

If $X = S^4$ we wish to show that $\Phi^* = \mp \Phi$ when $\phi^* = \pm \phi$. Noting that $H^1(x_3)^* = H^1(x_3)$ we have the following commutative diagram:

$$\begin{tikzcd}
H^1(\mathcal{E}(-2H)) \arrow[r, "H^1(x_3)"] \arrow[d, "SD"] & H^1(\mathcal{E}(-H)) \arrow[r, "H^1(\phi)"] & H^1(\mathcal{E}^*(-H)) \arrow[d, "SD"] \\
H^1(\mathcal{E}^*(-H))^* \arrow[r, "H^1(\phi^*)^*"] & H^1(\mathcal{E}(-H))^* \arrow[r, "H^1(x_3)"] & H^1(\mathcal{E}(-2H))^*
\end{tikzcd}$$

Following the diagram clockwise from the upper middle spot to the lower middle spot gives the map $-\Phi^*$ since $SD = -SD^*$ in this case. Following the diagram counterclockwise yields $\pm\Phi$ when $\phi^* = \pm\phi$ and so we have that $\Phi^* = \mp\Phi$.

We can prove the relation $\Phi a_i = \mp a_i^* \Phi^*$ in a similar fashion. We write the relations algebraically and suppress the diagram:

$$
\begin{aligned}
\Phi a_i &= H^1(\phi)^* \circ SD \circ H^1(x_3)^{-1} \circ H^1(z_i) \circ H^1(x_3)^{-1} \\
&= \pm H^1(\phi^*)^* \circ SD \circ H^1(x_3)^{-1} \circ H^1(z_i) \circ H^1(x_3)^{-1} \\
&= \pm SD \circ H^1(\phi) \circ H^1(x_3)^{-1} \circ H^1(z_i) \circ H^1(x_3)^{-1} \\
&= \mp H^1(x_3)^{-1} \circ H^1(z_i) \circ H^1(x_3)^{-1} \circ SD^* \circ H^1(\phi) \\
&= \mp a_i^* \Phi^*
\end{aligned}
$$

where $z_i = x_i$ in the S^4 case and $(z_1, z_2) = (-y_2, y_1)$ in the $\overline{\mathbf{CP}}^2$ case.

Finally, we also have

$$
\begin{aligned}
x^* \Phi &= H^1(s)^* \circ H^1(\phi)^* \circ SD \circ H^1(x_3)^{-1} \\
&= \pm (H^1(\phi^*) \circ H^1(s))^* \circ SD \circ H^1(x_3)^{-1} \\
&= \pm H^1(x_3)^{-1} \circ SD \circ H^1(\phi) \circ H^1(s) \\
&= \mp H^1(x_3)^{-1} \circ SD^* \circ H^1(\phi) \circ H^1(s) \\
&= \mp \Phi^* x.
\end{aligned}
$$

□

We are now in a position to define the inverse construction producing configurations from $(\mathcal{E}, \tau, \phi)$. Define a Sp or SO configuration $(\alpha_1, \alpha_2, \gamma)$ on S^4 by $\alpha_i = a_i$ and $\gamma = c$. Define a Sp or SO configuration $(\alpha_1, \alpha_2, \xi, \gamma)$ on $\overline{\mathbf{CP}}^2$ by $\alpha_i = a_i(\Phi^{-1})^*$, $\xi = \Phi^* x$ and $\gamma = c(\Phi^{-1})^*$. The proposition then implies that these are integrable configurations. This correspondence intertwines the action of the automorphism group and is well defined on the quotient. It is also the inverse to the monad construction and so completes the proof of Theorem 2.1.

