SYMPLECTIC GEOMETRY AND THE RELATIVE DONALDSON INVARIANTS OF $\overline{\mathbb{CP}}^2$

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Abstract. In this paper we construct an explicit differential form representative for $\mu(E) \in H^2(\mathcal{M}^0(\overline{\mathbb{CP}}^2))$ where $\mathcal{M}^0(\overline{\mathbb{CP}}^2)$ is the moduli space of framed instantons on $\overline{\mathbb{CP}}^2$ and $\mu : H_2(\overline{\mathbb{CP}}^2) \cong \mathbb{Z}[E] \to H^2(\mathcal{M}^0(\overline{\mathbb{CP}}^2))$ is Donaldson's μ-map. We extend this representative to an $SO(3)$-equivariant class and use Taubes' framework for Donaldson-Floer theory to express the relative Donaldson invariants of $\overline{\mathbb{CP}}^2$ in terms of explicit integrals on the monad construction of the moduli spaces. We prove a localization result to reduce the integrals to integrals of certain natural differential forms over the jumping line divisor in $\mathcal{M}^0(\overline{\mathbb{CP}}^2)$.

1. Introduction

Not long after Donaldson defined his celebrated polynomial invariants for smooth 4-manifolds, it was realized that the invariants are intimately connected with Floer's instanton homology groups for homology 3-spheres. Suppose that X can be written as the union along a homology 3-sphere Y of two 4-manifolds with respective boundaries Y and $\overline{Y}$ (where the bar indicates opposite orientation):

$$X = X_1 \coprod_Y X_2, \quad \partial X_1 = Y \quad \text{and} \quad \partial X_2 = \overline{Y}.$$

One can then define "relative" Donaldson invariants for X_1 and X_2 that take values in the Floer Homology of the respective boundaries. The Donaldson invariants of X are then recovered from the relative ones by using the natural pairing between $HF_*(Y)$ and $HF_*(\overline{Y})$. This suggests that Floer and Donaldson theory fit together into a "Topological Quantum Field Theory" (see [At]).

To complete the TQFT framework requires deciding what "Floer homology" and "relative Donaldson invariants" should mean in the general setting. Taubes has recently given a description of a general framework [Ta1], [Ta2] and Donaldson has been carrying out a similar program concurrently (see [Br-Do]). Broadly speaking, the difficulties fall into three categories. The first is dealing with the phenomenon that for arbitrary 3-manifolds the moduli space of flat connections can be very complicated. The second problem is dealing with homology classes of X that do not decompose into classes from X_1 and X_2 but involve $H_1(Y)$. The third problem is dealing with the possibility of reducible ASD connections on X_1 or X_2. This paper is concerned with the latter problem.

To examine the Donaldson invariants of a connected sum one tries to express anti-self-dual connections in terms of ASD connections on each constituent. When a constituent admits a reducible ASD connection, complications occur and consequently, Donaldson's vanishing theorem does not hold. The most basic example of this phenomenon is $X \# \overline{\mathbb{CP}}^2$, the connected sum of a 4-manifold X with $\overline{\mathbb{CP}}^2$. (I denote by $\overline{\mathbb{CP}}^2$ the complex projective plane equipped with an orientation opposite to the canonical one induced by the complex structure.) In particular, if X is a smooth algebraic surface, then $X \# \overline{\mathbb{CP}}^2$ is diffeomorphic to the blow-up of X at a point. The problem that motivated the work in this paper is to calculate the invariants for $X \# \overline{\mathbb{CP}}^2$ in terms of the invariants for X.

This problem has generated work by a number of authors and subsequent to the work in this paper, complete solutions to the problem have been obtained by Fintushel and Stern [Fi-St] and by Morgan and Ozsvath, each by methods quite different from ours.

The Donaldson invariant is a degree d homogeneous polynomial function $q_d(X)$ on $H_2(X) \oplus H_0(X)$ where generators of $H_2(X)$ have degree 1 and the generator $\wp \in H_0(X)$ has degree 2. Here we've adopted the recent practice of indexing the invariants by their degree rather than by the instanton number of the relevant moduli space (c.f. [Kr-Mr]).

We have $H_2(X \# \overline{\mathbb{CP}}^2) \cong H_2(X) \oplus \mathbb{Z}E$ where the class E is generated by the hyperplane $\mathbb{CP}^1 \subset \overline{\mathbb{CP}}^2$. The invariants are invariant under orientation-preserving diffeomorphisms and $X \# \overline{\mathbb{CP}}^2$ admits an orientation-preserving diffeomorphism that sends $E \mapsto -E$ and so $q_d(X \# \overline{\mathbb{CP}}^2)$ is an even function of E. One could thus hope for a formula of the form:

(1)
$$q_d(X \# \overline{\mathbb{CP}}^2)(x_1, \ldots, x_{d-2m}, \overbrace{E, \ldots, E}^{2m \text{ times}}) = \sum_{k=0}^{[\frac{m}{2}]} \alpha_{k,m} \cdot q_{d-4k}(X)(x_1, \ldots, x_{d-2m}, \overbrace{\wp, \ldots, \wp}^{m-2k})$$

where the $\alpha_{k,m}$'s are independent of d. (*c.f.* [Fr-Mo]).

Taubes proves that such a formula does exist and that the $\alpha_{k,m}$'s are universal numbers that have an interpretation as coefficients in the relative invariants of $\overline{\mathbb{CP}}^2$. We will henceforth call the $\alpha_{k,m}$'s the "blow-up coefficients" and equation 1 will be called the "blow-up formula".

The purpose of this paper is to use differential forms to express the blow-up coefficients as explicit integrals and compute the integrals directly in some cases. The plan is to realize Taubes' abstract framework by explicit constructions. We heavily incorporate the work of Cliff Taubes and Alistair King. Taubes' framework for Donaldson-Floer theory applied to the special case of $X \# \overline{\mathbb{CP}}^2$ provides the starting point; it is his definition of the relative Donaldson invariants of $\overline{\mathbb{CP}}^2$ that motivates the constructions and calculations herein. King's construction of the $\overline{\mathbb{CP}}^2$ moduli spaces provides the geometric setting in which we work.

The paper is organized as follows: Section 2 describes Taubes framework in our case and describes the features of his framework that motivate our constructions. Section 3 is concerned with some general constructions in symplectic geometry. It has three subsections. The theorems in the first subsection are essentially standard material but are included for completeness. The second subsection contains the constructions leading to our localization result. The third subsection incorporates theresults into the framework of equivariant DeRham theory. Section 4 has two subsections. The first subsection recalls the relevant parts of King's construction. The second subsection contains our theorem relating the Dirac line bundle to King's construction and it describes the construction of our representative for $\mu(E)$. The final section combines the work in the previous sections to find explicit equivariant representatives for $\mu_{SO(3)}(E)$ and to express their push-forwards as integrals. We also compute these integrals for $k = 1$ and discuss methods for higher k.

I would like to thank my advisor, Cliff Taubes for suggesting the topic of the relative invariants of $\overline{\mathbb{CP}}^2$ and for being helpful and supportive throughout graduate school. I would like to thank Alistair King for sending me his well written and useful disertation, and I would like to thank Richard Wentworth for helpful discussions and comments on this manuscript.

2. Taubes' Framework for the Donaldson-Floer Theory of $X \# \overline{\mathbb{CP}}^2$

Recall that the Donaldson invariants $q_d(M)$ are obtained by evaluating the various cup products

$$\mu(x_1) \cup \ldots \cup \mu(x_{d-2l}) \cup (\mu(\wp))^l \in H^{2d}(\mathcal{B}_M)$$

on the class[1]

$$[\mathcal{M}_k(M)] \in H_{2d}(\mathcal{B}_M), \tag{2}$$

where $\mu : H_*(M) \to H^{4-*}(\mathcal{B}_M)$ is Donaldson's μ-map.

Taubes has explicitly constructed differential form representatives for $\mu(x_i)$ [Ta2] with nice support properties so that the invariants can also be expressed as

$$q_d(M)(x_1, \ldots, x_d) = \int_{\mathcal{M}(X)} \mu(x_1) \wedge \cdots \wedge \mu(x_d).$$

[1] The moduli space $\mathcal{M}_k(M)$ is not compact so one typically compactifies the space and extends the cup product to the compactification (*c.f.* [Fr-Mo]). Alternatively, one can arrange so that the cup product has compact support when restricted to $\mathcal{M}_k(M)$ [Do-Kr].

Taubes' approach to relative invariants is to express the moduli space of $X_1 \coprod_Y X_2$ as a fiber product of a space $\mathcal{M}_1$ coming from X_1 with a space $\mathcal{M}_2$ coming from X_2 over a space $\mathcal{Z}$ coming from Y. The general strategy is to then evaluate the above integral by integrating first along the fibers of $\mathcal{M}_1 \to \mathcal{Z}$ and then pulling back by the map $\mathcal{M}_2 \to \mathcal{Z}$ and integrating over $\mathcal{M}_2$. Depending on the context, he then is able to interpret the intermediate steps as defining relative invariants that take their values in an appropriate cohomology theory depending only on $\mathcal{Z}$ and hence only on Y. For details see [Ta1] and [Ta2].

The goal of this section is to describe the outcome of such a description in the case of the blow-up formulas, *i.e.* $X_1 \cong X - \text{pt.}$, $X_2 = \overline{\mathbb{CP}}^2 - \text{pt.}$, and $Y = S^3$. To elucidate his differential form description we first make an imprecise but purely topological interpretation of the relative invariants and pairing.

By choosing a base point $x_0 \in X$ and a trivialization of the fiber $P|_{x_0}$ one gets the based moduli space $\mathcal{M}^0_k(X)$ of ASD connections modulo gauge transformations preserving $P|_{x_0}$. $SO(3)$ acts on $\mathcal{M}^0_k(X)$ and the quotient by the action is $\mathcal{M}_k(X)$. We must assume that $b^2_+(X) > 1$ in order for the Donaldson invariants of $X \# \overline{\mathbb{CP}}^2$ to be defined and so for a generic metric, $P_k \to X$ does not admit any reducible ASD connections. The $SO(3)$ action is consequently free and determines a principal $SO(3)$ fibration

$$\pi : \mathcal{M}^0_k(X) \to \mathcal{M}_k(X).$$

The class $\mu(\wp) \in H^4(\mathcal{M}_k(X))$ is the first Pontryagin class of this fibration. For $\mathcal{M}^0_k(\overline{\mathbb{CP}}^2)$ the action is not always free and will have a non-trivial stabilizer at reducible ASD connections. The class $\mu(e) \in H^2(\mathcal{M}^0_k(\overline{\mathbb{CP}}^2))$ is equivariant, *i.e.* it lifts to a class

$$\mu^{SO(3)}(e) \in H^2(\mathcal{M}^{SO(3)}_k(\overline{\mathbb{CP}}^2))$$

where

$$\mathcal{M}^{SO(3)}_k(\overline{\mathbb{CP}}^2) = (\mathcal{M}^0_k(\overline{\mathbb{CP}}^2)) \times_{SO(3)} ESO(3).$$

Consider $\mathcal{M}^{SO(3)}(\overline{\mathbb{CP}}^2) = \bigcup_k \mathcal{M}^{SO(3)}_k(\overline{\mathbb{CP}}^2)$. We then have the following maps:

$$\mathcal{M}^{SO(3)}(\overline{\mathbb{CP}}^2) \xrightarrow{\pi} BSO(3) \xleftarrow{f} \mathcal{M}(X), \tag{3}$$

π is the projection map onto the second factor and has fiber $\cup_k \mathcal{M}^0_k(\overline{\mathbb{CP}}^2)$. The map f is a representative in the homotopy class of maps from $\mathcal{M}_k(X)$ uniquely determined by the $SO(3)$ bundle $\mathcal{M}^0(X) \to \mathcal{M}(X)$. The real cohomology ring of $BSO(3)$ is a polynomial ring on one generator $p_1$of dimension 4:

$$H^*(BSO(3), \mathbb{R}) \cong \mathbb{R}[p_1],$$

and furthermore, $f^*(p_1) = \mu(\wp)$. The analogy with Taubes' formalism would then dictate that we consider the push-forward by π of powers of the equivariant μ-class which we can express then as a polynomial in p_1:

$$\pi_*(\overbrace{\mu^{SO(3)}(e) \cup \cdots \cup \mu^{SO(3)}(e)}^{2m \text{ times}}) = \sum_{k=0}^{[\frac{m}{2}]} \beta_{k,m} p_1^{m-2k}. \tag{4}$$

Note that the fibers of π have a component of dimension $8k$ for each k thus accounting for the inhomogeneity of the right hand side of equation 4. The problem is that the fibers are not compact and so the pushforward π_* is not defined.

This is handled on the level of differential forms by showing that the differential form representatives have good compactness properties and the push forward is well defined by integration.

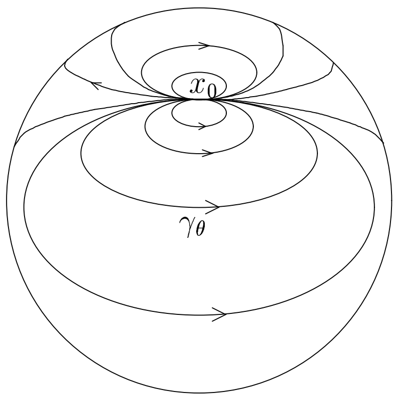
If we suppose that π_* does exist, the analogous topological statement to Taubes' theorem is

$$\begin{aligned} (5) \quad & q_d(X \# \overline{\mathbb{CP}}^2)(x_1, \ldots, x_{d-2m}, \overbrace{e, \ldots, e}^{2m \text{ times}}) \\ = & \langle f^* \pi_*(\overbrace{\mu^{SO(3)}(e) \cup \cdots \cup \mu^{SO(3)}(e)}^{2m \text{ times}}) \cup \mu(x_1) \cup \cdots \cup \mu(x_{d-2m}), \mathcal{M}(X)\rangle \\ = & \sum_{k=0}^{[\frac{m}{2}]} \beta_{k,m} \cdot q_{d-4k}(x_1, \ldots, x_{d-2m}, \overbrace{\wp, \ldots, \wp}^{m-2k \text{ times}}) \end{aligned}$$

so the $\beta_{k,m}$ are the blow-up coefficients $\alpha_{k,m}$.

In this case $H^*(BSO(3))$ is the "Floer cohomology" and equation 4 defines the "relative Donaldson invariant" for $\overline{\mathbb{CP}}^2$. Furthermore, equation 5 gives the pairing that recovers the Donaldson invariants of the connected sum.

The forementioned form representatives for $\mu(x)$ are Taubes' holonomy forms and we give a brief recount of the definition in the case when the homology class x is represented by a embedded 2-sphere $S \subset M^4$. Let $\mathcal{M}^0(M^4)$ be the moduli space of based instantons on a 4-manifold M^4 and assume that the base point x_0 lies on S. There is an obvious family of loops γ_θ based at x_0, filling out S and parameterized by S^1, so that the loops are mutually disjoint, except for the base point.



