SEIBERG-WITTEN THEORY AND $\mathbf{Z}/2^p$ ACTIONS ON SPIN 4-MANIFOLDS

JIM BRYAN

Abstract. Furuta's "10/8ths" theorem gives a bound on the magnitude of the signature of a smooth spin 4-manifold in terms of the second Betti number. We show that, in the presence of a $\mathbf{Z}/2^p$ action, this bound can be strengthened. As applications, we give new genus bounds on classes with divisibility, and we give a classification of involutions on rational cohomology $K3$'s.

We utilize the action of $Pin(2)\tilde{\times}\mathbf{Z}/2^p$ on the Seiberg-Witten moduli space. Our techniques also provide a simplification of the proof of Furuta's theorem.

1. Introduction

In early 1995, Furuta [6] proved that if X is a smooth, compact, connected spin 4-manifold with non-zero signature $\sigma(X)$, then

$$\frac{5}{4}|\sigma(X)|+2\leq b_2(X). \tag{1}$$

This estimate has been dubbed the "10/8ths" theorem in comparison with the "11/8ths" conjecture, which predicts the following bound:

$$\frac{11}{8}|\sigma(X)|\leq b_2(X).$$

Inequality (1) follows by a surgery argument from the non-positive signature, $b_1(X)=0$ case:

Theorem 1.1 (Furuta). *Let X be a smooth spin 4-manifold with $b_1(X)=0$ with non-positive signature. Let $k=-\sigma(X)/16$ and $m=b_2^+(X)$. Then,*

$$2k+1\leq m$$

if $m\neq 0$.

The main results of this paper improve the above bound by p under the assumption that X has a $\mathbf{Z}/2^p$ action satisfying some non-degeneracy conditions (the analogues of the condition $m\neq 0$ in the above theorem).

A $\mathbf{Z}/2^p$ action is called a *spin action* if the generator of the action $\tau: X\to X$ lifts to the spin bundle $\hat{\tau}: P_{Spin}\to P_{Spin}$. Such an action is of *even type* if $\hat{\tau}$ has order 2^p and is of *odd type* if $\hat{\tau}$ has order 2^{p+1}. Using [2], it is easy to determine if an action is spin and whether it is of even or odd type.

We assume throughout that X is a smooth spin 4-manifold with $b_1(X)=0$ and is oriented so that the signature is non-positive. We will continue with the notation $k=-\sigma(X)/16$ and $m=b_2^+(X)$. Our main results are the following:

Supported by M.S.R.I. postdoctoral fellowship.

Theorem 1.2. *Suppose that $\tau : X \to X$ generates a smooth $\mathbf{Z}/2^p$ action that is spin and of odd type. Let X_i denote the quotient of X by $\mathbf{Z}/2^i \subset \mathbf{Z}/2^p$. Then*

$$2k + 1 + p \leq m$$

if $m \neq 2k + b_2^+(X_1)$ and $b_2^+(X_i) \neq b_2^+(X_j) > 0$ for $i \neq j$.

For even actions, the easiest results to state are for involutions:

Theorem 1.3. *Suppose that $\sigma : X \to X$ is a smooth involution of even type. Then*

$$2k + 2 \leq m$$

if $m \neq b_2^+(X/\sigma) > 0$.

This is a special case of the following theorem:

Theorem 1.4. *Suppose that $\sigma_1, \dots, \sigma_q : X \to X$ are smooth involutions of even type generating an action of $(\mathbf{Z}/2)^q$. Then*

$$2k + 1 + q \leq m$$

if $m \neq b_2^+(X/g)$ for any non-trivial element $g \in (\mathbf{Z}/2)^q$ and $b_2^+(X/(\mathbf{Z}/2)^q) \neq 0$.

Remark 1.1. The non-degeneracy conditions in Theorems 1.2, 1.3, and 1.4 can be restated in terms of the fixed point set, although for our applications the conditions on the quotient are more convenient.

We also include the following:

Theorem 1.5. *Suppose that $\tau : X \to X$ generates a smooth $\mathbf{Z}/2^p$ action that is spin and of either type. Then $b_2^+(X/\tau) \neq 0$ if $k > 0$.*

One can apply Theorem 1.2 to a cover of a 4-manifold branched along a smoothly embedded surface. The inequality can be used to get a bound on the genus of the embedded surface (see Theorem 4.2). One special case is the following:

Theorem 1.6. *Let M be a smooth, compact, oriented, simply connected 4-manifold (not necessarily spin) with $b_2^+(M) > 1$, and let $\Sigma \hookrightarrow M$ be a smooth embedding of a genus g surface. Suppose that the homology class defined by Σ is divisible by 2 and that $[\Sigma]/2 \equiv w_2(M) \mod 2$. Then*

$$g \geq \frac{5}{4}\left(\frac{[\Sigma]^2}{4} - \sigma(M)\right) - b_2(M) + 2.$$

This bound is typically weaker than the adjunction inequality from the Seiberg-Witten invariants (see [9]), but the bound applies to manifolds even with zero Seiberg-Witten invariants. For examples where the bound is sharp and new we have (see also example 4.1):

Corollary 1.7. *The minimal genus of an embedded surface in $\mathbf{CP}^2\#\mathbf{CP}^2$ representing the class $(6,2)$ is 10. The minimal genus of an embedded surface in $S^2 \times S^2\#\mathbf{CP}^2$ representing the class $((4,4),6)$ is 19.*

Proof. In each case the lower bound of Theorem 1.6 is realized by the connected sum of smooth algebraic curves in each factor. □

As another application of the theorems, we give a "classification" of involutions on rational cohomology $K3$'s.

Theorem 1.8. *Let $\sigma : X \to X$ be a spin involution of a rational cohomology $K3$ (*i.e. $b_1(X) = 0$ *and $Q_X \cong Q_{K3}$). If σ is of even type then it has exactly 8 isolated fixed points and $b_2^+(X/\sigma) = 3$; if σ is of odd type then $b_2^+(X/\sigma) = 1$.*

This theorem recovers as a special case a theorem of Donaldson concerning involutions on the $K3$ ([5] Cor. 9.1.4) and is related to a theorem of Ruberman [11]. We also remark that both possibilities in the theorem actually occur.

The proof of Theorems 1.2, 1.3, 1.4, and 1.5 uses Furuta's technique of "finite dimensional approximation" for the Seiberg-Witten moduli spaces to reduce the problem to algebraic topology. The main innovation of our technique is our approach to the equivariant K-theory. In particular, we do not need the Adams' operations in equivariant K-theory and can thus simplify that part of Furuta's proof. In section 2 we introduce the equations and use Furuta's technique to study the moduli space; in section 3 we use equivariant K-theory and representation theory to study the G-equivariant properties of the moduli space and prove the main theorems; section 4 is devoted to applications, primarily genus bounds obtained by branched covers and our classification of involutions on rational cohomology $K3$'s.

We would like to thank R. Cohen, B. Gompf, C. Gordon, D. Kotschick, P. Kronheimer, T. Lawson, N. C. Leung, G. Matíc, P. Oszvath, D. Ruberman, R. Stern, P. Teichner, and R. Wentworth for helpful conversations, and we would like to thank M.S.R.I. for providing an excellent research environment where this work was carried out.

2. Seiberg-Witten Theory

In this section we use Furuta's "finite dimensional approximation" technique to study the Seiberg-Witten moduli space in the presence of our $\mathbf{Z}/2^p$ symmetry. The goal of the section is to use the Seiberg-Witten solutions to produce a certain G-equivariant map between spheres. This map will then be studied by algebraic topology in section 3 to produce a proof of the main theorems.

