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Existence of Locally Maximally Entangled Quantum States via Geometric Invariant Theory

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Abstract. We study a question which has natural interpretations both in quantum mechanics and in geometry. Let $V_1, \ldots, V_n$ be complex vector spaces of dimension $d_1, \ldots, d_n$ and let $G = \mathrm{SL}_{d_1} \times \cdots \times \mathrm{SL}_{d_n}$. Geometrically, we ask: Given $(d_1, \ldots, d_n)$, when is the geometric invariant theory quotient $\mathbb{P}(V_1 \otimes \cdots \otimes V_n)//G$ non-empty? This is equivalent to the quantum mechanical question of whether the multipart quantum system with Hilbert space $V_1 \otimes \cdots \otimes V_n$ has a locally maximally entangled state, i.e., a state such that the density matrix for each elementary subsystem is a multiple of the identity. We show that the answer to this question is yes if and only if $R(d_1, \ldots, d_n) \geqslant 0$ where

$$R(d_1, \ldots, d_n) = \prod_i d_i + \sum_{k=1}^{n} (-1)^k \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} (\gcd(d_{i_1}, \ldots, d_{i_k}))^2 .$$

We also provide a simple recursive algorithm which determines the answer to the question, and we compute the dimension of the resulting quotient in the non-empty cases.

1. Introduction

In a multipart quantum system, the space of pure states is described by a tensor product Hilbert space

$$V = V_1 \otimes \cdots \otimes V_n,$$

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where $V_i = \mathbb{C}^{d_i}$ are d_i-dimensional Hilbert spaces[1,2]. Given a pure state $\psi \in V$, the associated state of the ith elementary subsystem is described by the *reduced density operator* $\rho_i : V_i \to V_i$, a nonnegative unit-trace Hermitian operator defined by the action of the contraction map

$$V_1 \otimes \cdots \otimes V_n \otimes V_1^* \otimes \cdots \otimes V_n^* \to V_i \otimes V_i^*$$

on the operator $\rho_\psi = \psi \otimes \psi^* \in V \otimes V^*$. Equivalently, we have $\rho_i = \mathrm{tr}_{V_1 \otimes \cdots \otimes \widehat{V}_i \otimes \cdots \otimes V_n} \rho_\psi$.

In general, the structure of entanglement in a multipart quantum system is related to the eigenvalue spectra of the reduced density matrices for subsystems. A subsystem i is *entangled* with the rest of the system if its spectrum is different from $\{1, 0, 0, \ldots\}$ (i.e., if its density operator is not a projection operator associated with a single state). A subsystem i is *maximally entangled* with the rest of the system if all eigenvalues of ρ_i are equal, i.e., $\rho_i = \mathbb{1}/d_i$. For some $\{d_1, \ldots, d_n\}$, it is possible to find states where each elementary subsystem has maximal entanglement:

Definition 1.1. A pure state ψ in a multipart quantum system described by Hilbert space $V_1 \otimes \cdots \otimes V_n$ is said to be *locally maximally entangled*[3] if for each elementary subsystem i, the density operator ρ_i is a multiple of the identity operator, $\rho_i = \mathbb{1}/d_i$.

States that are locally maximally entangled (LME) have many applications in the field of quantum information theory and quantum computing. For example, this property is present in Bell states, GHZ states, quantum error correcting code states, cluster states, and graph states.

In this paper, our goal is to understand for which $\{d_1, \ldots, d_n\}$ such LME states exist, and in those cases to characterize the subset $V_{LME} \subset V$ of such states in the full Hilbert space[4]. While the necessary conditions $d_i \leqslant \prod_{j \neq i} d_j$ are well known and have been suggested to be sufficient, we will see that the necessary and sufficient condition is significantly more complicated.

Remarkably, the quantum mechanics problem we have described is equivalent to two very natural problems in geometry, the first related to symplectic geometry and the second related to algebraic geometry and geometric invariant theory; see Klyachko [10], [12, § 3], and Wallach [24, § 4].

Geometrically, the full space of pure states $\mathbb{P}(V)$ admits a natural symplectic structure defined by the Fubini–Study symplectic form. Consider the action of $K = \mathrm{SU}_{d_1} \times \cdots \times \mathrm{SU}_{d_n}$ on $\mathbb{P}(V)$, where SU_{d_i} acts via the fundamental representation on V_i. The spectra of the density operators ρ_i are invariant

[1] As usual, we require the states ψ to be normalized $(\psi, \psi) = 1$ and identify states which are related by an overall phase $\psi \sim e^{i\theta}\psi$. Thus, we can identify states ψ with points in the projective space $\mathbb{P}(V)$.

[2] It is natural to assume that $d_i \geqslant 2$ for every i; however, we will allow trivial subsystems (with $d_i = 1$), as long as there are at least two subsystems of dimension $d_i \geqslant 2$.

[3] Other names for locally maximally entangled states include "1-uniform states" and "maximally mixed entangled states", see, for example, [2,5,23]. For a more extensive review of results in this field, see the companion paper [3].

[4] Several authors have studied special cases of this problem, e.g., [8,13,15,26,27].

under the action of K, so we can group LME states into equivalence classes defined by the K orbits. Since the symplectic form is invariant under the action of K, we can define a moment map $\mu : \mathbb{P}(V) \to \mathfrak{k}^*$ from our space to the dual of the Lie algebra of K. The relation to symplectic geometry is provided by the result that $\psi \in V_{LME}$ if and only if $\mu(\psi) = 0$ (see Sect. 2). Thus, the space of equivalence classes of LME states under K is precisely the symplectic quotient $\mu^{-1}(0)/K$.

By the Kempf–Ness theorem,[5] this symplectic quotient is equivalent to the geometric invariant theory quotient $\mathbb{P}(V)//G$, where

$$G = \mathrm{SL}_{d_1} \times \cdots \times \mathrm{SL}_{d_n}$$

is the complexification of K. Thus for dimensions $\{d_1, \ldots, d_n\}$, we arrive at the central observation:

There exist LME states if and only if the quotient $\mathbb{P}(V)//G$ is non-empty.

Our main result provides a simple numerical criterion for when the quotient $\mathbb{P}(V)//G$ is non-empty, and we give a formula for the dimension of the non-empty quotients. For

$$\mathbf{d} = (d_1, \ldots, d_n)$$

we define the *expected dimension*

$$\begin{aligned}\Delta(\mathbf{d}) &= \dim \mathbb{P}(V) - \dim G \\ &= \prod_i d_i - 1 - \sum_i (d_i^2 - 1)\end{aligned}$$

as well as the arithmetic functions

$$g_{\mathsf{max}}(\mathbf{d}) = \max_{1 \leqslant i < j \leqslant n} (\gcd(d_i, d_j))$$

and

$$R(\mathbf{d}) = \prod_i d_i + \sum_{k=1}^{n} (-1)^k G_k(\mathbf{d}),$$

where

$$G_k(\mathbf{d}) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \left(\gcd\left(d_{i_1}, \ldots, d_{i_k}\right)\right)^2 .$$

Our main theorem is the following:

Theorem 1.2. *The GIT quotient $\mathbb{P}(V)//G$ is non-empty if and only if $R(\mathbf{d}) \geqslant 0$. Moreover:*

If $\Delta(\mathbf{d}) > -2$, then $R > 0$ and $\dim \mathbb{P}(V)//G = \Delta(\mathbf{d}) > 0$.

If $\Delta(\mathbf{d}) = -2$, then $R > 0$ and $\dim \mathbb{P}(V)//G = \max(g_{\mathsf{max}}(\mathbf{d}) - 3, 0)$.

If $\Delta(\mathbf{d}) < -2$, then $R \leqslant 0$ and the quotient is a single point for $R = 0$ and empty for $R < 0$.

[5] For a discussion of geometric invariant theory, symplectic geometry, and the Kempf–Ness theorem, see [16], or see [9] for a pedagogical introduction.

We remark that Theorem 1.2 implies that the quotient is of the expected dimension $\Delta(\mathbf{d})$ whenever this is nonnegative. If the expected dimension is negative, then the quotient can be empty, a single point, or positive dimensional.

Our theorem also has an amusing reformulation pointed out to us by David Savitt[6]:

Corollary 1.3. *A multipart quantum system admits a locally maximally entangled state if and only if the number of positive rationals between 0 and 1 whose denominator divides the product of the subsystem dimensions is not less than the number of rationals between 0 and 1 whose denominator divides the square of one of the subsystem dimensions.*

Our proof of Theorem 1.2 is based on solving the following simple algorithm which computes the dimension of $\mathbb{P}(V)//G$.

