THE MULTI-MONOPOLE EQUATIONS FOR KÄHLER SURFACES

JAMES A. BRYAN
RICHARD WENTWORTH

Abstract. The purpose of this paper is to introduce a natural generalization of the Seiberg-Witten equations having more than one Spinor field ("monopole") and to study the moduli space of solutions in the case of a Kähler surface. We find an explicit algebro-geometric construction of the moduli space. In the course of our construction, we prove an existence and uniqueness result for a generalization of the Kazdan-Warner equation.

1. Introduction

In the fall of 1994 Witten introduced invariants of a smooth four manifold using equations that arose in his work with Seiberg [Wi]. Since then a number of preprints on the subject have emerged, many of which include good exposition of the basic equations and results ([Br], [Sa],[Kr-Mr],[Fr-Mo],[Fi-St1], [Fi-St2], [Mo],[Ta1],[Ta2], [Ta3], etc ...). Because of the extensive recent literature, we mostly refer the reader to the papers for (the by now) standard results and limit ourselves to essentially new contributions.

The motivation for this work has been to construct topological invariants from the gauge theoretic moduli spaces of various equations that generalize the Seiberg-Witten equations. The particular equations we work with—the "multi-monopole" equations—differ from the one monopole case in that the associated moduli spaces fail to be compact. In this note we construct the moduli spaces explicitly for Kähler surfaces and we observe that there is an obvious compactification for this case. It is a first step in a program to find a compactification in a more general setting, particularly the case of almost Kähler surfaces.

The note is organized as follows. In the first section we introduce the multi-monopole equations for a general four manifold and derive the basic gauge theoretic properties of the moduli space and its configuration space. In the next section we specialize to the case of interest—namely compact, simply connected Kähler surfaces, and we formulate our main theorem constructing the moduli spaces (theorem 3.2). In the final section we prove the main theorem. In the course of the proof we show that generally the equation

$$\triangle u + Ae^{u} - Be^{-u} - w = 0$$

has a unique smooth solution for suitable A, B, and w (lemma 3.4).

2. The Equations

Let X be a compact, oriented 4-manifold with a Riemannian metric g and let L be a complex line bundle on X such that $c_1(L) \equiv w_2(TX) \mod 2$. L then defines a Spin$_{\mathbf{C}}$ structure on X consisting of a pair of U(2) bundles $W^{\pm}$ and a Clifford multiplication map $c : \Omega^1(X) \otimes \Gamma(W^{\pm}) \to \Gamma(W^{\mp})$.

Let $A \in \mathcal{A}(L)$ be a connection on L and let $\Phi_1, \dots, \Phi_N \in \Gamma(W^+)$ be N spinors. Along with the Levi-Civita connection, A induces a connection ∇_A on W^+, which, when composed with the map c, defines the Dirac operator $\partial\!\!\!/_A : \Gamma(W^+) \to \Gamma(W^-)$. The map c induces a map $c \wedge c : \Omega^2(X, \mathbf{C}) \otimes \Gamma(W^+) \to \Gamma(W^+)$. This map is zero on $\Omega^2_-(X)$ and for any 2-form, the induced endomorphism of $\Gamma(W^+)$ is tracefree and skew-hermitian on real forms and hermitian on pure imaginary forms. We call the resulting map

$$\rho : \Omega^2_+(X, i\mathbf{R}) \to \Gamma(\mathfrak{sl}(W^+)).$$

Let $(A, \Phi_1, \dots, \Phi_N)$ be as above and define *multi-monopole equations with perturbation* η to be the system

$$\begin{aligned} \partial\!\!\!/_A \Phi_i &= 0 \\ \rho(F_A^+ + i\eta) &= \sum_{j=1}^{N} (\Phi_j \otimes \Phi_j^* - \tfrac{1}{2}|\Phi_j|^2 \mathbb{1}) \end{aligned} \tag{1}$$

where $\mathbb{1}$ denotes the identity endomorphism on W^+ and $\eta \in \Omega^2_+(X)$ is a real self dual two form.