Let X be a smooth, compact, connected, spin 4-manifold with $b_1(X) = 0$ and fix an orientation so that $\sigma(X) \le 0$. By Rochlin's theorem, $\sigma(X)$ is divisible by 16, so let $k = -\sigma(X)/16$ and let $m = b_2^+(X)$. Let $\tau : X \to X$ be an orientation preserving diffeomorphism generating a $\mathbf{Z}/2^p$ action and fix an invariant Riemannian metric g. We assume that τ is a spin action. By definition this means that $\tau^*(\wp) - \wp$ is zero as an element of $H^1(X;\mathbf{Z}/2)$ where $\wp$ is the spin structure (the difference of two spin structures is naturally an element of $H^1(X;\mathbf{Z}/2)$). There is then a lift $\hat\tau$ of τ to the spin bundle:

$$\begin{array}{ccc}
P_{\mathrm{Spin}} & \xrightarrow{\hat\tau} & P_{\mathrm{Spin}} \\
\downarrow & & \downarrow \\
P_{\mathrm{SO}(4)} & \xrightarrow{d\tau} & P_{\mathrm{SO}(4)} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\tau} & X.
\end{array}$$

The order of $\hat\tau$ is either 2^p or 2^{p+1} depending on whether $(\hat\tau)^{2^p}$ is the identity or the non-trivial deck transformation of the double cover $P_{\mathrm{Spin}} \to P_{\mathrm{SO}(4)}$. We say that τ is of even type if $\hat\tau$ has order 2^p and of odd type if $\hat\tau$ has order 2^{p+1}. A lemma of Atiyah and Bott makes it easy to determine the type of τ: The 2^{p-1}-th iteration of τ is an involution σ of X and will have a fixed point set consisting of manifolds of constant even dimension. If the fixed point set of σ consists of points (or all of X), then τ is of even type; if the fixed point set is dimension 2, then τ is of odd type. In the case where σ acts freely, then τ is of even or odd type depending on whether or not the quotient X/σ is spin (c.f. [2] or [4]).

On a spin manifold, the Seiberg-Witten equations for the trivial spin$^{\mathbf{C}}$ structure take on a somewhat special form. In [6], Furuta gives a concise and elegant description of the equations. We give a slightly different description to avoid repetition and to make the exposition more closely match standard notation.

Let $S = S^+ \oplus S^-$ denote the decomposition of the spinor bundle into the positive and negative spinor bundles. Clifford multiplication induces an isomorphism

$$\rho : \Lambda^*_{\mathbf{C}} \to \mathrm{End}_{\mathbf{C}}(S)$$

between the bundle of complex valued forms and endomorphisms of S. Let $\partial\!\!\!/ : \Gamma(S^+) \to \Gamma(S^-)$ be the Dirac operator. The Seiberg-Witten equations are for a pair $(a, \phi) \in \Omega^1(X, \sqrt{-1}\mathbf{R}) \times \Gamma(S^+)$ and they are

$$\begin{aligned} \partial\!\!\!/\phi + \rho(a)\phi &= 0, \\ \rho(d^+a) - \phi \otimes \phi^* + \tfrac{1}{2}|\phi|^2 \mathbb{1} &= 0, \\ d^*a &= 0. \end{aligned}$$

This system of equations is elliptic as written—the last equation defines a slice for the (based) gauge group of the usual equations. The image of $\sqrt{-1}\Lambda^2_+$ under ρ is the tracefree, hermitian endomorphisms of S^+ which we denote by $\sqrt{-1}\mathfrak{su}(S^+)$. (We remark that by taking a slightly different gauge fixing condition, one could combine the last two equations in the single equation $\rho(d^+a + d^*a) = \phi \otimes \phi^*$.)

Following Furuta's notation, we regard the equations as the zero set of a map

$$D + Q : V \to W'$$

where V is the L^4_2-completion of $\Gamma(\sqrt{-1}\Lambda^1 \oplus S^+)$ and W' is the L^3_2 completion of $\Gamma(S^- \oplus \sqrt{-1}\mathfrak{su}(S^+) \oplus \sqrt{-1}\Lambda^0)$ and D and Q are the linear and quadratic pieces of the equations respectively, *i.e.*

$$\begin{aligned} D(a, \phi) &= (\partial\!\!\!/\phi, \rho(d^+a), d^*a), \\ Q(a, \phi) &= (\rho(a)\phi, \phi \otimes \phi^* - \tfrac{1}{2}|\phi|^2 \mathbb{1}, 0). \end{aligned}$$

The image of $D + Q$ is L^2-orthogonal to the constant functions in $\sqrt{-1}\Omega^0 \subset W'$. We define W to be the orthogonal complement of the constant functions in W' and consider $D + Q$ as defined on W:

$$D + Q : V \to W.$$

We now wish to determine the group of symmetries of the equations. As in Furuta, we will see that Pin (2) acts, and, as one would expect, there are additional symmetries arising from τ. The symmetry group turns out to depend on whether τ is of even or odd type.

Define Pin (2) $\subset$ SU(2) to be the normalizer of $S^1 \subset \mathrm{SU}(2)$. Regarding SU(2) as the group of unit quaternions and taking S^1 to be elements of the form $e^{\sqrt{-1}\theta}$, Pin (2) then consists of elements of the form $e^{\sqrt{-1}\theta}$ or $e^{\sqrt{-1}\theta}J$ (our quaternions are spanned by $\langle 1, \sqrt{-1}, J, \sqrt{-1}J\rangle$). We define the action of Pin (2) on V and W as follows: Since S^+ and S^- are SU(2) bundles, Pin (2) naturally acts on $\Gamma(S^\pm)$ by multiplication on the left. $\mathbf{Z}/2$ acts on $\Gamma(\Lambda^*_{\mathbf{C}})$ by multiplication by ± 1 and this pulls back to an action of Pin (2) by the natural map Pin (2) $\to \mathbf{Z}/2$. A calculation shows that this pullback also describes the induced action of Pin (2) on $\sqrt{-1}\mathfrak{su}(S^+)$. Both D and Q are then seen to be Pin (2) equivariant maps. Note that the action of $S^1 \subset$ Pin (2) is the ordinary action of the constant gauge transformations.

The isometry τ acts on V and W by pull back by $d\tau$ and $\hat{\tau}$; D and Q are equivariant with respect to this action. If τ is of even type, then it induces an action of $\mathbf{Z}/2^p$ on V and W; if τ is of odd type, then it induces an action of $\mathbf{Z}/2^{p+1}$. In the even case the symmetry group is thus

$$G_{ev} \equiv \mathrm{Pin}\,(2) \times \mathbf{Z}/2^p.$$

In the odd case Pin (2) $\times\, \mathbf{Z}/2^{p+1}$ acts but we see that the diagonal $\mathbf{Z}/2$ subgroup acts trivially: the 2^p-th iteration of $\hat{\tau}$ acts on the spinors by the action induced by the non-trivial deck transformation; this is the same as the action of the constant gauge transformation $-1 \subset S^1 \subset$ Pin (2). Thus the symmetry group for the odd case is

$$G_{odd} \equiv \frac{\mathrm{Pin}\,(2) \times \mathbf{Z}/2^{p+1}}{\mathbf{Z}/2}.$$

Throughout the sequel we write G to mean G_{odd} or G_{ev}.To state Furuta's finite dimensional approximation theorem, we define V_λ for any $\lambda \in \mathbf{R}$ to be the subspace of V spanned by the eigenspaces of D^*D with eigenvalues less than or equal to λ. Similarly, define W_λ using DD^* and write p_λ for the L^2-orthogonal projection from W to W_λ. Define

$$D_\lambda + Q_\lambda : V_\lambda \to W_\lambda$$

by the restriction of $D + p_\lambda Q$ to V_λ.