Theorem 1.4. *The GIT quotient $\mathbb{P}(V)//G$ has dimension $D(\mathbf{d})$, where $\mathbf{d} = (d_1, \ldots, d_n)$ and D is a function defined on weakly increasing tuples of integers by the following cases, depending on the size of d_n relative to $P = d_1 \cdots d_{n-1}$.*

(a) $d_n > P$ $\qquad D(\mathbf{d}) = -1.$

(b) $d_n = P$ $\qquad D(\mathbf{d}) = 0.$

(c) $\frac{P}{2} < d_n < P$ $\qquad D(\mathbf{d}) = D(\mathrm{sort}(d_1, \ldots, d_{n-1}, P - d_n)).$

(d) $d_n \leqslant \frac{P}{2}$ $\qquad D(\mathbf{d}) = \begin{cases} 0, & \text{if } \mathbf{d} = (1, \ldots, 1, 2, 2, 2) \\ d - 3, & \text{if } \mathbf{d} = (1, \ldots, 1, 2, d, d), d \geqslant 3 \\ \Delta(\mathbf{d}), & \text{otherwise.} \end{cases}$

In particular, the GIT quotient $\mathbb{P}(V)//G$ is empty if and only if $D(\mathbf{d}) = -1$ and is a single point if and only if $D(\mathbf{d}) = 0$.

In case (c), $D(\mathbf{d})$ is defined recursively in terms of the value of D on another dimension vector $\mathrm{sort}(d_1, \ldots, d_{n-1}, P - d_n)$, obtained by arranging the positive integers

$$d_1, \ldots, d_{n-1}, P - d_n$$

in weakly increasing order. Since the sum of these integers is strictly less than $\sum_i d_i$, the recursion stops after at most $\sum_i d_i$ steps. In particular, the function $D(\mathbf{d})$ is well defined.

Note that the condition that $\mathbb{P}(V)//G = \varnothing$ is equivalent to V being a pseudo-homogeneous space for G; see Corollary 3.2. Recall that an H-representation W is called a pseudo-homogeneous space if H has a Zariski dense orbit in W. Pseudo-homogeneous spaces for reductive groups were classified by Sato and Kimura [22]. The vectors $\mathbf{d}$ such that $\mathbb{P}(V)//G = \varnothing$ can, in

[6]Savitt pointed out that the function $F(a_1, \ldots, a_n) = \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} (-1)^{k+1} \gcd(a_{i_1}, \ldots, a_{i_k})$ is equal to the number of rationals $0 < \frac{p}{q} \leqslant 1$ such that q divides a_k for some k. The reformulation given in the corollary is then easily deduced.

principle, be described by appealing to this classification. Note that the passage from V to V' in Theorem 1.4(c) is an example of what Sato and Kimura called a castling operation.

Alternatively, the vectors $\mathbf{d}$ such that $\mathbb{P}(V)//G = \varnothing$ (or equivalently, $V//G$ is a point) can be described by appealing to the classification of coregular irreducible representations $\phi\colon H \to \mathrm{GL}(W)$ such that H is semisimple, due to Littelmann [14]. Here ϕ is called coregular if $W//H$ is an affine space $\mathbb{A}^m$. (We are only interested in the cases where $H = \mathrm{SL}_{d_1} \times \cdots \times \mathrm{SL}_{d_n}$, $V = V_{d_1} \otimes \cdots \otimes V_{d_n}$ and $m = 0$.) Note that the castling operation plays a prominent role in [14] as well.

Example 1.5. For $n = 2$, the GIT quotient $\mathbb{P}(V)//G$ is non-empty if and only if $(d_1, d_2) = (d, d)$. Indeed, if $d_2 > d_1$, then part (a) of Theorem 1.4 applies, and $\mathbb{P}(V)//G = \varnothing$. If $d_2 = d_1$, then part (b) applies, and $\mathbb{P}(V)//G$ is single point. □

For $n \geqslant 3$, the situation is considerably more complicated. For example, the quotient for dimension vectors of the form $(2, d_1, d_2)$ is non-empty iff $d_1 = d_2 \geqslant 2$ or $d_2/d_1 = (k+1)/k$ with integer $k \geqslant 2$; see Corollary 7.1. The general characterization of the set of dimension vectors (d_1, d_2, d_3) which admit LME states is arithmetically complicated and can be described in terms of generalized Fibonacci sequences, see [3].

The remainder of this paper is structured as follows: In Sect. 2, we provide some additional background on the quantum mechanics problem and its connection to symplectic geometry and geometric invariant theory. Theorem 1.4 is proved in Sects. 3 and 4. Our argument does not use the above-mentioned classifications due to Sato–Kimura and Littelmann; however, the proof of part (d) relies on the work of Elashvili [4]. Sections 5 and 6 are devoted to analyzing the recursive algorithm of Theorem 1.4 and proving Theorem 1.2.

In a companion paper [3], we discuss numerous explicit results and examples of locally maximally entangled states with a view to applications in quantum information theory.

2. Background

We begin with a short review of relevant background material on quantum mechanics, the connection to symplectic geometry, and basics of geometric invariant theory.

Density Operators and Entanglement

In a multipart quantum system with Hilbert space $V = V_1 \otimes \cdots \otimes V_d$, physical observables associated with subsystem i correspond to Hermitian operators $\mathcal{O} : V_i \to V_i$; the expectation value of the observable in a measurement on a state $\psi_i \in V_i$ is given by the inner product $(\psi_i, \mathcal{O}\psi_i)$. Any such observable can be promoted to an observable in the full multipart system; the associated Hermitian operator $\widehat{\mathcal{O}}$ acting on V is defined by $\widehat{\mathcal{O}} = \mathbb{1} \otimes \cdots \otimes \mathcal{O} \otimes \cdots \otimes \mathbb{1}$.

A crucial feature of multipart quantum systems is that their states are generally *entangled*; that is, they cannot be written as products $\psi_1 \otimes \cdots \otimes \psi_n$. Furthermore, for $\psi \in V$, there does not generally exist $\psi_i \in V_i$ for which $(\psi_i, \mathcal{O}\psi_i) = (\psi, \widehat{\mathcal{O}}\psi)$ for all $\mathcal{O}$ acting on V_i. Thus, in the context of a multipart system it is no longer possible to represent the state of an individual subsystem simply as a vector or *pure state* in Hilbert space V_i. Rather, the subsystem can be described by a *density operator*, defined as a nonnegative Hermitian operator $\rho_i : V_i \to V_i$ with unit trace. The density operator $\rho_i = \mathrm{tr}_{V_1 \otimes \cdots \widehat{V}_i \cdots \otimes V_n} \rho_\psi$ defined in the introductory section is the unique density operator ρ_i such that $(\psi, \widehat{\mathcal{O}}\psi) = \mathrm{tr}(\rho_i \mathcal{O})$ for all $\mathcal{O}$ acting on V_i.

A subsystem described by a density operator with eigenvalues/eigenvectors $\{(p_i, \psi_i)\}$ can be interpreted as being in a statistical ensemble or *mixed state* in which we have state ψ_i with probability p_i. This subsystem is entangled with the rest of the system unless $\{p_i\} = \{1, 0, \ldots, 0\}$. When the density matrix for the subsystem is a multiple of the identity operator, $\rho_i = \mathbb{1}/d_i$, the subsystem is in an equal mixture of all possible states for the subsystem, and we say that the subsystem is *maximally mixed* or *maximally entangled* with the rest of the system. The locally maximally entangled states that we characterize in this paper are defined by the condition that all elementary subsystems have this property.

The Quantum Marginal Problem

The existence question that we consider is a special case of the *quantum marginal problem*: which collections of density operators $\{\rho_\alpha\}$ associated with subsystems α of a multipart system can arise from a quantum state of the entire system? In our case where the subsystems are non-overlapping and the state of the full system is assumed to be pure, a general answer to this question has been provided by Klyachko [11] (see [25] for a review) via a set of inequalities on the spectra for the density operators, or equivalently, in terms of a criterion expressed in the language of representation theory of the symmetric group. These results provide an in-principle method to answer our question, but one that quickly becomes computationally intractable as the subsystem dimensions increase.

The Moment Map

We now briefly review the connection to symplectic geometry. The Fubini–Study symplectic form on $\mathbb{P}(V)$ is fixed up to overall scaling by its invariance under $U(V)$ transformations and is thus invariant under $K = \mathrm{SU}(V_1) \times \cdots \times \mathrm{SU}(V_n)$. The associated moment map $\mu : \mathbb{P}(V) \to \mathfrak{k}^*$ is given explicitly by

$$\mu(\psi) : k \mapsto (\psi, k\psi) \qquad k \in \mathfrak{k} \,. \tag{1}$$

Any k may be written as a linear combination of elements of the form $\mathbb{1} \otimes \cdots k_i \otimes \cdots \mathbb{1}$ with k_i a traceless, Hermitian operator acting on V_i. For an element of this form, we have

$$\mu(\psi)(\mathbb{1} \otimes \cdots k_i \otimes \cdots \mathbb{1}) = \mathrm{tr}(\rho_i k_i) \,. \tag{2}$$

For $\psi \in V_{LME}$, each ρ_i is proportional to the identity operator; tracelessness of k_i then implies that the moment map vanishes. Conversely, $\mathrm{tr}(\rho_i k_i)$ vanishes for arbitrary traceless Hermitian k_i if and only if ρ_i is proportional to the identity operator. Thus we have that $V_{LME} = \mu^{-1}(0)$.

