THE DONALDSON-THOMAS THEORY OF $K3 \times E$ VIA THE TOPOLOGICAL VERTEX.

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ABSTRACT. We give a general overview of the Donaldson-Thomas invariants of elliptic fibrations and their relation to Jacobi forms. We then focus on the specific case of where the fibration is $S \times E$, the product of a $K3$ surface and an elliptic curve. Oberdieck and Pandharipande conjectured [11] that the partition function of the Gromov-Witten/Donaldson-Thomas invariants of $S \times E$ is given by minus the reciprocal of the Igusa cusp form of weight 10. For a fixed primitive curve class in S of square $2h-2$, their conjecture predicts that the corresponding partition functions are given by meromorphic Jacobi forms of weight -10 and index $h-1$. We calculate the Donaldson-Thomas partition function for primitive classes of square -2 and of square 0, proving strong evidence for their conjecture.

Our computation uses reduced Donaldson-Thomas invariants which are defined as the (Behrend function weighted) Euler characteristics of the quotient of the Hilbert scheme of curves in $S \times E$ by the action of E. Our technique is a mixture of motivic and toric methods (developed with Kool in [4]) which allows us to express the partition functions in terms of the topological vertex and subsequently in terms of Jacobi forms. We compute both versions of the invariants: unweighted and Behrend function weighted Euler characteristics. Our Behrend function weighted computation requires us to assume Conjecture 18 in [4].

1. AN OVERVIEW OF DONALDSON-THOMAS THEORY FOR ELLIPTIC FIBRATIONS.

Let X be a quasi-projective non-singular Calabi-Yau threefold over $\mathbb{C}$. The Donaldson-Thomas invariants are a virtual count of sheaves on X. They are invariant under deformations and they are the mathematical counterpart of counting BPS states in type IIB topological string theory compactified on X. Of particular importance is the case where the sheaves are ideal sheaves of curves. Then the Donaldson-Thomas invariants "count curves", encoding subtle information about the enumerative geometry of X.

We can identify the moduli space of ideal sheaves of curves with

$$\mathrm{Hilb}^{\beta,n}(X) = \{Z \subset X : \quad [Z] = \beta \in H_2(X),\ n = \chi(\mathcal{O}_Z)\}$$

the Hilbert scheme of proper subschemes of X with fixed homology class and holomorphic Euler characteristic. To obtain a curve count, we would like to determine the "number of points" in $\mathrm{Hilb}^{\beta,n}(X)$. One way to assign an integer to a $\mathbb{C}$-scheme, which generalizes the number of points in a zero dimensional scheme, is the topological Euler characteristic of the scheme with its complex analytic topology. This gives rise to the "naïve Donaldson-Thomas invariant" which we denote by

$$\widehat{DT}_{\beta,n}(X) = e(\mathrm{Hilb}^{\beta,n}(X)).$$

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The term "invariant" is somewhat of a misnomer for this quantity since in general, it is not invariant under deformations[1] of X.

Topological Euler characteristic is insensitive to the singularities and non-reduced structure of the Hilbert scheme and to get a true deformation invariant, these must be taken into account. Instead, we define the Donaldson-Thomas invariant as a weighted Euler characteristic:

$$DT_{\beta,n}(X) = \sum_{k\in\mathbb{Z}} k\cdot e(\nu^{-1}(k))$$

where

$$\nu : \mathrm{Hilb}^{\beta,n}(X) \to \mathbb{Z}$$

is the Behrend function. The Behrend function is an integer-valued constructible function, defined for any $\mathbb{C}$-scheme, which is sensitive to non-reducedness and singularities. Behrend proves [1] that this definition of $DT_{\beta,n}(X)$ agrees with the traditional definition given by the degree of the virtual fundamental class [10], which is a deformation invariant[2].

It will be notationally convenient to treat an Euler characteristic weighted by a constructible function as a Lebesgue integral, where the measurable sets are constructible sets, the measurable functions are constructible functions, and the measure of a set is given by its Euler characteristic. In this language, one writes

$$DT_{\beta,n}(X) = \int_{\mathrm{Hilb}^{\beta,n}(X)} \nu\, de.$$

In general, Donaldson-Thomas invariants are extremely hard to compute. However, the connection to physics, has provided several conjectures suggesting that the invariants have a rich structure. A particularly beautiful instance of this is a conjecture of Huang, Katz, and Klemm [9] (see also [8]) which can be roughly expressed as the following slogan:

> *Generating functions for the Donaldson-Thomas invariants of elliptically fibered Calabi-Yau threefolds are given by the Fourier expansions of Jacobi forms.*

To give more details, suppose that X is elliptically fibered, that is $\pi : X \to S$ is a flat family of genus one curves over a smooth surface S with a section $\sigma : S \to X$. For a curve class β in S, we let

$$\beta + dE \in H_2(X)$$

be the curve class given by $\sigma_*(\beta) + d[\pi^{-1}(\mathrm{pt})]$, that is a class in the base plus d times the fiber class. We define a partition function

$$Z_\beta^{\mathsf{DT}}(X) = \sum_{n,d} DT_{\beta+dE,n}(X)\; q^{d+\frac{1}{2}K_S\cdot\beta}\,(-p)^n.$$

Then the Huang-Katz-Klemm conjecture is

Conjecture 1.1. *Let S be Fano[3] and let h be the arithmetic genus of β. Then*

$$\frac{Z_\beta^{\mathsf{DT}}(X)}{Z_0^{\mathsf{DT}}(X)}$$

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[1] Although, from general considerations, $\widehat{DT}_{\beta,n}(X)$ is a constructible function on the deformation space of X and hence it is an invariant of the generic member of each deformation type.

[2] The traditional definition is only for projective X. When X is only quasi-projective, the weighted Euler characteristic definition is well defined, but no longer guarenteed to be a deformation invariant.

[3] It is not clear what the optimal hypotheses on $\pi : X \to S$ should be. In [9], they primary consider the case of Fano S and where all fibers $\pi^{-1}(\mathrm{pt})$ are irreducible, but they discuss other cases, including the trivial fibrations studied in this paper.

is a Jacobi form of index $h-1$ and weight $l = l(S,\beta)$ and with variables

$$q = \exp(2\pi i \tau), \qquad p = \exp(2\pi i z).$$

We make several remarks:

- A Jacobi form is a two variable generization of modular forms which depend on both a modular parameter $\tau \in \mathbb{H}$ and an elliptic parameter $z \in \mathbb{C}$. Jacobi forms are meromorphic functions on $\mathbb{H} \times \mathbb{C}$ satisfying a modular transformation law under the action of $SL_2(\mathbb{Z})$ and an elliptic transformation law under lattice translations (see [7]).
- The denominator in the conjecture is $Z_0^{\mathsf{DT}}(X)$ is the partition function for the fiber class invariants. An explicit expression for this series has been computed by Toda [17].
- This conjecture reveals that the Donaldson-Thomas invariants have a beautiful internal structure and a suprising connection with number theory. Additionally, the conjecture is a powerful computational tool: since the Jacobi forms of a fixed weight and degree form a finite dimensional vector space, the conjecture reduces an infinite number of computations down to a finite number. In [9], several explicit predictions are made and partially checked.
- A proof of the elliptic transformation law was given by Oberdieck and Shen using a fiberwise Fourier-Mukai transform and wall-crossing techniques [14]. Oberdieck and Shen were able to use their result to confirm several cases of the Huang-Katz-Klemm conjecture.
- The variable change $p = \exp(2\pi i z)$ is close to the variable change $p = \exp(i\lambda)$ which (conjecturally) equates the Donaldson-Thomas partition function to the Gromov-Witten partition function[10]. So while Donaldson-Thomas invariants of an elliptically fibered Calabi-Yau threefold are the coefficients of the Fourier expansion of a Jacobi form, the Gromov-Witten invariants are essentially the coefficients of the Taylor expansion of the same Jacobi form.
- A proof for a the local version of the conjecture (where X is a local elliptic surface), was given in [4].

2. The case of $K3 \times E$: Overview

In this paper we focus on the simpliest possible elliptically fibered Calabi-Yau threefold, namely where $X = S \times E$ where S is a $K3$ surface and E is an elliptic curve. In [11], Oberdieck and Pandharipande conjectured that the partition function for the curve counting invariants of X is given by $-1/\chi_{10}$, minus the reciprocal of the weight 10 Igusa cusp form[4,5] . The relevant curve counting invariants include modified versions of Gromov-Witten invariants and stable pairs invariants. In this paper, we define modified Donaldson-Thomas invariants of X. Our definition is given by taking the Euler characteristic of the quotient of the Hilbert scheme of curves on X by the action of the elliptic curve (we consider both the Behrend function weighted and unweighted Euler characteristics). Our invariants are expected to be equal to the invariants defined via stable pairs in [11].

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[4]Oberdieck and Pandharipande were the first to formulate a mathematically precise conjecture, but the appearence of the Igusa cusp form in closely related partition functions in physics dates back to the mid-nineties [6].

[5]Since this paper was originally written in 2015, Oberdieck and Pixton have given a complete proof of this conjecture using methods (different from ours) from both Donaldson-Thomas theory and Gromov-Witten theory [13].

We employ an approach to computing these invariants which uses a mixture of motivic and toric methods (technology developed with M. Kool in [4]). We show that these methods yield complete computations for the partition functions of X in the case where S is $K3$ surface with a primitive curve class of square -2 or of square 0 (assuming Conj. 18 [4] in the Behrend function weighted case). The resulting partition functions are given by the Jacobi forms $F^{-2}\Delta^{-1}$ and $-24\,\wp\,\Delta^{-1}$ respectively where F is a Jacobi theta function, Δ is the discriminant modular form, and $\wp$ is the Weierstrass $\wp$ function. This agrees with the prediction from by the Oberdieck-Pandharipande conjecture thus proving their conjecture for primitive curve classes in the $K3$ of square -2 or 0.

