The Rank Stable Topology of Instantons on $\overline{\mathbf{CP}}^2$

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April 2, 1997

Abstract

Let $\mathcal{M}_k^n$ be the moduli space of based (anti-self-dual) instantons on $\overline{\mathbf{CP}}^2$ of charge k and rank n. There is a natural inclusion $\mathcal{M}_k^n \hookrightarrow \mathcal{M}_k^{n+1}$. We show that the direct limit space $\mathcal{M}_k^\infty$ is homotopy equivalent to $BU(k) \times BU(k)$. Let ℓ_∞ be a line in the complex projective plane and let $\widetilde{\mathbf{CP}}^2$ be the blow-up at a point away from ℓ_∞. $\mathcal{M}_k^n$ can be alternatively described as the moduli space of rank n holomorphic bundles on $\widetilde{\mathbf{CP}}^2$ with $c_1 = 0$ and $c_2 = k$ and with a fixed holomorphic trivialization on ℓ_∞.

1 Introduction

In his 1989 paper [Ta], Taubes studied the stable topology of the based instanton moduli spaces. He showed that if $\mathcal{M}_k^n(X)$ denotes the moduli space of based $SU(n)$-instantons of charge k on X, then there is a map $\mathcal{M}_k^n(X) \to \mathcal{M}_{k+1}^n(X)$ and, in the direct limit topology, $\mathcal{M}_\infty^n(X)$ has the homotopy type of $\mathrm{Map}_0(X, BSU(n))$.

There is also a map $\mathcal{M}_k^n(X) \hookrightarrow \mathcal{M}_k^{n+1}(X)$ given by the direct sum of a connection with the trivial connection on a trivial line bundle and one can consider the direct limit $\mathcal{M}_k^\infty(X)$. For the case of $X = S^4$ with the round metric, it was shown by Kirwan and also by Sanders ([Kir],[Sa]) that the direct limit has the homotopy type of $BU(k)$.

In this note we consider the case of $X = \overline{\mathbf{CP}}^2$ where $\overline{\mathbf{CP}}^2$ denotes the complex projective plane with the Fubini-Study metric and the opposite orientation of the one induced by the complex structure. Our result is:

Theorem 1.1. *$\mathcal{M}_k^\infty(\overline{\mathbf{CP}}^2)$ has the homotopy type of $BU(k) \times BU(k)$.*

The main tool in the proof of the theorem is a construction of the moduli spaces $\mathcal{M}_k^n(\overline{\mathbf{CP}}^2)$ due to King [Ki]. In general, Buchdahl [Bu] has shown that, for appropriate metrics on the N-fold connected sum $\#_N\overline{\mathbf{CP}}^2$, the moduli spaces $\mathcal{M}_k^n(\#_N\overline{\mathbf{CP}}^2)$ are diffeomorphic to certain spaces of equivalence classes of holomorphic bundles on $\mathbf{CP}^2$ blown-up at N points. The universal $U(k) \times U(k)$

bundle that appears giving the homotopy equivalence of theorem 1.1 can be constructed as higher direct image bundles (see section 3).

Remark 1.1. The cofibration $S^2 \to \overline{\mathbf{CP}}^2 \to S^4$ gives rise to the fibration of mapping spaces $\Omega^4 BSU(n) \to \mathrm{Map}_*(\overline{\mathbf{CP}}^2, BSU(n)) \to \Omega^2 BSU(n)$ which for K-theoretic reasons is a trivial fibration in the limit over n. The total space of this fibration is homotopy equivalent to the space of based gauge equivalence classes of all connections on $\overline{\mathbf{CP}}^2$. Thus, from Taubes' result, $\mathcal{M}_k^\infty$ must have the property that taking the limit over k gives $BU \times BU$. For S^4, similar remarks imply that $\lim_{k\to\infty} \mathcal{M}_k^\infty(S^4) \simeq BU$ and the inclusion of $\mathcal{M}_k^\infty(S^4)$ into this limit has been shown to be (up to homotopy) the natural inclusion $BU(k) \hookrightarrow BU$ ([Sa]). Theorem 1.1 and these results for S^4 suggest a general conjecture which is supported by the fact that the higher direct image bundle giving our homotopy equivalence generalizes in an appropriate way.