These equations are invariant under the gauge group $\mathcal{G} = \mathrm{Map}(X, S^1)$ of automorphisms of L and we define configuration space to be

$$\mathcal{C} = (\mathcal{A}(L) \times \oplus_N \Gamma(W^+))/\mathcal{G}.$$

We denote by $\mathcal{M}_L \subset \mathcal{C}$ the moduli space of solutions to the multi-monopole equations modulo gauge equivalence. (We will normally drop the explicit dependence on X, g, and η.)

As in the one monopole case, one gets a good deformation theory. The linearized equations at a solution, along with a local gauge fixing condition give rise to local Kuranishi model of $\mathcal{M}_L$ and a model of the (virtual) tangent space at a solution $(A, \Phi_1, \dots, \Phi_N)$ as the kernel of a first order elliptic operator. The operator is a coupled operator that is, to first order, equal to the uncoupled operator:

$$\oplus_N \partial\!\!\!/_A \oplus d^* \oplus d^+ : \oplus_N \Gamma(W^+) \oplus \Omega^1(X) \to \oplus_N \Gamma(W^-) \oplus \Omega^0(X) \oplus \Omega^2_+(X).$$

The real index of this operator gives the virtual dimension of $\mathcal{M}_L$ and it is given by

$$\begin{aligned} \mathrm{VirDim}(\mathcal{M}_L) &= \tfrac{N}{4}(c_1(L)^2 - \sigma) - \tfrac{1}{2}(\chi + \sigma) \\ &= \tfrac{1}{4}(N c_1(L)^2 - (2\chi + (N+2)\sigma)). \end{aligned} \tag{2}$$

Of course, in order to get a good local theory, one must complete the relevant function spaces in the appropriate Sobolev norms. As in the one spinor case, everything works with L^p_1 configurations and L^p_2 gauge transformations where $p > 2 = \dim X/2$.

Solutions to equations 1 can be described as the minimum an energy functional on configuration space. We define $\mathcal{E} : \mathcal{C} \to \mathbf{R}$ by

$$\mathcal{E}(A, \Phi_1, \dots, \Phi_N) = \int_X \tfrac{1}{4}|\rho(F_A^+ + i\eta) - \sum_{j=1}^{N} (\Phi_j \otimes \Phi_j^*)_0|^2 + \sum_{j=1}^{N} |\partial\!\!\!/_A \Phi_j|^2 \tag{3}$$

where we have used the notation $(\cdot)_0$ to indicate the traceless part of an endomorphism so to abbreviate the right hand side of the last equation in 1. As in the one monopole case, the Weitzenböck formula can be used to manipulate the integrand of the energy functional. One gets

$$\begin{aligned}\mathcal{E}(A,\Phi_1,\dots,\Phi_N) &= \int_X \tfrac{1}{4}|F_A^+ + i\eta|^2 + \sum_{j=1}^N |\nabla_A\Phi_j|^2 + \tfrac{s}{4}|\Phi_j|^2 \\ &\quad + \tfrac{1}{4}\left|\sum_{j=1}^N (\Phi_j\otimes\Phi_j^*)_0 - \rho(i\eta)\right|^2 - \tfrac{1}{4}|\eta|^2 \end{aligned} \tag{4}$$

where s is the scalar curvature.[1]

When $N=1$, one can complete the square in the integrand of equation 4 to get the famous *a priori* L^2 bound on the curvature and deduce the compactness of the moduli spaces. For $N>1$ the multi-monopole moduli spaces are *not* compact in general. We do have, though, the following

Lemma 2.1. *Let $f:\mathcal{M}_L\to\mathbf{R}^+$ be the total L^2 norm of the sections,* i.e.

$$f(A,\Phi_1,\dots,\Phi_N)=\sum_{j=1}^N ||\Phi_j||^2.$$

Then f is proper.