The Kempf–Ness Theorem and Geometric Invariant Theory

As discussed above, the space of LME states, up to equivalence, is given by the symplectic quotient

$$\mu^{-1}(0)/K.$$

The Kempf–Ness theorem identifies this space with an algebro-geometric quotient given by geometric invariant theory which we now briefly describe. Let

$$G = SL(V_1) \times \cdots \times SL(V_n).$$

Note that K is a maximal compact subgroup of G. G acts algebraically on $\mathbb{P}(V_1 \otimes \cdots \otimes V_n) = \mathbb{P}(V)$ and the geometric invariant theory (GIT) quotient $\mathbb{P}(V)//G$ is the projective variety defined by

$$\mathbb{P}(V)//G = \mathrm{Proj}\left(\mathbb{C}[V]^G\right).$$

Here $\mathbb{C}[V]$ is the ring of polynomial functions on V (graded by degree), and $\mathbb{C}[V]^G$ is the subring of invariant functions. Concretely, $\mathrm{Proj}\left(\mathbb{C}[V]^G\right)$ can be constructed as follows (cf. [19]). It is possible to choose homogeneous generators $f_0, \ldots, f_N$ for the ring of invariants $\mathbb{C}[V]^G$ (see [16, Thm. 1.1]), where we define w_i to be the degree of f_i. Consider the rational[7] map

$$\pi \colon \mathbb{P}(V) \dashrightarrow \mathbb{P}^N_{w_0,\ldots,w_N}$$

given by $v \mapsto (f_0(v) : \cdots : f_N(v))$, where $\mathbb{P}^N_{w_0,\ldots,w_N}$ stands for the N-dimensional weighted projective space with weights $w_0, \ldots, w_N$. Then the closure of the image of π is the GIT quotient $\mathbb{P}(V)//G$, and π is the quotient map. Geometrically, the points of $\mathbb{P}(V)//G$ correspond to closed orbits of G in $V \setminus \{0\}$, up to projective equivalence.

In this context, the Kempf–Ness theorem states that there is a homeomorphism

$$\mu^{-1}(0)/K \cong \mathbb{P}(V)//G$$

where $\mathbb{P}(V)//G$ is given the complex analytic topology. The Kempf–Ness theorem thus converts the problem of understanding $\mu^{-1}(0)/K$, the space of equivalence classes of LME states, into the purely algebraic problem of understanding the GIT quotient $\mathbb{P}(V)//G$. We study this quotient in depth in Sects. 3 and 4.

[7]Here, rational indicates that π is only defined on a dense open set $U^{ss} \subset \mathbb{P}^N$. The points of U^{ss} are called semistable points.

3. Invariant-Theoretic Preliminaries

Notational Conventions

In the sequel, we will denote the $\mathbb{C}$-algebra of regular functions on complex affine algebraic variety X by $\mathbb{C}[X]$. If X is irreducible (but not necessarily affine), then $\mathbb{C}(X)$ will denote the field of rational functions on X.

Recall that $\dim(X) = \operatorname{trdeg}_{\mathbb{C}}(\mathbb{C}(X))$. Here $\dim(X)$ denotes the dimension of X and $\operatorname{trdeg}_{\mathbb{C}}(\mathbb{C}(X))$ denotes the transcendence degree of $\mathbb{C}(X)$ over $\mathbb{C}$, i.e., the maximal number of elements $f_1, \ldots, f_n \in \mathbb{C}(X)$ which are algebraically independent over $\mathbb{C}$.

Finally, if G is a complex algebraic group acting on X, we will denote the ring of G-invariant regular functions on X by $\mathbb{C}[X]^G$ and the field of G-invariant rational functions by $\mathbb{C}(X)^G$.

Stabilizers in General Position

Let G be a reductive complex linear algebraic group and $\rho\colon G \to \mathrm{GL}(V)$ be a linear representation. By a theorem of Richardson [21], the action of G on V has a stabilizer S in general position. That is, there exists a closed subgroup $S \subset G$ and a G-invariant dense open subset $U \subset V$ such that the stabilizer G_x is conjugate to S for any $x \in U$. Note that here S is uniquely defined by ρ up to conjugacy and $\dim(G \cdot x) = \dim(G) - \dim(S)$ for every $x \in U$.

Lemma 3.1. *Let G be a semisimple linear algebraic group and $G \to \mathrm{GL}(V)$ be a finite-dimensional representation of G. Let $\mathbb{C}[V]$ be the ring of polynomial functions on V and let $\mathbb{C}(V)$ be the field of rational functions. Then*

(a) *$\mathbb{C}(V)^G$ is the field of fractions of $\mathbb{C}[V]^G$. In other words, every G-invariant rational function on V is a ratio of two G-invariant polynomials.*

(b) *The dimension of the GIT quotient $\mathbb{P}(V)//G$ equals $\dim(V) - \dim(G) + \dim(S) - 1$.*

(c) *If S is reductive, then $\mathbb{P}(V)//G \neq \varnothing$.*

Note that $\dim(\mathbb{P}(V)//G) = -1$ if and only if $\mathbb{P}(V)//G = \varnothing$.

Proof. Part (a) is an easy consequence of the fact that the polynomial ring $\mathbb{C}[V]$ is a unique factorization domain; see, e.g., [17, Lemma 1] or [18, Theorem 3.3].

(b) Recall that by definition, $\mathbb{P}(V)//G = \operatorname{Proj} \mathbb{C}[V]^G$. Thus

$$\dim P(V)//G = \operatorname{trdeg}_{\mathbb{C}} \mathbb{C}[V]^G - 1 = \operatorname{trdeg}_{\mathbb{C}} \mathbb{C}(V)^G - 1\,.$$

By Rosenlicht's theorem [20, Theorem 2], the transcendence degree of $\mathbb{C}(V)^G$ is given by

$$\operatorname{trdeg}_{\mathbb{C}} \mathbb{C}(V)^G = \dim(V) - \dim(G \cdot x) = \dim(V) - (\dim(G) - \dim(G_x)),$$

where $x \in V$ is a point in general position; see also [18, Corollary, p. 156]. Now recall that G_x is conjugate to S, so $\dim(G_x) = \dim(S)$, and part (b) follows.

(c) Suppose S is reductive. Then by a theorem of Popov [17, Theorem 1], there exists a $0 \neq x \in V$ such that the orbit $G \cdot x$ is closed. By a theorem of Hilbert, G-invariant polynomials separate closed orbits in V (see, e.g., [16,

Corollary 1.2]). In particular, there exists an $p \in \mathbb{C}[V]^G$ such that $p(x) \neq p(0)$. This shows that $\mathbb{C}[V]^G \neq \mathbb{C}$, and thus, $\mathbb{P}(V)//G = \operatorname{Proj} \mathbb{C}[V]^G \neq \varnothing$. □

Corollary 3.2 (cf. [22, §2, Propositions 2 and 3]). *Let G be a semisimple linear algebraic group and $G \to \mathrm{GL}(V)$ be a linear representation of G. Then the following conditions are equivalent.*

(a) *G has a dense orbit in V,*
(b) $\mathbb{C}(X)^G = \mathbb{C}$,
(c) $\mathbb{C}[X]^G = \mathbb{C}$,
(d) *The GIT quotient $\mathbb{P}(V)//G$ empty.*

Proof. The implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) are obvious.

(d) $\Longrightarrow$ (a). Assume that the GIT quotient $\mathbb{P}(V)//G$ empty, i.e., its dimension is -1. Then by Lemma 3.1(b), $\dim(V) = \dim(G) - \dim(S)$. On the other hand, for $x \in V$ in general position, $\dim(G \cdot x) = \dim(G) - \dim(S)$. This shows that $\dim(G \cdot x) = \dim(V)$ and thus $G \cdot x$ is dense in V. □

Corollary 3.3 ([22, §3, Proposition 1]). *Let $G \to \mathrm{GL}(V)$ be a finite-dimensional representation of a semisimple linear algebraic group G. If $\dim(V) > \dim(G)$, then $\mathbb{P}(V)//G \neq \varnothing$.*

Proof. If $\dim(V) > \dim(G)$, then by Lemma 3.1(b), $\dim(\mathbb{P}(V)//G) \geqslant \dim(S) \geqslant 0$, i.e., $\mathbb{P}(V)//G \neq \varnothing$. □

The Index of a Representation

Let $\rho\colon H \to \mathrm{GL}_d$ be a faithful finite-dimensional representation of a simple complex linear algebraic group H. Let $\mathrm{Lie}(H)$ be the lie algebra of H and

$$\rho^* = d\rho_{|e}\colon \mathrm{Lie}(H) \to \mathrm{Lie}(GL_d) = \mathrm{M}_d$$

be the induced map of Lie algebras. Following Andreev et al. [1], we define the index $l(\rho)$ by the formula

$$l(\rho) = \frac{\mathrm{Tr}\big(\rho^*(h)^2\big)}{\mathrm{Tr}\big(\,\mathrm{ad}(h)^2\big)}\,,$$

where $h \in \mathrm{Lie}(H)$. Note that the right-hand side of the formula is independent of the choice of h, as long as $\mathrm{Tr}\big(\,\mathrm{ad}(h)^2\big) \neq 0$. It is clear from this definition that the index is additive, i.e.,

$$l(\rho_1 \oplus \rho_2) = l(\rho_1) + l(\rho_2)\,. \tag{3}$$

Example 3.4. Consider the natural representation ρ_{nat} of $H = \mathrm{SL}_d$ on $V = \mathbb{C}^d$. Take

$$h = \mathrm{diag}(\lambda_1, \ldots, \lambda_d) \in \mathfrak{sl}_d, \text{ where } \lambda_1 + \cdots + \lambda_d = 0.$$

Here $\mathfrak{sl}_d = \mathrm{Lie}(\mathrm{SL}_d)$. Then $\rho^*_{\mathrm{nat}}(h) = h$, $\mathrm{Tr}(h^2) = \sum_{i=1}^d \lambda_i^2$ and

$$\begin{aligned}\mathrm{Tr}\left(\mathrm{ad}(h)^2\right) &= \sum_{i\neq j}(\lambda_i - \lambda_j)^2 = \sum_{i\neq j}(\lambda_i^2 + \lambda_j^2 - 2\lambda_i\lambda_j) \\ &= 2(d-1)\sum_{i=1}^d \lambda_i^2 - 2\left(\left(\sum_{i=1}^d \lambda_i\right)^2 - \sum_{i=1}^d \lambda_i^2\right) = 2d\sum_{i=1}^d \lambda_i^2.\end{aligned}$$

We conclude that $l(\rho_{\mathrm{nat}}) = \dfrac{1}{2d}$.