Since the eigenspaces of D^*D and DD^* are G invariant and p_λ is a G-map, $D_\lambda + Q_\lambda$ is a G equivariant map between finite dimensional G representations and it is an approximation to $D + Q$ in the following sense:

Lemma 2.1 (Lemma 3.4 of [6])**.** *There exists an $R \in \mathbf{R}$ such that for all $\lambda > R$ the inverse image of zero $(D_\lambda + Q_\lambda)^{-1}(0)$ is compact.*

The lemma allows Furuta to use $D_\lambda + Q_\lambda$ to construct a G-equivariant map on disks preserving boundaries

$$f_\lambda : (BV_{\lambda,\mathbf{C}}, SV_{\lambda,\mathbf{C}}) \to (BW_{\lambda,\mathbf{C}}, SW_{\lambda,\mathbf{C}}).$$

Here $V_{\lambda,\mathbf{C}} = V_\lambda \otimes \mathbf{C}$, $BV_{\lambda,\mathbf{C}}$ is homotopic to a ball in $V_{\lambda,\mathbf{C}}$, and $SV_{\lambda,\mathbf{C}}$ is the boundary of $BV_{\lambda,\mathbf{C}}$ with similar definitions for $W_{\lambda,\mathbf{C}}$, $BW_{\lambda,\mathbf{C}}$, and $SW_{\lambda,\mathbf{C}}$.

The virtual G-representation $[V_\lambda] - [W_\lambda] \in R(G)$, which is the G-index of D, can be determined by the G index theorem and is independent of λ. We discuss its computation in the next section.

3. Equivariant K-theory

In this section we use equivariant K-theory to deduce restrictions on the map $f_\lambda : BV_{\lambda,\mathbf{C}} \to BW_{\lambda,\mathbf{C}}$. Combining this with the index theorem determining $[V_\lambda] - [W_\lambda] \in R(G)$ we will prove the main theorems. Our K-theoretic techniques avoid Furuta's use of the equivariant Adams' operations and thus also provide a simplification in the proof of his "10/8ths" theorem.

3.1. **The representation ring of G_{ev} and G_{odd}.** We write $R(\Gamma)$ for the complex representation ring of a compact Lie group Γ and we write direct sum and tensor product of representations additively and multiplicatively respectively. The representation ring $R(\mathbf{Z}/2^p)$ of $\mathbf{Z}/2^p$ is isomorphic to the group ring $\mathbf{Z}(\mathbf{Z}/2^p)$ and is generated by the standard one dimensional representation ζ . We write 1 for the trivial representation, so for example $\zeta^{2^p} = 1$, and as a $\mathbf{Z}$ module, $R(\mathbf{Z}/2^p)$ is generated by $1, \zeta, \ldots, \zeta^{2^p-1}$.

The group $\mathrm{Pin}\,(2)$ has one non-trivial one dimensional representation (which we denote by $\tilde{1}$) given by pulling back the non-trivial $\mathbf{Z}/2$ representation by the map $\mathrm{Pin}\,(2) \to \mathbf{Z}/2$. It has a countable series of 2 dimensional irreducible representations $h_1, h_2, \ldots$. The representation h_1, which we sometimes write as h, is the restriction of the standard representation of $SU(2)$ to $\mathrm{Pin}\,(2) \subset SU(2)$. The representation h_i can be obtained using the relation $h_i h_j = h_{i+j} + h_{|i-j|}$ where by convention h_0 denotes $1 + \tilde{1}$. Note that $\tilde{1} \cdot h_i = h_i$. Let θ denote the standard 1 dimensional representation of S^1 so that $R(S^1) \cong \mathbf{Z}[\theta, \theta^{-1}]$. It is easy to see that h_i restricts to $\theta^i + \theta^{-i}$ as an S^1 representation.

Since $G_{ev} = \mathrm{Pin}\,(2) \times \mathbf{Z}/2^p$, the representation ring is simply the tensor product $R(G_{ev}) = R(\mathrm{Pin}\,(2)) \otimes R(\mathbf{Z}/2^p)$. We can thus write a general element $\beta \in R(G_{ev})$

$$\beta = \beta_0(\zeta)1 + \tilde{\beta}_0(\zeta)\tilde{1} + \sum_{i=1}^{\infty} \beta_i(\zeta) h_i$$

where β_0, $\tilde{\beta}_0$, β_1, $\ldots$ are degree $2^p - 1$ polynomials in ζ and all but a finite number of the β_i's are 0.

The irreducible representations of $G_{odd} = (\mathrm{Pin}\,(2) \times \mathbf{Z}/2^{p+1})/(\mathbf{Z}/2)$ are the representations of $\mathrm{Pin}\,(2) \times \mathbf{Z}/2^{p+1}$ that are invariant under $\mathbf{Z}/2$. To avoid confusion we will use ξ for the generator of $R(\mathbf{Z}/2^{p+1})$. The $\mathrm{Pin}\,(2) \times \mathbf{Z}/2^{p+1}$ representations ξ^i and h_i are non-trivial restricted to $\mathbf{Z}/2$ if and only if i is odd and the representation $\tilde{1}$ is trivial restricted to $\mathbf{Z}/2$. The subring $R(G_{odd}) \subset R(\mathrm{Pin}\,(2) \times \mathbf{Z}/2^{p+1})$ is therefore generated by $1, \tilde{1}, \xi^2$, and $\xi^i h_j$ where $i \equiv j \pmod 2$. We write a general element $\beta \in R(G_{odd})$ as

$$\beta = \beta_0(\xi) + \tilde{\beta}_0(\xi) + \sum_{i=1}^{\infty} \beta_i(\xi) h_i$$

where now $\beta_0, \tilde{\beta}_0$, and β_{2i} are *even* polynomials of degree $2^{p+1} - 2$ and the β_{2i+1}'s are *odd* polynomials of degree $2^{p+1} - 1$. In summary we have:

Theorem 3.1. *The ring $R(G_{ev})$ is generated by 1, $\tilde{1}$, h_1, $h_2, \ldots$, and ζ with the relations $\zeta^{2^p} = 1$ and $h_i h_j = h_{i+j} + h_{|i-j|}$, where $h_0 = 1 + \tilde{1}$.*

The ring $R(G_{odd})$ is generated by 1, $\tilde{1}$, ξ^2, h_2, $h_4, \ldots$, and ξh_1, $\xi h_3, \ldots$ with the relations $\xi^{2^{p+1}} = 1$ and $h_i h_j = h_{i+j} + h_{|i-j|}$, where $h_0 = 1 + \tilde{1}$.

For $G = G_{odd}$ or G_{ev} the restriction map $R(G) \to R(S^1) \cong \mathbf{Z}[\theta, \theta^{-1}]$ is given by $\tilde{1} \mapsto 1$, $\xi \mapsto 1$ (or $\zeta \mapsto 1$), and $h_i \mapsto \theta^i + \theta^{-i}$.

3.2. **The index of D.** The virtual representation $[V_{\lambda,\mathbf{C}}] - [W_{\lambda,\mathbf{C}}] \in R(G)$ is the same as $\mathrm{Ind}(D) = [\mathrm{Ker}\, D] - [\mathrm{Coker}\, D]$. Furuta determines $\mathrm{Ind}(D)$ as a $\mathrm{Pin}\,(2)$ representation; denoting the restriction map $r : R(G) \to R(\mathrm{Pin}\,(2))$, Furuta shows

$$r(\mathrm{Ind}(D)) = 2kh - m\tilde{1}$$

where $k = -\sigma(X)/16$ and $m = b_2^+(X)$. Thus $\mathrm{Ind}(D) = sh - t\tilde{1}$ where s and t are polynomials in ξ or ζ such that $s(1) = 2k$ and $t(1) = m$. In the case of $G = G_{ev}$ we write

$$s(\zeta) = \textstyle\sum_{i=1}^{2^p} s_i\zeta^i \quad \text{and} \quad t(\zeta) = \sum_{i=1}^{2^p} t_i\zeta^i$$

and for $G = G_{odd}$ we have

$$s(\xi) = \textstyle\sum_{i=1}^{2^p} s_i\xi^{2i-1} \quad \text{and} \quad t(\xi) = \sum_{i=1}^{2^p} t_i\xi^{2i}.$$

To compute the coefficients t_i and s_i, one can use the $\mathbf{Z}/2^p$-index theorem. We will only need information about the t_i coefficients.