Consider the map

$$H : S^1 \times \mathcal{M}^0(M^4) \to SU(2)$$

defined by $H(\theta, A) =$ holonomy of the connection A around the loop γ_θ. Now choose an Ad-invariant 3-form Ω on $SU(2)$ that is in the class $[1] \in H^3_{DR}(SU(2))$. Let $\pi : S^1 \times \mathcal{M}^0(M^4) \to \mathcal{M}^0(M^4)$ be projection and define $\nu \in H^2_{DR}(\mathcal{M}^0(M^4))$ to be

$$\nu = \pi_*(H^*(\Omega))$$

where π_* denotes integration over S^1. The form ν is the holonomy representative for $\mu(x)$.

$SO(3)$ acts on $\mathcal{M}^0(M^4)$ and we denote by $p, m : SO(3) \times \mathcal{M}^0(M^4) \to \mathcal{M}^0(M^4)$ the projection and the multiplication maps respectively. Taubes' shows that the form ν satisfies the equation

$$m^*(\nu) - p^*(\nu) = db \tag{6}$$

where $b \in \Omega^0(M^4) \otimes \mathfrak{so}(3)^* \subset \Omega^1(M^4 \times SO(3))$. If τ_i denotes the invariant 1-form on $SO(3)$ dual to the element a_i in a basis for $\mathfrak{so}(3)$, then we can expand $b = \sum b_i \tau_i$.

As explained in section 3.3, there are models for equivariant cohomology built out of differential forms and the Lie algebra of the group. The extension of ν to equivariant cohomology can be constructed using b. Specifically,

$$\nu^{\#} = \nu - \sum_i b_i \otimes a_i^* \in (\Omega^*(\mathcal{M}^0(M^4) \otimes S^*(\mathfrak{so}(3)^*))^{SO(3)}$$

defines an element of the Cartan complex (see section 3.3). The theorem that we take as our starting point can then be rigorously stated:

Theorem 2.1 (Taubes). *Let $M^4 = \overline{\mathbb{CP}}^2$ and $S = E \subset \overline{\mathbb{CP}}^2$ and construct ν and $\nu^{\#}$ as above. Then the map*

$$\pi_* : (\Omega^*(\mathcal{M}^0(M^4) \otimes S^*(\mathfrak{so}(3)^*))^{SO(3)} \to (S^*(\mathfrak{so}(3)^*))^{SO(3)} \cong H^*_{SO(3)}(pt.)$$

is well defined on powers of $\nu^{\#}$ by integration and has the significance that

$$\pi_*((\nu^{\#})^{2m}) = \sum_{k=0}^{[\frac{m}{2}]} \alpha_{k,m} p_1^{m-2k}$$

where the $\alpha_{k,m}$'s are the blow-up coefficients.

The holonomy forms have a number of good cohomological properties and so although the theorem is stated only for these forms, there should be a class of forms sharing similar properties for which this theorem holds. We list important properties of the holonomy forms:

(1) There are holonomy forms that are concentrated in a neighborhood of the divisor representative (in the sense of [Do-Kr]) of $\mu(E)$. By choosing Ω to have support near $-I \in SU(2)$, Taubes show that the corresponding holonomy form has support in a neighborhood of Donaldson's divisor.

(2) The forms have an extension to $SO(3)$-equivariant cohomology in the sense that they obey equation 6 .
(3) The forms behave well at the boundary. Roughly speaking, the holonomy forms near the boundary are approximated by the pull backs of forms from manifolds of dimension at most $\dim \mathcal{M}^0 - 4$.

It is reasonable to conjecture that there is a subcomplex of $\Omega^*(\mathcal{M}^0(\overline{\mathbb{CP}}^2))$ satisfying the above properties in some rigorous sense, that defines a cohomology theory for which theorem 2.1 is true on a cohomological level.

In the subsequent sections of this paper, we will construct an explicit form satisfying all the listed conditions and will thus conjecturally compute the blow-up coefficients.

3. Symplectic Geometry and Non-compact Kähler Manifolds

In this section we are concerned with certain Kähler manifolds with group actions. In subsection 3.1 we will describe the category of manifolds with group actions that we consider. We then discuss quotients by reductive groups and their relationship to symplectic reductions by the associated compact groups. In subsection 3.2 we develop a technique for altering the symplectic form on such manifolds without changing the topology of their resulting symplectic reductions. The technique allows us to "concentrate" the form near certain "divisors" and we prove a kind of localization theorem for certain integrals over these manifolds. In subsection 3.3 we recall the Cartan model for equivariant deRham cohomology and we rephrase the localization result of the previous section in terms of a push forward in equivariant cohomology.

3.1. Quotients and Reduction.

Definition 3.1. *We say that (M, L) is a* quantized Kähler manifold *if M is a (not necessarily compact) Kähler manifold with a holomorphic, hermitian line bundle $\pi : L \to M$ whose curvature defines the Kähler form.*

Explicitly, the hermitian structure defines a function $h : L \to \mathbb{R}^+$ by $h(\xi_p) = ||\xi_p||_p^2$ where $\xi_p \in \pi^{-1}(p)$ and $||\cdot||_p$ is the hermitian norm. Then let

$$\alpha = \frac{1}{4\pi i}(\partial - \bar{\partial})(\log(h)) = d^c \log(h).$$

α is a 1-form on $L - \{0\}$ and defines the canonical S^1 connection: if $\tilde{v}$ is the vector field generating the infinitesimal action of S^1 on L given by multiplication by $e^{2\pi i v}$ fiberwise, then $i_{\tilde{v}}\alpha = 1$ and $\mathcal{L}_{\tilde{v}}\alpha = 0$. Then $d\alpha$ is basic and defines the curvature form $\omega \in \Omega^{1,1}(M)$ such that

$$\pi^*(\omega) = d\alpha = \frac{1}{2\pi i}\bar{\partial}\partial \log h = dd^c \log h.$$

ω is the Kähler form by definition.

Remark 3.2. If M is compact, then 3.1 means M is a *Hodge manifold* which in turn, by the Kodaira embedding theorem, implies that M is a complex projective variety. Some sources in the literature do not require compactness for the definition of a Hodge manifold in which case such a structure is identical to 3.1. We use the term quantized Kähler manifold to avoid any potential ambiguity.

Let G be a compact Lie group and $G_{\mathbb{C}}$ be its complexification. We identify $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ where $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}$ are the Lie algebras of $G_{\mathbb{C}}$ and G respectively. We consider actions of $G_{\mathbb{C}}$ on M of the following type:

Assumption 3.3. *The action of $G_{\mathbb{C}}$ on M satisfies:*

(1) *$G_{\mathbb{C}} \times M \to M$ is holomorphic.*
(2) *The induced action $G \times M \to M$ acts by isometries with respect to the Kähler metric,* or
(3) *G acts symplectically with respect to ω.*

It follows from the definitions that (1) and (2) are equivalent to (1) and (3).

Throughout the rest of this section we assume that M is a quantized Kähler manifold and with a $G_{\mathbb{C}}$-action satisfying assumption 3.3.

Definition 3.4. *A* linearization *of the $G_{\mathbb{C}}$ action on M is a holomorphic action of $G_{\mathbb{C}}$ on L covering the action on M, and such that G acts unitarily on the fibers.*

We generally use the notation that if $a \in \mathfrak{g}$ then $\tilde{a} \in \mathrm{Vect}(M)$ is the vector field generated by the infinitesimal action of a and $\tilde{\tilde{a}} \in \mathrm{Vect}(L)$ is the vector field generated by its lift. Recall that $\tilde{}$ is a anti-homomorphism of Lie algebras so that $\widetilde{[a,b]} = -[\tilde{a}, \tilde{b}]$ (see [Ab-Ma], *e.g.*).

For any symplectic manifold (X, Ω) with a symplectic action of a compact group H we have the notion of a moment map. Let $\tilde{a}$ denote the vector field on X generated by the infinitesimal action of $a \in \mathfrak{h}$.

Definition 3.5. *An* Ad*-equivariant moment map *is a map $f : X \to \mathfrak{h}^*$ such that $df_a = i_{\tilde{a}}\Omega$ where f_a is the composition $X \xrightarrow{f} \mathfrak{h}^* \xrightarrow{\langle a, \cdot \rangle} \mathbb{R}$. Ad*-equivariance is the requirement that $f_a(g \cdot x) = f_{g^{-1}ag}(x)$.*

If the exponential map is onto then Ad*-equivariance is equivalent to the local condition $f_{[a,b]} = -\mathcal{L}_{\tilde{a}} f_b$. Unless otherwise stated we will always assume Ad*-equivariance and will just say *moment map*. Moment maps in general may not exist and if one does it may not be unique. In the case at hand though, we have the following:

Proposition 3.6. *A choice of a linearization uniquely determines a moment map and conversely a choice of a moment map uniquely determines a linearization.*

Proof: Assume we have a linearization. Then for $a \in \mathfrak{g}$ we have the generating vector field $\tilde{\tilde{a}}$ on L and $\pi_* \tilde{\tilde{a}} = \tilde{a}$. Furthermore, since the actions of S^1 and G on L commute, we have $[\tilde{\tilde{a}}, \tilde{v}] = 0$. Hence

$$\mathcal{L}_{\tilde{v}}(i_{\tilde{\tilde{a}}}\alpha) = i_{\tilde{\tilde{a}}}\mathcal{L}_{\tilde{v}}\alpha + i_{[\tilde{v},\tilde{\tilde{a}}]}\alpha = 0$$

and so the function $-i_{\tilde{\tilde{a}}}\alpha$ is basic. Define f componentwise by the equation $\pi^*(f_a) = -i_{\tilde{\tilde{a}}}\alpha$. Then

$$\begin{aligned} \pi^*(i_{\tilde{a}}\omega) &= \pi^*(i_{\pi_*\tilde{\tilde{a}}}\omega) = i_{\tilde{\tilde{a}}}(\pi^*\omega) = i_{\tilde{\tilde{a}}}d\alpha \\ &= -di_{\tilde{\tilde{a}}}\alpha + \mathcal{L}_{\tilde{\tilde{a}}}\alpha = \pi^*(df_a) + \mathcal{L}_{\tilde{\tilde{a}}}\alpha = \pi^*(df_a) \end{aligned}$$

where $\mathcal{L}_{\tilde{\tilde{a}}}\alpha = 0$ since G acts unitarily on the fibers. Thus f satisfies $df_a = i_{\tilde{a}}\omega$. To check equivariance we calculate

$$\begin{aligned} \pi^*(f_{[a,b]}) &= -i_{\widetilde{\widetilde{[a,b]}}}\alpha = -i_{-[\tilde{\tilde{a}},\tilde{\tilde{b}}]}\alpha \\ &= \mathcal{L}_{\tilde{\tilde{a}}}i_{\tilde{\tilde{b}}}\alpha - i_{\tilde{\tilde{b}}}\mathcal{L}_{\tilde{\tilde{a}}}\alpha \\ &= -\mathcal{L}_{\tilde{\tilde{a}}}\pi^*(f_b) = -\pi^*(\mathcal{L}_{\tilde{a}}f_b). \end{aligned}$$

When $A \in \mathfrak{g}$ then $\overline{f_a} = f_a$ and so f_a is real and f defines a moment map.

Now assume that we have a moment map f. We define a lift $\tilde{\tilde{a}}$ for $A \in \mathfrak{g}$ by

$$\tilde{\tilde{a}} = \hbar(\tilde{a}) - \pi^*(f_a)\tilde{v}$$

where $\hbar(\tilde{a})$ is the horizontal lift of $\tilde{a}$ defined by the connection α. To show that this extends to an action we need to know that $\tilde{\tilde{}}$ defines an anti-homomorphism of Lie algebras

$$\tilde{\tilde{}}: \mathfrak{g} \to \mathrm{Vect}(L).$$

Equivalently, we need

Claim 3.7. $[\widetilde{\tilde{a},\tilde{b}}] = [\tilde{\tilde{a}},\tilde{\tilde{b}}]$,

The lift $\tilde{\tilde{}}$ then extends to $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ by

$$\widetilde{\widetilde{(ia)}} = \hbar(\widetilde{ia}) - \pi^*(f_a)\widetilde{iv} = J_*(\tilde{\tilde{a}})$$

where $-\widetilde{iv} = J_*\tilde{v}$ generates multiplication by $e^{-2\pi v}$ fiberwise and J defines the complex structure. The vector fields $\widetilde{\widetilde{ia}}$ are complete, and so the infinitesimal action given by $\tilde{\tilde{}}$ generates a global action and hence a linearization.

To prove (3.7) is a straight forward calculation.

The following basic example exhibits the main features of the more complicated examples arising from the $\overline{\mathbb{CP}}^2$ moduli spaces.

Example 3.8. Let M be $\mathrm{End}(W)$ the space of all linear endomorphisms of a hermitian vector space W of dimension n. Let $L \to M$ be the product line bundle and define a hermitian structure on L by

$$h(x, \xi) = |\xi|^2 e^{\mathrm{tr}(xx^*)}$$

If we fix a unitary basis of W then there is a natural global coordinate system given by matrix elements which we write x_{ij}. Then

$$\alpha = \frac{1}{4\pi i}\left(\frac{d\xi}{\xi} - \frac{d\overline{\xi}}{\overline{\xi}} + \mathrm{tr}(dxx^* - xdx^*)\right)$$

$$\omega = \pi^*(d\alpha) = \frac{1}{2\pi i}\,\mathrm{tr}(dx^* dx) = \frac{i}{2\pi}\sum_{i,j} dx_{ij}d\overline{x}_{ij}$$

so X is isomorphic to $\mathbb{C}^{n^2}$ with the Euclidian Kähler structure. X is itself a hermitian vector space with inner product defined by $< a, b >= \mathrm{tr}(a^*b)$.

Let $G_{\mathbb{C}} = Gl(W)$ and $G = U(W)$. The action given by left multiplication *i.e.* $(g, x) \mapsto gx$ satisfies assumption 3.3. For any real number t, we can define a linearization by

$$g(x, \xi) = (gx, (\det g)^t \xi).$$

The Lie algebra of $U(W)$ is $\mathfrak{u} = \{a \in \mathrm{End}(W) : a = -a^*\}$ and we calculate:

$$\begin{aligned}
i_{\tilde{a}}dx &= \mathcal{L}_{\tilde{a}}x = \frac{d}{ds}(e^{sa}x)|_{s=0} = ax, \\
i_{\tilde{a}}dx^* &= x^*a^* = -x^*a, \\
i_{\tilde{\tilde{a}}}d\xi &= \frac{d}{ds}(\det(e^{sta})\xi)|_{s=0} = \frac{d}{ds}(e^{\mathrm{tr}(sta)}\xi)|_{s=0} = \mathrm{tr}(ta)\xi, \\
i_{\tilde{\tilde{a}}}d\overline{\xi} &= -\,\mathrm{tr}(ta)\overline{\xi}, \\
f_a &= -i_{\tilde{\tilde{a}}}\alpha = \frac{-1}{4\pi i}\Big\{2\,\mathrm{tr}(ta) + \mathrm{tr}\big(axx^* + xx^*a\big)\Big\} \\
&= \frac{-1}{2\pi i}\,\mathrm{tr}(ta + axx^*) =< a, \frac{1}{2\pi i}(tI + xx^*) >
\end{aligned}$$

so if $t = -1$ for example, then the zero set of the moment map are exactly the unitary matrices.