4. Proof of Theorem 1.4

Proof of case (a): We claim for every $v \in V$ the closure of the orbit $\mathrm{SL}(V_n)\cdot v$ in V contains 0. If we can prove this claim, then clearly $\mathbb{C}[V]^G = \mathbb{C}$, and thus $\mathbb{P}(V)//G$ is empty; see Corollary 3.2.

To prove the claim, let $U = V_1 \otimes \cdots \otimes V_{n-1}$, let $W = V_n$, and write $\sum_{i=1}^m u_i \otimes w_i$, where $u_1, \ldots, u_m$ form a basis of U and $w_i \in W$. By our assumption,

$$m = d_1 \cdots d_{n-1} < d_n = \dim(V_n)\,.$$

Hence, we can choose a basis $f_1, \ldots, f_{d_n}$ of V_n such that $w_1, \ldots, w_m \in \mathrm{Span}(f_1, \ldots, f_{d_n-1})$. Now define the 1-parameter subgroup $\lambda\colon \mathbb{G}_m \to \mathrm{SL}(V_n)$ by $\lambda(t)f_j = tf_j$ for $j = 1, \ldots, d_n - 1$ and $\lambda(t)f_n = t^{-d_n+1}f_n$. Then $\lambda(t)\cdot v = tv \to 0$, as $t \to 0$ and hence, 0 lies in the closure of $\mathrm{SL}(V_n)\cdot v$ in V, as claimed.

Proof of case (b): Let $U = V_1 \otimes \cdots \otimes V_{n-1}$ and let $W = V_n$. By our assumption, $\dim(U) = \dim(W)$. Note that

$$\mathrm{SL}(W) \subset G \subset \mathrm{SL}(U) \times \mathrm{SL}(W). \tag{4}$$

Identify U with W^* and thus $V = U \otimes W$ with the space of $d_n \times d_n$-matrices M_{d_n}, where $\mathrm{SL}(U) \simeq \mathrm{SL}_{d_n}$ acts by multiplication on the left and $\mathrm{SL}(W) \simeq \mathrm{SL}_{d_n}$ acts by multiplication on the right. Let $f\colon U \otimes W = \mathrm{M}_{d_n} \to \mathbb{C}$ be the determinant map. Then f is invariant under $\mathrm{SL}(U)\times\mathrm{SL}(W)$ and hence, under G; see (4). This shows that $\mathbb{C}[f] \subset \mathbb{C}[V]^G$. On the other hand,

$$\mathbb{C}[V]^G \subset \mathbb{C}[V]^{\mathrm{SL}(W)} = \mathbb{C}[\mathrm{M}_{d_n}]^{\mathrm{SL}_{d_n}} = \mathbb{C}[f]\,.$$

Thus $\mathbb{C}[V]^G = \mathbb{C}[f]$ is a polynomial ring in one variable. Consequently,

$$\mathbb{P}(V)//G = \mathrm{Proj}\,\mathbb{C}[f] = \mathbb{P}^0$$

is a single point, as claimed.

Proof of case (c): Let $W = V_1 \otimes \cdots \otimes V_{n-1}$ and $H = \mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_{n-1}) \subset \mathrm{SL}(W)$. Set $P = d_1 \cdots d_{n-1}$, $V_n' = \mathbb{C}^{P-d_n}$, $V' = W \otimes V_n'$, and $G' = H \times \mathrm{SL}(V_n')$. By [14, Lemma 2(a)], $V//G$ is isomorphic to $V'//G'$. Thus

$$\dim(\mathbb{P}(V)//G) = \dim(V//G) - 1 = \dim(V'//G') - 1 = \dim(\mathbb{P}(V')//G')\,,$$

as claimed.

Proof of case (d): Our argument will rely on the description of stabilizers in general position in irreducible representations $\rho\colon G \to \mathrm{GL}(V)$ of a semisimple group G, satisfying the condition that the index

$$l(\rho_{|\,H}) \geqslant 1 \text{ for every simple normal subgroup } H \text{ of } G, \tag{5}$$

due to Elashvili [4]. In order to apply this description to our representation of $G = \mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_n)$ on $V = V_1 \otimes \cdots \otimes V_n$ (which we will denote by ρ), we need to check that condition (5) is satisfied for this representation.

The simple normal subgroups of G are $H = \mathrm{SL}(V_i)$ for $i = 1, \ldots, n$. Clearly the restriction $\rho_{|\,\mathrm{SL}(V_1)}$ is isomorphic to the direct sum of $\dim(V_2 \otimes \cdots \otimes V_n) = d_2 \cdots d_n$ copies of the natural representation ρ_{nat} of $\mathrm{SL}(V_1)$. As we saw in Example 3.4, $l(\rho_{\mathrm{nat}}) = \dfrac{1}{2d_1}$. Hence, by (3), $l\big(\rho_{|\,\mathrm{SL}(V_1)}\big) = \dfrac{d_2 \cdots d_n}{2d_1}$. Similarly $l\big(\rho_{|\,\mathrm{SL}(V_i)}\big) = \dfrac{d_1 \cdots d_{i-1} d_{i+1} \cdots d_n}{2d_i}$ for any $i = 1, 2, \ldots, n$. Since $d_1 \leqslant \cdots \leqslant d_n$, the smallest of these indices is $l(\rho_n) = \dfrac{d_1 \cdots d_{n-1}}{2d_n}$. The assumption of part (d), that $d_n \leqslant \dfrac{1}{2} d_1 \cdots d_{n-1}$, is thus equivalent to (5).

Under this assumption, [4, Theorem 9] asserts that the connected component S^0 of the stabilizer S in general position for the action of G on V is as follows:

$$S^0 \simeq \begin{cases} (\mathbb{G}_m)^{d-1}, \text{ if } n = 3,\ (d_1, d_2, d_3) = (2, d, d) \text{ and } d \geqslant 3, \\ (\mathbb{G}_m)^2, \text{ if } n = 3 \text{ and } (d_1, d_2, d_3) = (2, 2, 2), \\ \{1\}, \text{ otherwise.} \end{cases}$$

In all cases, S^0 is reductive, and hence, so is S. By Lemma 3.1(c), we conclude that $\mathbb{P}(V)/\!/G$ is non-empty. Moreover, using the formula $\dim(\mathbb{P}(V)/\!/G) = \dim(V) - \dim(G) + \dim(S) - 1$ of Lemma 3.1(b) and remembering that $\dim(S) = \dim(S^0)$, we readily check that

$$\dim(\mathbb{P}(V)/\!/G) = \begin{cases} d - 3, \text{ if } n = 3,\ (d_1, d_2, d_3) = (2, d, d),\ d \geqslant 3, \\ 0, \text{ if } n = 3 \text{ and } (d_1, d_2, d_3) = (2, 2, 2), \text{ and} \\ \dim(V) - \dim(G) - 1 \text{ in all other cases.} \end{cases} \qquad \square$$

Remark 4.1. Suppose $1 \leqslant d_1 \leqslant \cdots \leqslant d_n$ and

$$d_n \leqslant d_1 \cdots d_{n-1}/2. \tag{6}$$

Theorem 1.4(d) tells us that then there exists a non-constant $G = \mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_n)$-invariant polynomial on $V = V_1 \otimes \cdots \otimes V_n$; see Corollary 3.2. Here, $\dim(V_i) = d_i$, as before. Equivalently, there exists a homogeneous G-invariant polynomial of degree $\geqslant 1$. It is natural to try to exhibit such a polynomial explicitly.

It is well known that the hyperdeterminant $h_{d_1,\ldots,d_n}\colon V \to \mathbb{C}$ of format $d_1 \times \cdots \times d_n$ is a homogeneous G-invariant polynomial. However, $h_{d_1,\ldots,d_n} \neq 0$ if and only if

$$d_n \leqslant d_1 + \cdots + d_{n-1} - (n - 2);$$

see [6, p. 446, Theorems 1.3 and Proposition 1.4]. This inequality is considerably weaker than (6). In other words, for many dimension vectors $\mathbf{d}$ satisfying (6), the hyperdeterminant is identically zero.