¿From the definition of the operator D we see that the polynomial $\sum_{i=1}^{2^p} t_i\zeta^i$ is the honest $\mathbf{Z}/2^p$ representation

$$\mathrm{Coker}((d^*, d^+) : \Omega^1 \to \Omega^0_\perp \oplus \Omega^2_+)$$

where $\Omega^0_\perp$ denotes the L^2-orthogonal complement of the constant functions (this is true for both the odd and even cases). The constant coefficient, t_{2^p} is the dimension of the invariant part of $H^2_+(X)$ which is just $b_2^+(X/\tau)$. More generally, b_2^+ of the quotients of X by subgroups of $\mathbf{Z}/2^p$ are given by the various sums of the t_i's. The dimension of the subspace of $\sum_{i=1}^{2^p} t_i\xi^i$ invariant under $\mathbf{Z}/2^j \subset \mathbf{Z}/2^p$ is given by

$$\sum_{i \equiv 0 \bmod 2^j} t_i = b_2^+(X_j)$$

where X_j is the quotient of X by $\mathbf{Z}/2^j$ and $X_0 = X$ by convention.

We summarize the discussion in the following

Theorem 3.2. *The index of D is given by*

$$[V_{\lambda,\mathbf{C}}] - [W_{\lambda,\mathbf{C}}] = sh - t\tilde{1} \in R(G)$$

where if $G = G_{ev}$ then

$$s(\zeta) = \textstyle\sum_{i=1}^{2^p} s_i\zeta^i \quad \textit{and} \quad t(\zeta) = \sum_{i=1}^{2^p} t_i\zeta^i$$

and if $G = G_{odd}$ then

$$s(\xi) = \textstyle\sum_{i=1}^{2^p} s_i\xi^{2i-1} \quad \textit{and} \quad t(\xi) = \sum_{i=1}^{2^p} t_i\xi^{2i}.$$

For either G_{ev} or G_{odd} we have

$$\begin{aligned} \sum_{i=1}^{2^p} s_i &= 2k \\ \sum_{i \equiv 0 \bmod 2^j} t_i &= b_2^+(X_j). \end{aligned}$$

3.3. **The Thom isomorphism and a character formula for the K-theoretic degree.** The Thom isomorphism theorem in equivariant K-theory for a general compact Lie group is a deep theorem proved using elliptic operators [1] . The subsequent character formula of this section uses only elementary properties of the Bott class. We follow tom Dieck [12] pgs. 254–255 for this discussion.

Let V and W be complex Γ representations for some compact Lie group Γ. Let BV and BW denote balls in V and W and let $f : BV \to BW$ be a Γ-map preserving the boundaries SV and SW. $K_\Gamma(V)$ is by definition $K_\Gamma(BV, SV)$, and by the equivariant Thom isomorphism theorem, $K_\Gamma(V)$ is a free $R(\Gamma)$ module with generator the Bott class $\lambda(V)$. Applying the K-theory functor to f we get a map

$$f^* : K_\Gamma(W) \to K_\Gamma(V)$$

which defines a unique element $\alpha_f \in R(\Gamma)$ by the equation $f^*(\lambda(W)) = \alpha_f \cdot \lambda(V)$. The element α_f is called the *K-theoretic degree* of f.

Let V_g and W_g denote the subspaces of V and W fixed by an element $g \in \Gamma$ and let $V_g^\perp$ and $W_g^\perp$ be the orthogonal complements. Let $f^g : V_g \to W_g$ be the restriction of f (well defined because of equivariance) and let $d(f^g)$ denote the ordinary topological degree of f^g (by definition, $d(f^g) = 0$ if $\dim V_g \neq \dim W_g$). For any $\beta \in R(\Gamma)$ let $\lambda_{-1}\beta$ denote the alternating sum $\sum(-1)^i \lambda^i \beta$ of exterior powers.

T. tom Dieck proves a character formula for the degree α_f:

Theorem 3.3. *Let $f : BV \to BW$ be a Γ-map preserving boundaries and let $\alpha_f \in R(\Gamma)$ be the K-theory degree. Then*

$$\mathrm{tr}_g(\alpha_f) = d(f^g)\,\mathrm{tr}_g(\lambda_{-1}(W_g^\perp - V_g^\perp))$$

where tr_g is the trace of the action of an element $g \in \Gamma$.

This formula will be especially useful in the case where $\dim W_g \neq \dim V_g$ so that $d(f^g) = 0$.

We also recall here that $\lambda_{-1}(\sum_i a_i r_i) = \prod_i (\lambda_{-1} r_i)^{a_i}$ and that for a one dimensional representation r, we have $\lambda_{-1} r = (1 - r)$. A two dimensional representation such as h has $\lambda_{-1} h = (1 - h + \Lambda^2 h)$. In this case, since h comes from an $SU(2)$ representation, $\Lambda^2 h = \det h = 1$ so $\lambda_{-1} h = (2 - h)$.

All the proofs in the following subsections proceed by using the character formula to examine the K-theory degree α_{f_λ} of the map $f_\lambda : BV_{\lambda,\mathbf{C}} \to BW_{\lambda,\mathbf{C}}$ coming from the Seiberg-Witten equations. We will abbreviate α_{f_λ} as just α and $V_{\lambda,\mathbf{C}}$ and $W_{\lambda,\mathbf{C}}$ as just V and W, and we will use the following elements of G. Let $\phi \in S^1 \subset \mathrm{Pin}\,(2) \subset G$ be an element generating a dense subgroup of S^1; let $\eta \in \mathbf{Z}/2^p$ and $\nu \in \mathbf{Z}/2^{p+1}$ be generators and recall that there is the element $J \in \mathrm{Pin}\,(2)$ coming from the quaternions. Note that the action of J on h has two invariant subspaces on which J acts by multiplication with $\sqrt{-1}$ and $-\sqrt{-1}$.

3.4. **Proof of Furuta's theorem (Theorem 1.1).** Consider $\alpha = \alpha_{f_\lambda} \in R(\mathrm{Pin}\,(2))$; it has the form

$$\alpha = \alpha_0 + \tilde{\alpha}_0 \tilde{1} + \sum_{i=1}^{\infty} \alpha_i h_i.$$

Since ϕ acts non-trivially on h and trivially on $\tilde{1}$, we have that $\dim V_\phi \neq \dim W_\phi$ as long as $m > 0$. The character formula then gives

$$\mathrm{tr}_\phi(\alpha) = 0 = \alpha_0 + \tilde{\alpha}_0 + \sum_{i=1}^{\infty} \alpha_i (\phi^i + \phi^{-i})$$

so $\alpha_0 = -\tilde{\alpha}_0$ and $\alpha_i = 0$ for $i \geq 1$.