Alternatively, define the action of $G_{\mathbb{C}}$ on M by conjugation, *i.e.* $(g, x) \mapsto gxg^{-1}$ and define a linearization by lifting the action trivially to the fibers. Then a similar calculation shows that

$$f_a = \frac{-1}{2\pi i}\,\mathrm{tr}(a[x^*, x]) =< a, \tfrac{1}{2\pi i}[x^*, x] >$$

so that the zero set of the moment map are matrices that commute with their conjugate transpose, *i.e.* matrices that can be diagonalized by a unitary automorphism.

We return to the general setting and we now wish to characterize a "nice" subset of M on which the quotient by $G_{\mathbb{C}}$ is well behaved. This subset will consist of the "analytically stable" points.

Definition 3.9. *We say a point $x \in M$ is* analytically stable *for a linearized action of $G_{\mathbb{C}}$ on M if for every non-zero $\xi_x \in L_x$ the function $g \mapsto h(g \cdot \xi_x)$ is proper.*

The following two lemmas give conditions equivalent to analytic stability. The first provides a convenient criterion for identifying analytically stable points. The second will be the key to relating the quotient by $G_{\mathbb{C}}$ to the symplectic reduction by G (see also [Ki]).

Lemma 3.10 (Analytic version of Hilbert's criteria). *A point $x \in M$ is analytically stable with respect to the action of $G_{\mathbb{C}}$ if and only if it is analytically stable with respect to the restricted action of every real 1-parameter subgroup $t \mapsto e^{ita} \in G_{\mathbb{C}}$ where $ita \in i\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$.*

Analytic stability is a property of $G_{\mathbb{C}}$ orbits in M since if x is analytically stable then clearly $g \cdot x$ must be as well.

Lemma 3.11. *An orbit $G_{\mathbb{C}} \cdot x \subset M$ is analytically stable if and only if $G_{\mathbb{C}} \cdot x$ has no infinitesimal stabilizers and $f = 0$ somewhere on $G_{\mathbb{C}} \cdot x$. Furthermore the set $G_{\mathbb{C}} \cdot x \cap f^{-1}(0)$ consists of exactly one G orbit.*

Proof of Lemmas 3.10 and 3.11:

The function $g \mapsto h(g \cdot \xi_x)$ is constant on cosets of G and so descends to a function on $G_{\mathbb{C}}/G \cong \mathfrak{g}$ and is given by $a \mapsto h(e^{ia} \cdot \xi_x)$. Consider the function $S_x : \mathfrak{g} \to \mathbb{R}$ defined by

$$S_x(a) = \frac{h(e^{ia} \cdot \xi_x)}{h(\xi_x)}.$$

S_x does not depend on the choice of ξ_x and is proper if and only if x is analytically stable. Lemma 3.10 is equivalent to the statement that S_x is proper if and only if it is proper on each line through the origin.

For lemma 3.11 assume first that S_x is proper. Then for any $a \in \mathfrak{g}$ the real function defined $q_a(t) = \log(S_x(ta))$ is proper. We calculate that

$$\begin{aligned} q_a'(t) &= \frac{\mathcal{L}_{J_*\tilde{\tilde{a}}}(L^*_{e^{ita}} h)}{L^*_{e^{ita}} h}(\xi_x) \\ &= \Big(\frac{\mathcal{L}_{J_*\tilde{\tilde{a}}} h}{h}\Big)(e^{ita} \cdot \xi_x) \end{aligned}$$

where $L_{e^{ita}}$ is left multiplication by e^{ita} (which preserves $J_*\tilde{\tilde{a}}$). Now by definition of complex structure, $J^*(\partial h) = i\partial h$ and $J^*(\bar\partial h) = -i\bar\partial h$ so we have

$$\begin{aligned}\mathcal{L}_{J_*\tilde{\tilde{a}}}h &= i_{J_*\tilde{\tilde{a}}}dh = -J^* i_{\tilde{\tilde{a}}} J^*(\partial h + \bar\partial h)\\ &= -J^* i_{\tilde{\tilde{a}}}(-4\pi h\alpha) = 4\pi h J^* i_{\tilde{\tilde{a}}}\alpha\\ &= -4\pi h \pi^*(f_a)\end{aligned}$$

and so

$$q_a'(t) = -4\pi f_a(e^{ita}\cdot x) \tag{7}$$

Similarly,

$$\begin{aligned}q_a''(t) &= (-4\pi \mathcal{L}_{J_*\tilde{a}} f_a)|_{e^{ita}\cdot x}\\ &= (-4\pi i_{J_*\tilde{a}} i_{\tilde{a}}\omega)|_{e^{ita}\cdot x}\\ &= 4\pi||\tilde{a}||^2_{e^{ita}\cdot x}. \end{aligned}\tag{8}$$

If $\tilde{a}_x = 0$ then $q_a(t)$ will be constant and not proper, hence x cannot have any infinitesimal stabilizers and q_a is strictly convex. The function e^q is also strictly convex since $(e^q)'' = (q'' + (q')^2)e^q > 0$. This implies that the Hessian of S_x at the origin, $(\mathfrak{H}S_x)(0)$, is positive definite. But $(\mathfrak{H}S_x)(a) = (\mathfrak{H}S_{e^{ia}\cdot x})(0)$ which by the same analysis must also be positive definite. S_x is thus a strictly convex, proper function and such functions have exactly one critical point which is a minimum. By equation 7, such a point determines a unique G-orbit in the $G_{\mathbb{C}}$-orbit of x on which $f \equiv 0$.

Now assume that x has no infinitesimal stabilizers and $G_{\mathbb{C}} \cdot x \cap f^{-1}(0)$ is a single G-orbit. We need to show that S_x is proper. But by the above analysis, S_x has a single critical point and has a semi-definite Hessian everywhere. Hence S_x must be proper. □

We now recall the notion of symplectic reduction (see [HKLR] *e.g.*):

Definition 3.12. *Let (X, Ω) be a symplectic manifold with a free action of a compact Lie group H preserving Ω. Let $f : X \mapsto \mathfrak{h}^*$ be a moment map for the action. The* symplectic reduction *of (X,Ω) by H is the symplectic manifold $(\hat{X}, \hat{\Omega})$ where $\hat{X} = f^{-1}(0)/H$ and $\hat\Omega$ is characterized by the condition $p^*(\hat\Omega) = \Omega|_{f^{-1}(0)}$. Here $p : f^{-1}(0) \mapsto \hat{X}$ is the natural projection.*

Remark 3.13. If one drops the non-degeneracy condition on Ω but still requires that $d\Omega = 0$ then (X, Ω) is a *presymplectic* manifold. The notion of a moment map still makes sense and if $f^{-1}(0)$ is a manifold for which H acts freely (or has a component with this property) then $f^{-1}(0)/H$ is a manifold (or has a component that is a manifold) which we can define to be $\hat{X}$ and there exists a presymplectic form $\hat\Omega$ such that $p^*(\hat\Omega) = \Omega|_{f^{-1}(0)}$.

We now state the relationship between quotients by $G_{\mathbb{C}}$ and reduction by G.

Theorem 3.14. *Let M be a quantized Kähler manifold with $G_{\mathbb{C}}$ acting on M as in assumption* 3.3*. Assume we also have a linearization of the action and let M^{as} be the subset of points that are analytically stable with respect to the linearization. Assume further that for every $x \in M^{as}$ the stabilizer $(G_{\mathbb{C}})_x$ of x is trivial (by lemma* 3.11 *$(G_{\mathbb{C}})_x$ is necessarily discrete). Then*

$$M^{as}/G_{\mathbb{C}} \cong f^{-1}(0) \cap M^{as}/G_{\mathbb{C}}$$

and the bundle $L \to M^{as}$ descends to a hermitian holomorphic bundle $\hat{L} \to M^{as}/G_{\mathbb{C}}$ that gives the quotient the structure of a quantized Kähler manifold.

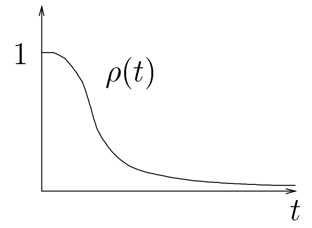
PROOF: Lemma 3.11 implies the diffeomorphism, for the quantized Kähler structure, see [HKLR].

3.2. Localization Techniques for Quantized Kähler Manifolds. There are two techniques developed in this section. The first is for altering the symplectic form on a quantized Kähler manifold with group actions as in section (3.1). We change it in such a way so that the symplectic reduction with respect to the altered form is diffeomorphic to reduction with respect to the original form. In particular, we can concentrate the symplectic volume near the zero set of certain sections of L. The second result uses the first technique to localize powers of the moment map integrated over the symplectic volume to integrals on the zero set of certain sections. We will see the intended application in section 3.3.

Let M be a quantized Kähler manifold. Let $s : M \to L$ be a section of L. Away from $s^{-1}(0)$ the section defines a trivialization ϕ. Let θ be the connection on $L|_{M-s^{-1}(0)}$ pulled back from the trivial connection by ϕ.

Fix a smooth decreasing function $\rho : \mathbb{R}^+ \to [0,1]$ with the following properties:

(1) $\rho(0) = 1$,
(2) $\rho'(0) = 0$, and
(3) $\rho(t) \to 0$ as $t \to \infty$.



Let $\beta = \rho \circ h \circ s$. We use β to interpolate between α and the flat connection θ. We define

$$\alpha' = \pi^*(\beta)\alpha + (1 - \pi^*(\beta))\theta.$$

α' defines a connection on L since $\mathcal{L}_{\tilde{v}}\alpha' = 0$ and $i_{\tilde{v}}\alpha' = \pi^*(\beta) + (1 - \pi^*(\beta)) = 1$. The curvature of this connection is a 2-form $\omega' \in \Omega^2(M)$ characterized by

$$\pi^*(\omega') = d\alpha' = \pi^*(\beta)d\alpha + \pi^*(d\beta) \wedge (\alpha - \theta).$$

We note that $\alpha - \theta$ is basic and so defines a 1-form $\gamma \in \Omega^1(M - s^{-1}(0))$ such that $\pi^*(\gamma) = \alpha - \theta$. In fact since $\pi \circ s = id$ we have

$$\gamma = s^*\pi^*\gamma = s^*(\alpha - \theta) = s^*(\alpha).$$

We see then that

$$\omega' = \beta w + d\beta \wedge \gamma \tag{9}$$

which is well defined on all of M since $d\beta = 0$ on $s^{-1}(0)$.

Remark 3.15. An alternative point of view presents itself by rewriting equation (9) as $\omega' = d(\beta\gamma) = s^* d((\rho \circ h)\alpha)$. The form $d((\rho \circ h)\alpha)$ is a representative for the Thom class of L as constructed in Bott and Tu (c.f. [Bo-Tu] pgs. 132-133) and the pull-back of the Thom class by any section is the Euler class.

Now let $G_{\mathbb{C}}$ act on M as in assumption (3.3) and assume we have a linearization. We assume that s is compatible with the G-action:

Assumption 3.16. *The section s is G-equivariant,* i.e. *for all $x \in M$,$g \in G$ we have $s(g \cdot x) = g \cdot s(x)$,* i.e. *the following diagram commutes:*

$$\begin{array}{ccc} L & \xrightarrow{\tilde{L}_g} & L \\ {\scriptstyle s}\uparrow & & {\scriptstyle s}\uparrow \\ M & \xrightarrow{L_g} & M \end{array}$$

Where $L_g, \tilde{L}_g$ denote the left action of g on M and L respectively.

The infinitesimal version of the above diagram is

$$\begin{array}{ccc} L & \xrightarrow{\tilde{\tilde{a}}} & T_*L \\ {\scriptstyle s}\uparrow & & {\scriptstyle ds}\uparrow \\ M & \xrightarrow{\tilde{a}} & T_*M \end{array}$$

for $a \in \mathfrak{g}$. From this we get the following useful reformulation of assumption 3.16:

$$i_{\tilde{a}} s^*(\theta) = s^*(i_{\tilde{\tilde{a}}}\theta) \tag{10}$$

where $a \in \mathfrak{g}$ and θ is any form in $\Omega^*(L)$. We note that equation 10 implies $\mathcal{L}_{\tilde{a}} s^*(\theta) = s^*(\mathcal{L}_{\tilde{\tilde{a}}}\theta)$.

Claim 3.17. *G acts symplectically with respect to ω'.*

PROOF: Since G preserves h we have $\mathcal{L}_{\tilde{a}}\beta = 0$ for $a \in \mathfrak{g}$. And so

$$\begin{aligned}\mathcal{L}_{\tilde{a}}\omega' &= \mathcal{L}_{\tilde{a}}(\beta\omega + d\beta \wedge \gamma) \\ &= d\beta \wedge \mathcal{L}_{\tilde{a}}(s^*(\alpha)) \\ &= d\beta \wedge s^*(\mathcal{L}_{\tilde{\tilde{a}}}\alpha) = 0 \quad \square\end{aligned}$$

Recall that the moment map f determined by the linearization is given by the condition (see section 3.1)

$$\pi^*(f_a) = -i_{\tilde{\tilde{a}}}\alpha.$$

By applying s^* to the above equation and using equation (10) we can express f_a directly on M as

$$f_a = -i_{\tilde{a}}\gamma.$$

Claim 3.18. *$f'_a = \beta f_a$ defines a moment map for the action of G with respect to the form ω'.*

PROOF: We have

$$\begin{aligned}i_{\tilde{a}}\omega' &= \beta i_{\tilde{a}}\omega + (i_{\tilde{a}}d\beta) \wedge \gamma - d\beta \wedge i_{\tilde{a}}\gamma \\ &= \beta df_a + \mathcal{L}_{\tilde{a}}\beta \wedge \gamma + d\beta f_a \\ &= d(\beta f_a), \text{ and} \\ \mathcal{L}_{\tilde{b}}f'_a &= \beta\mathcal{L}_{\tilde{b}}f_a = f'_{[a,b]}. \quad \square\end{aligned}$$

If $\beta > 0$ then the zero sets of f and f' coincide and the above claims imply:

Theorem 3.19. *The symplectic reduction of M by G with respect to ω is diffeomorphic to the reduction with respect to ω'.*

Remark 3.20. If we choose ρ to be supported in a compact interval $[0, \epsilon]$, then ω' is not symplectic but it is presymplectic and we will still have the presymplectic versions of claims 3.17 and 3.18 with the same proofs. Then the zero set of f' contains the zero set of f as a component and so theorem 3.19 holds for the presymplectic reduction of this component (c.f. remark 3.13).

The next theorem provides a localization technique for integrating powers of f'_a over the volume form defined by ω'. It is the main result of this section. Notice the explicit dependence on ρ vanishes. It is presently convenient to consider the presymplectic case and it will, in fact, be apt for our later applications.