We also note that for any dimension vector $\mathbf{d}$ and an integer $k \geqslant 1$, a procedure due to Gour and Wallach [7] produces a basis for the vector space $(\mathbb{C}[V]^G)_k$ of homogeneous G-invariant polynomials of degree $k \geqslant 1$. However, when $\mathbb{C}[V]^G \neq \mathbb{C}$, it is not a priori clear for which $k \geqslant 1$, $(\mathbb{C}[V]^G)_k \neq (0)$[8].

5. Proof of the First Part of Theorem 1.2

For each dimension vector $\mathbf{d} = (d_1, \ldots, d_n)$, with $d_1 \leqslant \cdots \leqslant d_n$, the recursive algorithm provided by Theorem 1.4 brings us to some terminal dimension vector $\mathbf{e} = (e_1, \ldots, e_n)$, where $e_1 \leqslant \cdots \leqslant e_n$. The GIT quotient $\mathbb{P}(V)//G$ is empty if the algorithm terminates on case (a), and non-empty if the algorithm terminates on case (b) or (d).

Remark 5.1. Recall from Introduction (see footnote 2, pg. 1) that our standing assumption is that $n \geqslant 2$ and $d_{n-1} \geqslant 2$. We now observe that the terminal vector $\mathbf{e}$ also satisfies these conditions. Obviously, n does not change; our claim is that there are at most $n-2$ ones among the integers $e_1, \ldots, e_n$. Indeed, in each recursion step, the number of 1s in the list of dimensions can increase by one, but with $n-2$ 1s, we will either be in case (a) or (b), and the recursion terminates.

Let us now define a new dimension vector $\mathbf{a} = (a_1, \ldots, a_m)$ by removing all 1s from $(e_1, \ldots, e_n)$.

Lemma 5.2. *Suppose that a terminal vector* $\mathbf{e} = (e_1, \ldots, e_n)$ *satisfying the conditions for case (a), (b), or (d), is obtained from* $\mathbf{d} = (d_1, \ldots, d_n)$ *by repeatedly performing the transformation of Theorem 1.4(c). Assume further that* $\mathbf{a} = (a_1, \ldots, a_m)$ *is obtained from* $\mathbf{e}$ *by removing all 1s, as above. Then*

(a) $\Delta(\mathbf{d}) = \Delta(\mathbf{e}) = \Delta(\mathbf{a})$,
(b) $g_{\mathsf{max}}(\mathbf{d}) = g_{\mathsf{max}}(\mathbf{e}) = g_{\mathsf{max}}(\mathbf{a})$, *and*
(c) $R(\mathbf{d}) = R(\mathbf{e}) = R(\mathbf{a})$.

Note that $m \geqslant 2$ by Remark 5.1. In particular, $g_{\mathsf{max}}(\mathbf{a})$ is well defined.

Proof. Let $\mathbf{d}' = (d'_1, \ldots, d'_n)$ be the set of dimensions obtained from $\mathbf{d}$ by a single application of the castling transformation of Theorem 1.4(c). Since $\Delta(d_1, \ldots, d_n)$, $R(d_1, \ldots, d_n)$, and $g_{\mathsf{max}}(d_1, \ldots, d_n)$ are all symmetric in $d_1, \ldots, d_n$, we may reorder $d'_1, \ldots, d'_n$ and thus assume that $d'_i = d_i$ for $i \leqslant n-1$ and $d'_n = P - d_n$, where $P = d_1 \cdots d_{n-1}$. The lemma follows by showing that each of the three quantities is invariant under $\mathbf{d} \to \mathbf{d}'$ and under eliminating a 1.

[8] Gour and Wallach show that $(\mathbb{C}[V]^G)_k = (0)$ unless k is divisible by the least common multiple $l = \mathrm{lcm}(d_1, \ldots, d_n)$. Thus we only need to consider k of the form lq, with $q = 1, 2, 3, \ldots$.

(a) An easy calculation shows that $\Delta(\mathbf{d}') = P(P-d_n) - (P-d_n)^2 - \sum_{i=1}^{n-1}(d_i^2-1)$ is equal to

$$\Delta(\mathbf{d}) = Pd_n - d_n^2 - \sum_{i=1}^{n-1}(d_i^2-1).$$

The identity $\Delta(1, d_2, \ldots, d_n) = \Delta(d_2, \ldots, d_n)$ is immediate from the definition.

(b) It is again obvious from the definition that $g_{\mathsf{max}}(1, e_2, \ldots, e_n) = g_{\mathsf{max}}(e_2, \ldots, e_n)$. Since $m \geqslant 2$, this implies that $g_{\mathsf{max}}(\mathbf{e}) = g_{\mathsf{max}}(\mathbf{a})$. To show invariance under the transformation in Theorem 1.4(c), we will show more generally that $\gcd(d'_{i_1}, \ldots, d'_{i_k}) = \gcd(d_{i_1}, \ldots, d_{i_k})$ for any $2 \leqslant k \leqslant n$ and $1 \leqslant i_1 < \cdots < i_k \leqslant n$. If $i_k \leqslant n-1$, this is obvious, since $d'_{i_j} = d_{i_j}$ for each j. If $i_k = n$, then

$$\gcd(d'_{i_1}, \ldots, d'_{i_{k-1}}, d'_n) = \gcd(d_{i_1}, \ldots, d_{i_{k-1}}, -d_n + P) = \gcd(d_{i_1}, \ldots, d_{i_{k-1}}, d_n)\,,$$

since P is divisible by d_{i_j} for $1 \leqslant j \leqslant n-1$. Thus $g_{\mathsf{max}}(\mathbf{d}) = g_{\mathsf{max}}(\mathbf{e})$.

(c) We have that

$$R(d_1, \ldots, d_n) = \Delta(d_1, \ldots d_n) - n + 1 + \sum_{k=2}^{n}(-1)^k G_k(d_1, \ldots, d_n).$$

Since Δ and the set of GCDs for all k-tuples with $k \geqslant 2$ are invariant under $\mathbf{d} \to \mathbf{d}'$, R is also invariant. If $d_1 = 1$, then all GCDs involving d_1 are equal to 1, so we have that $\sum_{k=1}^{n}(-1)^k G_k(d_1, \ldots, d_n)$ is equal to $\sum_{k=1}^{n}(-1)^k G_k(d_2, \ldots, d_n) + \sum_{k=1}^{n}(-1)^k \binom{n-1}{k-1} = \sum_{k=1}^{n}(-1)^k G_k(d_2, \ldots, d_n)$. Since also $\prod_i d_i = \prod_{i \neq 1} d_i$, R is unchanged if we remove dimensions equal to 1. □

We now show that the value of $R = R(\mathbf{d})$ predicts on which case the algorithm in Theorem 1.4 will terminate, and thus, whether or not the quotient is empty.

Proposition 5.3. *For the dimension vector $\mathbf{d} = (d_1, \ldots, d_n)$, the algorithm in Theorem 1.4 terminates on case (a) if and only if $R < 0$, on case (b) if and only if $R = 0$ and on case (d) if and only if $R > 0$.*

Proof. Since the algorithm always terminates on case (a), (b), or (d), we need only show that $R(\mathbf{d})$ is, respectively, negative, zero, and positive in these three cases. Let $\mathbf{e} = (e_1, \ldots, e_n)$ be the terminal vector, with $e_1 \leqslant \cdots \leqslant e_n$, and $\mathbf{a} = (a_1, \ldots, a_{m-1}, a_m)$ be obtained from $\mathbf{e}$ by removing all 1s, as above. By Lemma 5.2, it suffices to show that $R = R(\mathbf{a})$ is negative, positive, and zero in cases (a), (b) and (d), i.e., if $e_n > e_1 \cdots e_{n-1}$, $e_n = e_1 \cdots e_{n-1}$ and $e_n \leqslant \frac{1}{2} e_1 \cdots e_{n-1}$, or equivalently, if $a_m > a_1 \cdots a_{m-1}$, $a_m = a_1 \cdots a_{m-1}$ and $a_m \leqslant \frac{1}{2} a_1 \cdots a_{m-1}$, respectively. For the proof below, it will be useful to define B_k to be the sum of all the terms in G_k with k-tuples involving a_m and $A_k = G_k(a_1, \ldots, a_{m-1})$ to be the sum of the remaining terms. Then $G_k = A_k + B_k$, and

$$R = a_1 \cdots a_m - a_m^2 + \sum_{k=1}^{m-1}(-1)^k (A_k - B_{k+1})\,. \tag{7}$$

We consider the three cases in turn.

Case (b): $a_m = a_1 \cdots a_{m-1}$.

Here each term in B_{k+1} is equal to the corresponding term in A_k obtained by omitting a_m from the GCD, so we have $B_{k+1} = A_k$ for $k \geqslant 1$. Since $a_1 \cdots a_m - a_m^2 = 0$ in this case, Eq. (7) gives $R(d_1, \ldots, d_n) = 0$.

Case (a): $a_m > a_1 \cdots a_{m-1}$.