4. Lifting of maps to configurations

In this section we define the maps i and j and prove Theorem 2.2. They will be maps on configurations that descend to the maps on the moduli spaces. The map i will descend to the map induced by the inclusion $G_n \hookrightarrow G_{n'}$ for $n' > n$ and j will descend to the map induced by pulling back connections via the map $\overline{\mathbf{CP}}^2 \to S^4$. The maps will intertwine the action of the automorphism groups, *i.e.* i will be equivariant (the automorphism groups are independent of the rank), and j will intertwine the action with natural inclusions of the appropriate automorphism groups.

4.1. The rank inclusion map i.

We define i on the various kinds of configurations by:

$$
\begin{aligned}
i &: (a_1, a_2, b, c) \mapsto (a_1, a_2, b', c') \\
i &: (a_1, a_2, x, b, c) \mapsto (a_1, a_2, x, b', c') \\
i &: (\alpha_1, \alpha_2, \gamma) \mapsto (\alpha_1, \alpha_2, \gamma') \\
i &: (\alpha_1, \alpha_2, \xi, \gamma) \mapsto (\alpha_1, \alpha_2, \xi, \gamma')
\end{aligned}
$$

where $c' = \begin{pmatrix} 0 \\ c \end{pmatrix}$, $b' = \begin{pmatrix} 0 & b \end{pmatrix}$, $\gamma' = \begin{pmatrix} 0 \\ \gamma \end{pmatrix}$ and 0 is the appropriate zero map to (or from) $\mathbf{C}^{(n'-n)}$. The map is obviously equivariant with respect to the automorphism groups and from the monad constructions it is easy to see that i descends to the map $\mathcal{E} \mapsto \mathcal{E} \oplus \mathcal{O}^{(n'-n)}$. In terms of connections, this is the map $A \mapsto A \oplus \Theta$ where Θ is the trivial connection on the rank $n'-n$ bundle and this map is the natural one induced by the inclusion $G_n \hookrightarrow G_{n'}$.

4.2. **The pullback map j.** We define the map j as follows. For $G_n = Sp(n/2)$ or $SO(n)$ let

$$j : (\alpha_1, \alpha_2, \gamma) \mapsto (\alpha_1(\Phi^{-1})^*, \alpha_2(\Phi^{-1})^*, \Phi^*, \gamma(\Phi^{-1})^*).$$

This map intertwines the actions of the automorphism groups and the natural inclusions $SO(W) \hookrightarrow Gl(W)$ or $Sp(W) \hookrightarrow Gl(W)$ so it descends to a map on the moduli spaces.

For $G_n = SU(n)$ we have automorphism groups $Gl(W)$ and $Gl(W) \times Gl(U)$. Choose an isomorphism $\chi : W \to U$ so that we can define an inclusion $Gl(W) \hookrightarrow Gl(W) \times Gl(U)$ by $g \mapsto (g, \chi g \chi^{-1})$. Define j to be

$$j : (a_1, a_2, b, c) \mapsto (a_1\chi^{-1}, a_2\chi^{-1}, \chi, b, c\chi^{-1}).$$

We see that j then intertwines the action of the automorphism groups with the inclusion $Gl(W) \hookrightarrow Gl(W) \times Gl(U)$ induced by χ. It is clear that j commutes with i.

Lemma 4.1. *The map j induces the pull-back map on bundles.*

Proof. We will proceed by (1) pulling back the $\mathbf{CP}^2$ monad defined by an S^4-configuration to a $\widetilde{\mathbf{CP}}^2$ monad via the blow-down map $\widetilde{\mathbf{CP}}^2 \to \mathbf{CP}^2$ (*c.f.* subsection 2.3); (2) we use χ and a direct sum of the monad with an exact sequence to get an equivalent monad of the form of sequence 5; then (3) we will use a monad automorphism to arrive at the monad defined by the $\widetilde{\mathbf{CP}}^2$-configuration $(a_1\chi^{-1}, a_2\chi^{-1}, \chi, b, c\chi^{-1})$. The Sp and SO cases are similar and we leave them to the reader.

