THE ORBIFOLD QUANTUM COHOMOLOGY OF $\mathbb{C}^2/\mathbb{Z}_3$ AND HURWITZ-HODGE INTEGRALS

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Abstract. Let $\mathbb{Z}_3$ act on $\mathbb{C}^2$ by non-trivial opposite characters. Let $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_3]$ be the orbifold quotient, and let Y be the unique crepant resolution. We show the equivariant genus 0 Gromov-Witten potentials $F^{\mathcal{X}}$ and F^Y are equal after a change of variables — verifying the Crepant Resolution Conjecture for the pair $(\mathcal{X}, Y)$. Our computations involve Hodge integrals on trigonal Hurwitz spaces which are of independent interest. In a self contained Appendix, we derive closed formulas for these Hurwitz-Hodge integrals.

1. Introduction

The Crepant Resolution Conjecture predicts the Gromov-Witten theory of a Gorenstein orbifold $\mathcal{X}$ is equivalent to the Gromov-Witten theory of any crepant resolution Y. The conjecture was originally formulated in physics by Zaslow and Vafa [10, 9] and subsequently in mathematics by Chen and Ruan [3]. A precise statement of the general conjecture is given in [1].

One impediment to understanding the Crepant Resolution Conjecture is the dearth of non-trivial examples where the full Gromov-Witten theory (even in genus 0) of $\mathcal{X}$ and Y has been computed. In [1], the genus 0 (equivariant) Crepant Resolution Conjecture is verified in the cases

$$(\mathcal{X}, Y) = (\mathbb{C}^2/\mathbb{Z}_2, T^*\mathbb{P}^1), \quad (\mathcal{X}, Y) = (\mathrm{Sym}^d\,\mathbb{C}^2, \mathrm{Hilb}^d\,\mathbb{C}^2).$$

These examples, while highly non-trivial, are limited in their ability to exhibit many of the features of the general conjecture. In particular, since the Picard numbers are 1, the change of variables has a restricted form.

Our main result is the proof of the equivariant genus 0 Crepant Resolution Conjecture for the orbifold $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_3]$ with unique crepant resolution Y. Here, the Picard number is 2, and we see a more complicated transformation taking place. Our computations involve new integrals of Hodge classes over trigonal Hurwitz spaces.

1.1. **Notation.** Let $\mathbb{Z}_3 \subset SU(2)$ act on $\mathbb{C}^2$ via the standard representation of $SU(2)$. Let

$$\omega = e^{2\pi i/3}.$$

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We identify $\mathbb{Z}_3$ with $\{1, \omega, \overline{\omega}\}$. The $\mathbb{Z}_3$-action on $\mathbb{C}^2$ is

$$\omega \cdot (x, y) = (\omega x, \overline{\omega} y).$$

Let $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_3]$ be the quotient stack with coarse moduli space X. The singular variety X admits a unique crepant resolution

$$Y \to X.$$

The exceptional divisor is a chain of two rational curves E_1 and E_2. The action of the torus

$$T = \mathbb{C}^* \times \mathbb{C}^*$$

on $\mathbb{C}^2$ commutes with the $\mathbb{Z}_3$-action and induces T-actions on $\mathcal{X}$ and Y.

The potential F^Y is the generating function for equivariant genus 0 Gromov-Witten invariants of Y:

$$F^Y = \sum_{\beta = d_1[E_1]+d_2[E_2]} \sum_{n_0,n_1,n_2 \geq 0} \langle 1^{n_0} C_1^{n_1} C_2^{n_2} \rangle_\beta^Y \frac{y_0^{n_0}}{n_0!} \frac{y_1^{n_1}}{n_1!} \frac{y_2^{n_2}}{n_2!} q_1^{d_1} q_2^{d_2}.$$

The first sum ranges over effective curve classes β. The classes $C_i \in H_T^*(Y)$ are defined as

$$\begin{aligned} C_1 &= -\frac{2}{3}[E_1] - \frac{1}{3}[E_2] \\ C_2 &= -\frac{1}{3}[E_1] - \frac{2}{3}[E_1]. \end{aligned}$$

The images of the C_i in $H^*(Y)$ are Poincare dual to the proper transforms of the images of the two coordinate axes in $\mathbb{C}^2$ but the equivariant lifts here are chosen to make them dual to the $[E_i]$ with respect to the equivariant intersection form. The Gromov-Witten invariants $\langle \cdot \rangle_\beta^Y$ are multilinear functions on $H_T^*(Y)$ taking values in

$$H_T^*(pt) = \mathbb{Q}[t_1, t_2].$$

The unstable terms, where $d_1 = d_2 = 0$ and $n_0 + n_1 + n_2 < 3$, are defined to be zero.

For the orbifold $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_3]$, we have a basis for $H^*_{orb,T}(\mathcal{X})$, the T-equivariant *orbifold* cohomology of $\mathcal{X}$, given by classes $\{1, D_1, D_2\}$ corresponding to the elements $\{1, \omega, \overline{\omega}\}$ of $\mathbb{Z}_3$. The potential $F^{\mathcal{X}}$ generates the equivariant genus 0 orbifold Gromov-Witten invariants of $\mathcal{X}$:

$$F^{\mathcal{X}} = \sum_{n_0,n_1,n_2 \geq 0} \langle 1^{n_0} D_1^{n_1} D_2^{n_2} \rangle^{\mathcal{X}} \frac{x_0^{n_0}}{n_0!} \frac{x_1^{n_1}}{n_1!} \frac{x_2^{n_2}}{n_2!}.$$

The bracket $\langle \cdot \rangle^{\mathcal{X}}$ denotes the equivariant degree 0, genus 0 orbifold Gromov-Witten invariant of $\mathcal{X}$. Again, we set unstable terms (those with fewer than three insertions) equal to zero.

1.2. **Results.** The main result of the paper is the complete computation of the potential functions $F^{\mathcal{X}}$ and F^Y. The computations verify the Crepant Resolution Conjecture for the pair $(\mathcal{X}, Y)$.

Theorem 1.1. *The equivariant genus 0 Gromov-Witten potential of Y is:*

$$\begin{aligned} F^Y = \; & \frac{y_0^3}{18t_1t_2} - \frac{y_0}{3}(y_1^2 + y_1y_2 + y_2^2) \\ & + \frac{1}{3}(t_1 + 2t_2)\frac{y_1y_2^2}{2} + \frac{2}{3}(2t_1 + t_2)\frac{y_1^3}{6} + \frac{1}{3}(2t_1 + t_2)\frac{y_1^2y_2}{2} + \frac{2}{3}(t_1 + 2t_2)\frac{y_2^3}{6} \\ & + (t_1 + t_2)\sum_{d=1}^{\infty}\frac{1}{d^3}\left[(e^{y_1}q_1)^d + (e^{y_2}q_2)^d + (e^{y_1+y_2}q_1q_2)^d\right]. \end{aligned}$$

Theorem 1.2. *The equivariant genus 0 Gromov-Witten potential of $\mathcal{X}$ is:*

$$\begin{aligned} F^{\mathcal{X}} = \; & \frac{1}{18t_1t_2}x_0^3 + \frac{1}{3}x_0x_1x_2 + \frac{1}{18}t_1x_1^3 + \frac{1}{18}t_2x_2^3 \\ & + \frac{(t_1 + t_2)}{2}\sum_{g=2}^{\infty}\frac{(-1)^{g-1}A_g}{(g+2)!}\frac{1}{3}\Big[(x_1 + x_2)^{g+2} + (\omega x_1 + \overline{\omega}x_2)^{g+2} \\ & \qquad + (\overline{\omega}x_1 + \omega x_2)^{g+2}\Big], \end{aligned}$$

where the rational numbers A_g are determined by:

$$A(u) = \sum_{g=1}^{\infty} A_g \frac{u^{g-1}}{(g-1)!} = \frac{1}{\sqrt{3}}\tan\left(\frac{u}{\sqrt{12}} + \frac{\pi}{6}\right).$$

Geometrically, A_g arises as the integral of the Hodge class λ_{g-1} over any connected component of the Hurwitz scheme of curves in $\overline{M}_g$ which admit a cyclic triple cover of $\mathbb{P}^1$. In the Appendix, which is written to be self-contained, we prove A_g is independent of the choice of component (Proposition A.1) and is given by the above formula (Proposition A.2). We also prove formulas for related trigonal Hurwitz-Hodge integrals (Propositions A.3).

The series F^Y converges at $q_i = \omega$, in particular, the change of variables

$$y_0 = x_0 \tag{1}$$

$$y_1 = \frac{i}{\sqrt{3}}(\omega x_1 + \overline{\omega}x_2) \tag{2}$$

$$y_2 = \frac{i}{\sqrt{3}}(\overline{\omega}x_1 + \omega x_2) \tag{3}$$

$$q_i = \omega \tag{4}$$

is well-defined.

Theorem 1.3. *After the above change of variables,*

$$F^{\mathcal{X}} = F^Y$$

as power series in x_0, x_1, and x_2 up to unstable terms. Hence, the equivariant genus 0 Crepant Resolution Conjecture holds for $(\mathcal{X}, Y)$.