Proof. We wish to show that $f^{-1}([0,c])$ is compact. Consider a configuration $(A,\Phi_1,\dots,\Phi_N)\in f^{-1}([0,c])$. Equation 4 gives us a bound

$$\int |F_A^+|^2 \le C(s,\eta,\text{volume}(X),c).$$

Since $||F_A^+||^2-||F_A^-||^2$ is a topological constant proportional to $c_1(L)\cdot c_1(L)$, we get an *a priori* bound on $||F_A||^2$. Any sequence of solutions in $f^{-1}([0,c])$ will have a subsequence of connections that converge smoothly up to gauge by Uhlenbeck's theorem (which is just Hodge theory in this case). Then by standard elliptic regularity and bootstrapping, the Φ's will also have a smooth convergent subsequence. □

3. The Kähler Case

We restrict now to the case of interest. Assume that X is a simply connected Kähler surface with Kähler form ω. Let $\Lambda^{p,q}$ denote the bundle of differential forms of type (p,q). There is a canonical $\mathrm{Spin}_{\mathbf{C}}$ structure with connection on X associated to the anti-canonical line bundle $K^{-1}=\Lambda^{0,2}$. The associated $\mathrm{Spin}_{\mathbf{C}}$ bundles are

$$W_0^+\cong\Lambda^0\oplus\Lambda^{0,2}$$

and

$$W_0^-\cong\Lambda^{0,1},$$

and the canonical connection ∇_0 is given by restriction of the Levi-Civita connection. The following is well known ([Sa],[Br]):

Lemma 3.1. *For every $\mathrm{Spin}_{\mathbf{C}}$ structure L and connection $A_L\in\mathcal{A}(L)$ there is a unique pair E and $A_E\in\mathcal{A}(E)$ such that the $\mathrm{Spin}_{\mathbf{C}}$ structure with connection is given by $W_0^\pm\otimes E$ and $\nabla_0\otimes A_E$.*

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[1] The constants involved in this and similar formulas in the literature vary depending on various conventions that are used. We use conventions with $|\mathbb{1}|^2=\mathrm{tr}(\mathbb{1})=2$ and such that the map ρ is a pointwise isometry.

We can thus rewrite a spinor Φ_i as a pair (α_i, β_i) where $\alpha_i \in \Omega^0(E)$ and $\beta_i \in \Omega^{0,2}(E)$. Further we denote by $\beta_i^* \in \Omega^{2,0}(E^*) = \Gamma(K \otimes E^*)$ the E^*-valued $(2,0)$-form such that $\beta_i \wedge \beta_i^* = |\beta_i|^2 \mathrm{vol}$ and we denote by $\alpha_i^* \in \Omega^0(E^*)$ the section dual to α_i .

It is convenient to rewrite the equations in terms of connections on E and the α and β^* variables. We redefine configuration space so that

$$[A, \alpha_1, \dots, \alpha_N, \beta_1^*, \dots, \beta_N^*] \in \mathcal{C} = \mathcal{A}(E) \times (\oplus_N \Gamma(E) \oplus_N \Gamma(K \otimes E^*))/\mathcal{G}.$$

Now a gauge transformation $\lambda \in \mathcal{G} = \mathrm{Map}(X, S^1)$ acts on a configuration by

(5)

$$\lambda(A, \alpha_1, \dots, \alpha_N, \beta_1^*, \dots, \beta_N^*) = (A + \lambda^{-1} d\lambda, \lambda\alpha_1, \dots, \lambda\alpha_N, \lambda^{-1}\beta_1^*, \dots, \lambda^{-1}\beta_N^*).$$

On the canonical $\mathrm{Spin}_{\mathbf{C}}$ structure, the canonical connection induces a canonical Dirac operator $\partial\!\!\!/_0 : \Omega^0 \oplus \Omega^{0,2} \to \Omega^{0,1}$ which is given by $\sqrt{2}(\bar\partial + \bar\partial^*)$. The self-dual two forms split as $\Omega^2_+ \cong [\omega]\Omega^0 \oplus (\Omega^{2,0} \oplus \Omega^{0,2})_{\mathbf{R}}$ and the map ρ maps the ω factor into the diagonal and the other factors into the off diagonal terms (explicitly, $\rho(i\omega)\alpha = \alpha$ and $\rho(i\omega)\beta = -\beta$.) Consider the multi-monopole equations with perturbation $r\omega$ where r is a real number. With the above notation and identifications they can be rewritten

$$\bar\partial_A \alpha_i + \bar\partial_A^* \beta_i = 0 \tag{6}$$

$$2F_A^{0,2} = \alpha_1^* \beta_1 + \dots + \alpha_N^* \beta_N \tag{7}$$

$$-2i\Lambda_\omega F_A = \tfrac{1}{2} \sum_{j=1}^{N} (|\alpha_j|^2 - |\beta_j|^2) - i\Lambda_\omega F_{\nabla_0} - r. \tag{8}$$