The theorem will apply when the section is holomorphic and satisfies a certain transversality condition on the zero section. We first introduce some notation.

Definition 3.21. *N_ϵ is defined to be the set $\{x \in M$ such that $h(s(x)) \leq \epsilon\}$.*

Assumption 3.22. *The section s is holomorphic and transverse to the zero section. Furthermore, there exists an open cover $\{U_\alpha\}$ of N_ϵ for some ϵ such that there are holomorphic coordinates $(z_1^\alpha, \ldots, z_n^\alpha)$ on U_α with the property that $s^{-1}(0)|_{U_\alpha}$ is given by $z_n^\alpha = 0$.*

This assumption guarantees that $s^{-1}(0)$ is a complex submanifold and that N_ϵ is a tubular neighborhood with nice local coordinates.

Remark 3.23. An important case is if M is a quasi-affine or quasi-projective variety, *i.e.* $M \subset \overline{M}$ where $\overline{M} - M$ is a subvariety of a variety $\overline{M}$. The assumption will hold if the closure of $s^{-1}(0)$ in $\overline{M}$ intersects $\overline{M} - M$ transversely. We intuitively think of assumption 3.22 as saying $s^{-1}(0)$ is "transverse to infinity".

Theorem 3.24. *Let f' and ω' be defined as above taking ρ to be supported in $[0, \epsilon)$ for ϵ chosen as in assumption 3.22. Then*

$$\int_M (f_a')^k (\omega')^n = -\frac{n}{n+k} \int_{s^{-1}(0)} (f_a)^k \omega^{n-1}$$

whenever the integrals exist.

PROOF: We first derive an identity:

$$\begin{aligned} f_a \omega^n &= -\omega^n (i_{\tilde{a}} \gamma) \\ &= i_{\tilde{a}}(\omega^n) \wedge \gamma \\ &= n(i_{\tilde{a}} \omega) \wedge \omega^{n-1} \wedge \gamma \end{aligned}$$

and so we have

$$f_a \omega^n = n df_a \wedge \omega^{n-1} \wedge \gamma$$

Suppressing the subscript on f we then calculate

$$\begin{aligned} (f')^k (\omega')^n &= \beta^k f^k \Big(\beta^n \omega^n + n\beta^{n-1} d\beta \wedge \omega^{n-1} \wedge \gamma\Big) \\ &= \tfrac{n}{n+k} \beta^k f^k \Big((1 + \tfrac{k}{n})\beta^n \omega^n + (n+k)\beta^{n-1} d\beta \wedge \omega^{n-1} \wedge \gamma\Big) \\ &= \tfrac{n}{n+k} \Big(\beta^{n+k} f^k \omega^n + k\beta^{n+k} f^{k-1} df \wedge \omega^{n-1} \wedge \gamma + f^k d(\beta^{n+k}) \wedge \omega^{n-1} \wedge \gamma\Big) \\ &= \tfrac{n}{n+k} d\Big(\beta^{n+k} f^k \omega^{n-1} \wedge \gamma\Big) \end{aligned}$$

Since $\operatorname{supp}(\beta) \subseteq N_\epsilon$ we have

$$\int_M (f')^k (\omega')^n = \tfrac{n}{n+k} \int_{N_\epsilon} d\Big(\beta^{n+k} f^k \omega^{n-1} \wedge \gamma\Big).$$

If M were compact, our strategy at this point would be to cut out an even smaller neighborhood of the zero section and apply Stokes' theorem. Because of the potential non-compactness of $s^{-1}(0)$ we need to utilize assumption 3.22.

Let $\{V_\alpha\}$ be the open cover with coordinates $(z_1^\alpha, \ldots, z_n^\alpha)$ given by assumption 3.22. Define a cover $\{U_\alpha\}$ of $s^{-1}(0)$ by $U_\alpha = V_\alpha|_{s^{-1}(0)}$ and let $\{\rho_\alpha\}$ be a partition of unity on $s^{-1}(0)$ with compact support subordinate to $\{U_\alpha\}$. Since N_ϵ is a tubular neighborhood and s is transverse, there is a diffeomorphism of N_ϵ to $L|_{s^{-1}(0)}$ (see [Bo-Tu] *e.g.*) and hence a fibration $p : N_\epsilon \to s^{-1}(0)$. $p^*(\rho_\alpha)$ is a partition of unity on $\{p^{-1}(U_\alpha)\}$ and without loss of generality we can assume that $\{p^{-1}(U_\alpha)\}$ lies in the coordinate patch V_α. Then

$$\begin{aligned}\int_M (f')^k(\omega')^n &= \tfrac{n}{n+k}\sum_\alpha \int_{p^{-1}(U_\alpha)} p^*(\rho_\alpha)d\Big(\beta^{n+k}f^k\omega^{n-1}\wedge\gamma\Big)\\ &= \tfrac{n}{n+k}\sum_\alpha \int_{p^{-1}(U_\alpha)} d\Big(p^*(\rho_\alpha)\beta^{n+k}f^k\omega^{n-1}\wedge\gamma\Big) - p^*(d\rho_\alpha)\wedge\beta^{n+k}f^k\omega^{n-1}\wedge\gamma\end{aligned}$$

We examine the second term in the integrand:

$$\begin{aligned}\text{2nd term} &= -\tfrac{n}{n+k}\sum_\alpha \int_{p^{-1}(U_\alpha)} d\rho_\alpha \wedge p_*(\beta^{n+k}f^k\omega^{n-1}\wedge\gamma)\\ &= -\tfrac{n}{n+k}\int_{s^{-1}(0)} (\sum_\alpha d\rho_\alpha)\wedge p_*(\beta^{n+k}f^k\omega^{n-1}\wedge\gamma)\\ &= 0.\end{aligned}$$

To calculate the first term we use the coordinates: L restricted to $p^{-1}(U_\alpha)$ can be trivialized and let ξ be the fiber coordinate. Dropping the index from the notation we can write

$$\begin{aligned}s &= s(z_1,\dots,z_n) = z_n\tilde{s}(z_1,\dots,z_n), \text{ and}\\ h &= h(z_1,\dots,z_n,\xi) = |\xi|^2\tilde{h}(z_1,\dots,z_n)\end{aligned}$$

where $\tilde{s}$ is a non-vanishing holomorphic function and $\tilde{h}$ is a strictly positive real function. $\gamma = s^*(\alpha)$ can then be written

$$\gamma = d^c\log(|s|^2\tilde{h}) = d^c\log|z_n|^2 + d^c\log(\tilde{\tilde{h}})$$

where $\tilde{\tilde{h}} = |\tilde{s}|^2\tilde{h}$ is real and positive.

Then

$$\int_{p^{-1}(U_\alpha)} d\Big(p^*(\rho_\alpha)\beta^{n+k}f^k\omega^{n-1}\wedge\gamma\Big) = \int_{z_1\cdots z_n} d(F\wedge d^c\log|z_n|^2) + \int_{z_1\cdots z_n} d(F\wedge d^c\log\tilde{\tilde{h}})$$

where $F = p^*(\rho_\alpha)\beta^{n+k}f^k\omega^{n-1}$. F is compactly supported and so by Stokes' theorem the second term is zero and the first term is

$$\lim_{t\to 0}\int_{z_1\cdots z_{n-1};|z_n|^2=t} F\wedge d^c\log|z_n|^2.$$

The orientation on $|z_n|^2 = t$ is induced from bounding the set $|z_n|^2 \geq t$ and with this orientation $d^c\log|z_n|^2$ restricts to $\frac{-1}{2\pi}d\theta$. Then in the limit the above integral reduces to

$$-\int_{z_1\cdots z_{n-1}} F|_{z_n=0} = -\int_{U_\alpha}\rho_\alpha f^k\omega^{n-1}$$

and we can finally conclude that

$$\begin{aligned} \int_M (f')^k(\omega')^n &= -\tfrac{n}{n+k}\sum_\alpha \int_{U_\alpha} \rho_\alpha f^k\omega^{n-1} \\ &= -\tfrac{n}{n+k}\int_{s^{-1}(0)} f^k\omega^{n-1} \quad \square \end{aligned}$$

3.3. Equivariant DeRham Theory. In this section we briefly recall equivariant DeRham cohomology and its relationship with symplectic geometry à la Atiyah and Bott. Particularly we are concerned with when their ideas go through in a non-compact and/or pre-symplectic setting. The beginning of this subsection is really only a small perturbation of the ideas in Atiyah and Bott's paper [At-Bo]. In the last part of the subsection we will rephrase theorem 3.24 in terms of equivariant cohomology.

The objective of equivariant DeRham theory is to build a complex $(\Omega_G(M), D)$ out of the DeRham complex of M and the action of G on M so that the cohomology of the complex $H(\Omega_G(M), D)$ agrees with the usual equivariant cohomology taken with real coefficients $H^*_G(M;\mathbb{R}) \equiv H^*(M \times_G EG, \mathbb{R})$. The complex $(\Omega_G(M), D)$ models a "DeRham complex" for the infinite dimensional manifold $M \times_G EG$.

There are essentially two equivalent formulations of $(\Omega_G(M), D)$, the Cartan and the Weil models. For the sake of brevity we will only consider the Cartan model; for a nice treatment of the relationship between the Cartan and Weil points of view, see [Ka].

Definition 3.25. *We define the* Cartan complex *as follows. Let G be a compact Lie group acting on a manifold M and let $S^*(\mathfrak{g}^*)$ denote the graded algebra of symmetric polynomials on the Lie algebra of G. Let*

$$\Omega^*_G(M) = (\Omega^*(M) \otimes S^*(\mathfrak{g}^*))^G$$

where $(\cdot)^G$ denotes the subalgebra of invariants under the action of G (which on the $\mathfrak{g}^*$ factor is induced by the Ad^*-action). Define a grading on $\Omega^*_G(M)$ by defining elements of $S^1(\mathfrak{g}^*) \cong \mathfrak{g}^*$ to have degree 2 and using the natural grading on $\Omega^*(M)$. Define a differential

$$D : \Omega^i_G(M) \to \Omega^{i+1}_G(M)$$

by $D = d \otimes Id + \sum_\alpha i_{\tilde{a}_\alpha} \otimes a^*_\alpha$ where $\{a_\alpha\}$ is a fixed basis for $\mathfrak{g}$ and $\{a^*_\alpha\}$ is its dual basis.

It is easy to check that $(\Omega^*_G(M), D)$ is a complex, *i.e.* $D^2 = 0$.

For example if M is a single point then $D \equiv 0$ and so

$$H^*_G(\text{pt.}) = (S^*(\mathfrak{g}^*))^G.$$

The left hand side is the algebra of polynomial functions on $\mathfrak{g}$ that are invariant under the adjoint action.

Example 3.26. Let $G = SO(3)$, then $\mathfrak{g}$ is 3-dimensional and we fix a basis a_1, a_2, a_3. The adjoint action of $SO(3)$ on $\mathfrak{so}(3)$ is isomorphic to the standard action of $SO(3)$ on $\mathbb{R}^3$. Invariant functions are thus functions depending only on the radius, *i.e.*

$$H^*_{SO(3)}(\mathrm{pt.}) = \mathbb{R}[p_1],$$

where $p_1 = a_1^2 + a_2^2 + a_3^2$.

For M compact, there is a natural map

$$\pi_* : H^k_G(M) \to H^{k-\dim M}_G(\mathrm{pt.})$$

defined by

$$\pi_*(\omega \otimes a_\alpha) = \begin{cases} \int_M \omega \otimes a_\alpha & \text{if } \deg(\omega) = \dim(M) \\ 0 & \text{if } \deg(\omega) \neq \dim(M). \end{cases}$$

For the purposes of this paper it is important to note that one can replace $\Omega^*(M)$ in the Cartan complex by any subcomplex preserved by d and $i_{\tilde{a}}$. For example if M is not compact we might want to take the complex $\Omega^*_c(M)$ of forms with compact support so that π_* is well defined. However, this subcomplex is too restrictive for the type of non-compactness that arises in gauge theory applications. We will digress to discuss an alternate complex that is well suited for our applications.

3.3.1. *Digression on the DeRham Theory of Orbit Spaces.* Suppose M is contained a stratified space $\overline{M}$ and that $\overline{M} = X/K$ where X is a compact manifold on which a compact, connected group K acts. The orbit space $\overline{M}$ is potentially singular in the image of the points with non-trivial stabilizers. Suppose that the non-compact manifold M is exactly the image of the subset of X for which K acts freely. Assume further that the action of G on M is induced from an action of G on X and that the actions of G and K commute.

Consider a complex $\Omega^*(X|K) \subset \Omega^*(X)$ defined as follows:

$$\Omega^*(X|K) = \{\theta \in \Omega^*(X) : i_{\tilde{k}}\theta = \mathcal{L}_{\tilde{k}}\theta = 0 \text{ for all } k \in \mathfrak{k} = \mathrm{Lie}(K)\}.$$

This complex is preserved by d and it is called the complex of basic forms on X.

Theorem 3.27.

$$H^*(\Omega^*(X|K), d) = H^*(X/K, \mathbb{R})$$

where the cohomology on the right is ordinary cohomology (say sheaf cohomology with the constant $\mathbb{R}$ *sheaf.)*

Proof. [Ko]

The forms in $\Omega^*(X|K)$ descend to forms on M and so we can also view $\Omega^*(X|K)$ as a subcomplex of $\Omega^*(M)$, thinking of such forms as forms that "extend" to $\overline{M}$.

In the case where there are points with trivial stabilizers and the smallest non-trivial stabilizer is dimension 2 or greater, then the singular strata will have codimension at least 2 and $\overline{M}$ will have a top homology class and the pairing with forms is $\int_M$.

We end the digression by noting the following

Proposition 3.28. *If K and G commute then $(\Omega^*(X|K)\otimes S^*(\mathfrak{g}^*))^G$ is a DeRham model for $H^*_G(\overline{M})$ and π_* well defined by integration.*

PROOF: Since K and G commute, $i_{\tilde{a}}$ preserves on $\Omega^*(X|K)$.

Recall a general fact:

Proposition 3.29. *Let $T\subset G$ be a maximal torus and let $W=N(T)/T$ be the Weyl group, then*

$$H^*_G(M)\cong (H^*_T(M))^W$$

PROOF: [At-Bo].

Example 3.30. Let $G=SO(3)$ as in example 3.26. Let T be the circle subgroup generated by a_1. Then the Weyl group is $\mathbb{Z}/2\mathbb{Z}$ acting on a_1 as $a_1\mapsto \pm a_1$. $(H^*_T(\text{pt.}))^W$ is thus even polynomial functions in a_1, *i.e.* $\mathbb{R}[a_1^2]\cong H^*_{SO(3)}(\text{pt.})$.

Now suppose that M has a symplectic form and that G acts symplectically with a moment map f. Atiyah and Bott show that f determines a unique extension of ω to an equivariant class $\omega^\#$. Explicitly,

$$\omega^\# = \omega - \sum_\alpha f_{a_\alpha} a^*_\alpha$$

where a_α is a basis for $\mathfrak{g}$. $\omega^\#$ is obviously D closed and so represents a class $[\omega^\#]\in H^2_G(M)$.