Corollary 1.4. *The equivariant quantum cohomology rings $QH^*_{T,orb}(\mathcal{X})$ and $QH^*_T(Y)$ are isomorphic after the above change of variables.*

The correct definition of quantum cohomology for Gorenstein orbifolds requires the notion of quantum parameters in the twisted sector, see [1].

1.3. **DuVal singularities.** Let $G \subset SU(2)$ be a finite subgroup. Let

$$\mathcal{X} = [\mathbb{C}^2/G]$$

be the orbifold quotient, and let Y be the unique crepant resolution of the DuVal singularity X.

By the McKay correspondence, the cohomology of Y has a natural basis indexed by irreducible representations of G, where the trivial representation $\mathbb{C}$ corresponds to the identity in $H^0(Y)$ and non-trivial representations R correspond to classes in $H^2(Y)$. The orbifold cohomology of $\mathcal{X}$ has a natural basis indexed by conjugacy classes of G, where the trivial conjugacy class (e) corresponds to the identity in $H^0_{orb}(\mathcal{X})$ and non-trivial conjugacy classes (g) correspond to classes in $H^2_{orb}(\mathcal{X})$.

We speculate that the potential functions $F^{\mathcal{X}}$ and F^Y are identified by the change of variables:

$$\begin{aligned} y_{\mathbb{C}} &= x_{(e)}, \\ y_R &= \frac{1}{|G|}\sum_{g\in G} (\chi_\rho(g) - 2)^{1/2}\, \chi_R(g)\, x_{(g)}, \\ q_R &= \omega^{n_R}. \end{aligned}$$

Here ρ is the standard representation of $G \subset SU(2)$ on $\mathbb{C}^2$, ω is a primitive $|G|$-th root of unity, and n_R is the coefficient of R in the representation corresponding the the longest root of the associated Dynkin diagram.

The above change of variables specializes to equations (1)–(4) for the case of $\mathbb{C}^2/\mathbb{Z}_3$.

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2. The Gromov-Witten invariants of Y

We compute the equivariant genus 0 Gromov-Witten invariants of Y via an equivariant embedding into a Calabi-Yau threefold $\widetilde{Y}$ for which the Gromov-Witten invariants have been previously computed,

Consider the threefold $\widetilde{X} \subset \mathbb{C}^4$ given by the equation

$$xy = z(z-s)(z+s).$$

$\widetilde{X}$ admits a small resolution $\widetilde{Y} \subset \mathbb{C}^4 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by the closure of the graph of the rational map

$$\begin{gathered} \widetilde{X} \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ (x,y,z,s) \mapsto (x:z),(x:z(z-s)) \end{gathered}$$

see [2, 7].

The surfaces X and Y are isomorphic to the subvarieties of $\widetilde{X}$ and $\widetilde{Y}$ defined by $s=0$. This construction is T equivariant under the action

$$\begin{gathered} (x,y,z,s) \mapsto (t_1^3 x, t_2^3 y, t_1t_2 z, t_1t_2 s) \\ (u_1:v_1),(u_2:v_2) \mapsto (t_1^2 u_1 : t_2 v_1),(t_1 u_2 : t_2^2 v_2) \end{gathered}$$

The exceptional set of the resolution $\widetilde{Y} \to \widetilde{X}$ consists of two rational curves $E_1 \cup E_2$ meeting in a point p_1. $\widetilde{Y}$ is a Calabi-Yau threefold and the normal bundle of $E_i \subset \widetilde{Y}$ is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

The Gromov-Witten invariants of $\widetilde{Y}$ were computed by Bryan-Katz-Leung (see Proposition 2.10 of [2]). For

$$\beta = d_1E_1 + d_2E_2 \neq 0,$$

the genus 0 invariants of $\widetilde{Y}$ are given by

$$\langle\,\rangle_\beta^{\widetilde{Y}} = \begin{cases} \frac{1}{d^3} & \text{if } (d_1,d_2) = (d,d),\ (d,0), \text{ or } (0,d) \\ 0 & \text{otherwise.} \end{cases}$$

The normal bundle $Y \subset \widetilde{Y}$ is trivial with the T-action for which

$$c_1(N_{Y/\widetilde{Y}}) = t_1 + t_2.$$

The 0 point invariants of $\widetilde{Y}$ can be computed in terms of the 0 point invariants of Y as follows.

$$\begin{aligned} \langle\,\rangle_\beta^{\widetilde{Y}} &= \int_{[\overline{M}_{0,0}(\widetilde{Y},\beta)]^{vir}} 1 \\ &= \int_{[\overline{M}_{0,0}(Y,\beta)]^{vir}} \frac{1}{e(R^\bullet \pi_* f^*(N_{Y/\widetilde{Y}}))} \\ &= \frac{1}{t_1+t_2} \int_{[\overline{M}_{0,0}(Y,\beta)]^{vir}} 1 \\ &= \frac{1}{t_1+t_2} \langle\,\rangle_\beta^{Y} \end{aligned}$$

where $\pi : \mathcal{C} \to \overline{M}_{0,0}(Y,\beta)$ and $f : \mathcal{C} \to Y$ are the universal curve and universal map respectively. Combining the above with the divisor and point axioms, we see that the $\beta \neq 0$ part of F^Y is given by:

$$(t_1+t_2)\sum_{d=1}^{\infty} \frac{1}{d^3}\left[(e^{y_1}q_1)^d + (e^{y_2}q_2)^d + (e^{y_1+y_2}q_1q_2)^d\right].$$

To finish the proof of Theorem 1.1, we must to compute the $\beta = 0$ terms of F^Y. These consist solely of three point invariants given by triple intersections:

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle = \int_Y \gamma_1 \cup \gamma_2 \cup \gamma_3$$

which we define by localization and in general depend on the choice of the equivariant lifts of γ_i.

The surface Y has three T fixed points p_0, p_1, p_2. The T-invariant curve E_1 connects p_0 and p_1, and the T-invariant curve E_2 connects p_1 and p_2. The weights of the T-action on $T_{p_i}Y$ can be easily computed from our explicit description of Y and are given by

$$(3t_1, -2t_1 + t_2), \quad (2t_1 - t_2, -t_1 + 2t_2), \quad (t_1 - 2t_2, 3t_2)$$

for $T_{p_0}Y$, $T_{p_1}Y$, and $T_{p_2}Y$ respectively.

The basis $\{C_1, C_2\}$ is dual to the basis $\{E_1, E_2\}$. We can choose a lift of the T-action on Y to $L_i = \mathcal{O}(C_i)$ such that the weights of the T-action on $L_1|_{p_0}$, $L_1|_{p_1}$, $L_1|_{p_2}$ are

$$-2t_1, \quad -t_2, \quad -t_2$$

respectively, and the weights of the T-action on $L_2|_{p_0}$, $L_2|_{p_1}$, $L_2|_{p_2}$ are

$$-t_1, \quad -t_1, \quad -2t_2$$

respectively. We can then compute by localization:

$$\langle 1,1,1 \rangle = \frac{1}{3t_1t_2}, \quad \langle 1,1,C_1 \rangle = 0, \quad \langle 1,1,C_2 \rangle = 0,$$

$$\langle 1,C_1,C_1 \rangle = -\frac{2}{3}, \quad \langle 1,C_2,C_2 \rangle = -\frac{2}{3}, \quad \langle 1,C_1,C_2 \rangle = -\frac{1}{3},$$

$$\langle C_1,C_1,C_1 \rangle = \frac{2}{3}(2t_1 + t_2), \quad \langle C_1,C_1,C_2 \rangle = \frac{1}{3}(2t_1 + t_2),$$

$$\langle C_2,C_2,C_2 \rangle = \frac{2}{3}(2t_2 + t_1), \quad \langle C_2,C_2,C_1 \rangle = \frac{1}{3}(2t_2 + t_1),$$

completing the proof of Theorem 1.1.

3. The Orbifold Gromov-Witten invariants of X

The cubic terms of $F^{\mathcal{X}}$ can be computed directly. The higher degree terms are expressed here as trigonal Hurwitz-Hodge integrals and computed in the Appendix.

The inertia stack $I\mathcal{X}$ has three components corresponding to the three elements $\{1, \omega, \overline{\omega}\}$ of $\mathbb{Z}_3$. Each component is contractable and so the graded vector space

$$H^*_{orb}(\mathcal{X}) = H^*(I\mathcal{X})$$

has a canonical basis $\{1, D_1, D_2\}$ corresponding to the three components. Moreover, the grading for the twisted sectors is shifted by two:

$$1 \in H^0_{orb}(\mathcal{X}) \quad \text{and} \quad D_i \in H^2_{orb}(\mathcal{X}).$$

The invariant $\langle 1^{n_0} D_1^{n_1} D_2^{n_2} \rangle^{\mathcal{X}}$ is defined to be the integral

$$\int_{\overline{M}_{0,n_0+n_1+n_2}(\mathcal{X},0)} \prod_{i=1}^{n_0} \mathrm{ev}_i^*(1) \prod_{i=n_0+1}^{n_0+n_1} \mathrm{ev}_i^*(D_1) \prod_{i=n_0+n_1+1}^{n_0+n_1+n_2} \mathrm{ev}_i^*(D_2).$$

By the usual point axiom, $\langle 1^{n_0} D_1^{n_1} D_2^{n_2} \rangle^{\mathcal{X}} = 0$ if $n_0 > 0$ and $n_0+n_1+n_2 > 3$. Moreover, if $n_1 + n_2 > 0$, then there must be stacky points of the domain curves of the twisted stable maps. Consequently, the maps must factor through $B\mathbb{Z}_3 \subset \mathcal{X}$.