(1) Since in our notation $\langle x_1, x_2, x_3 \rangle$ and $\mathcal{O}(\pm H)$ on $\mathbf{CP}^2$ pull back to $\langle x_1, x_2, x_3 \rangle$ and $\mathcal{O}(\pm H)$ on $\widetilde{\mathbf{CP}}^2$, the pull-back of sequence 4 does not change notationally. We apply the monad isomorphism $(\chi, \chi \oplus \chi \oplus \mathbb{1}, \mathbb{1})$ to it to get

$$\mathcal{O}(-H) \xrightarrow{A_1} (W \oplus W \oplus \mathbf{C}^n) \otimes \mathcal{O} \xrightarrow{B_1} W \otimes \mathcal{O}(H) \tag{11}$$

where

$$\begin{aligned} A_1 &= \begin{pmatrix} x_1 - \chi a_1 \chi^{-1} x_3 \\ x_2 - \chi a_2 \chi^{-1} x_3 \\ c\chi^{-1} x_3 \end{pmatrix}, \\ B_1 &= \begin{pmatrix} -\chi^{-1}x_2 + a_2\chi^{-1}x_3 & \chi^{-1}x_1 - a_1\chi^{-1}x_3 & bx_3 \end{pmatrix}. \end{aligned}$$

(2) Since y_1 and y_2 do not vanish simultaneously on $\widetilde{\mathbf{CP}}^2$, the sequence

$$W \otimes \mathcal{O}(-H+E) \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} (W \oplus W) \otimes \mathcal{O} \xrightarrow{\begin{pmatrix} \chi y_1 & \chi y_2 \end{pmatrix}} U \otimes \mathcal{O}(H-E)$$

is exact. We can thus direct sum this sequence to the previous monad to obtain a monad with the same cohomology bundle. We get

$$\begin{matrix} U \otimes \mathcal{O}(-H) \\ \oplus \\ W \otimes \mathcal{O}(-H+E) \end{matrix} \xrightarrow{A_2} V \otimes \mathcal{O} \xrightarrow{B_2} \begin{matrix} W \otimes \mathcal{O}(H) \\ \oplus \\ U \otimes \mathcal{O}(H-E) \end{matrix} \tag{12}$$

where $V = W \oplus U \oplus W \oplus U \oplus \mathbf{C}^n$ and

$$A_2 = \begin{pmatrix} 0 & -y_2 \\ x_1 - \chi a_1 \chi^{-1} x_3 & 0 \\ 0 & y_1 \\ x_2 - \chi a_2 \chi^{-1} x_3 & 0 \\ c\chi^{-1} x_3 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & -\chi^{-1} x_2 + a_2 \chi^{-1} x_3 & 0 & \chi^{-1} x_1 - a_1 \chi^{-1} x_3 & b x_3 \\ \chi y_1 & 0 & \chi y_2 & 0 & 0 \end{pmatrix}.$$

(3) Finally we use an automorphism to put the monad into the form of the sequence 5 . Recall that $s = -x_1/y_2 = x_2/y_1$ is a well defined section in $H^0(\mathcal{O}(E))$. The automorphism we use is (η_1, η_2, η_3) where

$$\eta_1 = \begin{pmatrix} 1 & 0 \\ -\chi^{-1} s & 1 \end{pmatrix}, \eta_3 = \begin{pmatrix} 1 & \chi^{-1} s \\ 0 & 1 \end{pmatrix},$$

$$\eta_2 = \begin{pmatrix} 1 & -\chi^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\chi^{-1} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and matrix multiplication shows that $A = \eta_2 A_2 \eta_1^{-1}$ and $B = \eta_3 B_2 \eta_2^{-1}$ are exactly the monad maps defined by the $\overline{\mathbf{CP}}^2$-configuration $(a_1\chi^{-1}, a_2\chi^{-1}, \chi, b, c\chi^{-1})$. □

5. Proof of the Stabilization theorem

In section we prove Theorem 2.3.