Integrating the equation 8 over $\omega \wedge \omega$ we see that solutions to the multi-monopole equations satisfy

$$2\pi c_1(L) \cdot [\omega] + r[\omega] \cdot [\omega] = \sum_{i=1}^{N} ||\alpha_i||^2 - ||\beta_i||^2 \tag{9}$$

We see that if r is sufficiently large, then $(\alpha_1, \dots, \alpha_N) \neq (0, \dots, 0)$.

Remark 3.1. The decomposition of spinors into the alpha and beta components exist for any almost complex structure. It was first observed by Taubes [Ta1] that for *almost Kähler* manifolds, *i.e.* symplectic manifolds with a compatible almost complex structure, the Seiberg-Witten equations are still given by equations 6-8.

The main result of this note is the explicit construction of the moduli space:

Theorem 3.2. *Let X be a simply connected Kähler surface as above and let $\mathcal{M}_L$ be the moduli space of solutions to the multi-monopole equations with perturbation $r\omega$ (eq.'s 6-8). If $\mathcal{M}_L \neq \emptyset$, then E is a holomorphic line bundle. Let $V_1 = \oplus_N H^0(K \otimes E^*)$ and $V_2 = \oplus_N H^0(E)$. For r large enough such that $2\pi c_1(L) \cdot [\omega] + r[\omega] \cdot [\omega] > 0$, $\mathcal{M}_L$ has the following description:*

Consider the affine variety $Z \subset V_1 \oplus V_2$ given by the zero set of the map $S : V_1 \oplus V_2 \to H^0(K)$ defined by

$$S(\beta_1^*, \dots, \beta_N^*, \alpha_1, \dots, \alpha_N) = \alpha_1 \beta_1^* + \dots + \alpha_N \beta_N^*.$$

Let $Z^s \subset Z$ be the points with $(\alpha_1, \dots, \alpha_N) \neq (0, \dots, 0)$.

Then $\mathcal{M}_L = Z^s/\mathbf{C}^*$ where $\lambda \in \mathbf{C}^*$ acts on V_1 and V_2 by multiplication by λ and λ^{-1} respectively.

Remark 3.2. A similar result holds for $2\pi c_1(L)\cdot[\omega]+r[\omega]\cdot[\omega]<0$ with the roles of V_1 and V_2 reversing and the condition $(\alpha_1,\dots,\alpha_N)\neq(0,\dots,0)$ changing to $(\beta_1,\dots,\beta_N)\neq(0,\dots,0)$.

Remark 3.3. $\mathcal{M}_L$ is, by the theorem, a subvariety of the quotient

$$U=V_1\oplus(V_2-\{0\})/\mathbf{C}^*.$$

U is the total space of the bundle

$$\begin{array}{c}\mathcal{O}(-1)\otimes V_1\\ \Big\downarrow{\scriptstyle\pi}\\ \mathbf{P}(V_2).\end{array}$$

Consider the bundle $\pi^{-1}(\mathcal{O}_{\mathbf{P}(V_2)}(1)\otimes H^0(K))\to U$. S defines a section of this bundle and $\mathcal{M}_L$ is the zero set of this section.

Remark 3.4. Since the manifold U of the above remark is the total space of a bundle, it has a natural compactification $\overline{U}=\mathbf{P}(\mathcal{O}(-1)\otimes(V_1\oplus\mathbf{C}))$. We can thus define a compactification $\overline{\mathcal{M}}_L$ of $\mathcal{M}_L$ by taking the closure of $\mathcal{M}_L$ in $\overline{U}$.