Consider $\langle D_1^{n_1} D_2^{n_2} \rangle^{\mathcal{X}}$ where $n_1 + n_2 > 3$. Since the maps factor through $B\mathbb{Z}_3$, we can rewrite the integral in terms of stable maps to $B\mathbb{Z}_3$:

$$\langle D_1^{n_1} D_2^{n_2} \rangle^{\mathcal{X}} =$$
$$\int_{[\overline{M}_{0,n_1+n_2}(B\mathbb{Z}_3)]^{vir}} e(R^1\pi_* f^*(L_\omega \oplus L_{\overline{\omega}})) \prod_{i=1}^{n_1} \mathrm{ev}_i^*(D_1) \prod_{i=n_1+1}^{n_1+n_2} \mathrm{ev}_i^*(D_2),$$

where

$$\pi : \mathcal{C} \to \overline{M}_{0,n_1+n_2}(B\mathbb{Z}_3)$$

is the universal curve and

$$f : \mathcal{C} \to B\mathbb{Z}_3$$

is the universal map. The normal bundle of $B\mathbb{Z}_3 \subset \mathcal{X}$ is the sum of the line bundles $L_\omega \oplus L_{\overline{\omega}}$ determined by the $\mathbb{Z}_3$-representations where $\omega \in \mathbb{Z}_3$ acts by multiplication by ω and $\overline{\omega}$ respectively.

Concretely, $\overline{M}_{0,n_1+n_2}(B\mathbb{Z}_3)$ may be thought of as parameterizing curves $\overline{C}$ equipped with a $\mathbb{Z}_3$-action for which the quotient map is a cover $p : \overline{C} \to C$ of a $n_1 + n_2$ marked genus 0 curve C ramified over the marked points and possibly the nodes of C. The integral $\langle D_1^{n_1} D_2^{n_2} \rangle^{\mathcal{X}}$ is possibly non-zero only on the components of $\overline{M}_{0,n_1+n_2}(B\mathbb{Z}_3)$ where $p : \overline{C} \to C$ is ramified over all the marked points with monodromy ω around the first n_1 points and $\overline{\omega}$ around the last n_2 points.

Consider the diagram of universal structures:

$$\begin{tikzcd}
\overline{\mathcal{C}} \arrow[r] \arrow[d, "p"'] & pt \arrow[d] \\
\mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"'] & B\mathbb{Z}_3 \\
\overline{M}_{0,n_1+n_2}(B\mathbb{Z}_3) &
\end{tikzcd}$$

Let $\overline{\pi} : \overline{\mathcal{C}} \to \overline{M}_{0,n_1+n_2}(B\mathbb{Z}_3)$ be the composition $\pi \circ p$ and let

$$\mathbb{E}^\vee = R^1\overline{\pi}_*\mathcal{O}$$

be the dual Hodge bundle. By the Riemann-Hurwitz formula, $\mathbb{E}^\vee$ is a bundle of rank

$$g = n_1 + n_2 - 2.$$

The action of $\omega \in \mathbb{Z}_3$ on $\overline{\mathcal{C}}$ induces an action of ω on $\mathbb{E}^\vee$. This gives a decomposition of $\mathbb{E}^\vee$ into eigenbundles

$$\mathbb{E}^\vee = \mathbb{E}^\vee_1 \oplus \mathbb{E}^\vee_\omega \oplus \mathbb{E}^\vee_{\overline{\omega}}.$$

(Note that our convention throughout is that $\mathbb{E}^\vee_\omega$ is the ω eigenbundle of $\mathbb{E}^\vee$ and not the dual of $\mathbb{E}_\omega$.)

A chase through the definitions shows that

$$R^1\pi_* f^*(L_\omega) = \mathbb{E}^\vee_{\overline{\omega}}, \quad R^1\pi_* f^*(L_{\overline{\omega}}) = \mathbb{E}^\vee_\omega.$$

Moreover, $\mathbb{E}^\vee_1 = 0$ is empty since

$$\mathbb{E}^\vee_1 = R^1\pi_*\mathcal{O}$$

and π is a family of genus 0 curves.

Let $\overline{M}^\sigma_{0,n_1+n_2}$ be the component of $\overline{M}_{0,n_1+n_2}(B\mathbb{Z}_3)$ on which

$$\prod_{i=1}^{n_1} \mathrm{ev}_i^*(D_1) \prod_{i=n_1+1}^{n_1+n_2} \mathrm{ev}_i^*(D_2)$$

is possibly non-zero. We can identify $\overline{M}^\sigma_{0,n_1+n_2}(B\mathbb{Z}_3)$ with the Hurwitz scheme $\overline{H}^\sigma_g((3)^{g+2})$ defined in the Appendix.

So we have

$$\langle D_1^{n_1} D_2^{n_2} \rangle^{\mathcal{X}} = \int_{[\overline{M}^\sigma_{0,n_1+n_2}(B\mathbb{Z}_3)]} e(\mathbb{E}^\vee_\omega \oplus \mathbb{E}^\vee_{\overline{\omega}})$$

where e is the T-equivariant Euler class.

Since $\mathbb{E}^\vee_\omega \oplus \mathbb{E}^\vee_{\overline{\omega}}$ has rank $g = n_1+n_2-2$ and $\overline{M}^\sigma_{0,n_1+n_2}(B\mathbb{Z}_3)$ has dimension $n_1 + n_2 - 3$, $\langle D_1^{n_1} D_2^{n_2} \rangle^{\mathcal{X}}$ is a linear function of the equivariant parameters t_1 and t_2.

Lemma 3.1. *For $n_1 + n_2 > 3$, $\langle D_1^{n_1} D_2^{n_2} \rangle^{\mathcal{X}}$ is a multiple of $t_1 + t_2$.*

Proof: It suffices to prove that $\langle D_1^{n_1} D_2^{n_2} \rangle^{\mathcal{X}} = 0$ for

$$t_1 = -t_2 = t.$$

Let

$$r_1 = \mathrm{rk}\, \mathbb{E}^\vee_\omega, \quad r_2 = \mathrm{rk}\, \mathbb{E}^\vee_{\overline{\omega}}.$$

Then,

$$e(\mathbb{E}^\vee_\omega \oplus \mathbb{E}^\vee_{\overline{\omega}}) = (-1)^{r_2}\left(t^g + t^{g-1}c_1(\mathbb{E}^\vee_\omega \oplus \mathbb{E}_\omega) + \cdots + c_g(\mathbb{E}^\vee_\omega \oplus \mathbb{E}_\omega)\right).$$

The Lemma then follows from a $\mathbb{Z}_3$-version of Mumford's relation:

Proposition 3.2. *Let $\pi : C \to B$ be a flat family of prestable curves with the action of a finite group G. Let ω_π be the relative dualizing sheaf and let $\mathbb{E} = \pi_*\omega_\pi$ be the Hodge bundle. Let*

$$\mathbb{E} = \oplus_\rho \mathbb{E}_\rho$$

be the decomposition of summands corresponding to the irreducible representations of G. Then

$$c(\mathbb{E}_\rho \oplus \mathbb{E}_\rho^\vee) = 1 \in H^*(B, \mathbb{Q}).$$

PROOF: The following argument is known to experts and is referred to by Mumford in [8], but since it does not seem to be written down, we include it for the benefit of the reader. We may assume that B is smooth, proper, and that the boundary divisor $D \subset B$ over which C is singular has normal crossings. The Lemma follows from the decomposition of $R^1\pi_*\mathbb{C}$ into eigensheaves for the natural action of G. Over $B - D$ we have the standard sequence

$$0 \to \mathbb{E}^\vee \to R^1\pi_*\mathbb{C} \otimes \mathcal{O}_B \to \mathbb{E} \to 0$$

which admits an extension over all of B where the middle term is interpreted globally as

$$V = R^1\pi_*[\mathcal{O}_C \xrightarrow{d} \omega_{C/B}].$$

Moreover, the Gauss-Manin connection extends to a connection over all of B with logarithmic poles along D whose polar part is nilpotent (page 130, [6]). Because the Gauss-Manin connection respects the decomposition of $R^1\pi_*\mathbb{C}$ into eigenbundles, it follows that this extension does so as well. Thus, after splitting V into eigenbundles, we get a sequence

$$0 \to \mathbb{E}_\rho^\vee \to V_\rho \to \mathbb{E}_\rho \to 0$$

and on V_ρ we have a log connection with nilpotent residue. In [4, Appendix B], it is shown how to use a connection with log poles to compute the Atiyah class (and hence the Chern classes) of a bundle. Because the formula for the Chern classes is in terms of the eigenvalues of the residue of the connection, and these all vanish in our situation, it follows that the Chern classes of V_ρ all vanish.