We need to show that $\lim_{n\to\infty} A_k^{G_n}(X)$ is contractible. Since the $A_k^{G_n}$'s are all algebraic spaces and the inclusion maps are algebraic, they admit triangulations compatible with the maps. Thus $A_k^{G_\infty}$ inherits the structure of a CW-complex and so it suffices to show that the its homotopy groups are all zero. To this end we show that the inclusion

$$i : A_k^{G_n}(X) \to A_k^{G_{2k+n}}(X)$$

is null homotopic. The basic point is that in $A_k^{G_{2k}}(X)$ there are configurations whose only non-zero monad data consists of the maps to or from $\mathbf{C}^{2k}$, in other words the data a_i, α_i, x, or ξ are all zero. We will fix such a configuration in each case and show that the image of $A_k^{G_n}(X)$ in $A_k^{G_{n+2k}}(X)$ homotopes to the image of the fixed configuration.

Lemma 5.1. *There are configurations of the form* $(0, \ldots, 0, b_0, c_0) \in A_k^{SU(2k)}(X)$ *and* $(0, \ldots, 0, \gamma_0) \in A_k^{Sp(2k)}(X)$ *or* $A_k^{SO(2k)}(X)$.

Proof. The integrability and non-degeneracy conditions for SU configurations reduce to $b_0 c_0 = 0$ with c_0 injective and b_0 surjective. This can be easily accomplished by having c_0 map isomorphically onto the first k factors of $\mathbf{C}^{2k}$ and b_0 an isomorphism on the remaining k factors. For the Sp and SO cases we need a map γ_0 such that $\gamma_0^* J \gamma_0 = 0$ or $\gamma_0^* \gamma_0 = 0$ respectively, and so that γ_0 is injective. This is also easily done; for example, in the SO case choose an isomorphism $Q : W \to \mathbf{C}^k$ and let $\gamma_0 = (Q, \sqrt{-1}Q)$. We remark that configurations of this form correspond exactly to instantons on S^4 or $\overline{\mathbf{CP}}^2$ that are invariant under the natural S^1 action. □

Fix configurations as in the above lemma and define a homotopy $H_t : A_k^{G_n}(X) \to A_k^{G_{2k+n}}(X)$ by the following:

For $X = S^4$ and $G_n = SU(n)$

$$H_t(a_1, a_2, b, c) = ((1-t)a_1, (1-t)a_2, \begin{pmatrix} tb_0 & (1-t)b \end{pmatrix}, \begin{pmatrix} tc_0 \\ (1-t)c \end{pmatrix}).$$

For $X = \overline{\mathbf{CP}}^2$ and $G_n = SU(n)$

$$H_t(a_1, a_2, x, b, c) = ((1-t)^{2/3}a_1, (1-t)^{2/3}a_2, (1-t)^{2/3}x, \begin{pmatrix} tb_0 & (1-t)b \end{pmatrix}, \begin{pmatrix} tc_0 \\ (1-t)c \end{pmatrix}).$$

For $X = S^4$ and $G_n = Sp(n/2)$ or $SO(n)$

$$H_t(\alpha_1, \alpha_2, \gamma) = ((1-t)\alpha_1, (1-t)\alpha_2, \begin{pmatrix} t\gamma_0 \\ (1-t)\gamma \end{pmatrix}).$$

For $X = \overline{\mathbf{CP}}^2$ and $G_n = Sp(n/2)$ or $SO(n)$

$$H_t(\alpha_1, \alpha_2, \xi, \gamma) = ((1-t)^{2/3}\alpha_1, (1-t)^{2/3}\alpha_2, (1-t)^{2/3}\xi, \begin{pmatrix} t\gamma_0 \\ (1-t)\gamma \end{pmatrix})$$

Configurations in the image of H_t are integrable and non-degenerate so H_t is a well defined homotopy from the inclusion i to a constant map. We can thus conclude that in the $n \mapsto \infty$ limit $A_k^{G_n}(X)$ is contractible and Theorem 2.3 follows. □

Appendix A. Differentio-geometric construction of universal bundles

In this appendix we describe a differentio-geometric construction of the universal bundles $R^1\pi_*(\mathbb{E}(-H))$ and $R^1\pi_*(\mathbb{E}(-H+E))$ directly using anti-self-dual connections.