Our general computational strategy is the following. Donaldson-Thomas invariants are given by weighted Euler characteristics of Hilbert schemes. We stratify the Hilbert scheme using the geometric support of the curves and we compute Euler characteristics of strata separately. Many of the strata acquire actions of E or $\mathbb{C}^*$ (that were not present globally) and we restrict to the fixed point loci. We are able to further stratify the fixed point loci and those strata sometimes acquire further actions. Iterating this strategy, we reduce the computation to subschemes which are formally locally given by monomial ideals. These are counted using the topological vertex. New identities for the topological vertex lead to closed formulas. To incorporate the Behrend function into this strategy, we must assume the conjecture formulated in [4, Conj. 18]. This is a general conjecture regarding the behavior of the Behrend function at subschemes given locally by monomial ideals.

3. The case of $K3\times E$: definitions, conjectures, and results.

As above, let

$$X = S\times E$$

where S is a non-singular $K3$ surface with a primitive curve class β of square

$$\beta^2 = 2h-2.$$

We call h the *genus* of the $K3$ surface. Let

$$\beta + dE \in H_2(X)$$

denote the class $i_{S*}(\beta)+i_{E*}(d[E])$ where $i_S : S\to X$ and $i_E : E\to X$ are the inclusions obtained from choosing points $s\in S$ and $e\in E$.

It is worth emphasizing:

The ordinary Donaldson-Thomas invariants $DT_{\beta+dE,n}(X)$ are all zero.

This can be seen in two different ways:

(1) The action of E on $\mathrm{Hilb}^{\beta+dE,n}(X)$ is fixed point free, consequently its (Behrend function[6] weighted) Euler characteristic is zero.
(2) There exists deformations of S which make β non-algebraic. Under this deformation, the Hilbert scheme $\mathrm{Hilb}^{\beta+dE,n}(X)$ becomes empty. Since $DT_{\beta+dE,n}(X)$ is deformation invariant it must be zero.

Remarkably, the above two issues can be solved simultaneously by taking the weighted Euler characteristic of the quotient of the Hilbert scheme.

[6]The value of the Behrend function at a closed point of a scheme only depends on the local ring of that point, therefore the Behrend function of a scheme is invariant under any group action.

Definition 3.1. *The **reduced** Donaldson-Thomas invariants of X are defined by*

$$\mathsf{DT}_{\beta+dE,n}(X) = \int_{\mathrm{Hilb}^{\beta+dE,n}(X)/E} \nu\, de$$

where $\nu : \mathrm{Hilb}^{\beta+dE,n}(X)/E \to \mathbb{Z}$ is the Behrend function of the quotient. Note that we denote the reduced invariants with the san serif font DT, while the ordinary invariants have the ordinary font DT.

Conjecture 3.2. *The number $\mathsf{DT}_{\beta+dE,n}(X)$ is invariant under deformations of X which keep the class $\beta + dE$ algebraic.*

Proof sketch: The Hilbert scheme $\mathrm{Hilb}^{\beta+dE,n}(X)$ admits a (-1)-shifted symplectic structure coming from viewing it as a moduli space of rank 1 sheaves on X with trivialized determinant [16]. Taking the (-1)-symplectic quotient of the Hilbert scheme by the action of E yields a (-1)-symplectic space whose underlying space is $\mathrm{Hilb}^{\beta+dE,n}(X)/E$ (the moment map affects the derived structure, but not the classical space). As with any (-1)-shifted symplectic structure, this shifted symplectic structure gives rise to a symmetric obstruction theory whose associated virtual class has degree equal to the Behrend function weighted Euler characteristic of underlying scheme. The effect of taking the zeros of the moment map in the symplectic quotient construction is to remove from the obstruction space those obstructions to deforming the class β to a non-algebraic class. Note that these obstructions are dual to the deformations of a subscheme given by the action of E. The resulting virtual class on $\mathrm{Hilb}^{\beta+dE,n}(X)/E$ should be invariant under deformations preserving the algebraicity of β.

The analogous conjecture for reduced stable pairs invariants was proved in [12].

Up to deformation, a curve class on a $K3$ surface is determined by its square and divisibility, so by our assumption that β is primitive, it only depends on h up to deformation. We thus streamline the notation by writing:

$$\mathsf{DT}_{h,d,n}(X) := \mathsf{DT}_{\beta+dE,n}(X)$$

and we also write

$$\mathrm{Hilb}^{h,d,n}(X) := \mathrm{Hilb}^{\beta+dE,n}(X).$$

We also consider the naïve version given by unweighted Euler characteristics.

$$\widehat{\mathsf{DT}}_{h,d,n}(X) = \int_{\mathrm{Hilb}^{h,d,n}(X)/E} 1\, de.$$

Since the unweighted Euler characteristics are not expected to be deformation invariants, we must assume in this case that the $K3$ surface S and the elliptic curve E are generic. These Euler characteristics are constant on a dense open set in the deformation space of X and so under the genericity assumption, the above notation is justified.

We define[7] partition functions as follows

$$\begin{aligned}
\mathsf{DT}(X) &= \sum_{h=0}^{\infty} \mathsf{DT}_h(X)\,\tilde{q}^{\,h-1} \\
&= \sum_{\substack{h,d\geq 0\\ n\in\mathbb{Z}}} \mathsf{DT}_{h,d,n}(X)\,\tilde{q}^{\,h-1} q^{d-1}(-p)^n \\
\widehat{\mathsf{DT}}(X) &= \sum_{h=0}^{\infty} \widehat{\mathsf{DT}}_h(X)\,\tilde{q}^{\,h-1} \\
&= \sum_{\substack{h,d\geq 0\\ n\in\mathbb{Z}}} \widehat{\mathsf{DT}}_{h,d,n}(X)\,\tilde{q}^{\,h-1} q^{d-1} p^n
\end{aligned}$$

We remark that our convention for the $\tilde{q}$ and q variables is the opposite from Oberdieck and Pandharipande's, however there is a conjectural symmetry $\tilde{q} \leftrightarrow q$ and so this difference should not be seen in the formulas. To be precise, the Donaldson-Thomas version of Oberdieck and Pandharipande's conjecture is the following.

Conjecture 3.3. *Let χ_{10} be the Igusa cusp form of weight 10, then*

$$\mathsf{DT}(X) = -\frac{1}{\chi_{10}}\ .$$

Explicitly, we can write

$$\chi_{10}\left(p,q,\tilde{q}\,\right) = pq\tilde{q}\, \left(1-p^{-1}\right)^2 \prod_{n\in\mathbb{Z}} \prod_{(d,h)>(0,0)} \left(1-p^n q^d \tilde{q}^{\,h}\right)^{c\left(4dh-n^2\right)}$$

where the integers $c(k)$ are given as the coefficients of Z, the elliptic genus of the $K3$ surface:

$$Z(p,q) = -24\wp F^2 = \sum_{n\in\mathbb{Z}} \sum_{d\geq 0} c(4d-n^2) p^n q^d.$$

Here F is a Jacobi theta function and $\wp$ is the Weierstrass $\wp$-function, namely

$$-F^{-2} = \frac{p}{(1-p)^2} \prod_{m=1}^{\infty} \frac{(1-q^m)^4}{\left(1-pq^m\right)^2 \left(1-p^{-1}q^m\right)^2}$$

and

$$\wp = \frac{1}{12} + \frac{p}{(1-p)^2} + \sum_{d=1}^{\infty} \left(\sum_{k|d} k(p^k + p^{-k} - 2) \right) q^d.$$

Expanding $-\chi_{10}^{-1}$ as a series in $\tilde{q}$, one obtains predictions for each $\mathsf{DT}_h(X)$ in terms of Jacobi forms of weight -10 and index $h-1$ (see [11, page 10]). The main result of this paper is the following theorem.

[7] Our insertion of the sign on the p variable in the Donaldson-Thomas partition function is somewhat nonstandard. It makes subsequent formulas simpler to state. Geometrically, the extra sign can be interpreted as using the Behrend function coming from the moduli stack of structure sheaves of the curves as oppose to the moduli space of ideal sheaves (see [3, § 3,Thm 3.1]). Since the naïve Donaldson-Thomas invariants do not involve the Behrend function, no sign appears.

Theorem 3.4. *For a generic $K3$ surface S and elliptic curve E, the genus 0 and genus 1 partition functions for the unweighted Donaldson-Thomas invariants of $X = S \times E$ are given by*

$$\widehat{\mathsf{DT}}_0(X) = \frac{pq^{-1}}{(1-p)^2} \prod_{m=1}^{\infty} (1-q^m)^{-20}(1-pq^m)^{-2}(1-p^{-1}q^m)^{-2}$$

$$\widehat{\mathsf{DT}}_1(X) = 24q^{-1} \prod_{m=1}^{\infty} (1-q^m)^{-24} \left(\frac{1}{12} + \frac{p}{(1-p)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k \left(p^k + 2 + p^{-k}\right) q^d \right)$$

Moreover, assuming Conjecture 7.1 (i.e. [4, Conj. 18]*), the partition functions for the Behrend function weighted Donaldson-Thomas invariants are given by (note the sign differences with the above formulae, an overall sign on each, and on the 2 within the sum)*

$$\mathsf{DT}_0(X) = \frac{-pq^{-1}}{(1-p)^2} \prod_{m=1}^{\infty} (1-q^m)^{-20}(1-pq^m)^{-2}(1-p^{-1}q^m)^{-2}$$

$$\mathsf{DT}_1(X) = -24q^{-1} \prod_{m=1}^{\infty} (1-q^m)^{-24} \left(\frac{1}{12} + \frac{p}{(1-p)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k \left(p^k - 2 + p^{-k}\right) q^d \right)$$

The above two formulas verify the Oberdieck-Pandharipande conjecture for $K3$ surfaces with a primitive curve class of square -2 or 0. Namely, the series $\mathsf{DT}_h(X)$ for $h=0$ and $h=1$ are given by the following Jacobi forms

$$\mathsf{DT}_0(X) = \frac{1}{F^2\Delta},$$
$$\mathsf{DT}_1(X) = -24\frac{\wp}{\Delta}.$$

4. Preliminaries and notation.

Our aim is to compute $\mathsf{DT}_h(X)$ for $h=0$ and $h=1$. We begin by computing $\widehat{\mathsf{DT}}_h(X)$ and then discuss how to modify the argument to include the Behrend function in section 7.