Remark 3.5. In the Kähler case, the virtual dimension of $\mathcal{M}_L$ is even and can be computed by rewriting equation 2 or applying the index formula to the equations 6-8:

$$\begin{aligned}\mathrm{VirDim}_{\mathbf{C}}(\mathcal{M}_L) &= NE\cdot(E-K)+(N-1)(1+p_g)\\ &= N(h^0(E)-h^1(E)+h^2(E))-(1+h^0(K)).\end{aligned}$$

We see from above that $\mathcal{M}_L$ is smooth and has the expected dimension whenever $h^1(E)=0$ and 0 is a regular value of S.

3.1. **Proof of Theorem 3.2.** The proof of theorem 3.2 is a fairly straightforward application of what has become a common paradigm in gauge theory: generalizing the finite dimensional correspondence between symplectic and algebraic quotients to the infinite dimensional setting. This is formalized in lemma 3.3.

The moduli space is contained in the *harmonic configurations* which is defined to be the (gauge invariant) subspace of $\mathcal{A}\times\oplus_N\Gamma(W^+)$ defined by

$$\mathcal{H}=\{(A,\Phi_1,\dots,\Phi_N)\in\mathcal{A}\times\oplus_N\Gamma(W^+):\partial\!\!\!/_A\Phi_j=0\text{ for all }j\}.$$

In the case of a Kähler metric we will show that the moduli space actually lies in the *holomorphic-harmonic configurations* defined to be

$$\mathcal{H}^{1,1}=\{(A,\Phi_1,\dots,\Phi_N)\in\mathcal{H}:F_A\text{ is of pure type }(1,1)\}.$$

The standard proof that Seiberg-Witten solutions have vanishing $F_A^{0,2}$ directly generalizes. Applying $\bar\partial_A$ to equation 6 and taking the L^2 inner product with β_i we get

$$\int_X\langle F_A^{0,2}\alpha_i,\beta_i\rangle+\langle\bar\partial_A\bar\partial_A^*\beta_i,\beta_i\rangle=0.$$

Summing over i and applying the $(0,2)$ part of the curvature equation we get

$$\int_X\tfrac12|\alpha_1^*\beta_1+\cdots+\alpha_N^*\beta_N|^2+\sum_{i=1}^N|\bar\partial_A^*\beta_i|^2=0\tag{10}$$

so that $F_A^{0,2}$, $\bar\partial_A^*\beta_i$, and $\bar\partial_A\alpha_i$ must all identically vanish.

We have thus shown that

$$\mathcal{M}_L = \mathcal{H}^{1,1} \cap \{\text{solutions to equation 8 and } \sum_j \alpha_j \beta_j^* = 0\}/\mathcal{G}. \tag{11}$$

Now $\mathcal{H}^{1,1}$ is acted on by the *complex gauge group* which is defined to be $\mathcal{G}^{\mathbf{C}} = \mathrm{Map}(X, \mathbf{C}^*)$. A complex gauge transformation e^f acts on configurations by

$$\begin{array}{rcl} A & \mapsto & A + \partial \overline{f} - \bar{\partial} f \\ \alpha_j & \mapsto & e^f \alpha_j \\ \beta_j & \mapsto & e^{-\overline{f}} \beta_j. \end{array}$$

When f is pure imaginary this coincides with the usual action of $\mathcal{G}$, and the induced action $\beta_j^* \mapsto e^{-f}\beta_j^*$ is the usual action of $\mathcal{G}^{\mathbf{C}}$ on $K \otimes E^{-1}$. It is easy to verify that $\mathcal{H}^{1,1}$ is preserved by this action.

The main lemma is

Lemma 3.3. *Let $\mathcal{H}_s^{1,1} \subset \mathcal{H}^{1,1}$ be the set with $(\alpha_1, \ldots, \alpha_N) \neq 0$. Then*

$$\begin{array}{rcl} \mathcal{H}^{1,1} \cap \{\textit{solutions to equation 8}\}/\mathcal{G} & \cong & \mathcal{H}_s^{1,1}/\mathcal{G}^{\mathbf{C}} \\ & \cong & V_1 \oplus (V_2 - \{0\})/\mathbf{C}^* \end{array}$$

where V_1 and V_2 are defined as in theorem 3.2.

