Seiberg-Witten à la Furuta and genus bounds for classes with divisibility

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1. Introduction

An important classical question in 4-manifold topology asks for a lower bound on the genus of an embedded surface $\Sigma$ in terms of its homology class $[\Sigma] \in H_2(X, \mathbb{Z})$. For example, the classical Thom conjecture (proved by Kronheimer and Mrowka [5]) states that if $X = \mathbb{C}P^2$ and $\Sigma \hookrightarrow \mathbb{C}P^2$ is a smooth embedding with $[\Sigma] = d[H]$ where $[H] \in H_2(\mathbb{C}P^2)$ is the hyperplane class and $d > 0$ then

$$g(\Sigma) \geq \frac{(d - 1)(d - 2)}{2}.$$ 

A generalization of this inequality exists for manifolds with non-zero Seiberg-Witten invariants and gives bounds in terms of the self-intersection of $\Sigma$ and the pairings of $\Sigma$ with the Seiberg-Witten basic classes (see [7]).

These methods break down for manifolds with zero Seiberg-Witten invariants which includes those manifolds that decompose as connected sums $X = X_1 \# X_2$ with $b_+(X_i) > 0$. For example, an unknown question related to the $\frac{11}{4}$-th conjecture is

**Question 1.** Does the connected sum of $n$ copies of the K3 surface split off an $S^2 \times S^2$?

If one had strong enough genus bounds one could potentially give a negative answer to the above question by using the embedded surfaces in the $S^2 \times S^2$ summand.

Another interesting example is $\mathbb{C}P^2 \# \cdots \# \mathbb{C}P^2$. Mikhailin [6] has shown that the genus-minimizing surfaces in $\mathbb{C}P^2$ can have their genus reduced further after direct sum with additional copies of $\mathbb{C}P^2$. With better bounds on manifolds such as $\mathbb{C}P^2 \# \cdots \# \mathbb{C}P^2$ we could determine if Mikhailin’s examples are sharp.

Before gauge theoretic methods were introduced into 4-manifold topology, genus bounds were obtained by assuming divisibility conditions on the class of $\Sigma$ and studying the branched cover ([9],[8]). This idea was combined with the gauge theoretical methods of the Yang-Mills equations by Kotschick and Matic [4].

In this short note we apply Furuta’s “$\frac{10}{9}$-th’s theorem” to the classical techniques of branched covers to obtain genus bounds. We also outline a strategy for generalizing Furuta’s technique in this setting to improve the genus bounds. We lay some groundwork for implementing that strategy.

Furuta’s theorem is the following important non-existence result for spin four-manifolds:
Theorem 1.1 (Furuta [3]). Let $X$ be a smooth 4-manifold with intersection from

$$Q_X = 2kE_8 \oplus m\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $k > 0$. Then $2k + 1 \leq m$.

To obtain genus bounds on $\Sigma \hookrightarrow X$, we make homological assumptions on the class of $\Sigma$ so that there is a $q$-fold cover $Y \to X$ branched along $\Sigma$ and such that $Y$ is spin. We get a genus bound on $\Sigma$ by applying Theorem 1.1 to $Y$:

**Proposition 1.2.** Let $\Sigma \hookrightarrow X$ be a smooth embedding of a surface of genus $g$ and self-intersection $[\Sigma] : [\Sigma] = n$. Let $q = p^r$ be a prime power such that $q|[\Sigma]$ and let $Y \to X$ be the $q$-fold branched cover branched along $\Sigma$. Further suppose if $q$ is even that $PD([\Sigma]) \equiv w_2(X) \mod 2$ and if $q$ is odd suppose that $X$ is spin (these conditions guarantee that $Y$ is spin). Then

$$g(\Sigma) \geq 1 + \frac{5n(1 + q)}{24q} + \frac{2}{q - 1} - \frac{q}{2(q - 1)}(e_X + \frac{5}{4} \sigma_X)$$

where $e_X$ and $\sigma_X$ are the Euler characteristic and signature of $X$ respectively.

On a spin 4-manifold, the Seiberg-Witten moduli space for the trivial Spin$^C$ has an action of Pin(2). Furuta’s technique produces $Pin(2)$ representations $V$ and $W$ such that the virtual representation $[V] - [W]$ is equivalent to $[\text{Ker } D] - [\text{Coker } D]$ where $D$ is the operator obtained by linearizing the Seiberg-Witten equations at the trivial solution. Furthermore, his technique then produces a map

$$f : (B(V), S(V)) \to (B(W), S(W))$$

where $B(\cdot)$ and $S(\cdot)$ denote the unit ball and sphere respectively.

By applying the equivariant $K$-theory functor to the above map, and employing operations in $K$-theory, Furuta deduces Theorem 1.1.