Euler characteristic is motivic: it defines a homomorphism from $K_0(\mathrm{Var}_{\mathbb{C}})$, the Grothendieck group of varieties over $\mathbb{C}$, to the integers. We define

$$\mathrm{Hilb}^{h,\bullet,\bullet}(X)/E = \sum_{n,d} \left[\mathrm{Hilb}^{h,d,n}(X)/E\right] p^n q^d$$

which we regard as an element in $K_0(\mathrm{Var}_{\mathbb{C}})((p))[[q]]$. We will use this convention throughout:

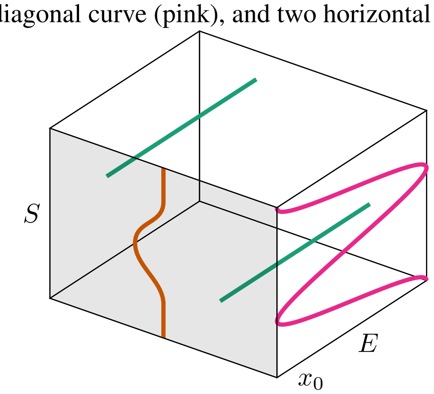
Convention 4.1. *When an index is replaced by a bullet, we will sum over the index, multiplying by the appropriate variable.*

We see that with our notation

$$\widehat{\mathsf{DT}}_h(X) = q^{-1} e\left(\mathrm{Hilb}^{h,\bullet,\bullet}(X)/E\right).$$

Definition 4.2. *Let p_S and p_E be the projections of $X = S \times E$ onto each factor and let $C \subset X$ be an irreducible curve. We say that C is **vertical** if $p_E : C \to E$ is degree zero and we say C is **horizontal** if $p_S : C \to S$ is degree zero. If both maps are of non-zero degree, we say C is **diagonal**. See Figure 1.*

FIGURE 1. A vertical curve (orange) contained in the slice $S \times \{x_0\}$ (light grey), a diagonal curve (pink), and two horizontal curves (green).



We will assume that $X = S \times E$ where S is generic among $K3$ surfaces admitting a primitive class β of square $2h-2$. In particular, β is an irreducible class.

Since β is an irreducible class, any subscheme Z corresponding to a point in $\mathrm{Hilb}^{h,d,n}(X)$ must have a unique component $C_0 \subset Z$ which is either a vertical or a diagonal curve with all other curve components of Z being horizontal. Subschemes with C_0 diagonal cannot deform to subschemes with C_0 horizontal and so we get a decomposition of the Hilbert scheme into disjoint components corresponding to subschemes with vertical and diagonal components respectively:

$$\mathrm{Hilb}^{h,d,n}(X) = \mathrm{Hilb}^{h,d,n}_{\mathsf{vert}}(X) \sqcup \mathrm{Hilb}^{h,d,n}_{\mathsf{diag}}(X)$$

Diagonal curves do not appear in the $h = 0$ case, but do occur for $h \geq 1$.

5. COMPUTING $\widehat{\mathsf{DT}}_h(X)$ IN THE CASE $h = 0$.

We now consider the case where $h = 0$. The $K3$ surface S has a single curve $C_0 \cong \mathbb{P}^1$ in the class β. There are no diagonal curves since such a curve would have geometric genus 0 but also admit a non-constant map to E.

We fix a base point $x_0 \in E$. We can fix a slice for the action of E on $\mathrm{Hilb}^{0,d,n}(X)$ by requiring that the unique vertical curve lies in $S \times \{x_0\}$. We denote the slice with the subscript "fixed".

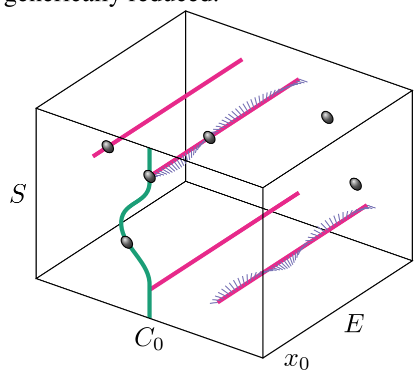
$$\mathrm{Hilb}^{0,d,n}(X)/E \cong \mathrm{Hilb}^{0,d,n}_{\mathsf{fixed}}(X) \subset \mathrm{Hilb}^{0,d,n}(X).$$

The points in $\mathrm{Hilb}^{0,d,n}_{\mathsf{fixed}}(X)$ correspond to subschemes $Z \subset X$ given by unions of the curve $C_0 \times \{x_0\}$ with horizontal curves whose support is of the form $\{\text{points} \times E\}$, but may have nilpotent thickenings. The subscheme Z also potentially has embedded points as well as zero dimensional components away from the curve support (see Figure 2).

As a consequence of the above geometric description, we see that any such subscheme is a disjoint union of a subscheme of $\widehat{X}_{C_0 \times E}$, the formal neighborhood of $C_0 \times E$ in X, and $X - (C_0 \times E)$. This leads to a decomposition of the Hilbert scheme into strata given by products of Hilbert schemes of subschemes of $\widehat{X}_{C_0 \times E}$ and subschemes of $X - (C_0 \times E)$. Using our bullet convention, this can be efficiently expressed as follows.

$$\mathrm{Hilb}^{0,\bullet,\bullet}(X)/E = \mathrm{Hilb}^{0,\bullet,\bullet}_{\mathsf{fixed}}\left(\widehat{X}_{C_0 \times E}\right) \cdot \mathrm{Hilb}^{0,\bullet,\bullet}(X - (C_0 \times E)) \tag{1}$$

FIGURE 2. Subschemes in $S \times E$ up to translation. Horizontal curves (pink) can have nilpotent thickenings (blue), and there can be embedded and floating points (gray). The unique vertical curve C_0 (green) lies in $S \times \{x_0\}$ and is generically reduced.



where as before the subscript "fixed" indicates that we are restricting to the sublocus

$$\mathrm{Hilb}^{0,d,n}_{\mathsf{fixed}}\left(\widehat{X}_{C_0\times E}\right) \subset \mathrm{Hilb}^{0,d,n}\left(\widehat{X}_{C_0\times E}\right) \subset \mathrm{Hilb}^{0,d,n}(X)$$

parameterizing subschemes where the unique vertical curve is $C_0 \times \{x_0\}$.

Note that d (the degree in the E direction) and n (the holomorphic Euler characteristic) are both additive under the disjoint union which allows us to express the decomposition as a product of Grothendieck group valued power series as above. Taking Euler characteristics of the above series, we find

$$q\,\widehat{\mathsf{DT}}_0(X) = e\left(\mathrm{Hilb}^{0,\bullet,\bullet}_{\mathsf{fixed}}\left(\widehat{X}_{C_0\times E}\right)\right)\cdot e\left(\mathrm{Hilb}^{0,\bullet,\bullet}(X - C_0\times E)\right). \tag{2}$$

Note that the action of E on $X - C_0 \times E$ induces an action on $\mathrm{Hilb}^{0,d,n}(X - C_0 \times E)$. This "new" E action is possible because the "fixed" condition lives entirely in the $\mathrm{Hilb}^{0,d,n}_{\mathsf{fixed}}\left(\widehat{X}_{C_0\times E}\right)$ factors (which do not have E actions).

The Euler characteristic of a scheme with a free E action is trivial and so

$$e\left(\mathrm{Hilb}^{0,d,n}(X - C_0\times E)\right) = e\left(\mathrm{Hilb}^{0,d,n}(X - C_0\times E)^E\right).$$

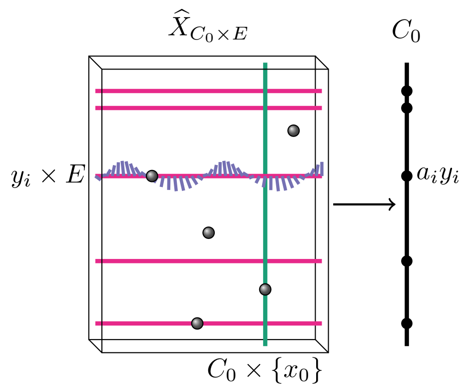
The E-fixed locus $\mathrm{Hilb}^{0,d,n}(X - C_0 \times E)^E$ parameterizes subschemes which are invariant under the E action. Such subschemes are of the form $Z \times E$ where $Z \subset S - C_0$ is a zero-dimensional subscheme of length d. Such subschemes have $n = \chi(\mathcal{O}_{Z\times E}) = 0$ and so

$$e\left(\mathrm{Hilb}^{0,\bullet,\bullet}(X - C_0\times E)^E\right) = e\left(\sum_{d=0}^{\infty} \mathrm{Hilb}^d(S - C_0)\, q^d\right)$$

$$= \prod_{m=1}^{\infty}(1-q^m)^{-22}. \tag{3}$$

Here we have used Göttsche's formula for the Euler characteristics of Hilbert schemes of points of surfaces; the 22 appearing in the exponent is the Euler characteristic of the surface $S - C_0$.

