On the rigidity of stable maps to Calabi-Yau threefolds.

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1 Introduction

In 1998, Gopakumar and Vafa proposed a duality between $SU(N)$ Chern-Simons theory on the 3-sphere and topological string theory on the resolved conifold [5]. As evidence, Gopakumar and Vafa showed the large N free energy in Chern-Simons theory exactly matches (after a change of variables) the topological string partition function on the resolved conifold.

Mathematically, the *topological string partition function* is just the natural generating function for the Gromov-Witten invariants. The *resolved conifold* is the total space of the bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, considered as a Calabi-Yau threefold. The Gromov-Witten theory of the non-compact total space $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is well-defined: all (non-constant) stable maps have image contained in the zero section and thus their moduli spaces are compact.

The Gromov-Witten invariants of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ are often regarded as the *local Gromov-Witten invariants* of $\mathbb{P}^1$. Indeed, if $X \subset Y$ is any smoothly embedded rational curve in a Calabi-Yau threefold Y with normal bundle isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then the contribution of X to the Gromov-Witten invariants of Y is well-defined and is given by the corresponding invariants of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

We consider here the local theory of higher genus curves. If $X \subset Y$ is a non-singular curve in a Calabi-Yau threefold with generic normal bundle $N_{X/Y}$, are the local Gromov-Witten invariants of X well defined? For X of genus two or higher, the issues are subtle. We will formulate a precise line of inquiry and present some results, some positive and some negative.

2 Definitions and Results

Let $X \subset Y$ be a nonsingular genus g curve in a threefold Y with normal bundle $N_{X/Y}$ of degree $2g-2$. If Y is Calabi-Yau, the condition on the normal bundle is always satisfied. We define the following notions of rigidity:

Definition 2.1.

- *A curve $X \subset Y$ is (d,h)-**rigid** if for every degree d, genus h stable map $f : C \to X$, we have $H^0(C, f^*N_{X/Y}) = 0$.*
- *A curve $X \subset Y$ is d-**rigid** if $X \subset Y$ is (d,h)-rigid for all genera h.*
- *A curve $X \subset Y$ is **super-rigid** if $X \subset Y$ is d-rigid for all $d > 0$.*

For example, a nonsingular rational curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is super-rigid. An elliptic curve $E \subset Y$ is d-rigid if and only if $N_{E/Y} \cong L \oplus L^{-1}$ where $L \to E$ is a flat line bundle which is not d-torsion (see [7]).

For a (d,h)-rigid curve $X \subset Y$, the contribution of X to $N^h_{d[X]}(Y)$, the genus h Gromov-Witten invariant of Y in the class $d[X]$, is well-defined and given by

$$\int_{[\overline{M}_h(X,d[X])]^{vir}} c_{\mathrm{top}}(R^1\pi_* f^* N), \tag{1}$$

where $\overline{M}_h(X,d)$ is the moduli space of degree d, genus h, stable maps to X,

$$\pi : U \to \overline{M}_h(X,d)$$

is the universal curve,

$$f : U \to X$$

is the universal map, and $[\quad]^{vir}$ denotes the virtual fundamental class. The (d,h)-rigidity of X guarantees that $R^1\pi_* f^*N$ is a *bundle*. See [3] for an expanded discussion.

By definition, (d,h)-rigidity is a condition on the normal bundle $N_{X/Y}$. Assuming $N_{X/Y}$ is generic, we may ask for which pairs (d,h) does (d,h)-rigidity hold? The 1-rigidity of a generic normal bundle is straightforward and was used in [7]. We prove the following positive result.

Theorem 2.2. *If $X \subset Y$ is a genus g curve in a threefold Y and $N_{X/Y}$ is a generic stable bundle of degree $2g-2$, then X is 2-rigid.*

However, 3-rigidity is *not* satisfied for genus 3 curves.

Theorem 2.3. *If $X \subset Y$ is a genus 3 curve in a threefold Y with*

$$\deg(N_{X/Y}) = 4,$$

then X is not 3-rigid.

Let $N \to X$ be a generic stable bundle of degree $2g-2$. By Theorem 2.2, the degree 2 Gromov-Witten theory of the total space of N considered as a non-compact threefold is well-defined defined by the integral (1).

In case X embeds in a threefold Y with normal bundle N, we may regard the above theory as the degree 2 local Gromov-Witten theory of $X \subset Y$. Such embeddings of X can always be found. For example, let Y be the threefold $\mathbb{P}(\mathcal{O}_X \oplus N)$ with the embedding $X \subset Y$ determined by the trivial factor. It would be interesting to construct a curve in a Calabi-Yau threefold with a 2-rigid normal bundle.

The degree 2 *local* theory of X is a *global* theory for N. Strong global integrality constraints, obtained from the Gopakumar-Vafa conjecture [4] and more recently from the conjectural Gromov-Witten/Donaldson-Thomas correspondence of [6], should therefore hold for the degree 2 local theory of X.