□

By Lemma 3.1, after setting

$$t_1 = t_2 = t,$$

we obtain

$$\begin{aligned}
\langle D_1^{n_1} D_2^{n_2}\rangle^{\mathcal{X}}\Big|_{t_1=t_2=t} &= \int_{\overline{M}^{\sigma}_{0,n_1+n_2}(B\mathbb{Z}_3)} e(\mathbb{E}^{\vee}_{\omega}\oplus\mathbb{E}^{\vee}_{\overline{\omega}})\\
&= t\int_{\overline{M}^{\sigma}_{0,n_1+n_2}(B\mathbb{Z}_3)} c_{g-1}(\mathbb{E}^{\vee})\\
&= t(-1)^{g-1}\int_{\overline{H}^{\sigma}_{g}((3)^{g+2})} \lambda_{g-1}\\
&= t(-1)^{g-1}A_g,
\end{aligned}$$

where the last equality is well defined by Proposition A.1 and the values of A_g are given by Proposition A.2.

For $g = n_1 + n_2 - 2 > 1$, we conclude

$$\langle D_1^{n_1} D_2^{n_2}\rangle^{\mathcal{X}} = \begin{cases} \frac{t_1+t_2}{2}(-1)^{g-1}A_g & \text{for } n_1 \equiv n_2 \mod 3\\ 0 & \text{for } n_1 \not\equiv n_2 \mod 3.\end{cases}$$

Let

$$F^{\mathcal{X}} = F^{\mathcal{X}}_{\text{cubic}} + \widehat{F}^{\mathcal{X}},$$

where $F^{\mathcal{X}}_{\text{cubic}}$ consists of all the cubic terms. Then,

$$\begin{aligned}
\widehat{F}^{\mathcal{X}} &= \frac{t_1+t_2}{2}\sum_{g=2}^{\infty}\sum_{\substack{n_1+n_2=g-2\\ n_1\equiv n_2 \mod 3}} (-1)^{g-1}A_g\frac{x_1^{n_1}}{n_1!}\frac{x_2^{n_2}}{n_2!}\\
&= \frac{t_1+t_2}{2}\sum_{g=2}^{\infty}(-1)^{g-1}\frac{A_g}{(g-2)!}\sum_{\substack{n_1+n_2=g-2\\ n_1\equiv n_2 \mod 3}}\binom{g-2}{n_1}x_1^{n_1}x_2^{n_2}\\
&= \frac{t_1+t_2}{2}\sum_{g=2}^{\infty}(-1)^{g-1}\frac{A_g}{(g-2)!}\frac{1}{3}\Big[(x_1+x_2)^{g-2}+(\omega x_1+\overline{\omega}x_2)^{g-2}\\
&\qquad\qquad + (\overline{\omega}x_1+\omega x_2)^{g-2}\Big].
\end{aligned}$$

To finish the proof of Theorem 1.2, we must remains to compute $F^{\mathcal{X}}_{\text{cubic}}$.

By the monodromy condition, the only non-vanishing 3-point invariants are

$$\left\langle 1^3\right\rangle^{\mathcal{X}}, \quad \left\langle 1D_1D_2\right\rangle^{\mathcal{X}}, \quad \left\langle D_1^3\right\rangle^{\mathcal{X}}, \quad \left\langle D_2^3\right\rangle^{\mathcal{X}}.$$

The moduli space for the first invariant is just $\mathcal{X}$ itself, so it is a trivial localization calculation. Each of the other invariants is an integral over a moduli space consisting of a single point with a $\mathbb{Z}_3$ automorphism group. In the first case, the corresponding cover is connected and genus 0, and in the last two cases, $\overline{C}$ is the elliptic curve with an order 3 automorphism (the two non-trivial automorphisms determining the two different cases). Thus the invariants are all 1/3 times the appropriate weight, namely

$$\frac{e(H^1(\overline{C},\mathcal{O})_{\omega}\oplus H^1(\overline{C},\mathcal{O})_{\overline{\omega}})}{e(H^0(\overline{C},\mathcal{O})_{\omega}\oplus H^0(\overline{C},\mathcal{O})_{\overline{\omega}})}$$

where the subscript indicates the eigenspace for the action of ω. These are easily computed by first principles or by the holomorphic Lefschetz formula. We get

$$\left\langle 1^3 \right\rangle^{\mathcal{X}} = \frac{1}{3t_1t_2}, \quad \langle 1 D_1 D_2\rangle^{\mathcal{X}} = \frac{1}{3}, \quad \left\langle D_1^3 \right\rangle^{\mathcal{X}} = \frac{t_1}{3}, \quad \left\langle D_2^3 \right\rangle^{\mathcal{X}} = \frac{t_2}{3}.$$

The proof of Theorem 1.2 is complete. □

Trigonometric evaluations of Hodge integrals over hyperelliptic Hurwitz spaces [5] play a basic role in the Crepant Resolution Conjecture for $\mathbb{C}^2/\mathbb{Z}_2$ studied in [1]. Hodge integrals over trigonal Hurwitz spaces arise in the study of $\mathbb{C}^2/\mathbb{Z}_3$. For $n \geq 4$, the Crepant Resolution Conjecture for $\mathbb{C}^2/\mathbb{Z}_n$ (where $\mathbb{Z}_n$ acts on the first factor via the standard representation ρ and the second factor via the dual representation $\rho^\vee$) predicts simple evaluations of certain Chern classes of $\mathbb{E}_\rho \oplus \mathbb{E}_{\rho^\vee}$ on Hurwitz spaces of $\mathbb{Z}_n$-covers.

4. Checking the series agree

We make the substitutions given by equations (1)–(4) into F^Y and compare with $F^{\mathcal{X}}$. The terms of homogeneous degree -2 and degree 0 in t_1 and t_2 are easily checked to agree with the corresponding terms in $F^{\mathcal{X}}$.

The remaining terms are linear in t_1 and t_2. Hence, agreement for the specializations $t_1 + t_2 = 0$ and $t_1 - t_2 = 0$ implies full agreement. The case of $t_1 + t_2 = 0$ is straightforward.

Let $\widetilde{F}^Y$ and $\widetilde{F}^{\mathcal{X}}$ denote the t linear term of $F^Y|_{t_1=t_2=t}$ and $F^{\mathcal{X}}|_{t_1=t_2=t}$ respectively. We need to prove that after making the substitution (1)–(4), $\widetilde{F}^Y$ and $\widetilde{F}^{\mathcal{X}}$ agree as power series in x_1 and x_2 up to terms of degree less than or equal to two. Equivalently, we must check that the third partial derivatives of $\widetilde{F}^Y$ and $\widetilde{F}^{\mathcal{X}}$ agree.

$$\begin{aligned}
\widetilde{F}^{\mathcal{X}} =& \frac{1}{18}(x_1^3 + x_2^3) \\
&+ \sum_{g=2}^{\infty} \frac{(-1)^{g-1} A_g}{(g+2)!} \frac{1}{3} \left[(x_1+x_2)^{g+2} + (\omega x_1 + \overline{\omega} x_2)^{g+2} + (\overline{\omega} x_1 + \omega x_2)^{g+2} \right] \\
=& \sum_{g=1}^{\infty} \frac{(-1)^{g-1} A_g}{(g+2)!} \frac{1}{3} \left[(x_1+x_2)^{g+2} + (\omega x_1 + \overline{\omega} x_2)^{g+2} + (\overline{\omega} x_1 + \omega x_2)^{g+2} \right].
\end{aligned}$$

Differentiation yields formulas for the partial derivatives (denoted by subscripts) :

$$\begin{aligned}
\widetilde{F}^{\mathcal{X}}_{111} =& \frac{1}{3} \left[A(-x_1 - x_2) + A(-\omega x_1 - \overline{\omega} x_2) + A(-\overline{\omega} x_1 - \omega x_2) \right], \\
\widetilde{F}^{\mathcal{X}}_{112} =& \frac{1}{3} \left[A(-x_1 - x_2) + \omega A(-\omega x_1 - \overline{\omega} x_2) + \overline{\omega} A(-\overline{\omega} x_1 - \omega x_2) \right].
\end{aligned}$$

