A SIMPLE IDENTITY

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Lemma 1. *Let $a > 1$ and s be integers and let $\omega = \exp\left(\frac{2\pi i}{a}\right)$. Then*

$$\frac{1}{a}\sum_{l=1}^{a-1}\frac{\omega^{ls}}{1-\omega^{-k}} = \left\lfloor \frac{s}{a} \right\rfloor - \frac{s}{a} + \frac{a-1}{2a}$$

Proof. Using

$$\frac{1}{a}\sum_{n=0}^{a-1}\omega^{ns} = \begin{cases} 0 & s \neq 0 \mod a \\ 1 & s = 0 \mod a, \end{cases}$$

the identity

$$\left\lfloor \frac{s}{a} \right\rfloor = \frac{1}{a^2}\sum_{k,l=0}^{a-1}(s-k)\omega^{l(s-k)}$$

follows easily from writing $s = s_0 + na$ with $0 \leq s_0 \leq a-1$.

Separating out the $l = 0$ term, we get

$$\left\lfloor \frac{s}{a} \right\rfloor = \frac{s}{a} - \frac{1}{a^2}\binom{a}{2} + \frac{1}{a^2}\sum_{l=1}^{a-1}\sum_{k=0}^{a-1}(s-k)\omega^{l(s-k)}.$$

Consequently, to prove the lemma, it suffices to show that

$$\frac{1}{a}\sum_{k=0}^{a-1}(s-k)\omega^{l(s-k)} = \frac{\omega^{ls}}{1-\omega^{-l}}.$$

To this end, we rewrite the corresponding polynomial function:

$$\begin{aligned} \frac{1}{a}\sum_{k=0}^{a-1}(s-k)x^{l(s-k)} &= \frac{1}{la}x\frac{d}{dx}\left(\sum_{k=0}^{a-1}x^{l(s-k)}\right) \\ &= \frac{x}{la}\frac{d}{dx}\left(\frac{x^{ls}-x^{l(s-a)}}{1-x^{-l}}\right) \\ &= \frac{(sx^{ls}-(s-a)x^{l(s-a)})(1-x^{-l}) - x^{-l}(x^{ls}-x^{l(s-a)})}{a(1-x^{-l})^2}. \end{aligned}$$

Evaluating the above at $x = \omega$, we get

$$\frac{a\omega^{ls}(1-\omega^{-l})}{(1-\omega^{-l})^2} = \frac{\omega^{ls}}{1-\omega^{-l}}$$

and the lemma is proved. $\square$