In [2, 3], a formal local theory is defined for in all degrees for $X \subset Y$. By Theorem 2.2, the above local theory coincides with the degree 2 formal local theory defined in [2, 3]. However by Theorem 2.3, the formal local theory of [2, 3] does *not* correspond exactly to a well-defined global theory of N is degree 3.

Using degeneration techniques, we have recently obtained complete computations of the formal local invariants of curves in degrees up to 5 [1]. The expected integrality holds for the degree 2 invariants. Integrality for the degree 3 formal local invariants fails, in fact the breakdown occurs for precisely the *same* domain genus h for which rigidity fails.

3 Proof of Theorem 2.2

Let X be a nonsingular curve of genus g. Let N be a bundle of rank 2 and degree $2g-2$ on X. The bundle N is 2-rigid if

$$H^0(C, f^*(N)) = 0 \tag{2}$$

for all stable maps $f : C \to X$ of degree 2. If $g = 0$, then

$$N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$$

is 2-rigid. For $g > 0$, we will prove 2-rigidity holds on an open set of the irreducible moduli space of semistable bundles on X. As the 2-rigidity statement was already proven for $g = 1$ in [7], we will assume $g > 1$.

Let Λ be a fixed line bundle of degree $2g-2$. Let $\overline{M}_X(2, \Lambda)$ be the moduli space of rank 2, semistable bundles with determinant Λ. Since 2-rigidity is a generic condition, we need only prove the existence of a 2-rigid bundle $[N] \in \overline{M}_X(2, \Lambda)$.

We first prove $H^0(X, N) = 0$ for an open set $V \subset \overline{M}_X(2, \Lambda)$. Since the vanishing of sections is an open condition and the moduli space is irreducible, we need only find an example. The bundle

$$N = L \oplus L^{-1}\Lambda$$

with $L \in \mathrm{Pic}^{g-1}(X)$ is semistable. Since the locus of Pic^{g-1} determined by bundles with nontrivial sections is a divisor, both L and $L^{-1}\Lambda$ do not have sections for generic L. Then, $H^0(X, L \oplus L^{-1}\Lambda) = 0$.

If there exists a nonzero section $s \in H^0(C, f^*(N))$ for a stable map f, then s must be nonzero on some dominant, irreducible component of $C' \subset C$. Also, s must be nonzero when pulled-back to the normalization of C'. Therefore, to check 2-rigidity for N, we need only prove

$$H^0(C, f^*(N)) = 0$$

for maps $f : C \to X$ where the domain is nonsingular and irreducible.

Let C be nonsingular and irreducible, and let $f : C \to X$ be a (possibly ramified) double cover. Let $B \subset X$ be the branch divisor of f. The map f is well-known to determine a square root of $\mathcal{O}(-B)$ in the Picard group of X by the following construction. Let $E = f_*(\mathcal{O}_C)$. E is a rank two bundle on X with a $\mathbb{Z}/2\mathbb{Z}$ action induced by the Galois group of f. E decomposes into a direct sum of $+1$ and -1 eigenbundles for the action:

$$E \cong \mathcal{O} \oplus Q.$$

The $+1$ eigenbundle is the trivial line bundle $\mathcal{O}$ while the -1 eigenbundle is a line bundle Q with $Q^2 \cong \mathcal{O}(-B)$.

If $[N] \in V$, a sequence of isomorphisms is obtained from the geometry of the double cover f:

$$H^0(C, f^*(N)) \cong H^0(X, f_*f^*(N)) \cong H^0(X, (\mathcal{O} \oplus Q) \otimes N) \cong H^0(X, Q \otimes N).$$

To prove the 2-rigidity of N, it therefore suffices to prove the vanishing of $H^0(X, Q \otimes N)$ for all double covers f.

Suppose $H^0(X, Q \otimes N) \neq 0$ for a double cover f. Then, there is a non-zero sheaf map

$$\iota : Q^{-1} \to N.$$

Let S be the saturation of the image of ι. S has the following properties:

(i) S is a sub-bundle of N,

(ii) S^2 has a section.

Property (ii) is proven as follows. The saturation short exact sequence of sheaves on X is

$$0 \to Q^{-1} \to S \to \mathcal{O}_D \to 0$$

for some effective divisor D on X. Hence $S \cong Q^{-1}(D)$, and

$$S^2 \cong Q^{-2}(2D) \cong \mathcal{O}(B + 2D).$$

The line bundle clearly has a section since both B and D are effective.

For $d \geq 0$, define $\Delta(d) \subset \overline{M}_X(2, \Lambda)$ to be the following locus in the moduli space:

$$\{[N] \mid \exists \text{ a subbundle } S \subset N \text{ such that } \deg(S) = d \text{ and } H^0(X, S^2) \neq 0.\}.$$

If $[N] \in V$ and N is not 2-rigid, then we have proven $[N]$ must lie in $\Delta(d)$ for some d. By Lemma 3.1 below, the proof of Theorem 2.2 is complete. $\square$

Lemma 3.1. *For all d,* $\dim \Delta(d) < \dim \overline{M}_X(2, \Lambda)$.