Since J acts non-trivially on both h and $\tilde{1}$, $\dim V_J = \dim W_J = 0$ so $d(f^J) = 1$ and the character formula says

$$\begin{aligned} \mathrm{tr}_J(\alpha) &= \mathrm{tr}_J(\lambda_{-1}(m\tilde{1} - 2kh)) \\ &= \mathrm{tr}_J((1-\tilde{1})^m(2-h)^{-2k}) \\ &= 2^{m-2k} \end{aligned}$$

using $\mathrm{tr}_J\, h = 0$ and $\mathrm{tr}_J\, \tilde{1} = -1$. On the other hand, $\mathrm{tr}_J(\alpha) = \mathrm{tr}_J(\alpha_0(1-\tilde{1})) = 2\alpha_0$ therefore the degree is

$$\alpha = 2^{m-2k-1}(1-\tilde{1})$$

and we can conclude $2k+1 \leq m$. □

The proofs of Theorems 1.2, 1.3, and 1.4 are a generalization of the above proof.

3.5. **Proof of Theorem 1.3.** In this case the group is $G = G_{ev} = \mathrm{Pin}\,(2) \times \mathbf{Z}/2$ and

$$V - W = (s_1\zeta + s_2)h - (t_1\zeta + t_2)\tilde{1}$$

where $t_1 + t_2 = m$ and $s_1 + s_2 = 2k$. The hypothesis $m \neq b_2^+(X/\sigma) > 0$ translates to $t_1 + t_2 \neq t_2 > 0$ which is equivalent to both t_1 and t_2 being non-zero.

Both ϕ and $\phi\nu$ act non-trivially on h and trivially on $\tilde{1}$ so (since $t_2 \neq 0$) we have $d(f^\phi) = d(f^{\phi\nu}) = 0$ so that $\mathrm{tr}_\phi(\alpha) = \mathrm{tr}_{\phi\nu}(\alpha) = 0$.

The general form of α is

$$\alpha = \alpha_0(\zeta) + \tilde{\alpha}_0(\zeta)\tilde{1} + \sum_{i=1}^{\infty} \alpha_i(\zeta)h_i$$

so the conditions $\mathrm{tr}_\phi(\alpha) = \mathrm{tr}_{\phi\nu}(\alpha) = 0$ imply that

$$\begin{aligned} \alpha_0(\pm 1) + \tilde{\alpha}_0(\pm 1) &= 0 \\ \alpha_i(\pm 1) &= 0 \end{aligned}$$

since α_0, $\tilde{\alpha}_0$, and α_i are degree 1 in ζ we see that $\alpha_0 = -\tilde{\alpha}_0$ and $\alpha_i = 0$ so

$$\alpha = (\alpha_0^1\zeta + \alpha_0^2)(1-\tilde{1}).$$

We kow that $J\nu$ acts trivially on $\zeta\tilde{1}$ and non-trivially on ζh and h so since $t_1 \neq 0$, $d(f^{J\nu}) = 0$. Thus we get

$$\mathrm{tr}_{J\nu}(\alpha) = 0 = (-\alpha_0^1 + \alpha_0^2) \cdot 2$$

so that $\alpha_0^1 = \alpha_0^2$.

Finally, as before, J acts non-trivially on W and V, so that $d(f^J) = 1$ and $\mathrm{tr}_J(\alpha) = 2^{m-2k}$ so it must be the case that

$$\alpha = 2^{m-2k-2}(1+\zeta)(1-\tilde{1})$$

and thus $2k+2 \leq m$. □

3.6. **Proof of Theorem 1.4.** We generalize the preceding proof. Now $G = \mathrm{Pin}\,(2) \times (\mathbf{Z}/2)^q$ and we write ζ_i for the non-trivial representation of the i-th copy of $\mathbf{Z}/2$. The index $\mathrm{Ind}(D)$ is given by

$$V - W = s(\zeta_1, \ldots, \zeta_q)h - t(\zeta_1, \ldots, \zeta_q)\tilde{1}$$

where s and t are polynomial functions in the variables $\zeta_1, \ldots, \zeta_q$ of multi-degree $(1, \ldots, 1)$.

The hypothesis $m \neq b_2^+(X/g)$ then implies that the representation $t(\zeta_1, \ldots, \zeta_q)$ contains a summand on which g acts as -1. Since J acts on $\tilde{1}$ by -1 we see that the representation $t(\zeta_1, \dots, \zeta_q)\tilde{1}$ has a positive dimensional subspace fixed by Jg for every non-trivial $g \in (\mathbf{Z}/2)^q$. On the other hand, Jg always acts non-trivially on $s(\zeta_1, \dots, \zeta_q)h$, so the character formula gives us

$$\mathrm{tr}_{Jg}(\alpha) = 0.$$

Since $b_2^+(X/(\mathbf{Z}/2)^q) \neq 0$, the coefficient of the trivial representation in $t(\zeta_1, \dots, \zeta_q)$ is non-zero and so ϕg always fixes a non-trivial subspace of $t\tilde{1}$. On the other hand, ϕg always acts non-trivially on sh, so again the character formula shows that

$$\mathrm{tr}_{\phi g}(\alpha) = 0.$$

The general form of α is

$$\alpha = \alpha_0 + \tilde{\alpha}_0 \tilde{1} + \sum_{i=1}^{\infty} \alpha_i h_i$$

where α_0, $\tilde{\alpha}_0$, and α_i are polynomial functions in $\zeta_1, \dots, \zeta_q$ of multi-degree $(1, \dots, 1)$.

Now for any polynomial function $\beta(x_1, \dots, x_q)$ of multi-degree $(1, \dots, 1)$, if we know that $\beta((-1)^{n_1}, \dots, (-1)^{n_q}) = 0$ for any arrangement of the signs, then $\beta \equiv 0$. Thus the formula $\mathrm{tr}_{\phi g}(\alpha) = 0$ for all $g \in (\mathbf{Z}/2)^q$ implies that $\alpha_0 + \tilde{\alpha}_0 \equiv 0$ and $\alpha_i \equiv 0$. We can therefore write

$$\alpha = \alpha_0(\zeta_1, \dots, \zeta_q)(1 - \tilde{1}).$$

Since J acts non-trivially on both U and V, we can compute as before:

$$\mathrm{tr}_J(\alpha) = \mathrm{tr}_J(\lambda_{-1}(W - V)) = 2^{m-2k}.$$

This equation, along with the $2^q - 1$ equations $\mathrm{tr}_{Jg}(\alpha) = 0$ for $g \neq 1 \in (\mathbf{Z}/2)^q$ give 2^q independent conditions on α_0 which determine it uniquely. It must be the following:

$$\alpha = 2^{m-2k-1-q}(1 + \zeta_1)(1 + \zeta_2) \cdots (1 + \zeta_q)(1 - \tilde{1})$$

and thus $2k + 1 + q \leq m$. □

3.7. **Proof of Theorem 1.2.** Recall that

$$[V] - [W] = s(\xi)h - t(\xi)\tilde{1} \in R(G_{odd})$$

with $s(\xi) = \sum_{i=1}^{2^p} s_i \xi^{2i-1}$ and $t(\xi) = \sum_{i=1}^{2^p} t_i \xi^{2i}$. The K-theory degree $\alpha = \alpha_{f_\lambda}$ of f_λ has the form

$$\alpha = \alpha_0(\xi) + \tilde{\alpha}_0(\xi) + \sum_{i=1}^{\infty} \alpha_i(\xi) h_i.$$

We compute the following characters of α:

Lemma 3.4. $\mathrm{tr}_{\phi\eta^j}(\alpha) = 0$ *for* $j = 1, \dots, 2^p, \mathrm{tr}_{J\eta^j}(\alpha) = 0$ *for* $j = 1, \dots, 2^p - 1$, *and* $\mathrm{tr}_J(\alpha) = 2^{m-2k}$.