FIGURE 3. The map $\rho_d : \mathrm{Hilb}^{0,d,n}_{\mathsf{fixed}}(\widehat{X}_{C_0\times E}) \to \mathrm{Sym}^d(C_0)$ records the location and multiplicity of the horizontal curve components.



To compute $e\left(\mathrm{Hilb}^{0,\bullet,\bullet}_{\mathsf{fixed}}(\widehat{X}_{C_0\times E})\right)$, we begin by noting that there is a morphism

$$\rho_d : \mathrm{Hilb}^{0,d,n}_{\mathsf{fixed}}(\widehat{X}_{C_0\times E}) \to \mathrm{Sym}^d(C_0)$$

given by the intersection (with multiplicity) of the horizontal components of a curve with the vertical curve C_0. In other words, a scheme whose curve support is $C_0 \cup_i (y_i \times E)$ with multiplicity a_i along $y_i \times E$ is mapped to $\sum_i a_i y_i \in \mathrm{Sym}^d(C_0)$ (see Figure 3).

We may compute the Euler characteristic of $\mathrm{Hilb}^{0,\bullet,\bullet}_{\mathsf{fixed}}(\widehat{X}_{C_0\times E})$ by computing the Euler characteristic of $\mathrm{Sym}^d(C_0)$, weighted by the constructible function given by the Euler characteristic of the fibers of ρ_d. In other words,

$$\begin{aligned} e\left(\mathrm{Hilb}^{0,d,n}_{\mathsf{fixed}}(\widehat{X}_{C_0\times E})\right) &= \int_{\mathrm{Hilb}^{0,d,n}_{\mathsf{fixed}}(\widehat{X}_{C_0\times E})} 1 \; de \\ &= \int_{\mathrm{Sym}^d C_0} (\rho_d)_*(1) \; de. \end{aligned}$$

Writing

$$\mathrm{Sym}^\bullet C_0 = \sum_{d=0}^{\infty} \mathrm{Sym}^d C_0 \, q^d$$

and extending the integration to the $\bullet$ notation in the obvious way, we get

$$e\left(\mathrm{Hilb}^{0,\bullet,\bullet}_{\mathsf{fixed}}(\widehat{X}_{C_0\times E})\right) = \int_{\mathrm{Sym}^\bullet C_0} \rho_*(1) \; de \tag{4}$$

where the constructible function $\rho_*(1)$ takes values in $\mathbb{Z}((p))$ and is given by

$$\rho_*(1)(\sum_i a_i y_i) = e\left(\rho^{-1}(\sum_i a_i y_i)\right).$$

We will prove that $\rho_*(1)$ only depends on the multiplicities of the points in the symmetric product, not their location.

Proposition 5.1. *There exists a universal series $F(a) \in \mathbb{Z}[[p]]$ such that the constructible function $\rho_*(1)$ is given by*

$$\rho_*(1)\left(\sum a_i y_i\right) = \frac{p}{(1-p)^2}\prod_i F(a_i).$$

Deferring the proof of the proposition for the moment, we apply the following lemma regarding weighted Euler characteristics of symmetric products.

Lemma 5.2. *Let T be a scheme and let $\mathrm{Sym}^\bullet(T) = \sum_{d=0}^\infty \mathrm{Sym}^d(T)\, q^d$. Suppose that G is a constructible function on $\mathrm{Sym}^d(T)$ of the form $G(\sum_i a_i y_i) = \prod_i g(a_i)$ where by convention $g(0) = 1$. Then*

$$\int_{\mathrm{Sym}^\bullet T} G\; de = \left(\sum_{a=0}^\infty g(a)\, q^a\right)^{e(T)}.$$

This lemma is a consequence of the fact that symmetric products define a pre-lambda ring structure on the Grothendieck group of varieties and the Euler characteristic homomorphism is compatible with that structure. An elementary proof is given in [4].

Applying Lemma 5.2 to Proposition 5.1 and combining with equations (2), (3), and (4) and we see that

$$q\, \widehat{\mathsf{DT}}_0(X) = \frac{p}{(1-p)^2}\left(\sum_{a=0}^\infty F(a)\, q^a\right)^2 \cdot \prod_{m=1}^\infty (1-q^m)^{-22}. \tag{5}$$

To finish the computation of $\widehat{\mathsf{DT}}_0(X)$, we need to prove Proposition 5.1 and compute the series $\sum_a F(a) q^a$.

5.1. **Proof of Proposition 5.1 and the computation of $\sum_a F(a)q^a$.** The fiber $\rho^{-1}(\sum a_i y_i)$ parameterizes subschemes supported on $\widehat{X}_{C_0 \times E}$ which have fixed curve support

$$C_0 \times x_0 \cup_i \{y_i\} \times E$$

where the multiplicity of the subscheme along $\{y_i\} \times E$ is a_i. Such a subscheme is *uniquely* determined[8] by its restriction to the formal neighborhoods $\widehat{X}_{\{y_i\}\times E}$ and their complement U in $\widehat{X}_{C_0\times E}$. The resulting stratification leads to a product decomposition for the Grothendieck group valued power series $\rho^{-1}(\sum a_i y_i)$ giving the product formula in Proposition 5.1. The factor $p(1-p)^{-2}$ comes from the contribution of U and it is the series for the Hilbert scheme of subschemes of $\widehat{X}_{C_0\times E}$ with fixed curve support $C_0 \times x_0$ (no curves in the E direction). The moduli for this Hilbert scheme comes from floating points and embedded points (see [4] for details).

The series $F(a)$ is given by

$$F(a) = (1-p)\cdot e\left(\mathrm{Hilb}^{0,a,\bullet}\left(\widehat{X}_{\{y_i\}\times E}\right)\right)$$

where

$$\mathrm{Hilb}^{0,a,\bullet}\left(\widehat{X}_{\{y_i\}\times E}\right) \subset \mathrm{Hilb}^{0,a,\bullet}(X)$$

is the locus parameterizing subschemes Z whose curve support is given by the union of $C_0 \times \{x_0\}$ and an a-fold thickening of $\{y_i\} \times E$ and such that all embedded points of

[8]This follows from fpqc descent since the set U and the sets $\widehat{X}_{\{y_i\}\times E}$ form a fpqc cover. Since $C_0 \times x_0$ is reduced there are no conditions on the overlaps of the cover. Thus the subscheme is *uniquely* determined by its restriction to the cover.

Z are supported on $\widehat{X}_{\{y_i\}\times E}$. The prefactor $(1-p)$ comes from the contribution of the complement U: the overall contribution of U is given by $p(1-p)^{-2+l}$ where l is the number of y_i's and so we have redistributed the l copies of $(1-p)$ into the $F(a_i)$ factors.

Since

$$\widehat{X}_{\{y_i\}\times E} \cong \operatorname{Spec}\left(\mathbb{C}[[u,v]]\right) \times E,$$

we get an action of $(\mathbb{C}^*)^2$ on the corresponding Hilbert scheme. Only the $(\mathbb{C}^*)^2$ fixed points contribute to the Euler characteristic so

$$\begin{aligned} F(a) &= (1-p)\cdot e\left(\operatorname{Hilb}^{0,a,\bullet}\left(\widehat{X}_{\{y_i\}\times E}\right)^{(\mathbb{C}^*)^2}\right) \\ &= (1-p)\sum_{\alpha\vdash a} e\left(\operatorname{Hilb}^{0,\alpha,\bullet}\left(\widehat{X}_{\{y_i\}\times E}\right)\right) \end{aligned}$$

where $\operatorname{Hilb}^{0,\alpha,\bullet}\left(\widehat{X}_{\{y_i\}\times E}\right)$ parameterizes subschemes whose curve component is the *unique* curve given by the union of $C_0\times\{x_0\}$ and $Z_\alpha\times E$ where $Z_\alpha\subset\operatorname{Spec}(\mathbb{C}[[u,v]])$ is the length a subscheme given by the monomial ideal determined[9] by the partition $\alpha\vdash a$.