Remark 3.31. This result does not use the non-degeneracy of the symplectic form and applies equally well in the pre-symplectic case.

Returning to our example, suppose that (M,ω) is a $2n$-dimensional (pre-)symplectic manifold with a symplectic action of $SO(3)$ and a moment map f. Let T be the torus generated by a_1. Ad-equivariance of the moment map implies that $\omega^\#_T=\omega - f_{a_1}a_1^*$ is invariant under the Weyl group and so $\omega^\#_T\in (H^2_T(M))^W$. Consider

$$\begin{aligned} \pi_*((\omega^\#_T)^k) &= \pi_*\Big(\sum_{j=0}^k \tbinom{k}{j}(-f_{a_1})^{k-j}\omega^j\otimes (a_1^*)^{k-j}\Big) \\ &= \Big(\tbinom{k}{n}\int_M (-f_{a_1})^{k-n}\omega^n\Big)\otimes (a_1^*)^{k-n}\in H^{2k-2n}_T(\text{pt.}). \end{aligned} \tag{11}$$

We see that $\pi_*((\omega_T^{\#})^k)$ is W invariant precisely when $k-n$ is even. Then for such k,

$$\pi_*((\omega_T^{\#})^k) = \pi_*((\omega^{\#})^k).$$

We see that in the $SO(3)$ case it is sufficient to understand π_* for the action restricted to some S^1. With this in mind we apply theorem 3.24 for the case of S^1-actions to equation 11 and get

Proposition 3.32. *Let (M, L, G, s, ω, f) be as in section 3.2 and assume $G = S^1$. Choose ρ as in theorem 3.24 so that we have ω' and f'. Then*

$$\pi_*^M((\omega'^{\#})^k) = -\pi_*^{s^{-1}(0)}((\omega^{\#})^{k-1}).$$

PROOF: Choose $a \in \mathrm{Lie}(S^1)$, then $\omega'^{\#} = \omega' - f'a^*$ and

$$\begin{aligned}
\pi_*^M((\omega'^{\#})^k) &= \left(\binom{k}{n}\int_M (-f')^{k-n}\omega^n\right) \otimes (a^*)^{k-n} \\
&= \left(\binom{k}{n}\frac{-n}{k}\int_{s^{-1}(0)} (-f)^{k-n}\omega^{n-1}\right) \otimes (a^*)^{k-n} \\
&= -\left(\binom{k-1}{n-1}\int_{s^{-1}(0)} (-f)^{k-n}\omega^{n-1}\right) \otimes (a^*)^{k-n} \\
&= -\pi_*^{s^{-1}(0)}((\omega^{\#})^{k-1}). \quad \square
\end{aligned}$$

Let us remark on the above proposition in various situations. First we note the following claim:

Claim 3.33. *$\omega'^{\#}$ and $\omega^{\#}$ are cohomologous in $\Omega_G^*(M)$.*

PROOF:

$$\begin{aligned}
\omega'^{\#} - \omega^{\#} &= (\omega' - \omega) - \sum_i (f'_{a_i} - f_{a_i}) \otimes a_i^* \\
&= d((\beta-1)\gamma) + \sum_i i_{\tilde{a}_i}((\beta-1)\gamma) \otimes a_i^* \\
&= D((\beta-1)\gamma)
\end{aligned}$$

and $(\beta-1)\gamma$ is defined on all of M. $\square$

This claim shows that for M compact, the procedure for concentrating w along $s^{-1}(0)$ constructing ω' is irrelevant to the content of proposition 3.32 .

The construction of ω' is relevant when M is not compact. As previously discussed, one must restrict the forms in the Cartan complex to some subcomplex in order for π_* to be defined. The point then is that ω' may lie in the subcomplex while ω does not.

We conclude with some formal comparisons of proposition 3.32 and the localization theorems in [At-Bo] for when M is compact. We show that when we have an abundance of transverse, equivariant sections, proposition 3.32 recovers the fixed point formulas.

Consider again $(M, L, G = S^1, \omega, f)$ as in section 3.2 and assume that M is compact so that $[\omega'^{\#}] = [\omega^{\#}]$. Assume also that we are lucky enough to have n sections $s_1, \dots, s_n$, each satisfying the equivariance and transversity assumptions and such that $\cap_i s_i^{-1}(0) = F$ is a transverse intersection consisting of a finite collection of points.

It is easy to see that F must be exactly the fixed point set of the action. The fixed point formula says

$$\pi_*^M(e^{\omega^{\#}}) = \pi_*^M(e^{\omega}e^{-fa^*}) = \sum_{p\in F}\frac{e^{-f(p)a^*}}{e(p)(a^*)^n} \tag{12}$$

where we only keep non-negative powers of a^* and $e(p)$ is up to sign the product of the integer weights of the action of S^1 on T_pM.

Successively applying our formula to the left hand side of the fixed point formula we get:

$$\begin{aligned}
\pi_*^M(e^{\omega^{\#}}) &= \sum_{k=0}^{\infty}\tfrac{1}{k!}\pi_*^M((\omega^{\#})^k)\\
&= \sum_{k=0}^{\infty}\tfrac{1}{k!}(-1)^n\pi_*^F((\omega^{\#})^{k-n})\\
&= \sum_{p\in F}\sum_{k=0}^{\infty}\tfrac{1}{k!}(-1)^n(-f(p)a^*)^{k-n}\\
&= \sum_{p\in F}\sum_{k=0}^{\infty}\frac{(-f(p)a^*)^k}{k!(f(p)a^*)^n}\\
&= \sum_{p\in F}\frac{e^{-f(p)}a^*}{(f(p)a^*)^n}.
\end{aligned}$$

We see that in this case $(f(p))^n = e(p)$. This can also be seen in the following way.

The normal bundles N_i of each $s_i^{-1}(0)$ are equivariantly isomorphic to $L|_{s_i^{-1}(0)}$ and so the action on $L|_{s_i^{-1}(0)}$ determined by the moment map is the same as the action on N_i. One can easily check that for a point $p \in s_i^{-1}(0)$, $f(p)$ is the weight of the action on $L|_p$. Thus if the n sections postulated exist, then all the weights at a fixed point must coincide and equal $f(p)$. Hence we see that $e(p) = (f(p))^n$.

In the general situation, where fixed points may have differing weights at each fixed point, we cannot hope for n equivariant sections. However, one can generalize theorem 3.24 by dropping the condition that s is transverse to the zero section. Then the order of vanishing of s, which is a well defined integer on each component of $s^{-1}(0)$, will appear multiplying the integral over the corresponding component of $s^{-1}(0)$. Then when we have n sections, the various orders of vanishing will correctly determine the weights on the fixed points and the fixed point formula will be recovered.

Our method of localizing at the zeros of equivariant sections gives an alternative to the fixed point formulas when one is in a non-compact setting and the usual fixed point formulas do not apply. There is a non-compact version of such formulas due to Prado and Wu [Pr-Wu], which apply when the moment map is proper. This condition does typically hold for non-compactness arising in situations such as the orbit spaces discussed in digression 3.3.1.

4. The Geometry of the Moduli Space $\mathrm{M}^0_k(\overline{\mathbb{CP}}^2)$

There is a limited set of examples of 4-manifolds for which the anti-self-dual (ASD) moduli spaces can be constructed explicitly. For *non-algebraic* 4-manifolds the only available construction uses the Ward correspondence and has a flavor of real algebraic geometry. The Ward correspondence translates the ASD equations into holomorphic equations on sections of certain bundles. The bundles are over the Penrose twistor space of the original four manifold and this twistor space needs to have a algebraic structure. This condition is equivalent to the base manifold having an anti-self-dual metric. The only tractable examples of such manifolds are S^4 and $\overline{\mathbb{CP}}^2$ for which the above construction yields the ADHM description of instantons on S^4 and Buchdahl's construction of $\mathcal{M}^0(\overline{\mathbb{CP}}^2)$ [Bu] respectively. Like the ADHM description, Buchdahl's construction is a finite dimensional set of linear algebra data subject to certain "integrability", "reality", and "non-degeneracy" conditions modulo the action of a compact group.

In his 1989 thesis, King constructs a certain moduli space of holomorphic bundles that produces that same space as Buchdahl's construction. He constructs his moduli space as the quotient of a quantized Kähler manifold by a reductive group and shows that the corresponding symplectic quotient (c.f. theorem 3.14) is exactly the twistor description. As a byproduct he gets a quantized Kähler structure on moduli space.

The geometric spaces involved arise from considering the affine complex plane blown-up at the origin which we denote $\widetilde{\mathbb{C}}^2$. The natural Kähler metric on $\widetilde{\mathbb{C}}^2$ is induced from the product of the flat metric and the Fubini-Study metric on $\mathbb{C}^2 \times \mathbb{P}^1 \supset \widetilde{\mathbb{C}}^2$. It is easy to check that $\widetilde{\mathbb{C}}^2$ is conformally equivalent to $\overline{\mathbb{CP}}^2 - \{x_0\}$ with the Fubini-Study metric (with the opposite orientation).

Fix a principal $SU(2)$ bundle P_k over $\overline{\mathbb{CP}}^2$ with $c_2(P_k)[\overline{\mathbb{CP}}^2] = k \geq 0$ and let V be the associated rank 2 complex vector bundle. Denote by $\mathcal{M}^0_k(\overline{\mathbb{CP}}^2)$ the moduli space of anti-self-dual connections on P_k modulo gauge transformations that are the identity on the fiber over x_0.

Let S be the complex projective plane blown up at one point. Fix a line ℓ_∞ not passing through the exceptional set. Note that if we collapse all the points of ℓ_∞ to one point the resulting space is diffeomorphic to $\overline{\mathbb{CP}}^2$. We abuse notation and denote also by V the bundle over S obtained by pulling back V via the projection

$$p : S \to S/_{\ell_\infty \sim x_0} \cong \overline{\mathbb{CP}}^2.$$

We continue to abuse notation and denote by E both the exceptional divisor in S and its image under the projection. Let $\mathcal{M}_k^{\mathrm{alg}}(S)$ denote the moduli space of rank 2 holomorphic bundles on S that are topologically isomorphic to V and are *holomorphically* trivial on ℓ_∞. We will describe a specific projective embedding of S (determining a polarization) and the condition of triviality on ℓ_∞ implies slope stability so $\mathcal{M}_k^{\mathrm{alg}}(S)$ naturally embeds in the moduli space of stable rank 2 vector bundles.

There is a natural map $\Phi : \mathcal{M}_k^0(\overline{\mathbb{CP}}^2) \longrightarrow \mathcal{M}_k^{\mathrm{alg}}(S)$ defined as follows. If $A \in \mathcal{M}_k^0(\overline{\mathbb{CP}}^2)$ then the $\bar\partial$ operator that defines the holomorphic bundle $\mathcal{V} = \Phi(A)$ is taken to be $(d_{p^*(A)})^{(0,1)}$, the anti-holomorphic part of the covariant derivative defined by the pullback of the connection. The anti-self-duality of A implies that the curvature of $p^*(A)$ is a $(1,1)$-form and so $\bar\partial^2 = 0$.

King's theorem is then

Theorem 4.1. *(King, 1989) The map $\Phi : \mathcal{M}_k^0(\overline{\mathbb{CP}}^2) \longrightarrow \mathcal{M}_k^{alg}(S)$ is a diffeomorphism.*

In subsection 4.1 we give a brief recount of King's construction. In subsection 4.2 we show that the Kähler form defined by the quantized structure on $\mathcal{M}_k^{\mathrm{alg}}(S)$ formally represents $\mu(E)$ but lacks the compactness properties dictated by the discussion in subsection 2. We use the techniques developed in section 3.2 to construct a representative for $\mu(E)$ that is supported in a neighborhood of the divisor of "jumping lines" on E.

4.1. King's construction. We give a quick recount of the construction in [Ki]. King considers configurations of linear maps:

$$\begin{tikzcd} W_0 \arrow[rr, shift right, "x"'] & & W_1 \arrow[ll, shift right, "a_1, a_2"'] \arrow[dl, "c"] \\ & V_\infty \arrow[ul, "b"] & \end{tikzcd}$$

where W_0, W_1 and V_∞ are hermitian inner product spaces of dimensions k, k, and 2 respectively. (We use x to denote the map King calls d to avoid confusion with exterior derivative later.) Following King's notation we define

$$\tilde{\mathcal{R}} = \mathrm{Hom}(W_1, W_0)^2 \oplus \mathrm{Hom}(V_\infty, W_0) \oplus \mathrm{Hom}(W_1, V_\infty) \oplus \mathrm{Hom}(W_0, W_1),$$

$$G_{\mathbb{C}} = GL(W_0) \times GL(W_1), \qquad G = U(W_0) \times U(W_1).$$

Definition 4.2. *A configuration* (a_1, a_2, b, c, x) *is called* integrable *if it satisfies the equation*

$$a_1 x a_2 - a_2 x a_1 + bc = 0.$$

Definition 4.3. *A configuration* (a_1, a_2, b, c, x) *is* non-degenerate *if it satisfies the following conditions:*

$$\forall\ (\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \mathbb{C}^2 \text{ such that } \lambda_1\mu_1 + \lambda_2\mu_2 = 0 \text{ and } (\mu_1, \mu_2) \neq (0,0),$$

$$\nexists v \in W_1 \text{ such that} \qquad \begin{cases} xa_1 v = \lambda_1 v & (\mu_1 a_1 + \mu_2 a_2)v = 0 \\ xa_2 v = \lambda_2 v & cv = 0 \end{cases}$$

$$\text{and } \nexists w \in W_0^\star \text{ such that} \qquad \begin{cases} x^* a_1^* w = \lambda_1 w & (\mu_1 a_1^* + \mu_2 a_2^*)w = 0 \\ x^* a_2^* w = \lambda_2 w & b^* w = 0 \end{cases}$$

Let $X \subset \tilde{\mathcal{R}}$ be the space of all integrable non-degenerate configurations. G and $G_{\mathbb{C}}$ act canonically on $\tilde{R}$ and descend to actions on X. The action is explicitly given by

$$(g_0, g_1) \cdot (a_1, a_2, b, c, x) = (g_0 a_1 g_1^{-1}, g_0 a_2 g_1^{-1}, g_0 b, c g_1^{-1}, g_1 x g_0^{-1}) \tag{13}$$

Theorem 4.4. *The moduli space* $\mathcal{M}_k^{alg}(S)$ *of holomorphic, rank 2 bundles on* S *with* $c_1 = 0$ *and* $c_2 = k$ *and that are holomorphically trivial on* ℓ_∞ *is isomorphic to* $X/G_{\mathbb{C}}$.

PROOF: We will outline the construction and proof to the extent that we need to understand the space. For details the reader is refered to [Ki].

King uses such configurations to determine a monads on S that in turn determine holomorphic bundles on S. Configurations in the same $G_{\mathbb{C}}$ orbit determine the same bundle.