Proof: We introduce the notation $(V)_g = \dim V_g$. By the character formula, $\mathrm{tr}_{\phi\eta^j}(\alpha) = 0$ if

$$(V)_{\phi\eta^j} - (W)_{\phi\eta^j} = (s(\xi)h - t(\xi)\tilde{1})_{\phi\eta^j} \neq 0.$$

Since $\phi\eta^j$ acts non-trivially on every $\xi^i h$ and $\phi\eta^j$ acts trivially on $\tilde{1}$, we have

$$(V)_{\phi\eta^j} - (W)_{\phi\eta^j} \leq -t_{2^p}.$$

By the hypothesis in Theorem 1.2, $t_{2^p} = b_2^+(X/\tau) > 0$ so $\mathrm{tr}_{\phi\eta^j}(\alpha) = 0$.

To show $\mathrm{tr}_{J\eta^j}(\alpha) = 0$ for $j = 1, \ldots, 2^p - 1$ we need to show $(V)_{J\eta^j} - (W)_{J\eta^j} \neq 0$. The 2 dimensional representation h decomposes into two complex lines on which J acts as $\sqrt{-1}$ and $-\sqrt{-1}$, so

$$(\xi^{2i-1}h)_{J\eta^j} = \begin{cases} 1 & \text{if } \eta^j \text{ acts on } \xi^{2i-1} \text{ by } \pm\sqrt{-1} \\ 0 & \text{otherwise.} \end{cases}$$

η^j acts as $\pm\sqrt{-1}$ on ξ^{2i-1} if and only if j and i satisfy

$$(2i-1)j \equiv \pm 2^{p-1} \bmod 2^{p+1}.$$

If $j = 2^{p-1}$, then the condition is satisfied for every i; if $j \neq 2^{p-1}$, the condition is never satisfied (divide both sides by the highest power of 2 that divides j to get an odd number on one side and an even on the other). Thus

$$(s(\xi)h)_{J\eta^j} = \begin{cases} 2k & \text{if } j = 2^{p-1}, \\ 0 & \text{if } j \neq 2^{p-1}. \end{cases}$$

$J\eta^j$ acts on $\tilde{1}$ by -1, therefore

$$(\xi^{2i}\tilde{1})_{J\eta^j} = \begin{cases} 1 & \text{if } \eta^j \text{ acts on } \xi^{2i} \text{ by } -1 \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Thus we see that

$$\begin{aligned} (t(\xi)\tilde{1})_{J\eta^j} &= \sum_{2ij \equiv 2^p \bmod 2^{p+1}} t_i \\ &= \sum_{ij \equiv 2^{p-1} \bmod 2^p} t_i. \end{aligned}$$

We want to see that the non-degeneracy conditions of the theorem imply $(V - W)_{J\eta^j} \neq 0$. The conditions are equivalent to

$$0 < b_2^+(X_p) < b_2^+(X_{p-1}) < \cdots < b_2^+(X_1)$$

and

$$b_2^+(X_1) \neq m - 2k$$

which in terms of the t_i's are

$$\begin{aligned} t_{2^p} &\neq 0 \\ t_{2^{p-1}} &\neq 0 \\ t_{2^{p-2}} + t_{3\cdot 2^{p-2}} &\neq 0 \\ t_{2^{p-3}} + t_{3\cdot 2^{p-3}} + t_{5\cdot 2^{p-3}} + t_{7\cdot 2^{p-3}} &\neq 0 \\ &\vdots \\ t_2 + t_{3\cdot 2} + t_{5\cdot 2} + \cdots &\neq 0 \end{aligned}$$

and

$$\sum_{i \text{ even}} t_i \neq m - 2k.$$

Since $m = \sum t_i$ the last condition is the same as

$$t_1 + t_3 + \cdots + t_{2^p-1} \neq 2k.$$

¿From the previous discussion we have in the case $j \neq 2^{p-1}$

$$\begin{aligned} (V-W)_{J\eta^j} &= (s(\xi)h - t(\xi)\tilde{1})_{J\eta^j} \\ &= -\sum_{ij \equiv 2^{p-1} \bmod 2^p} t_i \\ &= -(t_{2^a} + t_{3\cdot 2^a} + \cdots) \\ &\neq 0 \end{aligned}$$

where 2^{p-1-a} is the largest power of 2 dividing j.

For the case $j = 2^{p-1}$

$$\begin{aligned} (V-W)_{J\eta^j} &= 2k - \sum_{i2^{p-1} \equiv 2^{p-1} \bmod 2^p} t_i \\ &= 2k - \sum_{i \equiv 1 \bmod 2} t_i \\ &\neq 0. \end{aligned}$$

To complete the lemma we compute $\mathrm{tr}_J(\alpha)$. Once again, since J acts non-trivially on W and V,

$$\mathrm{tr}_J(\alpha) = \mathrm{tr}_J\, \Lambda_{-1}(W-V) = 2^{m-2k}.$$

Lemma 3.5. *Let $\beta(\xi)$ be of the form $\sum_{i=1}^{2^p} \beta_i \xi^{2i}$ or $\sum_{i=1}^{2^p} \beta_i \xi^{2i-1}$ in $R(\mathbf{Z}/2^{p+1})$. If $\mathrm{tr}_{\eta^j}\, \beta = 0$ for $j = 1, \ldots, 2^p$ then $\beta \equiv 0$.*

Proof: Let $\underline{\beta}(x) = \sum_{i=1}^{2^p} \beta_i x^i$. $\underline{\beta}$ is then a degree 2^p polynomial with roots at all of the 2^p-th roots of unity and at 0, so $\underline{\beta}$ is identically 0.

The general form of α is

$$\alpha = \alpha_0(\xi) + \tilde{\alpha}_0(\xi)\tilde{1} + \sum_{i=1}^{\infty} \alpha_i(\xi) h_i$$

so using the computation of Lemma 3.4

$$\begin{aligned} 0 &= \mathrm{tr}_{\phi\eta^j}(\alpha) \\ &= \mathrm{tr}_{\eta^j}(\alpha_0 + \tilde{\alpha}_0) + \sum_{i=1}^{\infty} \mathrm{tr}_{\eta^j}(\alpha_i)(\phi^i + \phi^{-i}) \end{aligned}$$

for $j = 1, \ldots, 2^p - 1$.

Since each ϕ^i term must vanish separately, Lemma 3.5 immediately implies that $\alpha_0 + \tilde{\alpha}_0 \equiv 0$ and $\alpha_i \equiv 0$. Thus α has the form $\alpha_0(\xi)(1-\tilde{1})$ where $\alpha_0(\xi) = \sum_{i=1}^{2^p} \alpha_0^i \xi^{2i}$. Since $\mathrm{tr}_{J\eta^j}(\alpha) = \mathrm{tr}_{\eta^j}(\alpha_0) \cdot 2 = 0$ for $j = 1, \ldots, 2^p - 1$, the degree 2^p polynomial

$$\underline{\alpha}_0(x) = \sum_{i=1}^{2^p} \alpha_0^i x^i$$

has 2^p known roots, namely the $2^p - 1$ non-trivial 2^p-th roots of unity and 0. Thus $\underline{\alpha}_0(x) = \text{const.} \sum_{i=1}^{2^p} x^i$ and we can use $\mathrm{tr}_J(\alpha) = 2^{m-2k}$ to determine the constant. Thus we can conclude that

$$\alpha = 2^{m-2k-p-1}(1 + \xi^2 + \xi^4 + \cdots + \xi^{2^{p+1}-2})(1 - \tilde{1}),$$

and so $2k + 1 + p \leq m$ and the theorem is proved. □

Remark 3.1. For a spin action of an arbitrary group Γ of order 2^p on X our methods should give the bound $2k+1+p \leq m$ as long as the action is subject to some non-degeneracy conditions. The proof would proceed as all the others, using the non-degeneracy conditions and the character formula to guarantee $\mathrm{tr}_{Jg}(\alpha) = 0$ for non-trivial $g \in \Gamma$ and $\mathrm{tr}_{\phi g}(\alpha) = 0$ for all $g \in \Gamma$. Then one can show that

$$\alpha = 2^{m-2k-1-p} \cdot \rho \cdot (1 - \tilde{1})$$

where ρ is the regular representation of Γ. Complications do occur: in order to guarantee that $\mathrm{tr}_{Jg}(\alpha) = 0$ one needs to incorporate information about the virtual representation s, *i.e.* the Γ-index of the Dirac operator. This can be computed via the index theorem, but in general the non-degeneracy conditions would then involve the fixed point set. A further complication occurs because the lift of the action to the spin bundle can be complicated; some group elements may lift with twice their original order while others may preserve their order. Nevertheless, our techniques could be applied in a case-by-case basis if the above issues are understood.