This lemma is an infinite dimensional analogue of a well understood finite dimensional principle: suppose V is a smooth projective variety with a (linearized) holomorphic action of a reductive group $G^{\mathbf{C}}$ and μ is the (in this case uniquely determined) moment map for the action of the compact group G, then

$$V \cap \mu^{-1}(0)/G \cong V^s/G^{\mathbf{C}}$$

where $V^s \subset V$ is a certain dense open set. Thus the "algebraic quotient" and the "symplectic quotient" agree.

In the case of lemma 3.3, $\mathcal{H}^{1,1}$ is formally an algebraic variety and equation 8 is formally the zero set of the moment map associated to the action of $\mathcal{G}$. Theorem 3.2 follows directly from the lemma.

Proof of lemma 3.3. The diffeomorphism $\mathcal{H}_s^{1,1}/\mathcal{G}^{\mathbf{C}} \cong V_1 \oplus (V_2 - \{0\})/\mathbf{C}^*$ follows from the fact that on a simply connected Kähler manifold $\mathcal{A}^{1,1}/\mathcal{G}^{\mathbf{C}}$ is a single point given by the equivalence class of the unique holomorphic structure on E. The stabilizer of the action of $\mathcal{G}^{\mathbf{C}}$ on $\mathcal{A}^{1,1}$ is exactly the constant complex gauge transformations. Furthermore, following the same argument that led to equation 10 we see that spinors in $\mathcal{H}^{1,1}$ actually consist of holomorphic sections of E and $K \otimes E^*$. Thus the projection of $\mathcal{H}_s^{1,1}$ onto $\mathcal{A}^{1,1}$ induces a map of $\mathcal{H}_s^{1,1}/\mathcal{G}^{\mathbf{C}}$ to a point with fiber $V_1 \oplus (V_2 - \{0\})/\mathbf{C}^*$.

To prove the first diffeomorphism of the lemma we must show that on every $\mathcal{G}^{\mathbf{C}}$-orbit in $\mathcal{H}_s^{1,1}$ there is exactly one $\mathcal{G}$-orbit of solutions to equation 8. (Equation 9 and the assumption of the theorem guarantee that solutions to equation 8 will have $\|\alpha\| \neq 0$.)

If we fix an arbitrary member $(A, \alpha_1, \ldots, \alpha_N, \beta_1^*, \ldots, \beta_N^*)$ of $\mathcal{H}_s^{1,1}$, we can write equation 8 for the gauge transformed solution and regard it as an equation for f. Since equation 8 is invariant for $e^f \in \mathcal{G}$ (*i.e.* f is pure imaginary), we only consider

$e^f \in \mathcal{G}^{\mathbf{C}}$ with f pure real. Equation 8 then becomes

$$-2i\Lambda_\omega F_{A+\partial f-\bar\partial f} = \tfrac{1}{2}\sum_{j=1}^{N}(e^{2f}|\alpha_j|^2 - e^{-2f}|\beta_j|^2) - i\Lambda_\omega F_{\nabla_0} - r.$$

If we set $A = |\alpha_1|^2 + \cdots + |\alpha_N|^2$, $B = |\beta_1|^2 + \cdots + |\beta_N|^2$, and $w = -4i\Lambda_\omega F_{\nabla_0} + 2i\Lambda_\omega F_A + 2r$ and we use the Kähler identity $-4i\Lambda_\omega\partial\bar\partial f = d^*df = \triangle f$, we can rewrite the above equation as

$$2\triangle f + Ae^{2f} - Be^{-2f} - w = 0. \tag{12}$$

We need to show that there is a unique f solving equation 12. Observe that A and B are non-negative functions and $\int w = 2\pi c_1(L)\cdot[\omega] + r[\omega]\cdot[\omega] > 0$. Also by equation 9 we have $\int A - B > 0$.