To compute $e\left(\operatorname{Hilb}^{0,\alpha,\bullet}\left(\widehat{X}_{\{y_i\}\times E}\right)\right)$ we can now integrate over the fibers of the constructible morphism

$$\sigma:\operatorname{Hilb}^{0,\alpha,\bullet}\left(\widehat{X}_{\{y_i\}\times E}\right)\to\operatorname{Sym}^\bullet E$$

which is defined by recording the length and locations of the embedded points. We thus get

$$\int_{\operatorname{Hilb}^{0,\alpha,\bullet}\left(\widehat{X}_{\{y_i\}\times E}\right)} de = \int_{\operatorname{Sym}^\bullet E}\sigma_*(1)\,de.$$

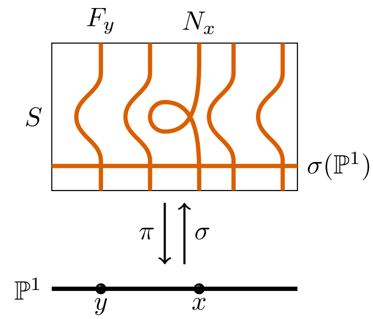
The constructible function $\sigma_*(1)$ is a product of local contributions which only depend on the length of the embedded point and whether or not the location of the embedded point is x_0 or not (recall that x_0 is where the curve $C_0\times\{x_0\}$ is attached to the curve $Z_\alpha\times E$). Writing the series for the local contributions at x_0 and at the general point as $\mathsf{V}_{\emptyset(1)\alpha}(p)$ and $\mathsf{V}_{\emptyset\emptyset\alpha}(p)$ respectively, and applying Lemma 5.2 we get

$$\begin{aligned} \int_{\operatorname{Sym}^\bullet E}\sigma_*(1)\,de &= \left(\mathsf{V}_{\emptyset(1)\alpha}(p)\right)\cdot\left(\mathsf{V}_{\emptyset\emptyset\alpha}(p)\right)^{e(E-x_0)} \\ &= \frac{\mathsf{V}_{\emptyset(1)\alpha}(p)}{\mathsf{V}_{\emptyset\emptyset\alpha}(p)}. \end{aligned}$$

The above naming of the local contributions is not accidental — the generating functions for the contributions are given by the topological vertex. In general, the topological vertex $\mathsf{V}_{\mu_1\mu_2\mu_3}(p)$ can be defined as the generating function of the Euler characteristics of the Hilbert schemes $\operatorname{Hilb}^n\left(\widehat{\mathbb{C}}^3_0,\{\mu_1,\mu_2,\mu_3\}\right)$, which by definition parameterize subschemes of $\mathbb{C}^3$ given by adding at the origin a length n embedded point to the fixed curve $Z_{\mu_1}\cup Z_{\mu_2}\cup Z_{\mu_3}$. Here Z_{μ_i} is supported on the ith coordinate axis and given by the monomial ideal determined by the partition μ_i in the transverse directions. Because $(\mathbb{C}^*)^3$ acts on these Hilbert schemes, their Euler characteristics can be computed by counting $(\mathbb{C}^*)^3$ fixed points, namely monomial ideals. This leads to the combinatorial interpretation of $\mathsf{V}_{\mu_1\mu_2\mu_3}(p)$ — it is the generating function for the number of 3D partitions with asymptotic legs given by $\{\mu_1,\mu_2,\mu_3\}$.

[9] i.e. identifying the partition α with its Ferrer's diagram $\alpha\subset(\mathbb{Z}_{\geq 0})^2$, the ideal of Z_α is generated by the monomials u^iv^j where $(i,j)\notin\alpha$.

FIGURE 4. $\pi : S \to \mathbb{P}^1$ is a elliptic fibration with 24 nodal fibers and a section σ. Figure depicts a smooth fiber F_y over a point $y \in \mathbb{P}^1$ and a nodal fiber N_x over a point $x \in \mathbb{P}^1$.



We thus get the following formula

$$\sum_{a=0}^{\infty} F(a)q^a = \sum_{\alpha} q^{|\alpha|}(1-p)\,\frac{\mathsf{V}_{\emptyset(1)\alpha}(p)}{\mathsf{V}_{\emptyset\emptyset\alpha}(p)}$$

which completes the proof of Proposition 5.1.

Lemma 5.3. *The generating function for the universal series $F(a)$ is given by the following formula*

$$\sum_{a=0}^{\infty} F(a)q^a = \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1-pq^m)(1-p^{-1}q^m)}.$$

Proof. Using the Okounkov-Reshetikhin-Vafa formula for the vertex [15, eqn 3.20], the sum

$$\sum_{\alpha} q^{|\alpha|}(1-p)\,\frac{\mathsf{V}_{\emptyset(1)\alpha}(p)}{\mathsf{V}_{\emptyset\emptyset\alpha}(p)}$$

can be expressed as the trace of a certain natural operator on Fock space. It can be evaluated explicitly by a theorem of Bloch-Okounkov [2, Thm 6.5]. The result is the product formula given by the lemma. See [5] for details. □

Substituting the formula of the lemma into equation (5) we get

$$\widehat{\mathsf{DT}}_0(X) = \frac{pq^{-1}}{(1-p)^2}\prod_{m=1}^{\infty}(1-q^m)^{-20}(1-pq^m)^{-2}(1-p^{-1}q^m)^{-2}$$

which proves the $g=0$ formula in Theorem 3.4, assuming that we can show

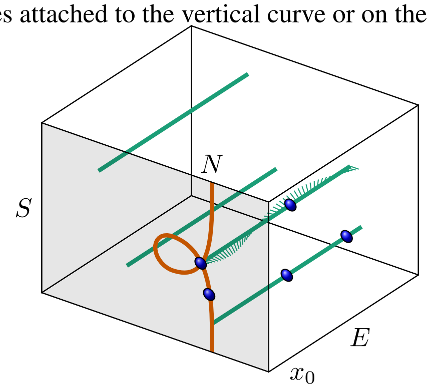
$$\mathsf{DT}_0(X) = -\widehat{\mathsf{DT}}_0(X).$$

We will address this issue in section 7.

6. THE CASE OF $h = 1$.

We now consider the case where S has a primitive curve class β with $\beta^2 = 0$. Such $K3$ surfaces are elliptically fibered with fiber class β. By our genericity assumption, we may assume that the elliptic fibration $\pi : S \to \mathbb{P}^1$ has 24 singular fibers, all of which are nodal, and we will further assume that the fibration has a section (see figure 4).

FIGURE 5. A configuration which includes a thickened horizontal curve (green) attached to the node of a nodal vertical curve (orange). For the contribution to be non-zero, embedded points (blue) must occur along horizontal curves attached to the vertical curve or on the vertical curve.



Recall that the Hilbert scheme decomposes into a disjoint union

$$\mathrm{Hilb}^{1,d,n}(X) = \mathrm{Hilb}^{1,d,n}_{\mathsf{vert}}(X) \sqcup \mathrm{Hilb}^{1,d,n}_{\mathsf{diag}}(X).$$

We can fix a slice for the E action on $\mathrm{Hilb}^{1,d,n}_{\mathsf{vert}}(X)$ by requiring that the unique vertical curve lies in $S \times \{x_0\}$. In the case where the subscheme has a diagonal curve, we require that the diagonal curve intersects the slice $S \times \{x_0\}$ somewhere on the section. Denoting the above conditions with the subscript fixed, we get

$$\mathrm{Hilb}^{1,d,n}(X)/E \cong \mathrm{Hilb}^{1,d,n}_{\mathsf{vert,fixed}}(X) \sqcup \mathrm{Hilb}^{1,d,n}_{\mathsf{diag,fixed}}(X)$$

and so

$$q\,\widehat{\mathsf{DT}}_1(X) = e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_{\mathsf{vert,fixed}}(X)\right) + e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_{\mathsf{diag,fixed}}(X)\right).$$

We get a map

$$\tau : \mathrm{Hilb}^{1,\bullet,\bullet}_{\mathsf{vert,fixed}}(X) \to \mathbb{P}^1$$

induced by the elliptic fibration $S \to \mathbb{P}^1$ since each subscheme parameterized by $\mathrm{Hilb}^{1,\bullet,\bullet}_{\mathsf{vert,fixed}}(X)$ has a unique vertical curve which is a fiber curve. Let F_y denote the fiber of $S \to \mathbb{P}^1$ over y. Let

$$\mathrm{Hilb}^{1,d,n}_{F_y}(X) \subset \mathrm{Hilb}^{1,d,n}_{\mathsf{vert,fixed}}(X)$$

denote the sublocus which parameterizes subschemes whose unique vertical component is $F_y \times \{x_0\}$.

We will see below that the Euler characteristic of $\mathrm{Hilb}^{1,d,n}_{F_y}(X)$ only depends on the topological type of the fiber, i.e. whether it is smooth or nodal. We write a generic smooth fiber as F and any nodal fiber as N. Integrating over the fibers of τ, we get

$$e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_{\mathsf{vert,fixed}}(X)\right) = -22e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_{F}(X)\right) + 24e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_{N}(X)\right)$$

where the -22 is $e(\mathbb{P}^1 - 24\text{pts})$. See figure 5 for a depiction of a curve configuration corresponding to a point in $\mathrm{Hilb}^{1,\bullet,\bullet}_{N}(X)$.

The computation of $e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_F(X)\right)$ and $e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_N(X)\right)$ follows the same strategy as the computation of $e\left(\mathrm{Hilb}^{0,\bullet,\bullet}_{\mathsf{fixed}}(X)\right)$ done in section 5. We use the product decompositions

$$\mathrm{Hilb}^{1,\bullet,\bullet}_F(X) = \mathrm{Hilb}^{1,\bullet,\bullet}_F\left(\widehat{X}_{F\times E}\right)\cdot \mathrm{Hilb}^{1,\bullet,\bullet}\left(X - F\times E\right) \tag{6}$$

$$\mathrm{Hilb}^{1,\bullet,\bullet}_N(X) = \mathrm{Hilb}^{1,\bullet,\bullet}_N\left(\widehat{X}_{N\times E}\right)\cdot \mathrm{Hilb}^{1,\bullet,\bullet}\left(X - N\times E\right) \tag{7}$$

and we use the extra E actions on the second factors to deduce

$$e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_F(X)\right) = e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_F\left(\widehat{X}_{F\times E}\right)\right)\cdot \prod_{m=1}^{\infty}(1-q^m)^{-24}$$

$$e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_N(X)\right) = e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_N\left(\widehat{X}_{N\times E}\right)\right)\cdot \prod_{m=1}^{\infty}(1-q^m)^{-23}$$

where $24 = e(S-F)$ and $23 = e(S-N)$.

Proceeding as we did in section 5, we use the maps

$$\rho : \mathrm{Hilb}^{1,\bullet,\bullet}_F\left(\widehat{X}_{F\times E}\right) \to \mathrm{Sym}^{\bullet}(F)$$

$$\rho : \mathrm{Hilb}^{1,\bullet,\bullet}_N\left(\widehat{X}_{N\times E}\right) \to \mathrm{Sym}^{\bullet}(N)$$

which record the location and multiplicity of the horizontal components. The argument proceeds exactly as it did in section 5 with F and N playing the role of C_0.