Similarly, for Y, we have

$$\begin{aligned}
\widetilde{F}^Y &= \frac{1}{2}y_1y_2^2 + \frac{1}{3}y_1^3 + \frac{1}{2}y_2y_1^2 + \frac{1}{3}y_2^3 \\
&\quad + 2\sum_{d=1}^{\infty}\frac{1}{d^3}\left[(e^{y_1}q_1)^d + (e^{y_2}q_2)^d + (e^{y_1+y_2}q_1q_2)^d\right] \\
&= \left(\tfrac{i}{\sqrt{3}}\right)^3\left[\frac{x_1^3}{6} + \frac{x_2^3}{6} - x_1^2x_2 - x_2^2x_1\right] \\
&\quad + 2\sum_{d=1}^{\infty}\frac{1}{d^3}\left[\left(\omega e^{\frac{i}{\sqrt{3}}(\omega x_1+\overline{\omega}x_2)}\right)^d + \left(\omega e^{\frac{i}{\sqrt{3}}(\omega x_2+\overline{\omega}x_1)}\right)^d + \left(\overline{\omega}e^{-\frac{i}{\sqrt{3}}(x_1+x_2)}\right)^d\right]
\end{aligned}$$

In the variables

$$\begin{aligned}
\xi &= x_1 + x_2, \\
\xi_\omega &= \omega x_1 + \overline{\omega}x_2, \\
\xi_{\overline{\omega}} &= \overline{\omega}x_1 + \omega x_2,
\end{aligned}$$

the partial derivative $\partial^3/\partial x_1^3$ is:

$$\widetilde{F}^Y_{111} = \left(\tfrac{i}{\sqrt{3}}\right)^3\left(1 + \frac{2\omega e^{\frac{i}{\sqrt{3}}\xi_\omega}}{1-\omega e^{\frac{i}{\sqrt{3}}\xi_\omega}} + \frac{2\omega e^{\frac{i}{\sqrt{3}}\xi_{\overline{\omega}}}}{1-\omega e^{\frac{i}{\sqrt{3}}\xi_{\overline{\omega}}}} - \frac{2\overline{\omega}e^{-\frac{i}{\sqrt{3}}\xi}}{1-\overline{\omega}e^{-\frac{i}{\sqrt{3}}\xi}}\right)$$

Applying the identity

$$\frac{2e^{2i\theta}}{1-e^{2i\theta}} = -i\tan\left(\theta + \tfrac{\pi}{2}\right) - 1,$$

we obtain

$$\begin{aligned}
\widetilde{F}^Y_{111} &= \left(\tfrac{1}{3\sqrt{3}}\right)\left\{-\tan\left(\tfrac{1}{\sqrt{12}}\xi_\omega + \tfrac{5\pi}{6}\right) - \tan\left(\tfrac{1}{\sqrt{12}}\xi_{\overline{\omega}} + \tfrac{5\pi}{6}\right) + \tan\left(-\tfrac{1}{\sqrt{12}}\xi + \tfrac{7\pi}{6}\right)\right\} \\
&= \frac{1}{3}\left(A(-\xi_\omega) + A(-\xi_{\overline{\omega}}) + A(-\xi)\right) \\
&= \widetilde{F}^{\mathcal{X}}_{111}.
\end{aligned}$$

A similar computation verifies $\widetilde{F}^{\mathcal{X}}_{112} = \widetilde{F}^Y_{112}$. The identities $\widetilde{F}^{\mathcal{X}}_{122} = \widetilde{F}^Y_{122}$ and $\widetilde{F}^{\mathcal{X}}_{222} = \widetilde{F}^Y_{222}$ are obtained by symmetry in the indices. Theorem 1.3 is proved.

The six unstable terms of $F^{\mathcal{X}}$ can be assigned values (expressed in terms of trilogarithms, dilogarithms, and logarithms) by imposing the equality $F^{\mathcal{X}} = F^Y$. It would be interesting to give a geometric interpretation of these unstable values.

Appendix A. Degree 3 Hurwitz Hodge Integrals

A.1. Consider the moduli spaces $\overline{H}_g(\mu^1, \ldots, \mu^n)$ of connected, genus g, degree 3 admissible covers of an unparameterized $\mathbf{P}^1$. We label the monodromy conditions μ^i in degree 3 by the size of the largest part of the associated partition. There are two natural maps:

$$\epsilon : \overline{H}_g(\mu^1, \ldots, \mu^n) \to \overline{M}_g,$$

$$\pi : \overline{H}_g(\mu^1, \ldots, \mu^n) \to \overline{M}_{0,n},$$

well defined if $g \geq 2$ and $n \geq 3$ respectively.

A.2. We will primarily be interested in the moduli spaces $\overline{H}_g((3)^{g+2})$ for $g \geq 1$. Consider a covering

$$[f : \overline{C} \to (C, p_1, \ldots, p_{g+2})] \in \overline{H}_g((3)^{g+2})$$

where $(C, p_1, \ldots, p_n)$ is a stable, n-pointed, genus 0 curve. Since all the monodromy conditions are 3-cycles, the covering f *must* be Galois with group $\mathbb{Z}_3$.

The monodromy around each ramification point determines a non-zero element of the Galois group of f. Hence, a canonical assignment

$$\sigma_f : \{p_1, \ldots, p_{g+2}\} \to \mathrm{Gal}(f) \setminus 0$$

is determined by f. Since $\mathrm{Gal}(f) \setminus 0$ has two elements, σ_f defines a two set partition of the markings,

$$\{p_1, \ldots, p_{g+2}\} = S_f \cup S'_f.$$

The parity condition

$$|S_f| = |S'_f| \mod 3 \tag{5}$$

must be satisfied by global monodromy considerations.

The connected components of $\overline{H}_g((3)^{g+2})$ are in bijective correspondence with *unordered* partitions $S \cup S'$ of the marking set satisfying the parity condition (5). Let

$$\overline{H}_g^{\sigma}((3)^{g+2})$$

be the connected component corresponding to a partition σ of the marking set satisfying the parity condition (5).

The total number γ_g of connected components of $\overline{H}_g((3)^{g+2})$ is given by the following formula:

$$\gamma_g = \frac{1}{2} \sum_{l=1-g \mod 3} \binom{g+2}{l}.$$

The prefactor $1/2$ occurs since the set partition σ is unordered.

A.3. We calculate the evaluations of λ_{g-1} against the components of the moduli space $\overline{H}_g((3)^{g+2})$. For $g \geq 1$, let

$$A_g^{\sigma} = \int_{\overline{H}_g^{\sigma}((3)^{g+2})} \lambda_{g-1}.$$

Proposition A.1. *The integral A_g^{σ} is independent of σ.*

Let A_g be the common value of the evaluations of λ_{g-1} over the components of $\overline{H}_g^{\sigma}((3)^{g+2})$. Consider the generating function

$$A(u) = \sum_{g \geq 1} A_g \frac{u^{g-1}}{(g-1)!}.$$

Proposition A.2. *The generating function A is determined by:*

$$A(u) = \frac{1}{\sqrt{3}} \tan\left(\frac{u}{\sqrt{12}} + \frac{\pi}{6}\right).$$

Let $A_g^\bullet$ denote the evaluation of λ_{g-1} against the full moduli space $\overline{H}_g((3)^{g+2})$,

$$A_g^\bullet = \int_{\overline{H}((3)^{g+2})} \lambda_{g-1}.$$

The relation

$$A_g^\bullet = \gamma_g \cdot A_g.$$

is a consequence of Propositions A.1 and A.2.

A.4. Our proofs of Propositions A.1 and A.2 require the study of a closely related Hodge integral series. For $g \geq 0$, let

$$B_g = \int_{\overline{H}_g((3)^{g+1}(2)^2)} \lambda_g,$$

and let

$$B(u) = \sum_{g \geq 0} B_g \frac{u^g}{g!}.$$

Proposition A.3. *The generating function B is determined by:*

$$B(u) = \frac{1}{\sqrt{3}} \tan\left(\frac{u}{\sqrt{12}} + \frac{\pi}{3}\right).$$

A.5. We start by considering the initial values B_0 and B_1. The first,

$$B_0 = \int_{\overline{H}_0((3)(2)^2)} 1 = 1,$$

is a genus 0 Hurwitz number. The second,

$$B_1 = \int_{\overline{H}_1((3)^2(2)^2)} \lambda_1 = 2/3,$$

calculated by the following geometric argument. Consider the map

$$\epsilon : \overline{H}_1((3)^2(2)^2) \to \overline{M}_{1,1}$$

obtained by marking the first triple ramification point. By definition,

$$B_1 = \int_{\overline{H}_1((3)^2(2)^2)} \epsilon^*(\lambda_1).$$

However, on $\overline{M}_{1,1}$, $\lambda_1 = \psi_1$. Hence,

$$B_1 = \int_{\overline{H}_1((3)^2(2)^2)} \epsilon^*(\psi_1) = \int_{\overline{H}_1((3)^2(2)^2)} \psi_1.$$

The last equality is *not* formal, but rather proven geometrically since the component of the admissible cover carrying the first marking is *never* contracted by ϵ. Now, ψ_1 on $\overline{H}_1((3)^2(2)^2)$ is easily seen to be given by pull-back via π,

$$\psi_1 = \frac{1}{3}\pi^*(\psi_1).$$

By applying the boundary relation to ψ_1 on $\overline{M}_{0,4}$ and the degeneration formula, we obtain the answer.