In this case, we can write $a_m = a_1 \cdots a_{m-1} + \alpha$ for some $\alpha > 0$. From (7), we have

$$\begin{aligned} &R(a_1, \ldots, a_{m-1}, a_m) \\ &\quad = -\alpha^2 - \alpha a_1 \cdots a_{m-1} - \sum_{k \geqslant 1 \ odd} (A_k - B_{k+1}) + \sum_{k \geqslant 2 \ even} (A_k - B_{k+1}) \end{aligned} \tag{8}$$

For each term in A_k, there is a corresponding term in B_{k+1} obtained by including a_m in the *gcd*. Since $gcd(a_{i_1}, \ldots, a_{i_k}, a_m) \leqslant gcd(a_{i_1}, \ldots, a_{i_k})$, we have $A_k \geqslant B_{k+1}$. Making use of this for odd k and assuming for now that $\alpha \geqslant 2$, we obtain from (8)

$$\begin{aligned} R(a_1, \ldots, a_{m-1}, a_m) &< -2a_1 \cdots a_{m-1} + \sum_{k \geqslant 2 \ even} A_k \\ &\leqslant -2a_1 \cdots a_{m-1} + \sum_{k \geqslant 2 \ even} \binom{m-1}{k} a_{m-2} a_{m-1} \\ &= -2a_1 \cdots a_{m-1} + (2^{m-2} - 1) a_{m-2} a_{m-1} \\ &= (2^{m-2} - 1 - 2a_1 \cdots a_{m-3}) a_{m-2} a_{m-1}. \end{aligned} \tag{9}$$

Here we obtain the second inequality by noting that that A_k has $\binom{m-1}{k}$ terms and

$$\gcd(a_{i_1} \cdots a_{i_k})^2 \leqslant a_{m-k}^2 \leqslant a_{m-2} a_{m-1},$$

for $k \geqslant 2$, because the GCD of k integers chosen from $(a_1, \ldots, a_{m-1})$ cannot exceed a_{m-k}, the kth largest number in this set. Since $a_i \geqslant 2$, the term in brackets in the final expression is negative, and we conclude that $R < 0$.

Next consider the case where $\alpha = 1$. Here, all GCDs involving a_m are equal to 1, so we have $B_k = \binom{m-1}{k-1}$ and the terms in (8) involving B sum up to

$$\sum_{k \geqslant 1} (-1)^{k-1} B_{k+1} = \sum_{k \geqslant 1} (-1)^{k-1} \binom{m-1}{k} = 1. \tag{10}$$

Then from (8), we get

$$R(a_1, \ldots, a_{m-1}, a_m) = -a_1 \cdots a_{m-1} - \sum_{k \geqslant 1 \ odd} A_k + \sum_{k \geqslant 2 \ even} A_k. \tag{11}$$

As above, we can now decompose $A_k = C_k + D_k$, where D_k represents all the terms in A_k with k-tuples involving a_{m-1} and C_k are the remaining terms.

The same argument as before shows that $C_k \geqslant D_{k+1}$. Starting from (11) and dropping negative terms $-D_{2l+1}$ and $-(C_{2l-1} - D_{2l})$ we obtain

$$\begin{aligned} R(a_1, \ldots, a_{m-1}, a_m) &< -a_1 \cdots a_{m-1} + \sum_{k \geqslant 2 \ even} C_k \\ &\leqslant (2^{m-3} - 1 - a_1 \cdots a_{m-3}) a_{m-2} a_{m-1}. \end{aligned}$$

Here the calculation is the same as in (9), except that we end up with 2^{m-3} instead of 2^{m-2}, since C_k involve only $m-2$ a_ks. Again, the term in brackets in the final expression is negative, and we conclude that $R < 0$.

Case (d): $a_m \leqslant \frac{1}{2} a_1 \cdots a_{m-1}$.

Note that this is only possible if $m \geqslant 3$. Starting from Eq. (7), we have

$$\begin{aligned} R &= \frac{1}{4} a_1^2 \cdots a_{m-1}^2 - (\frac{1}{2} a_1 \cdots a_{m-1} - a_m)^2 + \sum_{k \geqslant 1} (-1)^k (A_k - B_{k+1}) \\ &\geqslant a_{m-1}^2 (a_1 \cdots a_{m-2} - 1) + \sum_{k \geqslant 1} (-1)^k (A_k - B_{k+1}) \\ &> a_{m-1}^2 (a_1 \cdots a_{m-2} - 1) - \sum_{k \geqslant 1 \ odd} A_k, \end{aligned} \tag{12}$$

where in the second line we have used that the maximum value of $(\frac{1}{2} a_1 \cdots a_{m-1} - a_m)^2$ will be for $a_m = a_{m-1}$, and in the third line, we have discarded nonnegative terms $(A_k - B_{k+1})$ for k even and positive terms B_{k+1} for k odd. Since each of the $\binom{m-1}{k}$ GCDs contributing to A_k is less than or equal to a_{k-1}, we have that

$$\begin{aligned} R &> a_{m-1}^2 (a_1 \cdots a_{m-2} - 1) - \sum_{k \geqslant 1 \ odd} a_{m-1}^2 \binom{m-1}{k} \\ &= a_{m-1}^2 (a_1 \cdots a_{m-2} - 1 - 2^{m-2}) \end{aligned}$$

Since $a_1, \ldots, a_{m-2} \geqslant 2$, we see that $R > 0$ unless $(a_1, \ldots, a_{m-2}) = (2, \ldots, 2)$.

For this case, with $(a_1, \ldots, a_m) = (2, \ldots, 2, a_{m-1}, a_m)$, we can calculate the second line of (12) directly. Consider two cases.

Case 1: a_m is even. Here $A_1 = 4(m-2) + a_{m-1}^2$, $B_2 = 4(m-2) + \gcd(a_{m-1}, a_m)^2$, and $A_k = B_{k+1}$ for any $k \geqslant 2$. Thus

$$R \geqslant a_{m-1}^2 (2^{m-2} - 1) - (A_1 - B_2) = a_{m-1}^2 (2^{m-2} - 2) + \gcd(a_{m-1}, a_m)^2 > 0.$$

Case 2: a_m is odd. Here

$$B_2 = (m-2) + \gcd(a_{m-1}, a_m)^2 = \gcd(a_{m-1}, a_m)^2 - \binom{m-1}{0} + \binom{m-1}{1}$$

and $B_{k+1} = \binom{m-1}{k}$ for any $k \geqslant 2$. Thus

$$\sum_{k \geqslant 1} (-1)^k B_{k+1} = -\gcd(a_{m-1}, a_m)^2 + \sum_{i \geqslant 0} (-1)^i \binom{m-1}{i} = -\gcd(a_{m-1}, a_m)^2 .$$

Moreover, $A_1 = 4(m-2) + a_{m-1}^2 = a_{m-1}^2 - 4\binom{m-1}{0} + 4\binom{m-1}{1}$ and

$$\begin{aligned} A_k &= 4\binom{m-2}{k} + \gcd(2, a_{m-1})^2 \binom{m-2}{k-1} \\ &= 4\binom{m-1}{k} + (\gcd(2, a_{m-1})^2 - 4)\binom{m-2}{k-1} \end{aligned}$$

for any $k \geqslant 2$. Thus

$$\begin{aligned} R &\geqslant a_{m-1}^2(2^{m-2}-1) - \sum_{k\geqslant 1}(-1)^k B_{k+1} - A_1 + \sum_{k\geqslant 2}(-1)^k A_k \\ &= a_{m-1}^2(2^{m-2}-1) + \gcd(a_{m-1}, a_m)^2 - a_{m-1}^2 + 4\binom{m-1}{0} \\ &\qquad - 4\binom{m-1}{1} + \sum_{k\geqslant 2}(-1)^k A_k \\ &= a_{m-1}^2(2^{m-2}-2) + \gcd(a_{m-1}, a_m)^2 + 4\sum_{k\geqslant 0}(-1)^k\binom{m-1}{k} \\ &+ (\gcd(2, a_{m-1})^2 - 4)\sum_{k\geqslant 2}(-1)^k\binom{m-2}{k-1} \\ &= a_{m-1}^2(2^{m-2}-2) + \gcd(a_{m-1}, a_m)^2 + (\gcd(2, a_{m-1})^2 - 4)\,. \end{aligned}$$

If $m \geqslant 4$, then the first term is $\geqslant 8$ and thus $R > 0$. As we mentioned above, the inequality $a_m \leqslant a_1 \cdots a_{m-1}/2$ forces m to be $\geqslant 3$. Thus we may assume that $m = 3$. Since $a_1 = 2$, we have $a_3 \leqslant 2 \cdot a_2/2 = a_2$ and thus $a_2 = a_3$. In this case, $\gcd(a_{m-1}, a_m) = \gcd(a_2, a_3) = a_3$. Substituting $m = 3$ into the above inequality, and remembering that $a_1 = 2$ and $a_2 = a_3 \geqslant 2$ is odd, we see that $a_2 = a_3 \geqslant 3$ and thus

$$R \geqslant a_2^2(2^{3-2}-2) + a_2^2 + (\gcd(2, a_2)^2 - 4) \geqslant 0 + 9 + (1-4) > 0\,,$$

as desired. □

Proof of Theorem 1.2, first part. Suppose $R(\mathbf{d}) < 0$. By Proposition 5.3 the algorithm of Theorem 1.4 terminates on case (a) and the quotient $\mathbb{P}(V)//G$ is empty.