A monad is a three term complex of vector bundles over an algebraic space

$$\mathcal{U} \overset{A}{\hookrightarrow} \mathcal{V} \overset{B}{\twoheadrightarrow} \mathcal{W}$$

such that A is injective, B is surjective, and $B \circ A$ is 0. The monad defines a bundle $\mathcal{E} = \mathrm{Ker}(B)/\,\mathrm{Im}(A)$. The point is that one can build complicated holomorphic bundles from relatively simple bundles using monads.

Consider S as a subvariety of $\mathbb{P}^3 \times \mathbb{P}^3$:

$$S = \{([x_1, x_2, x_3], [y_1, y_2, y_3]) \in \mathbb{P}^3 \times \mathbb{P}^3 : \sum x_i y_i = 0, y_3 = 0\}.$$

If $\mathcal{E}$ is a bundle on S we denote by $\mathcal{E}(n,m)$ the bundle $\mathcal{E} \otimes i^*(\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}(n,m))$ where i^* is restriction to S.

Theorem 4.5. *Let $\mathcal{E}$ be a holomorphic vector bundle over S of rank 2, with $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = k$ and such that $\mathcal{E}|_{\ell_\infty}$ is trivial. Then $\mathcal{E}$ is the bundle associated with a monad of the form*

$$\bigoplus_{i=0}^{1} U_i \otimes \mathcal{L}_i \xrightarrow{A} \mathcal{O}_S \otimes V \xrightarrow{B} \bigoplus_{i=0}^{1} W_i \otimes \mathcal{L}_i^*$$

where U_i, W_i, and V are complex vector spaces of dimensions k, k, and $4k+2$ respectively, and the line bundles $\mathcal{L}_0$ and $\mathcal{L}_1$ are $\mathcal{O}_S(-1,0)$ and $\mathcal{O}_S(0,-1)$ respectively.

PROOF: See [Ki]

King fixes certain choices to reduce the symmetry group of the above monads and put them into a canonical form. He makes the identifications

$$U_0 = W_1,\ U_1 = W_0,\ V = W_0 \oplus W_1 \oplus W_0 \oplus W_1 \oplus V_\infty.$$

The maps $A \in \bigoplus_{i=0}^{1} \operatorname{Hom}(U_i, V) \otimes H^0(\mathcal{L}_i^*)$ and $B \in \bigoplus_{i=0}^{1} \operatorname{Hom}(V, W_i) \otimes H^0(\mathcal{L}_i^*)$ then have the canonical forms:

$$A = \begin{pmatrix} a_1x_3 & -y_2 \\ x_1 - xa_1x_3 & 0 \\ a_2x_3 & y_1 \\ x_2 - xa_2x_3 & 0 \\ cx_3 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} x_2 & a_2x_3 & -x_1 & -a_1x_3 & bx_3 \\ xy_1 & y_1 & xy_2 & y_2 & 0 \end{pmatrix}$$

noting that $\langle x_1, x_2, x_3\rangle$ spans $H^0(\mathcal{L}_0^*)$ and $\langle y_1, y_2\rangle$ spans $H^0(\mathcal{L}_1^*)$.

The condition $B \circ A = 0$ is then equivalent to the integrability condition of definition 4.2 and the non-degeneracy conditions in definition 4.3 are equivalent to the surjectivity and injectivity of B and A respectively. The residual symmetry of monads in canonical form is exactly the natural action of $G_{\mathbb{C}}$ on configurations in X. The vector spaces W_i are canonically isomorphic to $H^1(\mathcal{V} \otimes \mathcal{L}_i)$ and the vector space V_∞ is identified with the fiber over ℓ_∞.

In order to exploit the correspondence between quotients by reductive groups and symplectic reduction discussed in subsection 3.1, a quantized Kähler structure is needed on X. Consider a second space of linear maps

$$R' = \operatorname{Hom}(W_1, W_0)^2 \oplus \operatorname{Hom}(V_\infty, W_0) \oplus \operatorname{End}(W_1)^2 \oplus \operatorname{Hom}(V_\infty, W_1) \oplus \operatorname{Hom}(W_1, V_\infty)$$

and a map $\rho : \tilde{R} \to R'$

$$\rho : (a_1, a_2, b, c, x) \mapsto (a_1, a_2, b, xa_1, xa_2, xb, c).$$

ρ is compatible with the canonical action of $G_{\mathbb{C}}$ on $\tilde{R}$ and R' and ρ is injective on X. Let L be the trivial line bundle on X. Define a hermitian structure on L by

$$\|(\alpha, z)\|^2 = |z|^2 e^{\|\rho(\alpha)\|^2}$$

where the norm on R' is the direct sum of standard hermitian norms induced on homomorphisms of hermitian vector spaces, c.f. example 3.8.

Define a lifting of the $G_{\mathbb{C}}$-action to L by

$$(g_0, g_1) : (\alpha, z) \mapsto (g\alpha, \frac{\det g_1}{\det g_0} z)$$

where $(g_0, g_1) \in G_{\mathbb{C}} = U(W_0) \times U(W_1)$ and $(\alpha, z) \in X \times \mathbb{C}$.

One way to interpret this action is to think of the trivial bundle L as being canonically $X \times (\wedge^{\max} W_0^\star \otimes \wedge^{\max} W_1)$. The action described is then the induced natural action.

It is clear from the construction that the action on X satisfies assumption 3.3 and that the lifting defines a linearization. King shows that the non-degeneracy conditions guarantee analytic stability of all the points of X and so theorem 3.14 implies that

$$\mathcal{M}^{\mathrm{alg}}(S) = X \cap \mu^{-1}(0)/G.$$

The components of the moment map μ_{h_0,h_1} can be calculated explicitly as in example 3.8:

$$\begin{aligned} \mu_{h_0,h_1} &= \tfrac{1}{2\pi i}\langle h_0, a_1 a_1^* + a_2 a_2^* + bb^* - I\rangle \\ &+ \tfrac{1}{2\pi i}\langle h_1, [xa_1, (xa_1)^*] + [xa_2, (xa_2)^*] - a_1^* a_1 - a_2^* a_2 + xb(xb)^* - c_1^* c_1 + I\rangle \end{aligned} \tag{14}$$

The point is that the zero set of this moment map is described exactly by the equations that appear as the "reality" conditions in Buchdahl's twistor description of $\mathcal{M}^0(\overline{\mathbb{CP}}^2)$ [Bu] and thus King proves theorem 4.1. Furthermore, theorem 3.14 also tells us the quotient bundle $\hat{L} = L/G_{\mathbb{C}}$ defines a quantized Kähler structure on the quotient. $\square$

Since $W_0 = H^1(\mathcal{V}(-1,0))$ and $W_1 = H^1(\mathcal{V}(0,-1))$ and all the configurations in a single $G_{\mathbb{C}}$-orbit correspond to the same holomorphic bundle, the fiber of $\hat{L}$ over bundle $\mathcal{V} \in \mathcal{M}_k^{\mathrm{alg}}(S)$ can be canonically identified:

$$\hat{L}|_{\mathcal{V}} \equiv \wedge^{\max}(H^1(\mathcal{V}(-1,0)))^\star \otimes \wedge^{\max} H^1(\mathcal{V}(0,-1)). \tag{15}$$

Let ω and $\hat{\omega}$ denote the Kähler forms given by the curvature of L and $\hat{L}$ respectively.

4.2. The symplectic form and $\mu(E)$. In this section we show that $\hat{L}$, the natural line bundle defining the quantized Kähler structure on $\mathcal{M}^{\mathrm{alg}}(S)$ is the same as the Dirac line bundle defining Donaldson's μ-map on $\mathcal{M}^0(\overline{\mathbb{CP}}^2)$. We alter the connection on the bundle as in section 3.2 to define a new symplectic form on $\mathcal{M}^0(\overline{\mathbb{CP}}^2)$ and we show that this form is a representative for $\mu(E)$ and satisfies properties (1) and (3) on page 6.

Recall King's diffeomorphism Φ and let Ψ be the inverse map.

Theorem 4.6. *$\hat{L}$ is canonically isomorphic to $\Psi^{-1}(\mathcal{L}_E)$ where $\mathcal{L}_E$ is the Dirac bundle defining $\mu(E)$.*

PROOF: The bundle $\mathcal{L}_E$ is defined as follows. Let

$$\partial\!\!\!/ : \Omega^0(E, K^{\frac{1}{2}}) \to \Omega^1(E, K^{\frac{1}{2}})$$

be the Dirac operator for the Riemann surface $E \in \overline{\mathbb{CP}}^2$ ($K^{\frac{1}{2}}$ is the {unique in this case} square root of the canonical bundle over E). For a connection A representing an equivalence class $[A] \in \mathcal{M}^0_k(\overline{\mathbb{CP}}^2)$ we couple its restriction to E to the Dirac operator to get an operator

$$\partial\!\!\!/_A : \Omega^0(E, K^{\frac{1}{2}} \otimes V|_E) \to \Omega^1(E, K^{\frac{1}{2}} \otimes V|_E).$$

$\mathrm{Ind}(\partial\!\!\!/_A)$ is a virtual bundle (in the sense of K-theory) over $\mathcal{M}^0_k(\overline{\mathbb{CP}}^2)$ defined fiber-wise as $[\mathrm{Ker}\, \partial\!\!\!/_A)] - [\mathrm{Coker}(\partial\!\!\!/_A)]$. Consider then the line bundle over $\mathcal{M}^0_k(\overline{\mathbb{CP}}^2)$ whose fiber over A is $(\det(\mathrm{Ind}(\partial\!\!\!/_A))^\star$. A straight forward calculation using the Atiyah-Singer index theorem for families shows that $c_1(\mathcal{L}_E) = \mu(E)$.

The pullback by Ψ of the virtual bundle

$$\mathrm{Ind}\left\{\partial\!\!\!/_A : \Omega^0(E, K^{\frac{1}{2}} \otimes V|_E) \to \Omega^1(E, K^{\frac{1}{2}} \otimes V|_E)\right\}$$

is the virtual bundle

$$\mathrm{Ind}\left\{\bar{\partial} : \Omega^0(E, K^{\frac{1}{2}} \otimes \mathcal{V}|_E) \to \Omega^1(E, K^{\frac{1}{2}} \otimes \mathcal{V}|_E)\right\}$$

which is by definition

$$[H^0(\mathcal{V}|_E\,(-1))] - [H^1(\mathcal{V}|_E\,(-1))]$$

where $\mathcal{V}$ is twisted by $K^{\frac{1}{2}} = \mathcal{O}_E(-1)$. Hence $\Psi^{-1}(\mathcal{L}_E)$ is the bundle

$$\begin{array}{c} \bigwedge^{\max}(H^0(\mathcal{V}|_E\,(-1))^\star \otimes \bigwedge^{\max}(H^0(\mathcal{V}|_E\,(-1)) \\ \downarrow \\ \mathcal{M}^{\mathrm{alg}}_k(S). \end{array}$$

We construct a four term exact sequence. Note that $\mathcal{O}(E) = \mathcal{O}(1,-1)$ and that $H^0(\mathcal{V}(0,-1))$ and $H^2(\mathcal{V}(-1,0))$ are both 0 [Ki]. Consider the short exact sequence of sheaves:

$$0 \to \mathcal{O}(-E) \to \mathcal{O} \to \mathcal{O}|_E \to 0.$$

Tensoring the sequence with $\mathcal{V}(0,-1)$ and considering the associated long exact sequence in cohomology we get the sequence:

$$0 \to H^0(\mathcal{V}|_E\,(-1)) \to H^1(\mathcal{V}(-1,0)) \xrightarrow{x} H^1(\mathcal{V}(0,-1)) \to H^1(\mathcal{V}|_E\,(-1)) \to 0.$$

This sequence induces an isomorphism of virtual bundles

$$[H^0(\mathcal{V}|_E(-1))] - [H^1(\mathcal{V}|_E(-1))] \cong [H^{-1}(\mathcal{V}(-1,0))] - [H^1(\mathcal{V}(0,-1))].$$

Hence the line bundle

$$\Psi^{-1}(\mathcal{L}_E) \cong \wedge^{\max}(H^1(\mathcal{V}(-1,0)))^\star \otimes \wedge^{\max}(H^1(\mathcal{V}(0,-1))) \equiv \hat{L}. \quad \square$$

From the above four term sequence we see that $\det(x)$ determines a section of $\hat{L}$. From the sequence we see that if $\det(x) \neq 0$ so that x is an isomorphism, then

$$H^0(\mathcal{V}|_E(-1)) = H^1(\mathcal{V}|_E(-1)) = 0.$$

Since E is a copy of $\mathbb{P}^1$ and $c_1(\mathcal{V}) = 0$, $\mathcal{V}|_E$ must be isomorphic to $\mathcal{O}(m)\oplus\mathcal{O}(-m)$ for some m. Thus $H^0(\mathcal{V}|_E(-1)) = 0$ can only occur if $m = 0$. In general, $H^0(\mathcal{V}|_E(-1)) = m = \dim \operatorname{Ker}(x)$. The zero set of the section $\det(x)$ is thus the "jumping lines divisor" of E, *i.e.* the complex codimension 1 submanifold of $\mathcal{M}^{\mathrm{alg}}(S)$ consisting of bundles that restrict to E in a non-generic (in this case non-trivial) way.

We intend to apply the techniques of section 3.2 to the section $\det(x)$. It follows from theorem 4.6 that the Kähler form on $\mathcal{M}^0(\overline{\mathbb{CP}}^2)$ formally represents $\mu(E)$. However, we will show it lacks the requisite compactness properties and thus requires the use of our localization techniques. In particular, the form $\hat{\omega}$ does not have finite volume:

Proposition 4.7. *The symplectic volume of $\mathcal{M}_1^{alg}(S)$ is infinite,* i.e.

$$\int_{\mathcal{M}_1^{alg}(S)} (\hat{\omega})^4 = \infty.$$

Proof: See section 5.1 for the calculation.

The proposition implies that the form $\hat{\omega}$ cannot be in the class of forms discussed on page 6.

Let $s : X \to L \cong X \times \mathbb{C}$ be the map

$$s(a_1, a_2, b, c, x) = (a_1, a_2, b, c, x, \det(x)).$$

This defines a section of L that is clearly equivariant with respect to the linearization. Choose a function ρ as in section 3.2 so that the support of ρ is contained in $[0, \epsilon)$

for an $\epsilon > 0$ (to be specified later). Then let ω' be constructed as in section 3.2 . Using the results of that section, we know that the pre-symplectic reduction of X with respect to ω is diffeomorphic to $\mathcal{M}^{\text{alg}}(S)$ and has a descended pre-symplectic form $\hat{\omega}'$.

The form $\hat{\omega}'$ will be our representative for $\mu(E)$. We will show that it satisfies the conditions listed on page 6. By construction, it satisfies the first property—that it is supported in a neighborhood of the divisor representation of $\mu(E)$. In section 5.1 we will show the equivariance property. We will now show the third property— that $\hat{\omega}'$ extends, in a well defined way, to the bubbled off connections.