A.6. We now proceed to determine all the higher B_g. The method is a use of the WDVV relation in the context of Hodge integrals over the moduli spaces of admissible covers. We will prove the following recursion for $g \geq 2$,

$$B_{g-1} + \sum_{g=h_1+h_2} 3\binom{g-2}{h_1} B_{h_1}B_{h_2} = \sum_{g=h_1+h_2} 6\binom{g-2}{h_1-1} B_{h_1}B_{h_2} \tag{6}$$

Since the left side contains the summand $3B_0B_g$ and the right side does not contain B_g, all higher B_g are determined.

For $g \geq 2$, consider the space of Hurwitz covers of the rigid line

$$\overline{H}^r_g((2)(2)(3)(3)(2)^2(3)^{g-2}).$$

The dimension is $g+4$. The ramification conditions are written as above to distinguish the first 4. Let ξ be the class of a point on the rigid line. We may consider the integral

$$C_g = \int_{\overline{H}^r_g((2)(2)(3)(3)(2)^2(3)^{g-2})} \lambda_g \cup \prod_{i=1}^{4} \mathrm{ev}_i^*(p)$$

which fixes the positions of the first 4 ramification conditions. We then may specialize the 4 points to be in WDVV configurations

$$((2)(2)|(3)(3)) \quad \text{and} \quad ((2)(3)|(2)(3))$$

by breaking the rigid target.

We consider first the evaluation of C_g via the configuration $((2)(2)|(3)(3))$. We must now distribute the remaining ramifications $(2)^2(3)^{g-2}$ to either side. We focus our attention on the $(2)^2$.

(i) Both $(2)^2$ go to the left. By parity, the central partition over the node must be (1) or (3). If (1), then the resulting configuration must have a loop and is annihilated by λ_g. If (3), then the left moduli space is $\overline{H}_{h_1}((2)^4(3)^{h_1})$ and the right moduli space is $\overline{H}_{h_2}((3)(3)(3)^{h_2})$. However, the integrand distributes by λ_{h_1} and λ_{h_2} respectively. The dimension mismatch yields vanishing.

(ii) One (2) goes to left and one (2) goes to the right. By parity, the central partition must be (2). Since the final configuration can not have a loop, only one possibility is allowed: all $(3)^{g-2}$ are distributed to the right. The outcome is the term

$$2^2 \int_{\overline{H}_1((2)^4)} \lambda_1 \cdot B_{g-1}.$$

Here, one prefactor of 2 comes from the initial choice of (2) and one comes from the degeneration formula. The integral

$$\int_{\overline{H}_1((2)^4)} \lambda_1 = 1/4$$

is easily evaluated.

(iii) Both $(2)^2$ go to the right. By parity, the central partition must by (1) or (3). If (1), the the resulting configuration must have a loop and is annihilated by λ_g. If (3), we obtain the sum

$$\sum_{g=h_1+h_2} 3\binom{g-2}{h_1} B_{h_1}B_{h_2}.$$

Next, we consider the evaluation of C_g via the configuration $((2)(3)|(2)(3))$. Since loops must be avoided, the only possibility for the central partition is (3). Hence, among the distributed ramifications, one (2) must go to either side. The outcome is

$$\sum_{g=h_1+h_2} 2\cdot 3\binom{g-2}{h_1-1} B_{h_1}B_{h_2}$$

completing the derivation of equation (6). Here, a prefactor 2 comes from the initial choice of (2) and a prefactor of 3 comes from the degeneration formula. The determination of the integrals B_g is complete.

A.7. Multiplying equation (6) by $u^{g-2}/(g-2)!$ and summing over all $g \geq 2$, we easily derive the following differential equation for $B(u)$:

$$B' + 3BB'' = 6(B')^2.$$

With the initial conditions $B(0) = 1$ and $B'(0) = 2/3$, the above ODE is uniquely solved by

$$B(u) = \frac{1}{\sqrt{3}} \tan\left(\frac{u}{\sqrt{12}} + \frac{\pi}{3}\right)$$

which proves Proposition A.3.

A.8. We now turn to the integrals $A^\bullet_g$. A similar Hurwitz Hodge WDVV argument yield the following relation for $g \geq 1$,

$$\delta_{g,1} + \sum_{g=h_1+h_2} 3\binom{g-1}{h_1-1} A^\bullet_{h_1}B_{h_2} = \sum_{g-1=h_1+h2} 2\binom{g-1}{h_1} B_{h_1}B_{h_2}. \tag{7}$$

Certainly equation (7) determines all the integrals $A^\bullet_g$ from the integrals B_g.

For $g \geq 1$, consider the space of Hurwitz covers of the rigid line

$$\overline{H}^r_g((2)(2)(3)(3)(3)^{g-1}).$$

The dimension is $g+3$ The ramification conditions are written as above to distinguish the first 4. We may consider the integral

$$D_g = \int_{\overline{H}^r_g((2)(2)(3)(3)(3)^{g-1})} \lambda_{g-1} \cup \prod_{i=1}^{4} \mathrm{ev}_i^*(\xi)$$

which fixes the positions of the first 4 ramification conditions. We then may specialize the 4 points to be in WDVV configurations

$$((3)(3)|(2)(2)) \quad \text{and} \quad ((2)(3)|(2)(3))$$

by breaking the rigid target.

We consider first the evaluation of C_g via the configuration $((3)(3)|(2)(2))$. We must now distribute the remaining ramifications $(3)^{g-1}$ to either side. By parity the central partition must be (1) or (3).

(i) If the central partition is (1) and at least one (3) is distributed right, then the resulting configuration must have two loops and is then annihilated by λ_{g-1}. Hence, if the the central partition is (1), all $(3)^{g-1}$ must be distributed left. The left moduli space is then $\overline{H}_{g-1}((3)^{g+1}(1))$ For $g \geq 3$, the map

$$\epsilon : \overline{H}_{g-1}((3)^{g+1}(1)) \to \overline{M}_g$$

has 1-dimensional fibers and evaluated to 0 against any Hodge classes. For $g=2$, the vanishing still holds by the 1-dimensional fibers of

$$\epsilon : \overline{H}_1((3)^3(1)) \to \overline{M}_{1,1}.$$

The only contribution comes when $g=1$. Then configuration yields

$$\delta_{g,1}.$$

(ii) If the central partition is (3), the outcome is the term

$$\sum_{g=h_1+h_2} 3\binom{g-1}{h_1-1} A^\bullet_{h_1} B_{h_2}.$$

Next, we consider the evaluation of D_g via the configuration $((2)(3)|(2)(3))$. By parity, the only possibility for the central partition is (2). The outcome is

$$\sum_{g-1=h_1+h2} 2\binom{g-1}{h_1} B_{h_1} B_{h_2}$$

completing the derivation of equation (7). Here, a prefactor 2 comes from the degeneration formula. The determination of the integrals $A^\bullet_g$ is complete.

A.9. Let

$$A^{\bullet}(u) = \sum_{g=1}^{\infty} A_g^{\bullet} \frac{u^{g-1}}{(g-1)!}.$$

Multiplying equation (7) by $u^{g-1}/(g-1)!$ and summing over $g \geq 1$, we easily derive the following relation for $A^{\bullet}(u)$ in terms of $B(u)$:

$$1 + 3A^{\bullet}B = 2B^2$$

or equivalently

$$A^{\bullet} = \frac{2}{3}B - \frac{1}{3}B^{-1}.$$

We will now prove Proposition A.2 *assuming* Proposition A.1. We begin by finding a closed formula for γ_g, the number of components of $\overline{H}_g((3)^{g+2})$.

Lemma A.4. *The number of unordered set partitions* $S \cup S' = \{p_1, \dots, p_{g+2}\}$ *satisfying* $|S| \equiv |S'| \mod 3$ *is given by*

$$\gamma_g = \frac{1}{3}(2^{g+1} + (-1)^g).$$

Proof: Consider all unordered set partitions $S \cup S' = \{p_1, \dots, p_{g+2}\}$ and let $\overline{S} \cup \overline{S}'$ be the induced partition of $\{p_1, \dots, p_{g+1}\}$. The partitions fall into three mutually exclusive possibilities:

(i) $|S| \equiv |S'| \mod 3$,
(ii) $|\overline{S}| \equiv |\overline{S}'| \mod 3$, or
(iii) neither equality holds.

Set (i) has cardinality γ_g, set (ii) has cardinality $2\gamma_{g-1}$, and set (iii) has a bijection with set (i) obtained by moving p_{g+2} from one set in the partition to the other.

