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Evidence for a conjecture of Pandharipande

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1. Introduction

In [3], Pandharipande studied the relationship between the enumerative geometry of certain 3-folds and the Gromov-Witten invariants. In some good cases, enumerative invariants (which are manifestly integers) can be expressed as a rational combination of Gromov-Witten invariants. Pandharipande speculated that the same combination of invariants should yield integers even when they do not have any enumerative significance on the 3-fold. In the case when the 3-fold is the product of a complex surface and an elliptic curve, Pandharipande has computed this combination of invariants on the 3-fold in terms of the Gromov-Witten invariants of the surface [4]. This computation yields surprising conjectural predictions about the genus 0 and genus 1 Gromov-Witten invariants of complex surfaces. The conjecture states that certain rational combinations of the genus 0 and genus 1 Gromov-Witten invariants are always integers. Since the Gromov-Witten invariants for surfaces are often enumerative (as oppose to 3-folds), this conjecture can often also be interpreted as giving certain congruence relations among the various enumerative invariants of a surface.

In this note, we state Pandharipande's conjecture and we prove it for an infinite series of classes in the case of $\mathbf{CP}^2$ blown-up at 9 points. In this case, we find generating functions for the numbers appearing in the conjecture in terms of quasi-modular forms (Theorem 3.1). We then prove the integrality of the numbers by proving a certain a congruence property of modular forms that is reminiscent of Ramanujan's mod 5 congruences of the partition function (Theorem 3.2).

2. The conjecture

Let X be a smooth complex projective surface (or more generally, a symplectic 4-manifold), let K be its canonical class, and let $\chi(X)$ be its Euler characteristic. Let $\beta \in H_2(X,\mathbf{Z})$ and let $g(\beta)$ be defined by $2g(\beta)-2=\beta\cdot(K+\beta)$. Define $c(\beta)$ to be $-\beta\cdot K$ and assume that $c(\beta)>0$. Let $N^r(\beta)$ be the genus r Gromov-Witten invariant of X in the class β where we have imposed $c(\beta)+r-1$ point constraints. By convention we will say $N^r(0)=0$.

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Conjecture 2.1 (Pandharipande). *Define* $a(\beta)$ *by*

$$a(\beta)=-\frac{1}{12}g(\beta)N^{0}(\beta)$$

and define $b(\beta)$ *by*

$$\begin{aligned}
b(\beta)= &\ \frac{1}{2880}\left(12g(\beta)^{2}+g(\beta)c(\beta)-24g(\beta)\right)N^{0}(\beta)\\
&+\frac{1}{240}\chi(X)N^{1}(\beta)\\
&+\frac{1}{240}\sum_{\beta'+\beta''=\beta}\binom{c(\beta)-1}{c(\beta')}(\beta'\cdot\beta'')(\beta''\cdot\beta'')N^{1}(\beta')N^{0}(\beta'').
\end{aligned}$$

Then $a(\beta)$ *and* $b(\beta)$ *are integers.*

Remark 2.1. This conjecture is related to the proposal of Gopakumar and Vafa that relates the Gromov-Witten invariants of Calabi-Yau 3-folds to conjecturally integer valued invariants ("BPS state counts", or "BPS invariants"). Pandharipande has generalized the Gopakumar-Vafa formula to Fano classes in non-Calabi-Yau 3-folds (see [3]). In this formulation, the numbers $a(\beta)$ and $b(\beta)$ are respectively genus 1 and genus 2 "BPS invariants" for the surface cross an elliptic curve. The reason that these are expressible in terms of ordinary Gromov-Witten invariants of the surface is that the Hodge class in $\overline{M}_g$ (which appears in the computation of the virtual class) is readily expressible in terms of boundary classes for $g=1$ and $g=2$. For arbitrary g there will also be predictions for the invariants of the surface, but they will involve gravitational descendants in general.

3. The case of $\mathbf{CP}^2$ blown-up at 9 points

Let X be $\mathbf{CP}^2$ blown up at nine points. Let $F=-K$ be the anti-canonical class and let S be the exceptional divisor of one of the blow-ups (so if X is elliptically fibered, then F is the fiber and S is a section). Let $\beta_n=S+nF$. Then $N^r(\beta_n)$ was computed in [1]. We will find a nice generating functions for the numbers $a(\beta_n)$ and $b(\beta_b)$ and will prove that they are integers thus verifying Pandharipande's conjecture for X for this infinite series of classes.