To improve the bound of Theorem 1.2, we outline the following strategy: Choose a $\mathbb{Z}/q$-invariant metric on $Y$, and try to lift the action of $\mathbb{Z}/q$ to the spin bundle. In this note, we show that when $q$ is $2^r$ the action lifts to a $\mathbb{Z}/2q$ action on the spin bundle. Consequently, one obtains a Seiberg-Witten moduli space with a $\mathbb{Z}/q \times Pin(2)$-action where $\times$ denotes a twisted product (see Section 3).

Furuta’s technique in this setting should give a $\mathbb{Z}/q \times Pin(2)$ equivariant map $f : (B(V), S(V)) \to (B(W), S(W))$ where again $[V] - [W]$ represents the $K$-theoretic index of the linearized Seiberg-Witten equations. The characters of $[V] - [W]$ can be determined by the $G$-index theorem and computations show that the virtual representation $[V] - [W]$ can be completely determined and depends in on the genus of $\Sigma$ and its self-intersection $[\Sigma] : [\Sigma]$.

Equivariant $K$-theory methods applied to the map $f$ should then give a refinement of Proposition 1.2.
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2. Branched covers

Let $\Sigma \to X$ be an embedding of an oriented surface of genus $g = g(\Sigma)$ into an oriented, simply-connected smooth four manifold $X$. We assume that the class $[\Sigma] \in H_2(X, \mathbb{Z})$ is divisible by a prime power $q = p^r$ so we can consider the ramified cover

$$f : Y \to X$$

that is generically $q$-to-one and branched along $\Sigma$. $Y$ is constructed as follows:

Let $\nu$ denote a tubular neighborhood of $\Sigma$ and $\partial \nu$ its boundary; let $W = X \setminus \Sigma$. It is shown in [9] and [8] that $H_1(W, \mathbb{Z}) = \mathbb{Z}/d$ where $d = \max(a \in \mathbb{N} : a|\langle [\Sigma] \rangle)$ is the divisibility of $[\Sigma]$. Let $W \to W$ be the regular covering associated to the homomorphism

$$\pi_1(W) \to H_1(W) \to \mathbb{Z}/q.$$ 

Since the normal bundle $N_{\Sigma}$ has degree $[\Sigma]^2 = n$ and $q|n$ there is a $q$-fold cover of $N_{\Sigma}$ by the line bundle of degree $n/q$ given by $(v, x) \mapsto (v^q, x)$. Using the identification of $\nu$ with $N_{\Sigma}$ one thus gets a $q$-cover $\hat{\nu} \to \nu$ ramified along $\Sigma$. Using the Mayer-Vietoris sequence it is easy to see that over $\nu \cap W \hat{\nu}$ and $\hat{\nu}$ agree and so we can form $Y = W \cup \hat{\nu}$. It turns out that the assumption that $q$ is a prime power implies that $H_1(Y)$ is finite [9].

A manifold $Y$ admits a spin structure if and only if $w_2(Y) = 0$. Brand gives a general formula for the characteristic numbers of a general branched cover [2]. Let

$$\alpha = \frac{q-1}{q} PD(\Sigma) \mod 2.$$ 

In our case Brand’s formula then becomes

$$w_2(Y) = f^*(w_2(X) + \alpha).$$

Note that if $q$ is odd then $\alpha = 0$. To guarantee that $Y$ is spin we will always make the following assumption:

**Assumption 1.** If $q$ is odd we assume that $X$ is spin. If $q$ is even, we assume that $PD(\Sigma)/q$ is characteristic.

Since $H_1(Y)$ is torsion, $b_1 = b_2 = 0$ and so to determine the intersection form of $Y$ it is sufficient to compute the signature $\sigma_Y$ and Euler characteristic $\varepsilon_Y$. The Euler characteristic can be computed by an elementary counting argument using a triangulation of $X$ subordinate to $\Sigma \to X$. To compute the signature requires the $G$-signature theorem. The results are

$$\varepsilon_Y = q \varepsilon_X + (q - 1)(2g - 2),$$

$$\sigma_Y = q \sigma_X + \frac{1 - q^2}{3q} (\Sigma \cdot \Sigma).$$
Under Assumption 1, $Y$ is spin and the number $E_g$’s and hyperbolic pairs in the intersection form are determined by the above formulas and Furuta’s inequality translates into Proposition 1.2.

3. Lifting the $\mathbb{Z}/q$-action to the spin bundle

In this section we show that when $q = 2^r$ the Seiberg-Witten moduli spaces have an action of $\text{Pin}(2) \times \mathbb{Z}/q$.