Consequently, we obtain the following recursion for γ_g:

$$2^{g+1} = 2\gamma_g + 2\gamma_{g-1}.$$

The formula in the Lemma uniquely solves this recursion with the initial condition $\gamma_0 = 1$. □

Now we *assume* Proposition A.1 holds so $A^{\bullet} = A_g\gamma_g$. Then applying Lemma A.4, we see

$$\begin{aligned} A^{\bullet}(u) &= \sum_{g=1}^{\infty} A_g \gamma_g \frac{u^{g-1}}{(g-1)!} \\ &= \sum_{g=1}^{\infty} \frac{4}{3} A_g \frac{(2u)^{g-1}}{(g-1)!} - \frac{1}{3} A_g \frac{(-u)^{g-1}}{(g-1)!} \\ &= \frac{4}{3}A(2u) - \frac{1}{3}A(-u). \end{aligned}$$

To prove Proposition A.2 (assuming Proposition A.1), we must verify that the series

$$B(u) = \frac{1}{\sqrt{3}} \tan\left(\frac{u}{\sqrt{12}} + \frac{\pi}{3}\right)$$
$$A(u) = \frac{1}{\sqrt{3}} \tan\left(\frac{u}{\sqrt{12}} + \frac{\pi}{6}\right)$$

satisfy the functional equation

$$\frac{2}{3}B(u) - \frac{1}{3}B(u)^{-1} = \frac{4}{3}A(2u) - \frac{1}{3}A(-u).$$

Let

$$x = \frac{u}{\sqrt{12}} + \frac{\pi}{3}$$

Multiplying the functional equation by $3\sqrt{3}$, we get

$$2\tan(x) - 3\cot(x) = 4\tan\left(2x - \frac{\pi}{2}\right) + \tan\left(x - \frac{\pi}{2}\right).$$

Applying the trigonometric identities

$$\tan\left(\theta - \frac{\pi}{2}\right) = -\cot(\theta), \quad \cot(2\theta) = \frac{1}{2}\left(\cot(\theta) - \tan(\theta)\right),$$

the equality is easily seen to hold. We have proven:

Lemma A.5. *Proposition A.1 implies Proposition A.2.*

A.10. To prove Proposition A.1, we derive a set of recursions for the integrals A_g^σ. These recursions, *combined with* the determination of $A_g^\bullet$, uniquely determine the values of all the integrals A_g^σ. Since the recursions are indeed satisfied when $A_g^\sigma = A_g$, the integrals A_g^σ are independent of the component type and their values are given by the generating function in Proposition A.2.

The method requires a WDVV equation for Hodge integrals on the components of $\overline{H}_g((3)^{g+2})$.

Let σ be a two set partition of the markings $\{p_1, \ldots, p_{g+1}\}$ satisfying the parity condition. The integral A_g^σ depends only on the length 2 partition

$$|S_\sigma| + |S'_\sigma| = g + 2$$

as the geometry of the moduli space is symmetric under permutation of the markings. Let

$$A_g^{l,l'} = A_g^\sigma$$

where $l + l'$ is the associated length 2 partition of $g + 2$.

We must calculate all the integrals $A_g^{l,l'}$ where $l + l' = g + 2$ and

$$l \equiv l' \mod 3.$$

The constraints imply

$$l \equiv 1 - g \mod 3.$$

In particular,

$$A_g^{\bullet} = \frac{1}{2} \sum_{l \equiv 1-g \ (3)} \binom{g+2}{l} \cdot A_g^{l,g+2-l}$$

where the prefactor $1/2$ corrects for the double counting since

$$A_g^{l,l'} = A_g^{l',l} \tag{8}$$

correspond to the same class of components.

To simplify the notation, will we often write A_g^l for $A_g^{l,g+2-l}$. The equality

$$A_g^l = A_g^{g+2-l}$$

is obtained from (8)

A.11. For $g \leq 3$, only a single length 2 partition of $g+2$ occurs in each genus:

$$A_1^0, \quad A_2^2, \quad A_3^1.$$

Hence, Proposition A.1 is empty.

A.12. Let $g \geq 4$ and assume Proposition A.1 is proven for all lower genera. We will now prove Proposition A.1 for genus g.

Let $\omega, \overline{\omega}$ denote the non-zero elements of $\mathbb{Z}/3\mathbb{Z}$. Let $2 \leq l \leq g+1$ satisfy

$$l = g + 3 - l \mod 3. \tag{9}$$

Consider the connected component

$$\overline{H}_{g+1}(\omega^l \overline{\omega}^{g+3-l}) \subset \overline{H}_{g+1}((3)^{g+3})$$

corresponding to the monodromy ω for the first l markings and $\overline{\omega}$ for the last $g+3-l$. Equation (9) is the parity condition.

Let p_1, p_2, q_1, q_2 be the first and last two markings, and let

$$\pi : \overline{H}_{g+1}(\omega^l \overline{\omega}^{g+3-l}) \rightarrow \overline{M}_{0,4}$$

be the associated map. Let

$$E_{g+1}^l = \int_{\overline{H}_{g+1}(\omega^l \overline{\omega}^{g+3-l})} \lambda_{g-1} \cup \pi^{-1}(\xi)$$

where ξ is a class of a point in $\overline{M}_{0,4}$.

We may calculate E_{g+1}^l by specializing ξ to either of the two WDVV configurations

$$(p_1 p_2 | q_1 q_2), \quad \text{and} \quad (p_1 q_1 | p_2 q_2)$$

in $\overline{M}_{0,4}$. The resulting equation is easily derived:

$$\sum_{x-y \not\equiv 1 \ (3)} 3 \binom{l-2}{x} \binom{g+1-l}{y} A_{1+x+y}^{2+x+\phi(x,y)} A_{1+(l-x)+(g+3-l-y)}^{l-x+\overline{\phi}(x,y)} =$$

$$\sum_{x-y \not\equiv 0 \ (3)} 3 \binom{l-2}{x} \binom{g+1-l}{y} A_{1+x+y}^{2+x+\theta(x,y)} A_{1+(l-x)+(g+3-l-y)}^{l-x+\overline{\theta}(x,y)}.$$

The functions ϕ, $\overline{\phi}$, θ, $\overline{\theta}$ are defined as follows:

$$\begin{array}{lll} x-y\equiv 0 & \mod 3: & \phi(x,y)=1,\ \overline{\phi}(x,y)=0 \\ x-y\equiv 1 & \mod 3: & \theta(x,y)=0,\ \overline{\theta}(x,y)=1 \\ x-y\equiv 2 & \mod 3: & \phi(x,y)=0,\ \overline{\phi}(x,y)=1 \\ & & \theta(x,y)=1,\ \overline{\theta}(x,y)=0. \end{array}$$

Let $\mathcal{E}^l_{g+1}$ denote the equation obtained from E^l_{g+1}.

No terms of $\mathcal{E}^l_{g+1}$ contain A-integrals of genus greater than g. The *principal* terms of $\mathcal{E}^l_{g+1}$ are those which contain A-integrals of genus g. In fact, the principal terms of $\mathcal{E}^l_{g+1}$ occur only on the left side and are simply

$$A_g^{l-2} + A_g^{l+1}.$$

We now study the full linear system of principal terms. Let ν be the smallest non-negative integer congruent to $(1-g)$ mod 3. The set of A-integrals of genus g is

$$\{A_g^{\nu},\ A_g^{3+\nu},\ A_g^{6+\nu},\ \ldots,\ A_g^{g+2-\nu}\}.$$

To simplify notation, denote these A-integrals by the variables

$$x_i = A_g^{3i+\nu}.$$

We must solve for the variables

$$\{x_0,\ldots,x_n\}$$

for $n = \frac{g+2-2\nu}{3}$. Elementary considerations show the number of variables, $n+1$, is congruent to $(g+1)$ mod 2.

The set of principal terms of all the $\mathcal{E}^l_{g+1}$ equations is simply

$$\{x_0+x_1,\ x_1+x_2,\ x_2+x_3,\ \ldots,\ x_{n-1}+x_n\}. \tag{10}$$

These principal terms do not determine the variables: exactly one additional independent equation is required.

If g is odd, the equations $x_i = x_{n-i}$ from (8) provide an independent relation since the number of variables then is even. If g is even, the symmetry $x_i = x_{n-i}$ is redundant.

An additional linear equation is obtained from the completed calculation of $A_g^{\bullet}$:

$$\sum_{i=0}^{n} \binom{g+2}{3i+\nu} x_i = 2A_g^{\bullet} \tag{11}$$

Equation (11) is independent of the principal terms (10) if and only if

$$\delta_g = \sum_{i=0}^{n} \binom{g+2}{3i+\nu} (-1)^{3i+\nu}$$

does not vanish.