One motivation for the construction is the following. Let $\mathcal{FRED}$ denote the space of Fredholm operators on some Hilbert space. $\mathcal{FRED}$ is a classifying space for K-theory in the sense that

$$K(X) \cong [X, \mathcal{FRED}]$$

and the isomorphism is given by the index construction: if $f : X \to \mathcal{FRED}$ is a family of Fredholm operators, then $[\mathrm{Ker}\, f(x)] - [\mathrm{Coker}\, f(x)]$ pieces together to form a well defined element of $K(X)$. On the other hand, $BU(k)$ classifies rank k vector bundles:

$$Vect_k(X) \cong [X, BU(k)]$$

and one can ask if there is a natural, geometrically defined subset of $\mathcal{FRED}$ that is homotopic to $BU(k)$ such that the index construction induces the above equivalence. By coupling instantons to the Dirac operator and using our stabilization theorem we get such a family.

We define a rank k bundle over $\mathcal{M}_k^{SU(n)}(S^4)$ as follows: Since S^4 is spin there are spin bundles $S^\pm$ and the Dirac operator $\partial\!\!\!/$. We can couple the direct operator to a connection $A \in \mathcal{M}_k^{SU(n)}(S^4)$ to obtain an operator

$$\partial\!\!\!/_A : \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E).$$

Since the $\hat{A}(S^4) = 1$, the Atiyah-Singer index theorem shows that the index of $\not{\partial}_A$ is $-k$. The Bochner-Weitzenboch formula for coupled Dirac operators is

$$\not{\partial}_A^* \not{\partial}_A = \nabla_A^* \nabla_A + \frac{s}{4} + F_A^+.$$

Since $F_A^+ = 0$ and S^4 has positive scalar curvature s, the right hand side of the above equation is a positive operator. From the usual argument we deduce that $\mathrm{Ker}(\not{\partial}_A) = 0$ and so the vector space $\mathfrak{W}_A = \mathrm{Coker}(\not{\partial}_A)$ is always k-dimensional. The vector spaces $\mathfrak{W}_A$ vary smoothly with A and piece together to form the rank k vector bundle $\mathfrak{W} \to \mathcal{M}_k^{SU(n)}(S^4)$ we seek. The construction is natural with respect to the inclusion $SU(n) \hookrightarrow SU(n+1)$ and so $\mathfrak{W}$ ascends to $n \mapsto \infty$ direct limit. At the end of the section we will outline an argument showing this is the same bundle as $R^1\pi_*(\mathbb{E}(-H))$.

For $X = \overline{\mathbf{CP}}^2$, we consider the spin$_{\mathbf{C}}$ structure $W^{\pm}$ with $L = \det(W^{\pm})$ such that $c_1(L)$ is a generator of $H^2(\overline{\mathbf{CP}}^2; \mathbf{Z})$. Since $b_2^+(\overline{\mathbf{CP}}^2) = 0$ there is a unique (up to gauge) connection $a \in \mathcal{A}(L)$ such that $F_a^+ = 0$. Using the connection a we get a spin$_{\mathbf{C}}$ Dirac operator $\not{\partial}_a$ which we can couple to a connection $A \in \mathcal{M}_k^{SU(n)}(\overline{\mathbf{CP}}^2)$ to get an operator

$$\not{\partial}_{a,A} : \Gamma(W^+ \otimes E) \to \Gamma(W^- \otimes E).$$

The Bochner-Weitzenboch formula for this operator is

$$\not{\partial}_{a,A}^* \not{\partial}_{a,A} = \nabla_{a,A}^* \nabla_{a,A} + \frac{s}{4} + F_A^+ + \frac{F_a^+}{2}.$$