Note that $c(\beta_n)=1$, $g(\beta_n)=n$, and $\chi(X)=12$. Since for $N^0(\beta'')$ to be non-zero, we need $c(\beta'')=1$, the sum must have $c(\beta'')=1$ and $c(\beta')=0$. It follows that β'' and β'

are of the form $S+kF$ and $(n-k)F$ respectively. Thus we have

$$\begin{aligned}
a(\beta_n) = & \frac{1}{12} n N^0(\beta_n) \\
b(\beta_n) = & \frac{1}{2880}(12n^2-23n)N^0(\beta_n) \\
& + \frac{1}{20} N^1(\beta_n) \\
& + \frac{1}{240} \sum_{k=0}^{n-1} (n-k)(2k-1)N^1((n-k)F)N^0(\beta_k).
\end{aligned}$$

Define

$$\begin{aligned}
A(q) &= \sum_{n=0}^{\infty} a(\beta_n) q^n, \\
B(q) &= \sum_{n=0}^{\infty} b(\beta_n) q^n.
\end{aligned}$$

We will find an expression for $A(q)$ and $B(q)$ in terms of quasi-modular forms. Let $\sigma(k) = \sum_{d|k} d$ and let $p(k)$ be the number of partitions of k. Define

$$\begin{aligned}
G(q) &= \sum_{k=1}^{\infty} \sigma(k) q^k, \\
P(q) &= \sum_{k=1}^{\infty} p(k) q^k \\
&= \prod_{m=1}^{\infty} (1-q^m)^{-1}, \\
P_\alpha(q) &= (P(q))^\alpha, \\
D &= q\frac{d}{dq}.
\end{aligned}$$

Note that G and P are closely related to well known (quasi-) modular forms: $G - 1/24$ is the Eisenstein series G_2 and $q^{1/24}P_{-1}$ is the Dedekind η function.

With this notation, the results of [1] (Theorem 1.2) give

$$\begin{aligned}
\sum_{n=0}^{\infty} N^0(\beta_n) q^n &= P_{12} \\
\sum_{n=1}^{\infty} N^1(\beta_n) q^n &= P_{12} DG.
\end{aligned}$$

Furthermore, one can show that

$$N^1(lF) = \frac{1}{l}\sigma(l)$$

(when the blow-up points are generic, this comes from the multiple covers of the unique elliptic curve in the class F). We thus have

$$\begin{aligned}
A(q) &= \frac{1}{12}DP_{12} \\
B(q) &= \frac{1}{2880}(12D^2 - 23D)P_{12} + \frac{1}{20}P_{12}DG \\
&+ \frac{1}{240}\sum_{n\geq 1}\sum_{k=0}^{n-1}(2k-1)\sigma(n-k)N^0(\beta_k)q^{n-k}q^k \\
&= \frac{1}{240}D^2P_{12} - \frac{23}{2880}DP_{12} + \frac{1}{20}P_{12}DG \\
&+ \frac{1}{240}\sum_{m\geq 1}\sum_{k\geq 0}(2k-1)\sigma(m)N^0(\beta_k)q^kq^m \\
&= \frac{1}{240}D^2P_{12} - \frac{23}{2880}DP_{12} + \frac{1}{20}P_{12}DG + \frac{1}{240}G(2DP_{12} - P_{12})
\end{aligned}$$

Now, by a standard calculation, $G = P_{-1}DP$ and so $DP_{12} = 12P_{12}G$. Substituting and simplifying we arrive at:

Theorem 3.1. *The following equations holds:*

$$\begin{aligned}
A(q) &= P_{12}\cdot G \\
B(q) &= \frac{1}{10}P_{12}\left\{7G^2 - G + DG\right\}.
\end{aligned}$$

This theorem immediately shows that the coefficients of A are integers. On the other hand, the integrality of the coefficients of B requires the following theorem:

Theorem 3.2. *The following equation holds:*

$$7G^2 - G + DG \equiv 0 \pmod{10}.$$

Proof: By a simple calculation mod 5, we have:

$$7G^2 - G + DG \equiv 3P_{-2}(D^2 - D)P_2 \pmod 5$$

and so to prove that the above expression is 0 mod 5, it suffices to prove that $(D^2 - D)P_2 \equiv 0 \pmod 5$. Using the Jacobi triple product formula and the Euler inversion formula, it is easy to show that the kth coefficient of $P_2 = P_{-3}P_5$ is divisible by 5 unless k is 0 or 1 mod 5 (see [2]). In other words:

$$P_2(q) \equiv r(q^5) + qs(q^5) \pmod 5.$$

It follows that $DP_2 \equiv qs(q^5) \pmod 5$ and so $D^2P_2 \equiv DP_2 \pmod 5$ as desired.

On the other hand, it is easy to compute that

$$7G^2 - G + DG \equiv P_{-1}(D^2 + D)P \pmod{2}.$$

This expression is 0 mod 2 since the kth coefficient of $(D^2 + D)P$ is $k(k+1)p(k)$.

Thus we have established that $7G^2 - G + DG$ is 0 mod 2 and mod 5 and so the theorem is proved. □

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