The result for the smooth fiber case is the following:

$$\begin{aligned}
\int_{\mathrm{Hilb}^{1,\bullet,\bullet}_F\left(\widehat{X}_{F\times E}\right)} de &= \int_{\mathrm{Sym}^{\bullet}(F)} \rho_*(1)\, de \\
&= \left(p^{1/2}(1-p)^{-1}\right)^{e(F)}\cdot\left(\sum_{a=0}^{\infty} F(a)q^a\right)^{e(F)} \\
&= 1.
\end{aligned} \tag{8}$$

This result comports with the heuristic that F acts on $\widehat{X}_{F\times E}$ and hence on $\mathrm{Hilb}^{1,\bullet,\bullet}_F\left(\widehat{X}_{F\times E}\right)$ and so the Euler characteristic is 0 except for the unique F-fixed subscheme, i.e. the subscheme consisting of just the curve $F\times\{x_0\}$ with no added horizontal components or embedded points. However, this is only a heuristic: F does *not* act algebraically on the formal neighborhood $\widehat{X}_{F\times E}$ since the elliptic fibration is not isotrivial[10].

The situation for nodal fibers is a little different because of the presence of the nodal point $z\in N$. The constructible function $\rho_*(1)$, which is given by taking the Euler characteristic of the fibers of the map

$$\rho : \mathrm{Hilb}^{1,\bullet,\bullet}_N\left(\widehat{X}_{N\times E}\right) \to \mathrm{Sym}^{\bullet} N,$$

has the following form. Let $y_1,\dots,y_l$ be non-singular points of N and let $z\in N$ be the nodal point. Then $\rho^{-1}(bz+\sum a_i y_i)$ parameterizes subschemes of X, supported on $\widehat{X}_{N\times E}$, which have fixed curve support

$$N\times\{x_0\}\cup\{z\}\times E\cup_i\{y_i\}\times E$$

[10]However, see section 7 for the action of a related group.

where the multiplicity along $\{z\}\times E$ is b and the multiplicity along $\{y_i\}\times E$ is a_i. Such a subscheme is determined by its restriction to the formal neighborhoods $\widehat{X}_{\{z\}\times E}$, $\widehat{X}_{\{y_1\}\times E},\dots,\widehat{X}_{\{y_l\}\times E}$ and their complement U. The contribution of the Euler characteristic of U is given by

$$(1-p)^{-e(N^\circ)}=(1-p)^l$$

where $N^\circ=N-\{z,y_1,\dots,y_l\}$. Therefore we see that

$$\rho_*(1)(bz+\sum a_iy_i)=N(b)\prod_{i=1}^{l}F(a_i)$$

where $F(a)$ is as in section 5, and

$$N(b)=e\left(\mathrm{Hilb}^{1,b,\bullet}\left(\widehat{X}_{\{z\}\times E}\right)\right)$$

where

$$\mathrm{Hilb}^{1,b,n}\left(\widehat{X}_{\{z\}\times E}\right)\subset\mathrm{Hilb}^{1,b,n}(X)$$

is the sublocus parameterizing subschemes Z whose curve support is given by the union of $N\times\{x_0\}$ and a b-fold thickening of $\{z\}\times E$ and such that all embedded points are supported on $\widehat{X}_{\{z\}\times E}$.

So pushing the integral to $\mathrm{Sym}^\bullet N$ and applying lemma 5.2 we get

$$\begin{aligned}\int_{\mathrm{Hilb}^{1,\bullet,\bullet}_N\left(\widehat{X}_{N\times E}\right)}1\,de&=\int_{\mathrm{Sym}^\bullet N}\rho_*(1)de\\&=\int_{\mathrm{Sym}^\bullet(N-\{z\})}\prod_i F(a_i)\,de\cdot\int_{\mathrm{Sym}^\bullet(\{z\})}N(b)\,de\\&=\left(\sum_{a=0}^\infty F(a)q^a\right)^{e(N-\{z\})}\cdot\left(\sum_{b=0}^\infty N(b)q^b\right).\end{aligned}$$

Note that $e(N-\{z\})=0$ so that the $F(a)$ term doesn't contribute.

We compute the $N(b)$ contribution by using the $(\mathbb{C}^*)^2$ action on

$$\widehat{X}_{\{z\}\times E}\cong\mathrm{Spec}(\mathbb{C}[[u,v]])\times E$$

and arguing as in section 5. We find

$$\begin{aligned}\sum_{b=0}^\infty N(b)q^b&=\sum_{b=0}^\infty e\left(\mathrm{Hilb}^{1,b,\bullet}\left(\widehat{X}_{\{z\}\times E}\right)\right)q^b\\&=\sum_{b=0}^\infty\sum_{\beta\vdash b}e\left(\mathrm{Hilb}^{1,\beta,\bullet}\left(\widehat{X}_{\{z\}\times E}\right)\right)q^b\\&=\sum_\beta q^{|\beta|}\frac{\mathsf{V}_{(1)(1)\beta}(p)}{\mathsf{V}_{\emptyset\emptyset\beta}(p)}.\end{aligned}$$

We see that fact that the curve N has a node is manifest in the term in the numerator: the vertex $\mathsf{V}_{(1)(1)\beta}(p)$ is counting curve configurations which are locally monomial at the nodal point $\{z\}\times\{x_0\}$ where the curve is degree 1 along the two branches of the node and has the monomial thickening given by β along the E direction.

Putting this and the earlier computations together, we find that the total contribution of the components with vertical curves is given by the following:

$$\begin{aligned} e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_{\mathsf{vert,fixed}}(X)\right) &= -22\prod_{m=1}^{\infty}(1-q^m)^{-24} \quad + \quad 24\prod_{m=1}^{\infty}(1-q^m)^{-23}\cdot\sum_{\beta} q^{|\beta|}\frac{\mathsf{V}_{(1)(1)\beta}(p)}{\mathsf{V}_{\emptyset,\emptyset,\beta}(p)} \\ &= \quad 24\prod_{m=1}^{\infty}(1-q^m)^{-24}\left\{\frac{1}{12}-1+\prod_{m=1}^{\infty}(1-q^m)\sum_{\beta}q^{|\beta|}\frac{\mathsf{V}_{(1)(1)\beta}(p)}{\mathsf{V}_{\emptyset,\emptyset,\beta}(p)}\right\}. \end{aligned}$$

Proposition 6.1. *The following identity holds:*

$$\prod_{m=1}^{\infty}(1-q^m)\sum_{\beta}q^{|\beta|}\frac{\mathsf{V}_{(1)(1)\beta}(p)}{\mathsf{V}_{\emptyset\emptyset\beta}(p)} = 1+\frac{p}{(1-p)^2}+\sum_{d=1}^{\infty}\sum_{k|d}k(p^k+p^{-k})q^d.$$

Proof sketch: Using the Okounkov-Reshetikhin-Vafa formula for the topological vertex [15, Eqn 3.20], and some standard combinatorics, one can rewrite the left hand side of the above equation so that it is given in terms of Bloch-Okounkov's 2-point correlation function [2, Eqn 5.2]. Namely, one can show that it is given by $1-F(t_1,t_2)$ in the limit where t_1 and t_2 approach p and p^{-1} respectively. The limit can be evaluated explicitly using [2, Thm 6.1] and this leads to the right hand side of the formula. Details can be found in [5].

Plugging in the proposition's formula into the previously obtained equation, we see that the non-diagonal contribution to $\widehat{\mathsf{DT}}_1(X)$ is given as

$$e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_{\mathsf{vert,fixed}}(X)\right) = 24\prod_{m=1}^{\infty}(1-q^m)^{-24}\left\{\frac{1}{12}+\frac{p}{(1-p)^2}+\sum_{d=1}^{\infty}\sum_{k|d}k(p^k+p^{-k})q^d\right\}$$

6.1. **Diagonal contributions.** To finish our computation of $\widehat{\mathsf{DT}}_1(X)$, it remains to compute $e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_{\mathsf{diag,fixed}}(X)\right)$.

Let $C\subset X$ be a diagonal curve. The projections onto the factors of $X=S\times E$ induce maps

$$\begin{aligned} p_S &: C\to F_y \\ p_E &: C\to E \end{aligned}$$

where F_y is a fiber curve, and the maps have degree 1 and some $d>0$ respectively. F_y cannot be a nodal fiber since then C would have geometric genus 0 and consequently it would not admit a non-constant map to E. The above maps induce a map

$$f: F_y\to E$$

which must be unramified by the Riemann-Hurwitz formula. Thus the diagonal curve C is contained in the surface $F_y\times E$ and is given by the graph of the map f. Recall that we fixed a slice for the E action on $\mathrm{Hilb}^{1,d,n}_{\mathsf{diag}}(X)$ by requiring that the diagonal curve meets S_{x_0} at the section; this is equivalent to requiring that $f(s)=x_0$ where $s\in F_y$ is the section point on F_y. Up to automorphisms, such a map f must be a group homomorphism of the corresponding elliptic curves. Assuming that E is generic, so that the only non-trivial automorphism is given by $x\mapsto -x$, we see that every diagonal curve (with the fixed condition) is of the form

$$\{(z,f(z))\in F_y\times E\} \text{ or } \{(z,-f(z))\in F_y\times E\}$$

where $f : F_y \to E$ is a group homomorphism.