Lemma A.6. *The numbers δ_g are given by*

$$\delta_g = \begin{cases} -2(-3)^{g/2} & \text{if } g \text{ is even,} \\ 0 & \text{if } g \text{ is odd.} \end{cases}$$

Proof: Let $\omega = \exp(2\pi i/3)$, and note that $\omega - 1 = \sqrt{3}\exp(5\pi i/6)$. Define

$$\theta(x) = \begin{cases} 1 & \text{if } x \equiv 0 \mod 3, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned}
\delta_g &= \sum_{i=0}^{n} \binom{g+2}{3i+\nu}(-1)^{3i+\nu} \\
&= \sum_{k=0}^{g+2} \binom{g+2}{k}(-1)^k \theta(1-g-k) \\
&= \sum_{k=0}^{g+2} \binom{g+2}{k}(-1)^k \frac{1}{3}\left(1 + \omega^{1-g-k} + \overline{\omega}^{1-g-k}\right) \\
&= \frac{1}{3}(1-1)^{g+2} + \frac{1}{3}\omega^{-2g-1}(\omega-1)^{g+2} + \frac{1}{3}\overline{\omega}^{-2g-1}(\overline{\omega}-1)^{g+2} \\
&= \frac{1}{3}(\sqrt{3})^{g+2}\left(e^{\frac{2\pi i(-2g-1)}{3}} e^{\frac{(2\pi i)5(g+2)}{12}} + e^{\frac{2\pi i(2g+1)}{3}} e^{\frac{(-2\pi i)5(g+2)}{12}}\right) \\
&= 3^{g/2}\left(e^{2\pi i\frac{2-g}{4}} + e^{2\pi i\frac{g-2}{4}}\right) \\
&= 3^{g/2} \cdot \begin{cases} 0 & \text{if } g \text{ is odd,} \\ 2(-1)^{g/2-1} & \text{if } g \text{ is even.} \end{cases}
\end{aligned}$$

$\square$

We conclude the full set of component A-integrals is completely determined by the following three conditions:

(i) the initial values for $g \leq 3$,
(ii) the equations $\mathcal{E}^l_{g+1}$ for $g \geq 4$.
(iii) the additional equation (11) obtained from $A^\bullet_g$.

To complete the proof of Proposition A.1, we must simply check the compatibility of the proposed values for the component A-integrals with the conditions (i-iii).

Compatibility with (i) and (iii) has already been checked. Compatibility with (ii) is equivalent to the following set of relations for A_g: for every pair

(r,s) of non-negative integers congruent mod 3 and not equal (0,0),

$$\sum_{x-y\not\equiv 1\ (3)} \binom{r}{x}\binom{s}{y} A_{1+x+y}A_{1+(r-x)+(s-y)} = \sum_{x-y\not\equiv 0\ (3)} \binom{r}{x}\binom{s}{y} A_{1+x+y}A_{1+(r-x)+(s-y)}.$$

We define

$$\theta_{0,r,s} = \sum_{x-y\equiv 0\ (3)} \binom{r}{x}\binom{s}{y} A_{1+x+y}A_{1+(r-x)+(s-y)}$$
$$\theta_{1,r,s} = \sum_{x-y\equiv 1\ (3)} \binom{r}{x}\binom{s}{y} A_{1+x+y}A_{1+(r-x)+(s-y)}.$$

The above compatibility condition is equivalent to the condition that $\theta_{0,r,s} = \theta_{1,r,s}$ for all $r \equiv s \mod 3$, $(r,s) \neq (0,0)$. This is easily seen by subtracting the full sum over x and y from both sides of the compatibility equation. Let

$$\theta_i(v,w) = \sum_{\substack{r,s\geq 0\\ r\equiv s\ (3)}} \theta_{i,r,s}\frac{v^r}{r!}\frac{w^s}{s!}.$$

We need to prove that

$$\theta_0(v,w) - \theta_1(v,w) = \frac{1}{9}.$$

We expand θ_i and rearrange the sums:

$$\begin{aligned}
\theta_i(v,w) &= \sum_{\substack{r,s\geq 0\\ r\equiv s\ (3)}} \sum_{\substack{x,y\geq 0\\ x\equiv y+i\ (3)}} \frac{r!}{x!(r-x)!}\frac{s!}{y!(s-y)!} A_{1+x+y}A_{1+r+s-x-y}\frac{v^r}{r!}\frac{w^s}{s!}\\
&= \sum_{\substack{x,y\geq 0\\ x\equiv y+i\ (3)}} \sum_{\substack{n,m\geq 0\\ n\equiv m-i\ (3)}} \frac{(x+y)!}{x!\,n!}\frac{(n+m)!}{y!\,m!} A_{1+x+y}A_{1+n+m}\frac{v^{n+x}}{(x+y)!}\frac{w^{m+y}}{(n+m)!}\\
&= \sum_{\substack{x,y\geq 0\\ x\equiv y+i\ (3)}} \binom{x+y}{x} A_{1+x+y}\frac{v^x w^y}{(x+y)!} \sum_{\substack{n,m\geq 0\\ n\equiv m-i\ (3)}} \binom{n+m}{m} A_{1+n+m}\frac{v^n w^m}{(n+m)!}\\
&= Q_i(v,w)Q_{-i}(v,w)
\end{aligned}$$

where

$$Q_i(v,w) = \frac{1}{3}\left(\tilde{Q}_i(v,w) + \overline{\omega}^i\tilde{Q}_i(\omega v, \overline{\omega} w) + \omega^i \tilde{Q}_i(\overline{\omega} v, \omega w)\right)$$

and

$$\begin{aligned}\tilde{Q}_i(v,w) &= \sum_{x,y\geq 0} \binom{x+y}{x} v^x w^y \frac{A_{1+x+y}}{(x+y)!}\\ &= \sum_{k=0}^{\infty} A_{1+k}\frac{(v+w)^k}{k!}\\ &= A(v+w).\end{aligned}$$

Therefore

$$\begin{aligned}9(\theta_0-\theta_1) &= 9(Q_0^2 - Q_1Q_{-1})\\ &= (A[0]+A[1]+A[2])^2 - (A[0]+\overline{\omega}A[1]+\omega A[2])(A[0]+\omega A[1]+\overline{\omega}A[2])\\ &= 3(A[0]A[1]+A[1]A[2]+A[2]A[0])\end{aligned}$$

where $A[k]$ is defined by:

$$A[k](v,w) = A(\omega^k v + \overline{\omega}^k w).$$

Now

$$\begin{aligned}A(u) &= \left(\frac{1}{\sqrt{3}}\right)\tan\left(\frac{u}{\sqrt{12}}+\frac{\pi}{6}\right)\\ &= \left(\frac{-i}{\sqrt{3}}\right)\frac{e^{i\left(u/\sqrt{12}+\pi/6\right)} - e^{-i\left(u/\sqrt{12}+\pi/6\right)}}{e^{i\left(u/\sqrt{12}+\pi/6\right)} + e^{-i\left(u/\sqrt{12}+\pi/6\right)}}\\ &= \left(\frac{i}{\sqrt{3}}\right)\frac{1+\overline{\omega}e^{iu/\sqrt{3}}}{1-\overline{\omega}e^{iu/\sqrt{3}}}.\end{aligned}$$

Let

$$X = e^{iv/\sqrt{3}} \quad Y = e^{iw/\sqrt{3}},$$

then

$$A[k] = \left(\frac{i}{\sqrt{3}}\right)\frac{1+\overline{\omega}X^{\omega^k}Y^{\overline{\omega}^k}}{1-\overline{\omega}X^{\omega^k}Y^{\overline{\omega}^k}}$$

Define

$$\Theta_k^{\pm} = 1 \pm \overline{\omega}X^{\omega^k}Y^{\overline{\omega}^k}$$

then

$$\begin{aligned}-9(\theta_0-\theta_1) &= -3(A[0]A[1]+A[1]A[2]+A[2]A[0])\\ &= \frac{\Theta_0^+\Theta_1^+}{\Theta_0^-\Theta_1^-} + \frac{\Theta_1^+\Theta_2^+}{\Theta_1^-\Theta_2^-} + \frac{\Theta_2^+\Theta_0^+}{\Theta_2^-\Theta_0^-}\\ &= \frac{\Theta_0^+\Theta_1^+\Theta_2^- + \Theta_0^+\Theta_1^-\Theta_2^+ + \Theta_0^-\Theta_1^+\Theta_2^+}{\Theta_0^-\Theta_1^-\Theta_2^-}\end{aligned}$$

Applying the relations $1+\omega+\overline{\omega}=0$ and $\overline{\omega}^2=\omega$, we compute:

$$\begin{aligned}\Theta_0^-\Theta_1^-\Theta_2^- &= (1-\overline{\omega}XY)(1-\overline{\omega}X^{\omega}Y^{\overline{\omega}})(1-\overline{\omega}X^{\overline{\omega}}Y^{\omega})\\ &= \omega(X^{-1}Y^{-1}+X^{-\omega}Y^{-\overline{\omega}}+X^{-\overline{\omega}}Y^{-\omega}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\Theta_0^+\Theta_1^+\Theta_2^- &= \omega(-X^{-1}Y^{-1} - X^{-\omega}Y^{-\overline{\omega}} + X^{-\overline{\omega}}Y^{-\omega})\\
\Theta_0^+\Theta_1^-\Theta_2^+ &= \omega(-X^{-1}Y^{-1} + X^{-\omega}Y^{-\overline{\omega}} - X^{-\overline{\omega}}Y^{-\omega})\\
\Theta_0^-\Theta_1^+\Theta_2^+ &= \omega(+X^{-1}Y^{-1} - X^{-\omega}Y^{-\overline{\omega}} - X^{-\overline{\omega}}Y^{-\omega}),
\end{aligned}$$

and so we conclude that

$$-9(\theta_0 - \theta_1) = -1$$

as desired.

We've shown that the values of the integrals given by Propositions A.1 and A.2 are indeed the unique solution to the full set of recursions and so the proofs of Proposition A.1 and A.2 are complete. □

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