On the other hand, if $R(\mathbf{d}) \geqslant 0$, Proposition 5.3 shows that the algorithm of Theorem 1.4 terminates on case (b) or (d). Theorem 1.4 now tells us that the quotient $\mathbb{P}(V)//G$ is no-empty. This proves the first assertion of Theorem 1.2. □

6. Conclusion of the Proof of Theorem 1.2

We will prove the remaining statements in Theorem 1.2 using the following proposition.

Proposition 6.1. *Suppose that we perform the recursive procedure of Theorem 1.4, starting with the dimension vector* $\mathbf{d} = (d_1, \ldots, d_n)$*. Here, as always,* $d_1 \leqslant \cdots \leqslant d_n$*,* $n \geqslant 2$ *and* $d_{n-1} \geqslant 2$*. Denote the terminal dimension vector by* $\mathbf{e} = (e_1, \ldots, e_n)$*.*

1. *If* $\Delta(\mathbf{d}) < -5$*, then* $e_1 \cdots e_{n-1} \leqslant e_n$*. That is, the recursion in Theorem 1.4 terminates on case (a) or (b). The quotient* $\mathbb{P}(V)//G$ *is empty or a single point.*
2. *If* $-5 \leqslant \Delta(\mathbf{d}) < -2$*, then* $e_1 \cdots e_{n-1} = e_n$*. That is, the recursion in Theorem 1.4 terminates on case (b). The quotient* $\mathbb{P}(V)//G$ *is a single point.*
3. *If* $\Delta(\mathbf{d}) = -2$*, then the recursion in Theorem 1.4 terminates on case (d) with* $\mathbf{e} = (1, \ldots, 1, 2, a, a)$*. Here* $a = g_{\mathsf{max}}(\mathbf{d}) \geqslant 2$*. The quotient* $\mathbb{P}(V)//G$ *is a point if* $a = 2$ *and has dimension* $a - 3$ *if* $a \geqslant 3$*.*
4. *If* $\Delta(\mathbf{d}) > -2$*, then the recursion in Theorem 1.4 terminates on case (d) with* $\mathbf{e} \neq (1, \ldots, 1, 2, a, a)$ *for any* $a \geqslant 2$*. In this case* $\Delta(\mathbf{d}) \geqslant 2$*, and the quotient* $\mathbb{P}(V)//G$ *has dimension* $\Delta(\mathbf{d})$*.*

Proof. Let $\mathbf{a} = (a_1, \ldots, a_m)$ be obtained by removing all 1s from $\mathbf{e} = (e_1, \ldots, e_n)$, as in the previous section. By Remark 5.1, $m \geqslant 2$. By Lemma 5.2, $\Delta(\mathbf{d}) = \Delta(\mathbf{e}) = \Delta(\mathbf{a})$ and $g_{\mathsf{max}}(\mathbf{d}) = g_{\mathsf{max}}(\mathbf{e}) = g_{\mathsf{max}}(\mathbf{a})$. Moreover, one readily sees that $\mathbf{a}$ is also a terminal vector for the recursive procedure of Theorem 1.4, and that $\mathbf{e}$ and $\mathbf{a}$ correspond to the same terminal case in Theorem 1.4, i.e., case (a), (b), or (d). Thus, for the purpose of proving Proposition 6.1, we may replace $\mathbf{d}$ by $\mathbf{a}$. That is, we may assume that $\mathbf{d} = (d_1, \ldots, d_n)$ is terminal, $2 \leqslant d_1 \leqslant \cdots \leqslant d_n$, and $n \geqslant 2$.

Set $P = d_1 \cdots d_{n-1}$. To prove the proposition, we will show that

(i) if $\mathbf{d}$ corresponds to case (a), i.e., $d_n > P$, then $\Delta(\mathbf{d}) < -5$,

(ii) if $\mathbf{d}$ corresponds to case (b), i.e., $d_n = P$, then $\Delta(d_1, \ldots, d_n) < -2$,

(iii) if $\mathbf{d} = (2, a, a)$, then $\Delta(\mathbf{d}) = -2$ and $g_{\mathsf{max}}(\mathbf{d}) = a$.

(iv) if $\mathbf{d}$ corresponds to case (d), i.e., $d_n \leqslant P/2$, and moreover $\mathbf{d} \neq (2, a, a)$ for any integer $a \geqslant 2$, then $\Delta(\mathbf{d}) \geqslant 2$.

If we can establish (i)–(iv), Proposition 6.1 will follow directly from Theorem 1.4.

To prove (i), let us fix $d_1, \ldots, d_{n-1}$ and view

$$\phi(x) = \Delta(d_1, \ldots, d_{n-1}, x) = xP - d_1^2 - \cdots - d_{n-1}^2 - x^2 + n - 1$$

as a quadratic polynomial in x. Note that $\phi'(x) = P - 2x$, so $\phi(x)$ is increasing for $x \leqslant P/2$ and decreasing for $x \geqslant P/2$. In particular, if $d_n \geqslant P + 1$, then

$$\Delta(\mathbf{d}) = \phi(d_n) \leqslant \phi(P + 1) = -P - 1 - \sum_{i=1}^{n-1} (d_i^2 - 1)\,.$$

Since $n \geqslant 2$, each $d_i \geqslant 2$ and in particular, $P \geqslant d_1 \geqslant 2$. This yields

$$\Delta(\mathbf{d}) \leqslant -2 - 1 - (2^2 - 1) = -6,$$

as claimed.

To prove (ii), assume $d_n = P$. Then $\Delta(\mathbf{d}) = \phi(P) = -\sum_{i=1}^{n-1}(d_i^2 - 1) < -2$, since $n \geqslant 2$ and each $d_i \geqslant 2$.

(iii) is easy: $\Delta(2, a, a) = 2a^2 - 4 - a^2 - a^2 + 3 - 1 = -2$ by the definition of Δ, and $g_{\mathsf{max}}(2, a, a) = a$ by the definition of g_{max}.

To prove (iv), assume that $2 \leqslant d_1 \leqslant \cdots \leqslant d_{n-1} \leqslant d_n \leqslant P/2$. We are interested in the value of $\Delta(\mathbf{d}) = \phi(d_n)$. Since $\phi'(x) \leqslant 0$ for any x in the interval $[d_{n-1}, P/2]$, we have

$$\Delta(\mathbf{d}) = \phi(d_n) \geqslant \phi(d_{n-1}) = d_{n-1}P - d_1^2 - ... - d_{n-2}^2 - 2d_{n-1}^2 + n - 1\,.$$

Remembering that $P = d_1 \cdots d_{n-1}$ and $d_i \leqslant d_{n-1}$ for $i = 1, \ldots, n-1$, we conclude that

$$\Delta(\mathbf{d}) \geqslant d_{n-1}P - n(d_{n-1})^2 + n - 1 = d_{n-1}^2(d_1 \cdots d_{n-2} - n) + n - 1\,. \quad (13)$$

By our assumption, $n \geqslant 3$ and $(d_1, \ldots, d_n) \neq (2, d, d)$ for any $d \geqslant 2$. Let us now consider two cases.

Case 1: $n \geqslant 4$. In this case $d_1 \cdots d_{n-2} - n \geqslant 2^{n-2} - n \geqslant 0$ and (13) tells us that $\Delta(\mathbf{d}) \geqslant n - 1 \geqslant 3$.

Case 2: $n = 3$ but $(d_1, d_2, d_3) \neq (2, d, d)$ for any $d \geqslant 2$. Here our assumption that $d_3 \leqslant P/2 = d_1 d_2/2$ implies $d_1 \geqslant 3$. In this case (13) yields $\Delta(\mathbf{d}) \geqslant d_2^2(d_1 - 3) + 2$, and $d_1 \geqslant 3$ implies $\Delta(\mathbf{d}) \geqslant 2$. This completes the proof of (iv) and thus of Proposition 6.1. □

Proof of Theorem 1.2, second part. To prove the second assertion of Theorem 1.2, let us consider three cases, where $\Delta(\mathbf{d}) < -2$, $\Delta(\mathbf{d}) = -2$, and $\Delta(\mathbf{d}) > -2$, respectively.

When $\Delta(\mathbf{d}) > -2$, Proposition 6.1(4) tells us that the recursion terminates on case (d) and the dimension of $\mathbb{P}(V)//G$ is $\Delta(\mathbf{d}) \geqslant 2$. In this case $R > 0$ by Proposition 5.3.

When $\Delta(\mathbf{d}) = -2$, we are in case (3) of Proposition 6.1. Here the recursion terminates on case (d) and the dimension of $\mathbb{P}(V)//G$ is $(g_{\mathsf{max}}(\mathbf{d}) - 3)$ for $g_{\mathsf{max}}(\mathbf{d}) \geqslant 3$ and 0 otherwise.