Conjecture 1.1. *For appropriate metrics on $\#_N \overline{\mathbf{CP}}^2$, $\mathcal{M}_k^\infty(\#_N \overline{\mathbf{CP}}^2)$ has the homotopy type of a product $BU(k) \times \cdots \times BU(k)$ with $N+1$ factors.*

Remark 1.2. Combining theorem 1.1 with Taubes' stabilization result leads to an alternate proof of Bott periodicity for the unitary group. There is a natural map from instantons on S^4 to those on $\overline{\mathbf{CP}}^2$ (given by pull-back) which in the limit as $n \to \infty$ is homotopy equivalent to the diagonal $BU(k) \to BU(k) \times BU(k)$. Taking the limit as $k \to \infty$ and applying Taubes' result, the diagonal map appears as the inclusion of fibers in the fibration $BU \simeq \Omega_k^4 BU \to BU \times BU \to \Omega^2(BSU)$ (see remark 1.1). However, the map $BU \to BU \times BU$ has homotopy fiber $U \times U \big/ U \simeq U$, and therefore, given the above fibration we must have $U \simeq \Omega^3 BSU \simeq \Omega^2 SU \simeq \Omega^2 U$.

Tian ([Ti]) noticed that results for limits of instantons on S^4 (see [Kir] or [Sa]) already imply the four-fold periodicity $\Omega^4 BU \simeq Z \times BU$. The $\overline{\mathbf{CP}}^2$ case thus gives the finer two-fold periodicity $\Omega^2 BU \simeq Z \times BU$, as one might expect due to the nontrivial $S^2 \hookrightarrow \overline{\mathbf{CP}}^2$.

In a future paper we will study limits of $Sp(n)$ and $SO(n)$ instantons on $\overline{\mathbf{CP}}^2$ and their relationships to those on S^4. As an amusing corollary, we will be able to rederive many of the Bott periodicity relationships among Sp, U, SO, and their homogeneous spaces.

2 The construction of $\mathcal{M}_k^n(\overline{\mathbf{CP}}^2)$

Let $x_0 \in \overline{\mathbf{CP}}^2$ be the base point. Since $\overline{\mathbf{CP}}^2 - \{x_0\}$ is conformally equivalent to $\widetilde{\mathbf{C}}^2$, the complex plane blown-up at the origin, $\mathcal{M}_k^n(\overline{\mathbf{CP}}^2)$ can be regarded as instantons on $\widetilde{\mathbf{C}}^2$ based "at infinity". Buchdahl [Bu] proved an analogue in this non-compact setting of Donaldson's theorem relating instantons to holomorphic bundles: Let $\widetilde{\mathbf{C}}^2_N$ be the complex plane blown-up at N points with a Kähler metric. Then $\widetilde{\mathbf{C}}^2_N$ has a "conformal compactification" to $\#_N\overline{\mathbf{CP}}^2$ and a "complex compactification" to $\widetilde{\mathbf{CP}}^2_N$ (the projective plane blown-up at N points). We have added a point x_0 in the former case and a complex projective line ℓ_∞ in the latter.

Define $\mathcal{M}^n_{\mathrm{alg},k}(\widetilde{\mathbf{CP}}^2_N)$ to be the moduli space consisting of pairs $(\mathcal{E},\tau)$ where $\mathcal{E}$ is a rank n holomorphic bundle on $\widetilde{\mathbf{CP}}^2_N$ with $c_1(\mathcal{E})=0$, $c_2(\mathcal{E})=k$, and where $\tau:\mathcal{E}|_{\ell_\infty}\to\mathbf{C}^n\otimes\mathcal{O}_{\ell_\infty}$ is a holomorphic trivialization of $\mathcal{E}$ on ℓ_∞.