3.8. **Proof of Theorem 1.5.** This has a slightly different flavor than the previous proofs. The proof is essentially the same for G_{odd} and G_{ev}; we will use the G_{ev} notation. The index has the form

$$\mathrm{Ind}(D) = s(\zeta)h - t(\zeta)\tilde{1}$$

and the hypothesis $k > 0$ means that $s(\zeta) = \sum_{i=1}^{2^p} s_i \zeta^i$ has at least one positive coefficient. From Theorem 1.1, $k > 0$ also implies $m > 0$. We will prove that $b_2^+(X/\tau) \neq 0$ by contradiction. Suppose $b_2^+(X/\tau) = 0$ so that the constant coefficient of $t(\zeta)$ is 0, *i.e.* $t_{2^p} = 0$. Since $m > 0$ we have $(W - V)_\phi \neq 0$ so $\mathrm{tr}_\phi(\alpha) = 0$ and since $t_{2^p} = 0$, we know that $\phi\nu$ acts non-trivially on all of V and W so

$$\begin{aligned} \mathrm{tr}_{\phi\nu}(\alpha) &= \lambda_{-1}(W - V) \\ &= \frac{(1-\nu)^{t_1}(1-\nu^2)^{t_2}\cdots(1-\nu^{2^p-1})^{t_{2^p-1}}}{(1-\nu\phi)^{s_1}(1-\nu\phi^{-1})^{s_1}\cdots}. \end{aligned}$$

Since at least one of the s_i's is positive, the above expression has arbitrarily high powers of ϕ in it. On the other hand

$$\begin{aligned} \mathrm{tr}_{\phi\nu}(\alpha) &= \mathrm{tr}_{\phi\nu}\left(\alpha_0(\zeta) + \tilde{\alpha}_0(\zeta)\tilde{1} + \sum_{i=1}^{\infty}\alpha_i(\zeta)h_i\right) \\ &= \alpha_0(\nu) + \tilde{\alpha}_0(\nu) + \sum_{i=1}^{\infty}\alpha_i(\nu)(\phi^i + \phi^{-i}) \end{aligned}$$

has only finitely many non-zero α_i terms, which is our contradiction. □

4. Applications

4.1. **Genus bounds.** Our original motivation for this work is our application to genus bounds. We refer the reader to our paper [4] for details of the set up.

Let M be a smooth 4-manifold (not necessarily spin) and let $\Sigma \hookrightarrow M$ be a smoothly embedded surface representing a homology class $a \in H_2(M; \mathbf{Z})$. A basic question in 4-manifold topology asks for the minimal genus $g_{min}(a)$ of a smoothly embedded surface representing a given class a. In order to determine $g_{min}(a)$ one needs good lower bounds on the genus of embedded surfaces and constructions realizing those bounds.

The basic ideas we use go back to Hsiang-Szczarba [13], Rochlin [10], and Kotschick-Matíc [7] (see also T. Lawson [8]).

Suppose the class a is divisible by some number d, then one can construct the d-fold branched cover $X \to M$ branched along Σ. Under favorable hypotheses, X will be spin and the covering transformation is a spin action. The signature and Euler characteristic of X can be computed in terms of $\Sigma \cdot \Sigma$ and $g(\Sigma)$ and so the bounds of Theorems 1.1 and 1.2 give genus bounds.

The most straightforward implementation of this idea is for double branched covers and results in Theorem 1.6. We will prove the general result coming from 2^p covers in this section.

Proposition 4.1. *Let $\Sigma \hookrightarrow M$ be a smoothly embedded surface of genus g in a smooth, simply-connected, compact, oriented 4-manifold. Suppose that $[\Sigma] \in H_2(M)$ has the property that $2^p|[\Sigma]$ and $[\Sigma]/2^p \equiv w_2(M) \bmod 2$.*

Then there is a spin 4-manifold X with a spin $\mathbf{Z}/2^p$ action $\tau : X \to X$ of odd type such that $X/\tau = M$. Furthermore, $k = -\sigma(X)/16$ and $m = b_2^+(X)$ are given by:

$$\begin{aligned} k &= \frac{1}{16}\left(-2^p\sigma(M) + \frac{4^p-1}{3\cdot 2^p}[\Sigma]\cdot[\Sigma]\right), \\ m &= 2^p b_2^+(M) + (2^p-1)g - \frac{4^p-1}{6\cdot 2^p}[\Sigma]\cdot[\Sigma]. \end{aligned}$$

Proof: This is in [4]; a sketch of the proof is the following. The condition $2^p|[\Sigma]$ allows one to construct the 2^p-fold branched cover $X \to M$ branched along Σ. One can compute $w_2(X)$ using a formula of Brand [3] and the condition

of $[\Sigma]/2^p \equiv w_2(M) \bmod 2$ guarantees $w_2(X)$ vanishes.

$\pi_1(X)$ is finite (see [10], [7]) so $b_1(X) = b_3(X) = 0$ and so m and k can be determined by $e(X)$ and $\sigma(X)$ which can be computed by the G-index theorem. The covering action $\tau : X \to X$ is automatically spin because $H_1(X;\mathbf{Z}/2) = 0$ (see [7]) and the action is of odd type since the fixed point set is two dimensional. □

Here is our general genus bound (Theorem 1.6 is a special case):

Theorem 4.2. *Let $\Sigma \hookrightarrow M$ be a smoothly embedded surface of genus g in a smooth, simply-connected, compact, oriented 4-manifold. Suppose that $[\Sigma] \in H_2(M)$ has the property that $2^p|[\Sigma]$ and $[\Sigma]/2^p \equiv w_2(M) \bmod 2$.*

Suppose $b_2^+(M) > 1$ and $g \neq [\Sigma]^2(1 + 2^{2i-2p+1})/6 - b_2^+(M)$ for $i = 1, \dots, p-1$. Then

$$g \geq \frac{1}{2^p-1}\left[\frac{5}{4}\left(\frac{4^p-1}{6\cdot 2^p}[\Sigma]^2 - 2^{p-1}\sigma(M)\right) + 1 + p - 2^{p-1}b_2(M)\right]. \tag{2}$$

Remark 4.1. The conditions on $b_2^+(M)$ and g are so that the non-degeneracy conditions of Theorem 1.2 are met. The condition on g is not present for $p = 1$. Although it is aesthetically unpleasing to have restrictions on g in the hypothesis, in practice the conditions are easy to dispense with (see example 4.1). Even without the conditions, one still gets an inequality coming from Furuta's theorem that is the same as the bound (2) but without the p term in the brackets. Our theorem thus improves the Furuta bound by $p/(2^p-1)$. Probably the best applications occur in the case $p = 1$ (as in Theorem 1.6 where

we improve the Furuta bound by 1). However, we will give examples where Theorem 4.2 provides sharp bounds unobtainable from the Furuta bound.