We choose an invariant metric on $Y$ so that the action of $\mathbb{Z}/q$ is an isometry. Let $g : Y \to Y$ generate the $\mathbb{Z}/q$-action. Then $dg$ induces a $\mathbb{Z}/q$-action on the frame bundle $P \to Y$ that covers the action on $Y$. Since $Y$ is spin, there is a double cover $\hat{P}$ of $P$ that restricts to each fiber as $\text{Spin}(4) \to SO(4)$. We utilize the following lemma (c.f. [1]):

**Lemma 3.1.** An isometry $g : Y \to Y$ will have a lift $\hat{\tilde{g}}$ of $dg$ to $\hat{P}$ if $g^* : H^1(Y, \mathbb{Z}/2) \to H^1(Y, \mathbb{Z}/2)$ is the identity map. There are exactly two such lifts which we will denote $\pm \hat{\tilde{g}}$ and if $u : \hat{P} \to \hat{P}$ denotes the deck transformation then $u \hat{\tilde{g}} = -\hat{\tilde{g}}$.

To answer the question of whether a lift exists we solicit the help of a proposition in Kotschick and Matic ([4] Prop. 2.1 and note the remark following the proof):

**Proposition 3.2.** If $q = p^r$ is a power of a prime $p$, then $H_1(Y)$ has no $p$-torsion.

Thus when $p = 2$, $dg$ automatically lifts; however, $\hat{\tilde{g}}$ may have order $2q$ rather than $q$ since $(\hat{\tilde{g}})^q$ is either $u$ or the identity. The case of lifting involutions is considered in [1] and they show (Proposition 8.46):

**Proposition 3.3.** Let $Y$ be a spin manifold, $f : Y \to Y$ an involution preserving the orientation and spin structure, and let $\sigma_i$ be the connected components of the fixed point set of $f$. Then

\[
\text{codim } \Sigma_i \equiv 0 \mod 4 \text{ if } \hat{\tilde{f}} \text{ is order 2} \\
\text{codim } \Sigma_i \equiv 2 \mod 4 \text{ if } \hat{\tilde{f}} \text{ is order 4}.
\]

Applying the proposition to $f = g^{q/2}$ in our case, we see that $\pm \hat{\tilde{g}}$ has order $2q$ and $\hat{\tilde{g}}^q = u$.

The Seiberg-Witten equations for the trivial spin$_C$ structure on $Y$ can be written as equations for the pair

\[(A, \phi) \in \Omega^1(Y, \mathbb{R}) \times \Gamma(S^+).\]

They are (c.f. [3]):

\[
\hat{\partial} \phi + i a \cdot \phi = 0, \\
\rho(id^+ a) - (\phi \otimes \phi^*)_0 = 0, \\
d^* a = 0.
\]
The final equation is a gauge fixing condition. Consider $\text{Pin}(2)$ as the subgroup of the unit quaternions given by elements of the form $e^{i\theta}$ or $e^{i\theta}j$. Since $S^+$ is a quaternionic bundle, $\text{Pin}(2)$ naturally acts on $\Gamma(S^+)$ and we define the action of $\text{Pin}(2)$ on $\Omega^1(Y, \mathbb{R})$ to be multiplication by 1 or $-1$ for $e^{i\theta}$ or $e^{i\theta}j$ respectively. The above action can be seen to preserve the solution space of the Seiberg-Witten equations. The action of the $e^{i\theta}$ subgroup is just the usual action of the constant gauge transformations.

Since $g$ is an isometry, $dg$ also preserves the solution space and so we get a natural action of $\text{Pin}(2) \times \mathbb{Z}/2q$ on the solution space. We wish to show that $(-1, q) \in \text{Pin}(2) \times \mathbb{Z}/2q$ acts trivially so that we have an action of

$$\frac{(\text{Pin}(2) \times \mathbb{Z}/2q)}{\mathbb{Z}/2} = \text{Pin}(2) \times \mathbb{Z}/q.$$  

From Proposition 3.3, we know that $\tilde{dg}$ is the deck transformation $u$. Fiberwise, $u$ acts by $-1$ on $\text{Spin}(4) = SU(2) \times SU(2)$ and since $S^+$ is the bundle associated to the standard representation of the first $SU(2)$ factor, $u$ acts by $-1$ on sections of $S^+$. This is just the action of the constant gauge transformation $-1 \in \text{Pin}(2)$ thus $(-1, q)$ acts as $u^2 = 1$ on configurations.

In summary we have

**Theorem 3.4.** Let $Y$ and $q$ be as in Proposition 1.2 and further assume that $q = 2r$. Then the solution space to Seiberg-Witten equations for the trivial $\text{spin}_C$ structure has an action of $\text{Pin}(2) \times \mathbb{Z}/q$.

**References**


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