There is a natural map $\Phi:\mathcal{M}^n_k(\#_N\overline{\mathbf{CP}}^2)\longrightarrow\mathcal{M}^n_{\mathrm{alg},k}(\widetilde{\mathbf{CP}}^2_N)$ defined as follows. Let $p:\widetilde{\mathbf{CP}}^2_N\to\#_N\overline{\mathbf{CP}}^2$ be the map that collapses $\ell_\infty\mapsto x_0$. If $[A]\in\mathcal{M}^n_k$ then the $\bar\partial$ operator that defines the holomorphic bundle $\mathcal{V}=\Phi(A)$ is taken to be $(d_{p^*(A)})^{(0,1)}$, the anti-holomorphic part of the covariant derivative defined by the pullback of the connection. The anti-self-duality of A implies that the curvature of $p^*(A)$ is a $(1,1)$-form and so $\bar\partial^2=0$.

Buchdahl's theorem is then

Theorem 2.1. *The map* $\Phi:\mathcal{M}^n_k(\#_N\overline{\mathbf{CP}}^2)\longrightarrow\mathcal{M}^n_{alg,k}(\widetilde{\mathbf{CP}}^2_N)$ *is a diffeomorphism.*

The case $N=1$ was first proved by King [Ki]. We now restrict ourselves to that case and simply write $\mathcal{M}^n_k$ for $\mathcal{M}^n_k(\overline{\mathbf{CP}}^2)$ and $\mathcal{M}^n_{\mathrm{alg},k}(\widetilde{\mathbf{CP}}^2)$.

King constructed $\mathcal{M}^n_k$ explicitly in terms of linear algebra data. We recall his construction. Consider configurations of linear maps:

$$\begin{tikzcd}
W_0 \arrow[rr, shift right, "x"'] & & W_1 \arrow[ll, shift right, "a_1, a_2"'] \arrow[dl, "c"] \\
& V_\infty \arrow[ul, "b"] &
\end{tikzcd}$$

where W_0, W_1 and V_∞ are complex vector spaces of dimensions k, k, and n respectively.

A configuration (a_1,a_2,b,c,x) is called *integrable* if it satisfies the equation

$$a_1xa_2-a_2xa_1+bc=0.$$

A configuration (a_1,a_2,b,c,x) is *non-degenerate* if it satisfies the following conditions:

$$\forall\ (\lambda_1,\lambda_2),(\mu_1,\mu_2)\in\mathbf{C}^2 \text{ such that } \lambda_1\mu_1+\lambda_2\mu_2=0 \text{ and } (\mu_1,\mu_2)\neq(0,0),$$

$$\nexists v\in W_1 \text{ such that } \quad \left\{\begin{array}{ll} xa_1v=\lambda_1v & (\mu_1a_1+\mu_2a_2)v=0 \\ xa_2v=\lambda_2v & cv=0 \end{array}\right.$$

$$\text{and } \nexists w\in W_0^\star \text{ such that } \quad \left\{\begin{array}{ll} x^*a_1^*w=\lambda_1w & (\mu_1a_1^*+\mu_2a_2^*)w=0 \\ x^*a_2^*w=\lambda_2w & b^*w=0 \end{array}\right.$$

Let A_k^n be the space of all integrable non-degenerate configurations. $G = Gl(W_0) \times Gl(W_1)$ acts canonically on A_k^n. The action is explicitly given by

$$(g_0, g_1) \cdot (a_1, a_2, b, c, x) = (g_0 a_1 g_1^{-1}, g_0 a_2 g_1^{-1}, g_0 b, c g_1^{-1}, g_1 x g_0^{-1})$$

Theorem 2.2. *The moduli space $\mathcal{M}_k^n$ is isomorphic to A_k^n/G.*

Proof. King uses such configurations to determine monads that in turn determine holomorphic bundles. Configurations in the same G orbit determine the same bundle. For the sake of brevity we refer the reader to [Ki] or [Br] for details. The construction identifies the vector spaces W_0 and W_1 canonically as $H^1(\mathcal{E}(-\ell_\infty))$ and $H^1(\mathcal{E}(-\ell_\infty + E))$ respectively, where $E \subset \widetilde{\mathbf{CP}}^2$ is the exceptional divisor. The vector space V_∞ is identified with the fiber over ℓ_∞.

3 Proof of theorem 1.1

We prove the theorem in two steps: We first show that the space of monad data A_k^n forms a principal $G = Gl(k) \times Gl(k)$ bundle over $\mathcal{M}_k^n$. We then show that the induced G-equivariant inclusion $A_k^n \hookrightarrow A_k^{n+2k}$ is null-homotopic so that we can conclude that A_k^∞ is contractible.