Example 4.1. Let $M = \#_N \mathbf{CP}^2$ with $N > 1$ and consider $\Sigma \hookrightarrow M$ representing the class $(4, \ldots, 4)$. We can apply Theorem 4.2 with $p = 2$. It tells us

$$g \geq \tfrac{1}{3}(8N + 3)$$

if $g \neq 3N$. When $N \geq 6$ the condition on g is no condition and for $N = 2, \ldots, 5$ the theorem gives $g \geq 3N$ since if $g = 3N - 1$ it would contradict $g \geq \frac{1}{3}(8N + 3)$. Since the connected sum of the algebraic representative in each factor has genus $3N$, the theorem gives a sharp bound for $N = 2, \ldots, 5$.

Proof of Theorem 4.2 : Let $X \to M$ be the 2^p-th branched cover and, to match the notation of Theorem 1.2, let X_i be the 2^{p-i} branched cover. Then $X_0 = X$, $X_p = M$ and X_i is the quotient of X by $\mathbf{Z}/2^i \subset \mathbf{Z}/2^p$. Let $m_i = b_2^+(X_i)$ which is computable by essentially the same formula as in Proposition 4.1

$$m_i = 2^{p-i} m_p + (2^{p-i} - 1)g - \frac{4^{p-i} - 1}{6 \cdot 2^{p-i}} [\Sigma]^2.$$

Note that $m_0 \geq m_1 \geq \cdots \geq m_p > 1$. The inequality of the theorem is equivalent to $2k + 1 + p \leq m_0$, so the only way the theorem can fail is if $2k + \delta = m_0$ for $\delta \in \{1, \ldots, p\}$ *and* $m_0 - 2k = m_1$ or $m_i = m_{i-1}$ for $i = 2, \ldots, p$. The hypothesis $g \neq [\Sigma]^2(1 + 2^{2i-2p+1})/6 - b_2^+(M)$ is equivalent to $m_i \neq m_{i+1}$, so we know that $m_1 > m_2 > \cdots > m_p = b_2^+(M)$. This implies $m_1 \geq p - 1 + b_2^+(M)$, which, combined with the hypothesis $b_2^+(M) > 1$, gives $m_1 > p$. Then we see that we cannot have $2k + \delta = m_0$ and $m_0 - 2k = m_1$, since that would imply $\delta > p$. □

4.2. **Involutions on rational cohomology $K3$'s.** The proof of Theorem 1.8 follows readily from our main theorems (Theorems 1.2, 1.3, and 1.5). Suppose X is a rational cohomology $K3$ so that $H^*(X; \mathbf{Q}) \cong H^*(K3; \mathbf{Q})$ and suppose $\tau : X \to X$ is a spin involution.

First suppose that τ is of odd type. Since the inequality of Theorem 1.2 is violated, one of the non-degeneracy conditions must also be violated and so $b_2^+(X/\tau)$ is either 1 or 0. Theorem 1.5 shows that $b_2^+(X/\tau)$ is not 0 so it must be 1. This can easily occur, for example $K3$ is a double branched cover over $\mathbf{CP}^2$, $S^2 \times S^2$, and $E(1) \cong \mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$ and it has a free involution covering the Enriques surface.

Now suppose that τ is even so it must violate the non-degeneracy conditions of Theorem 1.3. In conjunction with Theorem 1.5 we see that $b_2^+(X/\tau)$ must be 3. Since the fixed point set is at most points, the G-signature theorem tells us that

$$\sigma(X/\tau) = \tfrac{1}{2}\sigma(X) = -8.$$

If N is then the number of fixed points, then the Lefschetz formula tells us the Euler characteristic

$$\chi(X/\tau) = \tfrac{1}{2}(\chi(X) + N) = 12 + N/2.$$

Since $b_1(X/\tau) = 0$ and we know $b_2^+(X/\tau) = 3$, we can solve the above equations to get $b_2^-(X/\tau) = 11$ and $N = 8$.

It was pointed out to us by P. Kronheimer that this does occur. The construction is as follows. Let $\tilde{Y}$ be the $K3$ surface. There are 8 disjoint -2 spheres $S_1, \ldots, S_8$ such that $S = \sum_{i=1}^{8} S_i$ is divisible by 2. Let $\tilde{X}$ be the double branched cover of $\tilde{Y}$ branched along S. The preimage of S in $\tilde{X}$ is then 8 disjoint (-1) spheres which we can blow down to obtain X, a smooth manifold with an involution fixing 8 points covering Y, the orbifold obtained by collapsing the S_i's in $\tilde{Y}$:

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d] & X \arrow[d] \\ \tilde{Y} \arrow[r] & Y. \end{tikzcd}$$

We will show that X is a rational cohomology $K3$ (with a little more work, one can see that $X \cong K3$) and $b_2^+(Y) = 3$, so this is the example we seek. From the signature theorem and Lefschetz formula, we compute that $\chi(\tilde{X}) = 2\chi(\tilde{Y}) - \chi(S) = 32$ and $\sigma(\tilde{X}) = 2\sigma(\tilde{Y}) - S^2 = 24$,so after blowing down $\chi(X) = 24$ and $\sigma(X) = -16$. Since the preimage of S is characteristic in $\tilde{X}$, we know that X is spin and since $b_1(\tilde{X}) = b_1(X) = 0$, X is a rational cohomology $K3$.

Remark 4.2. It is easy to construct a *homeomorphism* $\tau : X \to X$ generating an involution with $b_2^+(X/\tau) = 2$ (for example) but our classification implies this will not be smoothable, *i.e.* there is no smooth structure on X so that this τ is a diffeomorphism.

References

[1] M. F. Atiyah. Bott periodicity and the index of elliptic operators. *Quart. J. Math. Oxford*, 19(2), 1968.

[2] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes II. Applications. *Annals of Mathematics*, 88:451–491, 1968.

[3] N. Brand. Necesary conditions for the existence of branched coverings. *Inventiones Math.*, 54, 1979.

[4] Jim Bryan. Seiberg-Witten à la Furuta and genus bounds for classes with divisibility. In S. Akbulut and T. Onder, editors, *Proceedings of the 5^{th} Gökava Geometry and Topology conference*, 1996.

[5] S. K. Donaldson and P. B. Kronheimer. *The Geometry of Four-Manifolds.* Oxford Mathematical Monographs. Oxford University Press, 1990.

[6] M. Furuta. Monopole equation and the $\frac{11}{8}$-conjecture. Preprint., 1995.

[7] D. Kotschick and G. Matíc. Embedded surfaces in four-manifolds, branched covers, and $SO(3)$-invariants. *Mathematical Proceedings of the Cambridge Philosophical Society*, 117, 1995.

[8] T. Lawson. The minimal genus problem. Preprint.

[9] J. W. Morgan, Z. Szabó, and C. H. Taubes. The generalized Thom conjecture. Preprint.

[10] Rokhlin. Two-dimensional submanifolds of four-dimensional manifolds. *Functional Analysis and Applications.*, 6:93–48, 1971.

[11] D. Ruberman. Involutions on spin 4-manifolds. *Proceedings of the American Mathematical Society*, 123(2), 1995.

[12] Tammo tom Dieck. *Transformation Groups and Representation Theory.* Number 766 in Lecture Notes in Mathematics. Springer-Verlag, 1970.

[13] W.C.Hsiang and R.H.Szczarba. On embedding surfaces in four-manifolds. *Proceedings of the Symposium of Pure Math*, 22:97–103, 1971.

M.S.R.I., 1000 Centennial Dr., Berkeley, CA 94720
E-mail address: jbryan@msri.org