The number of group homomorphisms of degree d to a fixed elliptic curve E is given by $\sum_{k|d} k$. This classical fact can be seen by counting index d sublattices of $\mathbb{Z}\oplus\mathbb{Z}$. For each such cover, $F \to E$, the domain elliptic curve will occur exactly 24 times in the fibration $S \to \mathbb{P}^1$. So we find that the total number of diagonal curves having degree d in the E direction is

$$2 \cdot 24 \sum_{k|d} k.$$

Each such diagonal curve can be accompanied by horizontal curves (with thickenings) as well as embedded points. The contribution of these components of the Hilbert scheme is computed in exactly the same way as the contribution of the curves with a smooth vertical component F. Recall that $e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_F(X)\right) = \prod_{m=1}^\infty (1-q^m)^{-24}$. Taking into account the degree of the diagonal curves, we thus find

$$e\left(\mathrm{Hilb}^{1,\bullet,\bullet}_{\mathsf{diag},\mathsf{fixed}}(X)\right) = \prod_{m=1}^\infty (1-q^m)^{-24}\left(2\cdot 24 \cdot \sum_{d=1}^\infty \sum_{k|d} kq^k\right).$$

Finally, adding the vertical and diagonal contributions together we arrive at

$$\widehat{\mathsf{DT}}_1(X) = 24q^{-1}\prod_{m=1}^\infty (1-q^m)^{-24}\left\{\frac{1}{12} + \frac{p}{(1-p)^2} + \sum_{d=1}^\infty\sum_{k|d} k(p^k+p^{-k}+2)q^d\right\}.$$

Note that this formula is off from the desired formula for $\mathsf{DT}_{h=1}(X)$ by an overall minus sign and a minus sign on the 2. In fact we will see in section 7 that due to the Behrend function, the contribution of the diagonal components carry the opposite sign of the contribution of the vertical components. Denoting the contribution to $\widehat{\mathsf{DT}}_1(X)$ coming from $\mathrm{Hilb}^{1,\bullet,\bullet}_{\mathsf{vert},\mathsf{fixed}}(X)$ and from $\mathrm{Hilb}^{1,\bullet,\bullet}_{\mathsf{diag},\mathsf{fixed}}(X)$ by $\widehat{\mathsf{DT}}_{1,\mathsf{vert}}(X)$ and $\widehat{\mathsf{DT}}_{1,\mathsf{diag}}(X)$ respectively, we find that we need to show

$$\mathsf{DT}_1(X) = -\widehat{\mathsf{DT}}_{1,\mathsf{vert}}(X) + \widehat{\mathsf{DT}}_{1,\mathsf{diag}}(X).$$

7. PUTTING IN THE BEHREND FUNCTION

The goal of this section is to prove, assuming Conjecture 7.1, the relations

$$\mathsf{DT}_0(X) = -\widehat{\mathsf{DT}}_0(X)$$
$$\mathsf{DT}_1(X) = -\widehat{\mathsf{DT}}_{1,\mathsf{vert}}(X) + \widehat{\mathsf{DT}}_{1,\mathsf{diag}}(X)$$

which is all that is needed to complete the proof of Theorem 3.4.

7.1. **Overview.** Our general strategy for computing $\widehat{\mathsf{DT}}(X)$, the unweighted Euler characteristics of the Hilbert schemes, utilized the following general scheme.

(1) Using the geometric support of curves (and/or points) of the subschemes, we stratified $\mathrm{Hilb}(X)$ such that the strata could be written as products of simpler Hilbert schemes.
(2) We utilized actions of $\mathbb{C}^*$ or E which could be defined on individual factors in the stratification to discard strata not fixed by the action and restrict to fixed points.
(3) We found that some strata were parameterized by symmetric products, and we pushed forward the Euler characteristic computation to the symmetric products where we used Lemma 5.2.

(4) After possibly iterating steps (1)–(3), we reduced the computation to counting discrete subscheme configurations, namely those which are given formally locally by monomial ideas. These we counted with the topological vertex.

In order to incorporate the weighting by ν, the Behrend function, into the Euler characteristics in the above strategy, we need to modify steps (2) and (4).

For (2) to apply to ν-weighted Euler characteristics, we need to know that ν, restricted to the relevant stratum $S \subset \mathrm{Hilb}(X)$, is invariant under the action of the group. We can do this by showing that the group (or possibly a modification of the group) acts on the formal neighborhood of S inside of $\mathrm{Hilb}(X)$. This works since the value of the Behrend function at a point only depends on the formal neighborhood of the point.

To modify step (4), the final step, we will need to know the value of the Behrend function at subschemes given formally locally by monomial ideals. In particular, we want to show the value is given by ± 1, where the sign alternates as n increases. This will account for the relatively simple relationship between $\mathsf{DT}(X)$ and $\widehat{\mathsf{DT}}(X)$.

We are only partially able to succeed with the above modification. In the first two iterations of steps (1), (2), and (3) of the strategy, we succeed in extending the actions of the group (or related group) to the formal neighborhood of the strata. However, in the final iteration of (1)-(3), the $(\mathbb{C}^*)^3$ action we used on the strata does not obviously extend to their formal neighborhoods (quite possibly such an extension does not exist). To remedy this situation, we must assume a conjecture first formulated in [4, Conj 18].

7.2. **Elaboration.** Let us expand on the above discussion to highlight the issues.

In the first two iterations of steps (1), (2), and (3), we stratified by the curve support of the subschemes. These strata decompose into the product given by the equations (1), (6), and (7) where the second factors correspond to connected components of the curve with pure vertical support. This factor admits an extra E action, and we wish to show that this action extends to the formal neighborhoods of the strata.

The formal neighborhood of a closed point in $\mathrm{Hilb}(X)$ parameterizes infinitesimal deformations of the subscheme corresponding to the closed point. Connected components of a subscheme deform independently from each other and consequently the products given in equations (1), (6), and (7) extend to their formal neighborhoods, as do the E actions.

The first factor in the products (1), (6), and (7) correspond to subschemes supported in $\widehat{X}_{C\times E}$ where C is the curve in the slice $S \times x_0$, which is either C_0 (for $h=0$), or N or F (for $h=1$). This stratum was further stratified by the location of the vertical components of such curves. Each of these strata admits a $\mathbb{C}^* \times \mathbb{C}^*$ induced by the action

$$\widehat{X}_{\{y\}\times E} \cong \operatorname{Spec} \mathbb{C}[[x,y]] \times E,$$

the formal neighborhood of a vertical component. This action does *not* obviously extend to the formal neighborhood of the strata in the Hilbert scheme. The basic issue is the following.

A subscheme in X whose reduced support is

$$C \times x_0 \cup y \times E$$

and whose multiplicies along $C \times x_0$ and $y \times E$ is 1 and some $a \geq 1$ on E respectively is uniquely determined by its restrictions to

$$\widehat{X}_{y\times E} \quad \text{and} \quad \widehat{X}_{C\times X_0} - \{y \times x_0\}.$$

The action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\widehat{X}_{y\times E}$ thus induces an action on this strata. However, it is not clear if the action on this stratum extends to an action on its formal neighborhood due to (for example) the possibility of obstructed infinitesimal deformations of the subscheme smoothing the node.

We can circumvent this problem by using somewhat different group actions. In the case of $h=0$, the $\mathbb{C}^*\times\mathbb{C}^*$ action on $\widehat{X}_{y\times E}$ *does* extend to an action on $\widehat{X}_{C_0\times E}$. The reason is that C_0 is a smooth -2 curve in a surface and consequently its formal neighborhood in the surface is isomorphic to the formal neighborhood of the zero section in the total space of $T^*\mathbb{P}^1$. This formal scheme carries an $\mathbb{C}^*\times\mathbb{C}^*$ action which can be choosen to be compatible with the one on $\widehat{X}_{y\times E}$. This then induces an action on $\operatorname{Hilb}(\widehat{X}_{C_0\times E})/E$ which naturally extends to its formal neighborhood in $\operatorname{Hilb}(X)/E$.

In the case of $h=1$, we can also construct global group actions on $\widehat{X}_{C\times E}$, but different from the $\mathbb{C}^*\times\mathbb{C}^*$ action previously used. For $C=F$, a smooth elliptic curve, we previously noted that the action of F on $F\times E$ given by translation on the first factor, does not extend to $\widehat{X}_{F\times E}$. The reason is that the linear system of F in S induces a non-trivial elliptic fibration $S\to\mathbb{P}^1$ and thus a non-trivial fibration $\hat{S}_F\to\operatorname{Spec}\mathbb{C}[[t]]$. However, after choosing a section of the above map, $\hat{S}_F$ is a group scheme over $\operatorname{Spec}\mathbb{C}[[t]]$ and the Mordell-Weil group of sections acts (freely) on $\hat{S}_F$, and thus on $\widehat{X}_{F\times E}=\hat{S}_F\times E$. This induces an action of the Mordell-Weil group on $\operatorname{Hilb}(\widehat{X}_{F\times E})/E$ which extends to its formal neighborhood and is free whenever the degree in the vertical direction is non-zero. The Mordell-Weil group is an an extension of the group F by an additive group and so its orbits have Euler characteristic zero. Consequently the conclusion expressed by equation (8) for the usual Euler characteristics also applies to the ν-weighted Euler characteristics.