Recall that we have implicitly taken Banach space completions of the various configuration spaces and gauge groups in the appropriate norms, and we would prove existence and uniqueness of equation 12 in the Sobolev setting. However, by evoking the diffeomorphism $\mathcal{H}^{1,1}_s/\mathcal{G}^{\mathbf{C}} \cong V_1 \oplus (V_2 - \{0\})/\mathbf{C}^*$ (which uses regularity for the $\bar\partial$ Laplacian) we can assume that our representative for the $\mathcal{G}^{\mathbf{C}}$-orbit is smooth. Thus it only remains for us to prove the following

Lemma 3.4. *Let X be a compact Riemannian manifold (of any dimension) and let A, B, and w be smooth functions with A and B non-negative, $\int A - B > 0$, and $\int w > 0$. Let $\triangle = d^*d$ be the positive definite Laplacian on X. Then the equation*

$$\triangle u + Ae^{u} - Be^{-u} - w = 0 \tag{13}$$

has a unique C^∞ solution.

When $B \equiv 0$, equation 13 reduces to the equation that arises in the usual Seiberg-Witten theory and was studied by Kazdan and Warner in [Ka-Wa] (who were primarily interested in the two dimensional case where the equation has geometric significance in the prescribed curvature problem). When $B \neq 0$, the equation has somewhat different characteristics, but our proof is inspired by the methods of [Ka-Wa].

The outline of our proof is as follows:

1. We construct a sub-solution and a super-solution, *i.e.* smooth functions u_- and u_+ such that $u_+ > u_-$ everywhere and
$$\begin{aligned}\triangle u_- + Ae^{u_-} - Be^{-u_-} - w &< 0\\ \triangle u_+ + Ae^{u_+} - Be^{-u_+} - w &> 0.\end{aligned}$$
2. We define a sequence of functions inductively by setting $u_0 = u_-$ and defining u_{i+1} to be the unique solution to
$$Lu_{i+1} = -Ae^{u_i} + Be^{-u_i} + w + ku_i \tag{14}$$
where L is the linear operator $L\phi = \triangle\phi + k\phi$ and k is a suitably defined non-negative function.
3. We show that $u_- = u_0 \le u_1 \le \cdots \le u_i \le \cdots \le u_+$ and we show that $\{u_i\}$ converges to a smooth unique solution of equation 13. [2]

[2]For a general theorem producing solutions to equations of the form $\triangle u = F(u,x)$ on compact manifolds, given the existence of sub- and super-solutions, see Chapt. 5, Prop. 1.1 in [Sc-Ya]. Though their method works in our case, our method will produce a solution to 13 in the non-compact case as well (given the existence of sub- and super-solutions).

$\triangle v = f$ has a solution whenever $\int f = 0$. To construct u_+, let v_1 and v_2 be solutions to $\triangle v_1 = w - \overline{w}$ and $\triangle v_2 = c - A + B$ where $c = \int A - B > 0$ and $\overline{w} = \int w$. Choose a to be a constant large enough so that $ac > \overline{w}$ and then choose a constant b large enough so that $e^{v_1 + av_2 + b} - a > 0$ and $a - e^{-v_1 - av_2 - b} > 0$. Let $u_+ = v_1 + av_2 + b$ and we have

$$\triangle u_+ + Ae^{u_+} - Be^{-u_+} - w = (ac - \overline{w}) + A(e^{v_1 + av_2 + b} - a) + B(a - e^{-v_1 - av_2 - b}) > 0.$$

We can then simply define $u_- = v_1 - n$ with n a large enough constant so that $u_- < u_+$ and $-\overline{w} + Ae^{v_1 - n} < 0$. u_- will then be a sub-solution.

We now wish to define our monotonic sequence of functions u_i. Two fundamental facts concerning the linear operator $L(*) = \triangle(*) + k \cdot (*)$ when k is a smooth quasi-positive function are:

- **Bijection:** For any given smooth function g the equation $L(f) = g$ has a unique smooth solution f.
- **Maximum Principle:** If $L(f) \geq 0$ then $f \geq 0$.

These are proven in section 3 of [Ka-Wa], though beware that our Laplacian has the opposite sign as the one used in [Ka-Wa].