Finally, when $\Delta(\mathbf{d}) < -2$ we are in case (1) or (2) of Proposition 6.1. The proposition tells us that the recursion terminates on case (a) or (b) of Theorem 1.4. By Proposition 5.3, the recursion terminates on case (a) if $R < 0$ and on case (b) if $R = 0$. Combining this with Theorem 1.4, we see that $\mathbb{P}(V)//G$ is a single point if $\Delta(\mathbf{d}) < -2$ and $R = 0$ and is empty if $\Delta(\mathbf{d}) < -2$ and $R < 0$, as desired. □

7. Examples

To conclude, we describe a few explicit results implied by Theorems 1.2 and 1.4. For a more detailed discussion of examples, see [3].

Corollary 7.1. *For dimension vectors $(2, d_2, d_3)$, with $2 \leqslant d_2 \leqslant d_3$, the quotient $\mathbb{P}(V)//G$ is non-empty if and only if*

(i) $(d_1, d_2, d_3) = (2, b, b)$ for $b \geqslant 2$ or

(ii) $(d_1, d_2, d_3) = (2, kb, (k+1)b)$ *for positive integers* k, b *with* $kb > 1$.

In case (i), the quotient $\mathbb{P}(V)//G$ *is of dimension* $\max(b-3, 0)$. *In case (ii),* $\mathbb{P}(V)//G$ *is a single point.*

Proof. First assume that (d_1, d_2, d_3) is as in (i) and (ii). Note that $\Delta(2, d_2, d_3) = -(d_3 - d_2)^2 - 2$. In particular, in case (i), $\Delta(d_1, d_2, d_3) = -2$, and the desired conclusion follows from Proposition 6.1(3). In case (ii), the recursive procedure of Theorem 1.4 yields

$$(2, kb, (k+1)b) \mapsto (2, (k-1)b, kb) \mapsto \cdots \mapsto (2, 2b, 3b) \mapsto (2, b, 2b). \tag{14}$$

The terminal triple $(2, b, 2b)$ is covered by Theorem 1.4(b), for any $b \geqslant 1$, except that for $b = 1$, we should write it as $(1, 2, 2)$, rather than $(2, 1, 2)$. (Note that we can also check directly that $R(\mathbf{d}) = b^2 - 4 + \gcd(2, b)^2 > 0$ in case (i) and $R(\mathbf{d}) = 0$ in case (ii).)

Conversely, suppose $\mathbb{P}(V)//G$ is non-empty for some dimension vector $(2, d_2, d_3)$. Denote the terminal triple by $\mathbf{e} = (e_1, e_2, e_3)$. Then either $e_1 = 1$ and $e_2 = 2$, or $e_1 = 2$. Moreover, either $e_1 e_2 = e_3$, as in Theorem 1.4(b) or $e_3 \leqslant e_1 e_2/2$, as in Theorem 1.4(d). This leaves us with

(1) $(e_1, e_2, e_3) = (2, b, b)$ or

(2) $(e_1, e_2, e_3) = (2, b, 2b)$, where $b \geqslant 2$ or

(3) $(e_1, e_2, e_3) = (1, 2, 2)$.

In cases (2) and (3), we recover $(d_1, d_2, d_3) = (2, kb, (k+1)b)$, for $b \geqslant 2$ and $b = 1$, respectively, by reversing the recursive procedure (14). □

Remark 7.2. For $\{n = 3, d_1 > 2\}$ or $n \geqslant 3$, the expected dimension $\Delta(d_1, \ldots d_n)$ considered as a function of d_n is a downwards parabola that is positive for $d_n = d_{n-1}$, increases to a maximum at $d_n = \frac{1}{2} d_1 \cdots d_{n-1}$, and then decreases to -2 at some $d_* \in (P/2, P)$ where $P = d_1 \cdots d_{n-1}$. Thus, by Proposition 6.1, the quotient is non-empty and has dimension $\Delta(d_1, \ldots d_n)$ for all d_n in the range $[d_{n-1}, d_*)$. If d_* is an integer, the quotient is non-empty for $d_n = d_*$ and has dimension governed by case (3) of Proposition 6.1. The remaining values of d_n for which the quotient is non-empty are a set of sporadic cases with $d_* < d_n \leqslant P$ satisfying $R(\mathbf{d}) = 0$ for which the quotient is a point. We provide a more detailed analysis of these sporadic cases for $n = 3$ in [3].

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References

[1] Andreev, E.M., Vinberg, E.B., Elashvili, A.G.: Orbits of highest dimension of semisimple linear Lie groups. Funkcional. Anal. i Priložen. **1**(4), 3–7 (1967). English translation in Functional Anal. Appl. **1**, 257–261 (1967)

[2] Arnaud, L., Cerf, N.J.: Exploring pure quantum states with maximally mixed reductions. Phys. Rev. A **87**, 012319 (2013)

[3] Bryan, J., Leutheusser, S., Reichstein, Z., Van Raamsdonk, M.: Locally maximally entangled states of multipart quantum systems. arXiv:1801.03508

[4] Elashvili, A.G.: Stationary subalgebras of points of general position for irreducible linear Lie groups. Funkcional. Anal. i Priložen. **6**(2), 65–78 (1972). English translation in Functional Anal. Appl. **6**, 139–148 (1972)

[5] Facchi, P., Florio, G., Parisi, G., Pascazio, S.: Maximally multipartite entangled states. Phys. Rev. A **77**(6), 060304 (2008)

[6] Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V.: Discriminants, Resultants, and Multidimensional Determinants. Mathematics: Theory and Applications. Birkhäuser Boston, Inc., Boston (1994)

[7] Gour, G., Wallach, N.R.: Classification of multipartite entanglement of all finite dimensionality. Phys. Rev. Lett. **111**, 060502 (2013)

[8] Goyeneche, D., Bielawski, J., Zyczkowski, K.: Multipartite entanglement in heterogeneous systems. Phys. Rev. A **94**, 012346 (2016)

[9] Hoskins, V.: Geometric Invariant Theory and Symplectic Quotients. Lecture Notes (2012). http://userpage.fu-berlin.de/hoskins/GITnotes.pdf. Accessed May 2017

[10] Klyachko, A.A.: Coherent states, entanglement, and geometric invariant theory (2002). arXiv:quant-ph/0206012

[11] Klyachko, A.A.: Quantum marginal problem and representations of the symmetric group (2004). arXiv:quant-ph/0409113

[12] Klyachko, A.A.: Dynamical symmetry approach to entanglement. In: Physics and Theoretical Computer Science, Volume 7 of NATO Secur. Sci. Ser. D Inf. Commun. Secur., pp. 25–54. IOS, Amsterdam (2007). arXiv:0802.4008v1

[13] Li, J.-L., Qiao, C.-F.: Classification of the entangled states $2 \times M \times N$. Quantum Inf. Process. **12**(1), 251–268 (2013)

[14] Littelmann, P.: Koreguläre und äquidimensionale Darstellungen. J. Algebra **123**(1), 193–222 (1989)

[15] Miyake, A., Verstraete, F.: Multipartite entanglement in $2 \times 2 \times n$ quantum systems. Phys. Rev. A **69**, 012101 (2004)

[16] Mumford, D., Fogarty, J., Kirwan, F.: Geometric Invariant Theory, Third edition, Ergebnisse der Mathematik und ihrer Grenzgebiete (2), **34**, Springer, Berlin (1994)

[17] Popov, V.L.: Criteria for the stability of the action of a semisimple group on the factorial of a manifold. Izv. Akad. Nauk SSSR Ser. Mat. **34**, 523–531 (1970)

[18] Popov, V.L., Vinberg, E.B.: Invariant theory. In: Algebraic Geometry, IV, Encyclopaedia of Mathematical Sciences, **55**, Springer, Berlin, pp. 123–284 (1994)

[19] Reid, M.: Graded Rings and Varieties in Projective Space. http://homepages.warwick.ac.uk/~masda/surf/more/grad.ps. Accessed May 2017

[20] Rosenlicht, M.: Some basic theorems on algebraic groups. Am. J. Math. **78**, 401–443 (1956)

[21] Richardson Jr., R.W.: Principal orbit types for algebraic transformation spaces in characteristic zero. Invent. Math. **16**, 6–14 (1972)

[22] Sato, M., Kimura, T.: A classification of irreducible prehomogeneous vector spaces and their relative invariants. Nagoya Math. J. **65**, 1–155 (1977)

[23] Scott, A.J.: Multipartite entanglement, quantum-error-correcting codes, and entangling power of quantum evolutions. Phys. Rev. A **69**, 052330 (2004)

[24] Wallach, N.R.: Quantum computing and entanglement for mathematicians. In: Representation Theory and Complex Analysis, Lectures Given at the C.I.M.E. Summer School held in Venice, Italy, June 10–17, 2004, pp. 345–376, Lecture Notes in Mathematics, 1931. Springer, Berlin (2008)

[25] Walter, M.: Multipartite Quantum States and their Marginals. Ph.D. Thesis, ETH Zurich (2014). arXiv:1410.6820

[26] Wang, S., Lu, Y., Long, G.-L.: Entanglement classification of $2 \times 2 \times 2 \times d$ quantum systems via the ranks of the multiple coefficient matrices. Phys. Rev. A **87**, 062305 (2013)

[27] Yu, C.-S., Zhou, L., Song, H.-S.: Genuine tripartite entanglement monotone of $(2 \otimes 2 \otimes n)$-dimensional systems. Phys. Rev. A **77**, 022313 (2008)

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