Recall that the conformal equivalence of $\overline{\mathbb{CP}}^2 - \{x_0\}$ and the blown-up plane $\tilde{\mathbb{C}}^2$ implies that $\mathcal{M}^0_k(\overline{\mathbb{CP}}^2)$ has the alternate description as $8\pi^2 k$-energy instantons on $\tilde{\mathbb{C}}^2$ modulo gauge transformations that are "fixed at infinity". The Uhlenbeck completion by ideal instantons is not a compactification in this case because charge can drift off to infinity.

King shows that the metric completion $\overline{\mathcal{M}}^{\text{alg}}_k$ of $\mathcal{M}^{\text{alg}}_k(S)$ is a stratified space that coincides with the Uhlenbeck completion of $\mathcal{M}^0_k(\tilde{\mathbb{C}}^2)$ in the sense described above. $\mathcal{M}^{\text{alg}}(S)$ is the symplectic reduction of X by G and its metric completion is the reduction of $\overline{X}$, the completion of X in R' (recall that the metric on X is induce from the metric on R'). The different strata in $\overline{\mathcal{M}}^{\text{alg}}_k$ then correspond to the different stabilizer types G_x of $x \in \mu^{-1}(0) \cap \overline{X}$. $\overline{X}$ can be obtained from X by dropping the non-degeneracy conditions 4.3.

$\overline{X} \cap \mu^{-1}(0)$ is smooth and so as described in digression 3.3.1 , there is a DeRham theory for the orbit space $\overline{\mathcal{M}}^{\text{alg}} = \overline{X} \cap \mu^{-1}(0)$ given by the cohomology of the complex of differential forms on $\overline{X} \cap \mu^{-1}(0)$ that are basic relative to the action of G. The lift of $\hat{\omega}'$ to $X \cap \mu^{-1}(0)$ is the restriction of ω' to $\mu^{-1}(0)$. This form certainly extends to $\overline{X} \cap \mu^{-1}(0)$ since it is the curvature of a connection on L which is defined on all of R'.

4.2.1. *Digression on the restriction of $\hat{\omega}'$ to the singular strata:* We have shown that $\hat{\omega}'$ extends to $\overline{\mathcal{M}}^{\text{alg}}_k$ in the sense that it descends from a basic form on $\overline{X} \cap \mu^{-1}(0)$. It is natural then to ask how the form restricts to the various strata of $\overline{\mathcal{M}}^{\text{alg}}_k$. King shows that the strata are determined by stabilizer type of G and are isomorphic to

$$S_{k,l} = \mathcal{M}^{\text{alg}}_{k-l}(S) \times S^l(\tilde{\mathbb{C}}^2)$$

for each $l = 1, \dots, k$ where S^l denotes symmetric product. Let $G_{k,l} \subset G$ be the stabilizer corresponding to $S_{k,l}$ and let $X_{k,l}$ be the points of $\overline{X} \cap \mu^{-1}(0)$ that have $G_{k,l}$ as stabilizer so that $S_{k,l} = X_{k,l}/G_{k,l}$. We show that

Proposition 4.8. *The restriction of $\hat{\omega}'$ to $S_{k,l}$ which by definition is the descent of $\omega'|_{X_{k,l}}$ is*

$$\pi^*_{k-l}(\hat{\omega}') + \pi^*_l(S^l(\tau_E))$$

where π_{k-l} and π_k are the projections of $S_{k,l}$ to $\mathcal{M}^{alg}_{k-l}(S)$ and $S^l(\widetilde{\mathbb{C}}^2)$ respectively. $\tau_E \in \Omega^2_c(\widetilde{\mathbb{C}}^2)$ is a form representing the Poincaré dual of the embedded surface E and has support in a tubular neighborhood of E.

PROOF: King shows that for $(a_1, a_2, b, c, x) \in X_{k,l}$ there exists a decomposition

$$W_0 = V_0' \oplus V_0,\ W_1 = V_1' \oplus V_1,$$

such that

$$a_i = \begin{pmatrix} a_i' & 0 \\ 0 & a_i^{\triangle} \end{pmatrix},\ x = \begin{pmatrix} x' & 0 \\ 0 & x^{\triangle} \end{pmatrix},\ c = (c', 0), \text{ and } b = \begin{pmatrix} b' \\ 0 \end{pmatrix}, \tag{16}$$

where (a_1', a_2', b', c', x') is a non-degenerate, integrable configuration for V_0' and V_1' and $(a_1^{\triangle}, a_2^{\triangle}, x^{\triangle}, 0, 0)$ is simultaneously diagonalizable in the following sense:

There exists $\{v_0^\alpha, v_1^\alpha\}_{\alpha=1,\dots,l}$ spanning V_0 and V_1 such that $a_i^{\triangle}(v_1^\alpha)$ is proportional to v_0^α and $x^{\triangle}(v_0^\alpha)$ is proportional to v_1^α. Each "eigenpair" (v_0^α, v_1^α) determines a unique point

$$((\lambda_1, \lambda_2), [z_1, z_2]) \in \widetilde{\mathbb{C}}^2 = \{(\lambda, z) \in \mathbb{C}^2 \times \mathbb{P}^1 : \lambda_1 z_1 + \lambda_2 z_2 = 0\}$$

by $x a_i(v_1^\alpha) = \lambda_i v_1^\alpha$, and $(z_1 a_1 + z_2 a_2) v_1^\alpha = 0$ (see lemma 4.1.1 [Ki]).

Furthermore there are hermitian metrics on V_0', V_0, V_1', and V_1 such that

(1) the decompositions $V_i' \oplus V_i$ respect the metric,
(2) (a', b', c', x') satisfy the moment map equations 14 with respect to the metrics on V_0' and V_1',
(3) $(a^{\triangle}, 0, 0, x^{\triangle})$ satisfy the same equations with respect to the metric on V_0 and V_1, and
(4) the $\{v_0^\alpha, v_1^\alpha\}$ are mutually orthogonal.

The fiber of $\hat{L}$ over (a, b, c, x) is canonically $\wedge^{\max} W_0^\star \otimes \wedge^{\max} W_1$ which under the above metric decomposition can be expressed as

$$(\wedge^{\max} V_0'^\star \otimes \wedge^{\max} V_1) \bigotimes_\alpha (V_0^{\alpha\star} \otimes V_1^\alpha).$$

where V_i^α is the span of v_i^α. We thus see that

$$\hat{L}|_{S_{k,l}} = \pi_{k-l}^*(\hat{L}_{k-l}) \otimes \pi_l^*(S^l(L^{\triangle})) \tag{17}$$

where $\hat{L}_{k-l}$ denotes the Kähler bundle over $\mathcal{M}^{\mathrm{alg}}_{k-l}(S)$ and $S^l(L^{\triangle})$ is the bundle induced on $S^l(\hat{\mathbb{C}}^2)$ from a bundle $L^{\triangle} \to \hat{\mathbb{C}}^2$. The connection defining $\hat{\omega}'$ is induced by the metric and the section $\hat{s}$. The metric and the section both decompose naturally with respect to the decomposition 17 and so to prove the proposition we only need to show that the connection on $L^{\triangle} \to \widetilde{\mathbb{C}}^2$ constructed from the metric and the section has curvature τ_E.

Restricting $(a_1^{\triangle}, a_2^{\triangle}, x^{\triangle})$ to $(V_0^{\alpha\star} \otimes V_1^\alpha)$ we see that the points $(\lambda, z) \in \widetilde{\mathbb{C}}^2$ corresponding to $\det x^{\triangle} = 0$ are $(0, z) = E \subset \widetilde{\mathbb{C}}^2$. By remark 3.15, $\hat{\omega}'$ is the pull back of the Thom class of $L^\triangle$ by $\hat{s} = \det x^\triangle$ which, for transverse sections is Poincaré dual to the zero set of the section. $\square$

Remark 4.9. If we analyze the behavior of $\hat{\omega}$ in the above way, we find that $\hat{\omega}$ restricted to $\widetilde{\mathbb{C}}^2$ is the Kähler from induced from the embedding $\widetilde{\mathbb{C}}^2 \subset \mathbb{C}^2 \times \mathbb{P}^1$ and so has infinite volume. This is another indication that the form $\hat{\omega}$ has bad compactness behavior.

To conclude this section we will show that the section $\hat{s}$ satisfies the technical assumption 3.22 so that we can use the localization result in the next chapter.

Lemma 4.10. *$\hat{s}$ satisfies assumption* 3.22.

PROOF: The section is clearly holomorphic. Let $\hat{D} \subset \overline{\mathcal{M}}^{\mathrm{alg}}$ be $\hat{s}^{-1}(0)$ (as we've seen $\hat{s}$ extends to $\overline{\mathcal{M}}^{\mathrm{alg}}$). Let $\hat{\mathcal{S}} \subset \overline{\mathcal{M}}^{\mathrm{alg}}$ be the singular set of $\overline{\mathcal{M}}^{\mathrm{alg}}$, *i.e.* $\hat{\mathcal{S}} = \overline{\mathcal{M}}^{\mathrm{alg}} - \mathcal{M}^{\mathrm{alg}}$. $\overline{\mathcal{M}}^{\mathrm{alg}}$ is algebraic and is the algebraic quotient of $\overline{X}$ by $G_{\mathbb{C}}$ (theorem 5.3.7 [Ki]). By remark 3.23 we need to show that $\hat{D}$ meets $\hat{\mathcal{S}}$ transversely in $\overline{\mathcal{M}}^{\mathrm{alg}}$ and the lemma will follow.

Let $\mathcal{S}, D \subset \overline{X}$ be the preimages of $\hat{\mathcal{S}}$ and $\hat{D}$ by the map $\overline{X} \to \overline{\mathcal{M}}^{\mathrm{alg}}$. We first show that $\mathcal{S}$ intersects D transversely in $\overline{X}$:

Since $\overline{X}$ is smooth and D is a hypersurface, we need to show that for a configuration $\alpha \in \mathcal{S} \cap D$ there is at least one vector $v \in T_\alpha \mathcal{S} \subset T_\alpha \overline{X}$ such that $v \notin T_\alpha D$. Since $\alpha = (a_1, a_2, b, c, x)$ is in $\mathcal{S}$, there is a decomposition $W_0 = V_0' \oplus V_0, W_1 = V_1' \oplus V_1$ such that α can be written as in equation 16. Since $\alpha \in D$ either $\det x' = 0$ or $\det x^\triangle = 0$. Suppose that $\det x^\triangle = 0$, then there is a zero entry on the diagonal of $x^\triangle$ that we can take to be in the bottom right corner. Define a path $\gamma : (-\epsilon, \epsilon) \to \mathcal{S}$ by

$$\gamma(t) = (a_1, a_2, b, c, x + te_{k,k})$$

where $e_{k,k}$ is the matrix with a 1 in the (k,k)-th slot and zero elsewhere. $\gamma(t)$ satisfies the integrability condition and has the same decomposition so lies in $\mathcal{S}$ and $\gamma'(0)$ does not lie in $T_\alpha \overline{X}$. Now suppose that $\det x' = 0$, then (a', b', c', x') defines a (rank $k - l$) integrable, non-degenerate configuration in $s^{-1}(0) \subset X_{k-l}$. We can pick a path $(a'(t), b'(t), c'(t), x'(t)) \subset X_{k-l}$ generating a vector $v' \in T_{(a',b',c',x')} X_{k-l}$ not contained in $T(s^{-1}(0))$. Then

$$\begin{pmatrix} a_i'(t) & 0 \\ 0 & a_i^\triangle \end{pmatrix}, \begin{pmatrix} x'(t) & 0 \\ 0 & x^\triangle \end{pmatrix}, \ldots$$

is in $\mathcal{S}$ and generates a vector $v \in T_\alpha \overline{X}$ that is not in $T_\alpha D$.

We have shown that D intersects $\mathcal{S}$ transversely in $\overline{X}$. The fact that $\hat{D}$ intersects $\hat{\mathcal{S}}$ transversely in $\overline{\mathcal{M}}^{\mathrm{alg}}$ follows from the Luna slice theorem [Lu]:

Let O_α be the $G_{\mathbb{C}}$-orbit of α and let $G_{\mathbb{C}}^{\alpha}$ be the stabilizer of α. Let $N_\alpha(O_\alpha, \overline{X})$, $N_\alpha(O_\alpha, D)$, and $N_\alpha(O_\alpha, \mathcal{S})$ be the normal bundle of O_α in $\overline{X}$, D, and $\mathcal{S}$ respectively. Since $\overline{X}$ is smooth the Luna slice theorem implies that

$$\begin{aligned} T_\alpha \overline{\mathcal{M}}^{\mathrm{alg}} &= N_\alpha(O_\alpha, \overline{X})/G_{\mathbb{C}}^{\alpha}, \\ T_\alpha \hat{D} &= N_\alpha(O_\alpha, D)/G_{\mathbb{C}}^{\alpha}, \\ T_\alpha \hat{\mathcal{S}} &= N_\alpha(O_\alpha, \mathcal{S}) \end{aligned}$$

Since $N_\alpha(O_\alpha, D)$ and $N_\alpha(O_\alpha, \mathcal{S})$ span $N_\alpha(O_\alpha, \overline{X})$, it follows directly from these local models that $T_\alpha \hat{D}$ and $T_\alpha \hat{\mathcal{S}}$ span $T_\alpha \overline{\mathcal{M}}^{\mathrm{alg}}$. $\square$

5. Calculating the Relative Invariant

In this section we combine the results of sections 2, 3 and 4. In subsection 5.1 we study the action of $SO(3)$ on $\mathcal{M}^{\mathrm{alg}}(S)$ and determine an equivariant extension of $\hat{\omega}'$. We use the techniques of section 3 to express the relative invariant coefficients as explicit integrals. In section 5.2 we explicitly compute the integrals for $k = 1$.

5.1. The representative for $\mu_{SO_3}(E)$ and a formula for the pushforward. The action of $SO(3)$ on $\mathcal{M}^0(\overline{\mathbb{CP}}^2)$ is induced by automorphisms of the fiber E_{x_0} over x_0. For $\mathcal{M}^{\mathrm{alg}}(S)$, the action is induced by automorphisms of the trivial bundle $V_\infty \times \ell_\infty \to \ell_\infty$.

The $SO(3)$ action is induced by the standard action of $SU(2)$ on V_∞. In terms of the monad configurations, the action is given by

$$h \cdot (a_1, a_2, x, b, c) = (a_1, a_2, x, bh^{-1}, hc)$$

where $h \in SU(2)$.

This action clearly preserves the integrability and non-degeneracy conditions (4.2 and 4.3) and so induces an action of $SU(2)$ on X. The action commutes with the action of $G_{\mathbb{C}}$ and so descends to an action on $X/G_{\mathbb{C}} = \mathcal{M}^{\mathrm{alg}}(S)$. $-I \in SU(2)$ acts trivially on the quotient since $-I(a_1, a_2, x, b, c) = (a_1, a_2, x, -b, -c)$ is in the same $G_{\mathbb{C}}$ orbit as (a_1, a_2, x, b, c). It follows that $SU(2)/\pm I$ acts on $\mathcal{M}^{\mathrm{alg}}(S)$ and this is the $SO(3)$ action.