Let $k = Ae^{u_+} + Be^{-u_-}$. We define $u_0 = u_-$ and inductively define u_{i+1} to be the unique solution to equation 14. We will prove the monotonicity of the sequence $\{u_i\}$ by inductively applying the maximum principle. We compute

$$\begin{aligned}
L(u_+ - u_i) &= \triangle u_+ + (Ae^{u_+} + Be^{-u_-})(u_+ - u_{i-1}) \\
&\quad + Ae^{u_{i-1}} - Be^{-u_{i-1}} - w \\
&> A(e^{u_{i-1}} - e^{u_+}) + B(e^{-u_+} - e^{-u_{i-1}}) \\
&\quad + (Ae^{u_+} + Be^{-u_-})(u_+ - u_{i-1}).
\end{aligned}$$

We rewrite this inequality in a convenient form and do similar computations and rearrangements to obtain the following inequalities:

$$\begin{aligned}
L(u_+ - u_i) &> Ae^{u_+}\left\{e^{-(u_+ - u_{i-1})} - e^{-(u_+ - u_+)} + (u_+ - u_{i-1}) - (u_+ - u_+)\right\} \\
&\quad + Be^{-u_-}\left\{e^{-(u_+ - u_-)} - e^{-(u_{i-1} - u_-)} + (u_+ - u_-) - (u_{i-1} - u_-)\right\} \\
L(u_i - u_-) &> Ae^{u_+}\left\{e^{-(u_+ - u_-)} - e^{-(u_+ - u_{i-1})} + (u_+ - u_-) - (u_+ - u_{i-1})\right\} \\
&\quad + Be^{-u_-}\left\{e^{-(u_{i-1} - u_-)} - e^{-(u_- - u_-)} + (u_{i-1} - u_-) - (u_- - u_-)\right\} \\
L(u_{i+1} - u_i) &> Ae^{u_+}\left\{e^{-(u_+ - u_{i-1})} - e^{-(u_+ - u_i)} + (u_+ - u_{i-1}) - (u_+ - u_i)\right\} \\
&\quad + Be^{-u_-}\left\{e^{-(u_i - u_-)} - e^{-(u_{i-1} - u_-)} + (u_i - u_-) - (u_{i-1} - u_-)\right\}
\end{aligned}$$

We will use the fact that $e^{-P} - e^{-Q} + P - Q \geq 0$ whenever $P, Q \geq 0$ and $P \geq Q$ (all the bracketed terms in the above inequalities have this form). It can thus be proven that $u_+ \geq u_i$ and $u_i \geq u_-$ for i simultaneously by induction on i using the first two inequalities and the maximum principle. $u_{i+1} \geq u_i$ subsequently follows in the same manner.

Since we have uniform upper and lower bounds on u_i, we additionally get a uniform bound on the derivative of u_i using the elliptic estimate for L and the

Sobolev inequalities: let $p > 4 = \dim X$, then

$$\begin{aligned} \|\nabla u_i\|_\infty &\leq C_1 \|u_i\|_{p,2} \\ &\leq C_2(\|Lu_i\|_p + \|u_i\|_p) \\ &= C_2(\| - Ae^{u_{i-1}} + Be^{-u_{i-1}} + w + ku_{i-1}\|_p + \|u_i\|_p) \\ &\leq C_3. \end{aligned}$$

It then follows that the sequence $\{u_i\}$ converges uniformly to a smooth function u_∞ and by construction it is a solution to equation 13.

The solution must be unique: suppose there are two smooth solutions to equation 13, u and u'. We have

$$\begin{aligned} 0 &\leq \|d(u-u')\|^2 \\ &= \int_X \langle \triangle(u-u'), u-u' \rangle \\ &= \int_X -A(e^u - e^{u'})(u-u') + B(e^{-u} - e^{-u'})(u-u') \end{aligned}$$

but both terms in the integral are everywhere non-positive so we can conclude that $d(u-u') \equiv 0$. But if u and u' differ by a non-zero constant then the above integral is strictly negative and thus it must be the case that $u = u'$.

□

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