Proof of Lemma 3.1: By stability, $\Delta(d)$ is empty if $d \geq g - 1$. We may assume

$$0 \leq d < g - 1.$$

If $[N] \in \Delta(d)$, then N is given by an extension:

$$0 \to S \to N \to S^{-1} \otimes \Lambda \to 0.$$

The number of parameters for S is g, the dimension of $\mathrm{Pic}^d(X)$. The number of parameters of the space of extensions is

$$h^1(X, S^2 \otimes \Lambda^{-1}) - 1 = 3g - 3 - 2d - 1$$

by Riemann-Roch. Therefore,

$$\dim \Delta(d) \leq g + 3g - 3 - 2d - 1 = 4g - 4 - 2d.$$

If $d > (g-1)/2$, then $\dim \Delta(d) < \dim M_X(2, \Lambda) = 3g - 3$.

By the above analysis, we may assume $0 \leq d \leq (g-1)/2$. We will now redo the dimension count for $\Delta(d)$ using the condition $H^0(S^2) \neq 0$. By Brill-Noether theory, the dimension of the space of of line bundles of degree $2d < g$ having nonzero sections is $2d$. Recomputing, we find

$$\dim \Delta(d) \leq 2d + 3g - 3 - 2d - 1 = 3g - 4.$$

Therefore, $\dim \Delta(d) < \dim \overline{M}_X(2, \Lambda)$ for all d and the Lemma is proved. $\square$

4 Proof of Theorem 2.3

Let X be a nonsingular genus 3 curve. For any rank 2 bundle N on X of degree 4, we will show there exists a degree 3 stable map

$$f : C \to X$$

for which $H^0(C, f^*(N)) \neq 0$.

If N contains a rank 1 sub-bundle $L \subset N$ of positive degree, then $H^0(X, L^3) \neq 0$ by Riemann-Roch. Let s be a nonzero section of L^3, and let $f : C \to X$ be the cube root of s in L. Since $H^0(C, f^*(L))$ has a canonical section, we have shown $H^0(C, f^*(N)) \neq 0$. If N is unstable, then N has a destabilizing rank 1 sub-bundle of positive degree and N is not 3-rigid.

Let Λ be a line bundle on X of degree 4. Let $\overline{M}_X(2, \Lambda)$ be the moduli space of rank 2, semistable bundle with determinant Λ. We will prove for general

$$[N] \in \overline{M}_X(2, \Lambda),$$

N contains a rank 1 sub-bundle of degree 1. Hence, the general semistable bundle N with determinant Λ is not 3-rigid. By semicontinuity and the irreducibility of the moduli of semistable bundles, we conclude no semistable bundle on X is 3-rigid.

The dimension of the moduli space $\overline{M}_X(2,\Lambda)$ is 6. The extensions

$$0 \to L \to N \to L^{-1}\Lambda \to 0, \tag{3}$$

where L has rank 1 and degree 1 are parameterized (up to scale) by a projective bundle B with fiber $\mathbb{P}(H^1(X, L^2\Lambda^{-1}))$ over $\mathrm{Pic}^1(X)$. Since dimensions of $\mathrm{Pic}^1(X)$ and $\mathbb{P}(H^1(X, L^2\Lambda^{-1}))$ are both 3, the dimension of B is 6. An elementary argument shows the map to moduli,

$$\epsilon : B \to \overline{M}_X(2,\Lambda),$$

is well-defined on a (nonempty) open set of B. If ϵ is dominant, the Theorem is proven.

It suffices to prove that the tangent space to B generically surjects onto the tangent space of $\overline{M}_X(2,\Lambda)$. The following argument was provided by Michael Thaddeus.

The tangent space to $\overline{M}_X(2,\Lambda)$ at N is given by $H^1(X, \mathrm{End}_0 N)$. The exact sequence (3) induces a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset E_3 = \mathrm{End}_0 N$$

where

$$E_1/E_0 = L^2\Lambda^{-1}, \quad E_2/E_1 = \mathcal{O}, \quad E_3/E_2 = L^{-2}\Lambda.$$

For generic L,

$$H^0(X, L^{-2}\Lambda) = H^1(X, L^{-2}\Lambda) = 0.$$

Hence, the inclusion $E_2 \subset E_3$ induces isomorphisms

$$H^1(X, E_2) \cong H^1(X, \mathrm{End}_0 N), \quad H^0(X, E_2) \cong H^0(X, \mathrm{End}_0 N) = 0.$$

Consequently, the exact sequence for the inclusion $E_1 \subset E_2$ reduces to:

$$0 \to H^1(X, L^2\Lambda^{-1})/H^0(X, \mathcal{O}) \to H^1(X, \mathrm{End}_0 N) \to H^1(X, \mathcal{O}) \to 0.$$

The tangent space of $\overline{M}_X(2,\Lambda)$ at N is therefore identified with the tangent space of B at N. Indeed, $H^1(X, L^2\Lambda^{-1})/H^0(X, \mathcal{O})$ is the space of deformations of the extension class and $H^1(X, \mathcal{O})$ is the space of deformations of L. $\square$

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