Claim 5.1. *$SO(3)$ acts symplectically on (X, ω) and $(\mathcal{M}^{alg}(S), \hat{\omega})$.*

Proof: The action of $SU(2)$ on X lifts to L by acting trivially on the fibers. It clearly preserves the hermitian norm h and so it preserves ω. The action also preserves the moment map equations 14 and commutes with the action of G and so the descended action preserves $\hat{\omega}$ and so $SO(3) = SU(2)/\{\pm I\}$ must also. $\square$

We compute the moment map $f : X \to \mathfrak{so}(3)^*$. Let $B \in \mathfrak{so}(3)$, then the moment map is given (see section 3.1) by

$$f_B = -i_{\tilde{B}}\alpha.$$

The following equations determine all the non-zero components of the vector field $\tilde{\tilde{B}}$:

$$i_{\tilde{\tilde{B}}}db = -bB \qquad i_{\tilde{\tilde{B}}}db^* = Bb^*$$
$$i_{\tilde{\tilde{B}}}dc = Bc \qquad i_{\tilde{\tilde{B}}}dc^* = -c^*B,$$

and

$$\alpha = d^c \log|\xi|^2 + \\ d^c \operatorname{tr}(a_1a_1^* + a_2a_2^* + bb^* + cc^* + (xb)(xb)^* + (xa_1)(xa_1)^* + (xa_2)(xa_2)^*).$$

Computing (c.f. example 3.8) we get:

$$f_B = \tfrac{1}{2\pi i}\langle B, -b^*b - (xb)^*(xb) + cc^*\rangle. \tag{18}$$

Since $SU(2)$ has no central factors, the above moment map is the unique one. Again since the actions of $SU(2)$ and G commute, f descends to a moment map $\hat{f}$ on $\mathcal{M}^{\mathrm{alg}}(S)$ defined by $f|_{\mu^{-1}(0)} = p^*(\hat{f})$.

Our section $s = \det(x)$ is $SU(2)$ equivariant and so by the constructions of section 3.2, $SU(2)$ acts pre-symplectically on ω' and $\hat{\omega}'$ with moment maps f' and $\hat{f}'$.

$\hat{f}'$ determines an equivariant extension $(\hat{\omega})'^{\#}$ of $\hat{\omega}'$ which is our representative for $\mu_{SO(3)}(E)$. As Taubes' theorem dictates, we wish to calculate the numbers $\Xi_{k,n}$ defined by

$$\Xi_{k,n}p_1^{n-2} = \pi_*((\mu_{SO(3)}(E))^{2n}) = \pi_*((\hat{\omega}'^{\#})^{2n})$$

where we are considering π_* defined on the level of forms by integration.

We have demonstrated that our form $\hat{\omega}'$ and it's extension $\hat{\omega}'^{\#}$ are good representatives in the sense that they satisfy conditions 1–3 from page 6. Then from the discussion there we have good reason to

Conjecture 5.2. $\Xi_{k,n} = \alpha_{k,n}$ *where* $\alpha_{k,n}$ *are the blow-up coefficients.*

From the discussion in section 3.3 we can calculate $\Xi_{k,n}$ by choosing an $S^1 \subset SO(3)$ and computing the pushforward of $\hat{\omega}'^{\#}_{S^1}$ in S^1-equivariant cohomology then restricting to the Weyl group invariants. From the parity of the dimension of the moduli space and the power we are taking it is clear that $\pi_*((\hat{\omega}'^{\#})^{2n})$ will be invariant under the Weyl group.

We now choose $S^1 \subset SO(3)$. This is equivalent to choosing a vector $B \in \mathfrak{so}(3) \cong \mathfrak{su}(2)$. Fix

$$B = \begin{pmatrix} \pi i & 0 \\ 0 & -\pi i \end{pmatrix} \tag{19}$$

so that the generated $S^1 \subset SO(3)$ lifts to $SU(2)$ matrices of the form $\begin{pmatrix} e^{\pi i\theta} & 0 \\ 0 & e^{-\pi i\theta} \end{pmatrix}$.

Fix a hermitian basis for V_∞ and decompose $b = (b_1, b_2)$, $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. Then

$$\begin{aligned} f_B &= \tfrac{\pi i}{2\pi i}\,\mathrm{tr}\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -b_1^*b_1-(xb_1)^*(xb_1)+c_1c_1^* & -b_1^*b_2-(xb_1)^*(xb_2)+c_1c_2^* \\ -b_2^*b_1-(xb_2)^*(xb_1)+c_2c_1^* & -b_2^*b_2-(xb_2)^*(xb_2)+c_2c_2^* \end{pmatrix}\right\} \\ &= \tfrac{1}{2}(||b_2||^2 - ||b_1||^2 + ||xb_2||^2 - ||xb_1||^2 + ||c_1||^2 - ||c_2||^2) \end{aligned} \tag{20}$$

We have shown (lemma 4.10) that we can apply the localization techniques of section 3 to $\hat{\omega}'_{S^1}\#$. Applying the equivariant formulation of the localization (proposition 3.32) we get

$$\begin{aligned} \Xi_{k,n} p_1^{n-2} &= \pi_*^{\mathcal{M}}((\hat{\omega}'^{\#}_{S^1})^{2n}) \\ &= -\pi_*^{s^{-1}(0)}((\hat{\omega}'^{\#}_{S^1})^{2n-1}) \\ &= -\pi_*^{s^{-1}(0)}((\hat{\omega} - \hat{f}_B B^*)^{2n-1}) \\ &= -\binom{2n-1}{4k-1}\int_{s^{-1}(0)} (-\hat{f}_B)^{2n-4k}(\hat{\omega})^{4k-1}(B^*)^{2n-4k}. \end{aligned}$$

We identify $(B^*)^2$ with the generator p_1 of $H^*_{SO(3)}(\mathrm{pt.}) \cong \mathbb{R}[p_1]$ and we get the result:

Theorem 5.3.

$$\Xi_{k,n} = -\binom{2n-1}{4k-1}\int_{s^{-1}(0)} (\hat{f}_B)^{2n-4k}(\hat{\omega})^{4k-1}.$$

We thus have a formula expressing the coefficients $\Xi_{k,n}$ as integrals of an explicit, canonical differential forms over explicitly constructed spaces. We carry out examples of these integrals in the next section.

5.2. Pushing forward $\mu_{SO_3}(E)$ for $k = 1$. In this section we apply theorem 5.3 to calculate the coefficients $\Xi_{k,n}$ for $k = 1$ and all n. To do this we will write down an explicit open dense coordinate patch for $\mathcal{M}_1^{\mathrm{alg}}(S)$.

We first examine the complex structure of $\mathcal{M}_1^{\mathrm{alg}}(S)$ by regarding it as the complex quotient $X/G_{\mathbb{C}}$. Identifying W_0 and W_1 with $\mathbb{C}$ we have $a = (a_1, a_2) \in \mathbb{C}^2$, $x \in \mathbb{C}$, $b \in (\mathbb{C}^2)^\star$, and $c \in \mathbb{C}^2$. The integrability condition is $bc = 0$ and the non-degeneracy conditions reduce to $c \neq 0$ and $b \neq 0$.

The action of $(\lambda_0, \lambda_1^{-1}) \in G_{\mathbb{C}} = \mathbb{C}^* \times \mathbb{C}^*$ is given by

$$(a, b, c, x) \mapsto (\lambda_0\lambda_1 a, \lambda_0 b, \lambda_1 c, (\lambda_0\lambda_1)^{-1}x).$$

The kernel of b and the image of c coincide and hence determine a line in $\mathbb{C}^2$ up to automorphisms. We thus can think of (b, c) determining a point in $\mathbb{P}^1$ and the projection

$$\{(b, c) \in (\mathbb{C}^2)^\star \times \mathbb{C}^2 : bc = 0, b \neq 0, c \neq 0\} \to \mathbb{P}^1$$

as the tautological principal $\mathbb{C}^* \times \mathbb{C}^*$-bundle over $\mathbb{P}^1$. We then see that as a complex manifold, $\mathcal{M}_1^{\mathrm{alg}}(S)$ is the total space of the bundle

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2).$$

Calculating the curvature form of L we see

$$\omega = \tfrac{1}{2\pi i}\bar{\partial}\partial\{(||a||^2 + ||b||^2)(1 + ||x||^2) + ||c||^2\}. \tag{21}$$

From equation 20 we get

$$f_B = f = \tfrac{1}{2}\{(1 + |x|^2)(|b_2|^2 - |b_1|^2) + |c_1|^2 - |c_2|^2\}.$$

To compute $\hat{\omega}$ and $\hat{f}$ we must restrict to the zero set of the moment map for the $U(1) \times U(1)$-action and then restrict to a slice of that action.

To find a dense coordinate patch of $\mathcal{M}_1^{\mathrm{alg}}(S)$ we further restrict to an affine set in the $\mathbb{P}^1$ determined by b and c: let $\xi = b_1/b_2 = -c_2/c_1$ be the affine coordinate on $\mathbb{P}^1$. Fix a slice through the orbit of $U(1) \times U(1)$ by requiring that b_2 and c_1 to be real and positive. We wish to express $\hat{\omega}$ and $\hat{f}$ in terms of the coordinates a_1, a_2, x, and ξ.

The moment map conditions (equations 14) reduce to

$$||a||^2 + ||b||^2 = 1, \text{ and } ||b||^2(1 + |x|^2) = ||c||^2$$

which, restricted to our slice can be rewritten as

$$\begin{aligned}
b_2 &= \sqrt{\frac{1 - ||a||^2}{1 + |\xi|^2}},\\
b_1 &= \xi b_2,\\
c_1 &= \sqrt{\frac{(1 - ||a||^2)(1 + |x|^2)}{1 + |\xi|^2}},\\
c_2 &= -\xi c_1.
\end{aligned}$$

The above equations are now enough to express $\hat{\omega}$ and $\hat{f}$ in the coordinates of our patch

$$\mathcal{U} = \{(a_1, a_2, x, \xi) \in \mathbb{C}^4 : ||a||^2 < 1\}.$$

Substituting we see that

$$\hat{f} = (1 + |x|^2)(1 - ||a||^2)\frac{1 - |\xi|^2}{1 + |\xi|^2}$$

and similarly we can express $\hat{\omega}$ as a rather long expression in terms of our coordinates. We then can calculate, for instance, that

$$\hat{\omega}^4 = (2\pi i)^{-4}\frac{48(1 - ||a||^2)(1 + |x|^2)^3}{(1 + |\xi|^2)^2} d\bar{a}_1 da_1 d\bar{a}_2 da_2 d\bar{x} dx d\bar{\xi} d\xi$$

which, when integrated over $\mathcal{M}_1^{\mathrm{alg}}(S)$, is clearly infinite, thus proving claim 4.7 .

Theorem 5.3 prescribes that we restrict to $s^{-1}(0)$, *i.e.* $x = 0$. On this set we compute that

$$\hat{\omega}^3|_{x=0} = 12(2\pi i)^{-3}\frac{1-||a||^2}{(1+|\xi|^2)^2}d\bar{a}_1 da_1 d\bar{a}_2 da_2 d\bar{\xi}d\xi.$$

So finally we can write

$$\Xi_{1,n} = -12(2\pi i)^{-3}\tbinom{2n-1}{3}\int_{\mathcal{U}|_{x=0}}\frac{(1-|\xi|^2)^{2n-4}(1-||a||^2)^{2n-3}}{(1+|\xi|^2)^{2n-2}}d\bar{a}_1 da_1 d\bar{a}_2 da_2 d\bar{\xi}d\xi.$$

Now $\frac{d\bar{\xi}d\xi}{2\pi i} = \frac{d|\xi|^2 d(\arg\xi)}{2\pi}$ for example, so defining $r = |\xi|^2$ and $r_i = |a_i|^2$ we can reduce the integral to

$$\Xi_{1,n} = -12\tbinom{2n-1}{3}\int_0^1\int_0^{1-r_2}(1-r_1-r_2)^{2n-3}dr_1 dr_2\int_{r=0}^{\infty}\frac{(1-r)^{2n-4}}{(1+r)^{2n-2}}dr. \tag{22}$$

We now calculate each integral separately:

$$\int_0^1\int_0^{1-r_2}(1-r_1-r_2)^{2n-3}dr_1 dr_2 = \frac{1}{(2n-2)(2n-1)}$$

$$\begin{aligned}\int_0^{\infty}\frac{(1-r)^{2n-4}}{(1+r)^{2n-2}}dr &= \int_1^{\infty}\frac{(2-v)^{2n-4}}{v^{2n-2}}dv\\ &= \int_1^{\infty}(\tfrac{2}{v}-1)^{2n-4}\tfrac{dv}{v^2}\\ &= -\tfrac{1}{2}\int_1^{-1}w^{2n-4}dw\\ &= \tfrac{1}{2n-3}.\end{aligned}$$

Theorem 5.4. $\Xi_{1,n} = -2$ *for all* n.

PROOF: Combining the above computations and substituting back into equation 22 we get

$$\begin{aligned}\Xi_{1,n} &= -12\tbinom{2n-1}{3}\tfrac{1}{(2n-3)(2n-2)(2n-1)}\\ &= -2. \quad\square\end{aligned}$$

Recall that the numbers $\Xi_{k,n}$ require a choice of a generator p_1 for $H^*_{SO(3)}(\mathrm{pt.})$. In our computation this choice was selected by the length of our fixed vector

$$B = \begin{pmatrix}\pi i & 0\\ 0 & -\pi i\end{pmatrix}.$$

This is a natural choice of a length in that B is the shortest non-zero vector in the kernel of the exponential map. If instead we had chosen the shortest vector in the kernel of the exponential map to $SU(2)$, *i.e.* $2B$, then we would get

$$\Xi_{1,n} = -2(4)^{n-2}.$$

This is the normalization used by Kronheimer and Mrowka and is in particular necessary to recover their convention for the case of simple type.

To evaluate the integrals in coordinates for $k > 1$, one needs to find a slice for the action of $G = U(k) \times U(k)$. For $k = 2$, the requirement that b and c have non-negative real eigenvalues fixes a slice of the action and makes it easy to invert the moment map equations. One can thus procede as with the $k = 1$ case, however the symbolic calculus becomes intractible. These integrals could, in principle be computed numerically.

The form of the integrand in theorem 5.3 suggests an alternate approach to computing the higher coefficients. For compact symplectic manifolds, the integral over the symplectic volume of some power of the moment map can be computed in terms of fixed point data for the action. As discussed in digression 4.2.1, the non-compactness arises from non-trivial stabilizers in the action of G. One can resolve such singularities by blowing up the strata associated to non trivial stabilizers and then performing symplectic reduction. The symplectic form changes in an understandable way and one should be able to relate the integral in theorem 5.3 to an integral of the same shape over a smooth space. The hope would then be to apply fixed point formulas to these integrals.

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