Lemma 3.1. *G acts freely on the space of monad data A_k^n.*

Proof. This is essentially proved in [Ki] where it is implicitly shown that the non-degeneracy conditions are precisely the conditions that guarantee freeness. We point out that this also follows more conceptually from the existence of a universal family $\mathbb{E} \to \mathcal{M}_k^n \times \widetilde{\mathbf{CP}}^2$ and the cohomological interpretation of W_0 and W_1:

First, the existence of a universal family can be shown via the gauge theoretic construction: Let V be a smooth hermitian vector bundle on $\widetilde{\mathbf{CP}}^2$ with $c_1(V) = 0$ and $c_2(V) = k$. Let $\mathcal{A}_0^{1,1}$ denote unitary connections on V with curvature of pure type $(1,1)$ and that restrict to the trivial connection on ℓ_∞ and let $\mathcal{G}_0^{\mathbf{C}}$ denote the complex gauge transformations of V that are the identity restricted to ℓ_∞. Then $\mathcal{M}_k^n = \mathcal{A}_0^{1,1}/\mathcal{G}_0^{\mathbf{C}}$. The quotient

$$(\mathcal{A}_0^{1,1} \times V)/\mathcal{G}_0^{\mathbf{C}} \to \mathcal{M}_k^n \times \widetilde{\mathbf{CP}}^2$$

will form a universal bundle if the moduli space is smooth and no $\mathcal{E} \in \mathcal{M}_k^n$ has non-trivial automorphisms (*c.f.* [Fr-Mo] Chapt. IV):

Lemma 3.2. *$\mathcal{M}_k^n$ is smooth and any $\mathcal{E} \in \mathcal{M}_k^n$ has no non-trivial automorphisms preserving $\tau : \mathcal{E}|_{\ell_\infty} \to \mathbf{C}^n \otimes \mathcal{O}_{\ell_\infty}$.*

By Serre duality $H^2(\mathcal{E} \otimes \mathcal{E}^*) = H^0(\mathcal{E} \otimes \mathcal{E}^* \otimes K)^*$. Since $\mathcal{E} \otimes \mathcal{E}^*$ is trivial on ℓ_∞, it is also trivial on nearby lines. Any section of $\mathcal{E} \otimes \mathcal{E}^* \otimes K$ restricts to a section of $\mathbf{C}^{n^2} \otimes \mathcal{O}_{\ell_\infty}(-3)$ and so must vanish on ℓ_∞. Likewise, it must vanish on nearby lines and so it is 0 on an open set and must be identically

0. Thus $H^2(\mathcal{E}\otimes\mathcal{E}^*)=0$ and smoothness follows once we show there are no automorphisms.

Suppose that there exists an automorphism $\phi\in H^0(\mathcal{E}\otimes\mathcal{E}^*)$ such that $\phi\neq\mathbb{1}$ and ϕ preserves τ so that $\phi|_{\ell_\infty}=\mathbb{1}|_{\ell_\infty}$. Then $\phi-\mathbb{1}$ is a non-zero section of $\mathcal{E}\otimes\mathcal{E}^*$ vanishing on ℓ_∞. We then get an injection $0\to\mathcal{O}(\ell_\infty)\to\mathcal{E}\otimes\mathcal{E}^*$. Restricting this sequence to ℓ_∞ we get an injection $0\to\mathcal{O}_{\ell_\infty}(1)\to\mathcal{O}_{\ell_\infty}\otimes\mathbf{C}^{n^2}$ which is a contradiction.

Let $\pi:\mathcal{M}_k^n\times\widetilde{\mathbf{CP}}^2\to\mathcal{M}_k^n$. The higher direct image sheaves $R^1\pi_*(\mathbb{E}(-\ell_\infty))$ and $R^1\pi_*(\mathbb{E}(-\ell_\infty+E))$ are locally free and rank k. This follows from the index theorem and the vanishing of the H^0 and H^2 cohomology of $\mathcal{E}(-\ell_\infty)$ and $\mathcal{E}(-\ell_\infty+E)$. The H^0 vanishing follows by again considering the restriction of a section of the bundles to lines nearby to ℓ_∞. Using Serre duality and the same argument, one gets the vanishing for H^2.