Similarly, the Mordell-Weil group of sections of $\hat{S}_N\to\operatorname{Spec}\mathbb{C}[[t]]$ acts on $\operatorname{Hilb}(\widehat{X}_{N\times E})/E$ and its formal neighborhood. This group is an extension of $\mathbb{C}^*$ (the group of the nodal fiber) by an additive group and hence necessarily splits. Thus we get an action of $\mathbb{C}^*$ on $\operatorname{Hilb}(\widehat{X}_{N\times E})/E$ which is compatible with the Behrend function. The fixed points are subschemes whose vertical part has support in $\widehat{X}_{\{z\}\times E}$ where $z\in N$ is the node. Moreover, the $\mathbb{C}^*$ action here is given by $\lambda(x,y)=(\lambda x,\lambda^{-1}y)$ for a suitable choice of isomorphism $\widehat{X}_{\{z\}\times E}\cong E\times\operatorname{Spec}\mathbb{C}[[x,y]]$. The fixed subschemes under this action correspond to subschemes whose maximal Cohen-Macaulay subscheme is formally locally given by monomial ideals.

Via the above, we have shown that the ν-weighted Euler characteristics of the Hilbert schemes are equal to the ν-weighted Euler characteristics of the locus in the Hilbert scheme parameterizing subschemes whose maximal Cohen-Macaulay subscheme is given formally locally by monomial ideals.

In the final iteration of steps (1)-(3), we were able to further reduce to subschemes whose embedded points are also local monomial. To achieve this we further stratified our strata by the location of the embedded points. We then used the action of $(\mathbb{C}^*)^3$ on the formal neighborhoods of the embedded points to construct a $(\mathbb{C}^*)^3$ action on these substrata. Unfortunately, we are unable to show that this action on the substrata extends to a formal neighborhood of the substrata. Consequently, we cannot show that the Behrend function is compatible with the action.

To circumvent this problem, we will assume the validity of the following conjecture, restated from [4, Conj 18]. Let Y be any quasi-projective Calabi-Yau threefold. Let $C\subset Y$ be a (not necessarily reduced) Cohen-Macaulay curve with proper support. Assume that the singularities of C_{red} are locally toric[11]. Let

$$\mathrm{Hilb}^n(Y,C) = \{Z \subset Y \text{ such that } C \subset Z \text{ and } I_C/I_Z \text{ has finite length } n\}.$$

Note that $\mathrm{Hilb}^n(Y,C) \subset \mathrm{Hilb}(Y)$ and let ν denote the Behrend function on $\mathrm{Hilb}(Y)$.

Conjecture 7.1 ([4]).

$$\int_{\mathrm{Hilb}^n(Y,C)} \nu\, de = (-1)^n \nu([C]) \int_{\mathrm{Hilb}^n(Y,C)} de,$$

where $\nu([C])$ is the value of the Behrend function at the point $[C] \in \mathrm{Hilb}(Y)$.

This conjecture allows us to promote the remaining part of our Euler characteristic computation to ν-weighted Euler characteristic, once we compute the value of the Behrend function at locally monomial Cohen-Macaulay subschemes. Suppose we could show that value was always $(-1)^n$ where n is the holomorphic Euler characteristic of the subscheme, then, because of the $(-p)$ built directly into the definition of $\mathsf{DT}_g(X)$, we could conclude that $\mathsf{DT}_g(X) = \widehat{\mathsf{DT}}_g(X)$ for $h = 0$ and $h = 1$. We will show this is nearly true: in fact the Behrend function has value $-(-1)^n$ for all curves in the $h = 0$ case and for those without diagonal components in the $h = 1$ case. Curves with diagonal components have the sign $(-1)^n$.

7.3. **The Behrend function at Cohen-Macaulay subschemes.** The only Cohen-Macaulay subschemes which contribute to the invariants in our counting scheme are all of the following form:

(1) $Z \times E$,
(2) $C_0 \times x_0 \cup Z \times E \qquad (h = 0)$,
(3) $N \times x_0 \cup Z \times E \qquad (h = 1)$, and
(4) C, a diagonal curve (necessarily reduced)

where $Z \subset S$ is a zero dimensional subscheme of length d. For each of these cases, we show these subschemes lie in a smooth, open locus of the corresponding component of the Hilbert scheme and hence the value of Behrend function is given by $(-1)^D$ where D is the dimension of that open set.

This is entirely parallel to the analysis in sections 8 and 9 of [4]. Namely, we construct an explicit D-dimensional family of such subschemes and we then show that the family is smooth by showing the Zariski tangent space also has dimension D.

Proposition 7.2. *The following families of subschemes have the given dimensions and are open sets of Hilbert scheme which are smooth at subschemes given locally by monomial ideals.*

(1) *The locus of schemes of the form $Z \times E$ has dimension $2d$, where $Z \subset S$ is a length d zero-dimensional subscheme.*
(2) *The locus of schemes of the form $C_0 \times x_0 \cup Z \times E$ has dimension $2d - k$, where $Z \subset S$ is a length d zero-dimensional subscheme such that* $\mathrm{length}(C_0 \cap Z) = k$.
(3) *The locus of schemes of the form $C \times x_0 \cup Z \times E$ has dimension $2d - k + 1$, where $Z \subset S$ is a length d zero-dimensional subscheme such that* $\mathrm{length}(C \cap Z) = k$ *and $C \in |F|$ is any curve in the class F (including N).*
(4) *The locus of diagonal curves has dimension 0.*

[11]This means that formally locally C_{red} is either smooth, nodal, or the union of the three coordinate axes. That is at $p \in C_{\mathrm{red}} \subset Y$ the ideal $\widehat{I}_{C_{\mathrm{red}}} \subset \widehat{\mathcal{O}}_{Y,p}$ is given by (x_1, x_2), (x_1, x_2x_3), or (x_1x_2, x_2x_3, x_1x_3) for some isomorphism $\widehat{\mathcal{O}}_{Y,p} \cong \mathbb{C}[[x_1, x_2, x_3]]$.

Proof. The family (1) is paramterized by $\mathrm{Hilb}^d(S)$ which has dimension $2d$. The family (2) is parameterized by the locus of points $[Z] \in \mathrm{Hilb}^d(S)$ given by the codimension condition that $\mathrm{length}(Z \cap C_0) = k$. This is smooth of dimension $2d-k$ by [4, Thm 20]. The family (3) is parameterized by the locus of points $[Z] \in \mathrm{Hilb}^d(S)$ such that $\mathrm{length}(Z \cap C) = k$ for some $C \in |F|$. This family maps to $|F| \cong \mathbb{P}^1$ with fibers of dimension $2d-k$ (again by [4, Thm 20]).

To complete the proof of the proposition, we want to show these sets are open and smooth at the monomial ideals. It suffices to show that the dimension of the Zariski tangent space is of the given dimension. The tangent space of the Hilbert scheme at a given subscheme Z is given by the group $\mathrm{Hom}(I_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^1(I_Z, I_Z)$. These Ext groups can be computed at monomial ideals using the technique of [4, § 9]. Indeed, the proof of Thm 21 in [4] applies with minor modifications to the cases (1)-(3). Finally it is easy to see that a diagonal curve $C \subset X$ is scheme-theoretically isolated up to translation by E. For example, since C is smooth and reduced, the Zariski tangent space is given by $H^0(C, N_{C/X})$. Here C is given as the graph of some homomorphism $f : F_y \to E$ and thus $N_{C/X}$ is given as the non-trivial extension of $f^*N_{F/S} \cong \mathcal{O}_C$ by $N_{C/F_y\times E} \cong \mathcal{O}_C$. Therefore $H^0(C, N_{C/X}) \cong \mathbb{C}$ which corresponds to the translations by E. □

Using the normalization exact sequence, one easily computes n, the holomorphic Euler characteristic of the subschemes given by the four families in Proposition 7.2. Namely,

$$n = \begin{cases} 0 & \text{for family (1)} \\ 1-k & \text{for family (2)} \\ -k & \text{for family (3)} \\ 0 & \text{for family (4)} \end{cases}$$

Since the value of the Behrend function at a smooth point of dimension D is $(-1)^D$, the above formulas, along with Proposition 7.2 gives us

$$\nu = \begin{cases} (-1)^{2d} = 1 = (-1)^n & \text{for family (1)} \\ (-1)^{2d-k} = (-1)^{-k} = -(-1)^n & \text{for family (2)} \\ (-1)^{2d-k+1} = (-1)^{-k+1} = -(-1)^n & \text{for family (3)} \\ (-1)^0 = 1 = (-1)^n & \text{for family (4)} \end{cases}$$

In the $h=0$ case, the locally monomial Cohen-Macaulay subschemes which contribute to the ν-weighted Euler characteristics are disjoint unions of curves in family (1) and a single curve in family (2). Thus, they always come with the sign $-(-1)^n$ and we can conclude

$$\mathsf{DT}_0(X) = -\widehat{\mathsf{DT}}_0(X).$$

In the $h=1$ case, the locally monomial Cohen-Macaulay subschemes without a diagonal component which contribute to the ν-weighted Euler characteristics are disjoint unions of curves in family (1) with a single curve in family (3). Thus the contribution of the non-diagonal curves to $\mathsf{DT}_{h=1}(X)$ is given by $-\widehat{\mathsf{DT}}_{1,\mathsf{vert}}(X)$.

Finally, locally monomial Cohen-Macaulay subschemes with a diagonal curve which contribute to the ν-weighted Euler characteristic are disjoint unions of curves in family (1) with a single curve in family (4). Thus the contribution of the diagonal curves to $\mathsf{DT}_{h=1}(X)$ is given by $\mathsf{DT}_{1,\mathsf{diag}}(X)$.

Putting it all together we see that

$$\mathsf{DT}_0(X) = -\widehat{\mathsf{DT}}_0(X)$$

$$\mathsf{DT}_1(X) = -\widehat{\mathsf{DT}}_{1,\mathsf{vert}}(X) + \widehat{\mathsf{DT}}_{1,\mathsf{diag}}(X)$$

as desired which completes the proof of Theorem 3.4.

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