Once again the right hand side of this equation is positive so $\mathrm{Ker}(\not{\partial}_{a,A}) = 0$. We compute the index of the operator:

$$\begin{aligned}
\mathrm{Ind}_{\mathbf{C}}(\not{\partial}_{a,A}) &= ch(E) \cdot ch(L/2) \cdot \hat{A}(\overline{\mathbf{CP}}^2)[\overline{\mathbf{CP}}^2] \\
&= (n - c_2(E))(1 + c_1(L/2) + \tfrac{1}{2}c_1(L/2)^2)(1 - \tfrac{1}{24}p_1)[\overline{\mathbf{CP}}^2] \\
&= -k + n\left(\frac{c_1(L)^2 - \sigma}{8}\right) \\
&= -k.
\end{aligned}$$

We define then a rank k bundle $\mathfrak{W}_+ \to \mathcal{M}_k^{SU(n)}(\overline{\mathbf{CP}}^2)$ whose fiber over A is $\mathrm{Coker}(\not{\partial}_{a,A})$. We get a second rank k bundle by the same construction with the spin$_{\mathbf{C}}$ structure associated to $-c_1(L)$. We will not provide a complete proof that these bundles are $R^1\pi_*(\mathbb{E}(-H))$ and $R^1\pi_*(\mathbb{E}(-H+E))$ but we will outline the argument. The case for S^4 is discussed in Section 3.3.3 and 3.3.4 in Donaldson and Kronheimer [9] so we will focus on the $\overline{\mathbf{CP}}^2$ case.

Let $\widetilde{\mathbf{C}}^2$ denote the complex plane blown-up at the origin with the standard Kähler structure. $\widetilde{\mathbf{C}}^2$ is biholomorphic to $\widetilde{\mathbf{CP}}^2 - H$ and conformally equivalent to $\overline{\mathbf{CP}}^2 - x_0$. We wish to bring the operators $\not{\partial}_{a,A}^*$ and $\bar{\partial}_{\mathcal{E}} + \bar{\partial}_{\mathcal{E}}^*$ to $\widetilde{\mathbf{C}}^2$ in order to compare them. Care must be taken on the non-compact manifold to impose the correct decay conditions.

The conformally changed Dirac operator is $\widetilde{\not{\partial}}_{A,a}^* = e^{3f/2}\not{\partial}_{A,a}^* e^{-3f/2}$ where the conformal factor is $f = (1+r^2)$ and r is the radial coordinate in $\widetilde{\mathbf{C}}^2$. Thus the kernel of $\widetilde{\not{\partial}}_{A,a}^*$ consists of sections of order $\mathcal{O}(r^{-3})$ and this turns out to be the same as L^2 harmonic spinors on $\widetilde{\mathbf{C}}^2$. On $\widetilde{\mathbf{C}}^2$ the Dirac operator is $\bar{\partial}+\bar{\partial}^*$ on $\Omega^{01}(E\otimes(K\otimes L)^{1/2})$.

One can then identify the kernel of $\bar\partial+\bar\partial^*$ on L^2 section-valued $(0,1)$-forms on $\widetilde{\mathbf{C}}^2$ with the kernel of $\bar\partial+\bar\partial^*$ on $\widetilde{\mathbf{CP}}^2$ where the bundle has been twisted by a certain multiple of H (the "divisor at infinity") determined by the decay conditions. In our case the result is that $\mathrm{Coker}(\not\partial^*_{A,a}) = \mathrm{Ker}(\not\partial_{A,a}) = \mathrm{Ker}(\widetilde{\not\partial}_{A,a}) = \mathrm{Ker}(\bar\partial+\bar\partial^*)$ is $H^1(\mathcal{E}(-2H+E))$ or $H^1(\mathcal{E}(-2H))$ depending on the sign of L.

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