Thus the vector spaces W_0 and W_1 are the fibers of the vector bundles $R^1\pi_*(\mathbb{E}(-\ell_\infty))$ and $R^1\pi_*(\mathbb{E}(-\ell_\infty+E))$. The G-orbit of a configuration giving a bundle $\mathcal{E}$ can be identified with the group of isomorphisms $g_0:H^1(\mathcal{E}(-\ell_\infty))\to\mathbf{C}^k$ and $g_1:H^1(\mathcal{E}(-\ell_\infty+E))\to\mathbf{C}^k$. Thus A_k^n is realized precisely as the total space of the principal $Gl(k)\times Gl(k)$ bundle associated to $R^1\pi_*(\mathbb{E}(-\ell_\infty))\oplus R^1\pi_*(\mathbb{E}(-\ell_\infty+E))$.

□

Recall that the map $\mathcal{M}_k^n\hookrightarrow\mathcal{M}_k^{n+1}$ is defined by the direct sum with the trivial connection: $[A]\mapsto[A\oplus\theta]$. In terms of holomorphic bundles this is $\mathcal{E}\mapsto\mathcal{E}\oplus\mathcal{O}$. Tracing through the monad construction, it is easy to see that the inclusion induces the G-equivariant map $A_k^n\hookrightarrow A_k^{n+1}$ given by $(a_1,a_2,x,b,c)\mapsto(a_1,a_2,x,b',c')$ where b' is b with an extra first column of zeroes and c' is c with an extra first row of zeroes. Define A_k^∞ to be the direct limit $\lim_{n\to\infty}A_k^n$ so that there is a homeomorphism between $\mathcal{M}_k^\infty$ and A_k^∞/G

Lemma 3.3. *A_k^∞ is a contractible space.*

Proof. Since the A_k^n's are algebraic varieties and the maps $A_k^n\to A_k^{n+1}$ are algebraic, they admit triangulations compatible with the maps. Thus A_k^∞ inherits the structure of a CW-complex and so it is sufficient to show that all of its homotopy groups are zero. To this end we prove that for any k and l there is an $r>l$ such that the natural inclusion from $A_k^n\hookrightarrow A_k^r$ is homotopically trivial.

Consider the homotopy $H_t:A_k^n\to A_k^{2k+n}$ defined as follows:

$$H_t((a_1,a_2,x,b,c))=((1-t)a_1,\ (1-t)a_2,\ (1-t)x,\ b_t,\ c_t)$$

where

$$c_t=\begin{pmatrix} tI_k\\ 0_{k,k}\\ (1-t)c\end{pmatrix}\ ,\ b_t=(0_{k,k},\ tI_k,\ (1-t)^2b),$$

I_k is the $k\times k$ identity matrix and $0_{k,k}$ is the $k\times k$ zero matrix. To see that $H_t(v)\in A_k^{n+2k}$ for any $v\in A_k^n$, we check that the integrability and non-degeneracy conditions are satisfied for all $0\le t\le 1$. Integrability holds because

$b_t c_t = (1-t)^3 bc$. Non-degeneracy is satisfied for all $t \neq 0$ because there is a full rank $k \times k$ block, tI_k, in both c_t and b_t. Furthermore, H_0 is just the inclusion $A_k^n \hookrightarrow A_k^{n+2k}$, so non-degeneracy also holds when $t = 0$. Finally, note that H_1 is a constant map. $\square$

These lemmas show that A_k^∞ is a contractible space acted on freely by $G = Gl(k) \times Gl(k)$ and $A_k^\infty / G = \mathcal{M}_k^\infty$. Thus $\mathcal{M}_k^\infty$ is homotopic to BG which in turn has the homotopy type of $BU(k) \times BU(k)$. We end by remarking that the proof shows that the universal $U(k) \times U(k)$ bundle is the bundle that restricts to any of the finite $\mathcal{M}_k^n$'s as $R^1\pi_*(\mathbb{E}(-\ell_\infty)) \oplus R^1\pi_*(\mathbb{E}(-\ell_\infty + E))$.

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