THE ENUMERATIVE GEOMETRY AND ARITHMETIC OF BANANA NANO-MANIFOLDS

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ABSTRACT. A banana manifold is a Calabi-Yau threefold fibered by Abelian surfaces whose singular fibers contain banana configurations: three rational curves meeting each other in two points. A nano-manifold is a Calabi-Yau threefold X with very small Hodge numbers: $h^{1,1}(X) + h^{2,1}(X) \leq 6$. We construct four rigid banana nano-manifolds $\widetilde{X}_N$, $N \in \{5, 6, 8, 9\}$, each with Hodge numbers given by $(h^{1,1}, h^{2,1}) = (4, 0)$.

We compute the Donaldson-Thomas partition function for banana curve classes and show that the associated genus g Gromov-Witten potential is a genus 2 meromorphic Siegel modular form of weight $2g - 2$ for a certain discrete subgroup $P_N^* \subset \mathrm{Sp}_4(\mathbb{R})$.

We also compute the weight 4 modular form whose pth Fourier coefficient is given by the trace of the action of Frobenius on $H^3_{\text{ét}}(\widetilde{X}_N, \mathbb{Q}_\ell)$ for almost all prime p. We observe that it is the unique weight 4 cusp form on $\Gamma_0(N)$.

1. INTRODUCTION: THE GEOGRAPHY, ENUMERATIVE GEOMETRY, AND ARITHMETIC OF CALABI-YAU THREEFOLDS.

In this paper, a *Calabi-Yau threefold (CY3)* is a smooth, projective threefold X over $\mathbb{C}$ with $K_X \cong \mathcal{O}_X$ and $H^1(X, \mathbb{C}) = 0$. We are interested three aspects of a CY3 X. Namely, geography (the Hodge numbers of X), enumerative geometry (curve counting on X), and arithmetic (for rigid X, counting points over $\mathbb{F}_p$ gives rise to a weight 4 modular form).

In this paper, we construct four new CY3s which are interesting from all three points of view.

1.1. **Geography.**

The Hodge numbers of a CY3 X are determined by the two values $h^{1,1}(X)$ and $h^{2,1}(X)$ which have geometric significance: $h^{1,1}(X)$ is the Picard number of X and $h^{2,1}(X)$ is the dimension of the space of deformations of X. The *geography problem for CY3s* asks which pairs of numbers $(h^{1,1}, h^{2,1})$ occur as the Hodge numbers of a CY3. There has been some recent interest in the physics community in the existence of CY3s with small Hodge numbers. Candelas, Constantine, and Mishra [6] list all known (at the time) CY3s with height $h^{1,1} + h^{2,1} \leq 24$. We call a CY3 with height satisfying

$$h^{1,1} + h^{2,1} \leq 6$$

a *nano-manifold*. We construct four new examples of rigid nano-manifolds of height 4.

Theorem 1. *There exist CY3s $\widetilde{X}_N$, $N \in \{5, 6, 8, 9\}$ with*

$$h^{1,1}(\widetilde{X}_N) = 4, \quad h^{2,1}(\widetilde{X}_N) = 0.$$

There is an Abelian surface fibration $\widetilde{X}_N \to \mathbb{P}^1$, with four singular fibers, whose generic fiber is an Abelian surface of Picard number three. $\mathrm{Pic}(\widetilde{X}_N)$ *is spanned by the class of the fiber and the three divisor classes of the generic fiber. $\widetilde{X}_N$ has fundamental group of order N (c.f. Theorem 9).*

1.2. **Enumerative Geometry.**

The enumerative geometry of a CY3 X is the study of curve counting on X. There are several equivalent curve counting theories on a CY3. We are particularly interested in *Donaldson-Thomas (DT)* invariants, *Gromov-Witten (GW)* invariants, and *Gopakumar-Vafa (GV)* invariants.

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1.2.1. *DT theory.* As usual, we may define the DT invariants of a CY3 X as the Behrend function weighted Euler characteristic of the Hilbert scheme

$$\mathsf{DT}_{n,\beta}(X) = e(\mathrm{Hilb}^{n,\beta}(X), \nu).$$

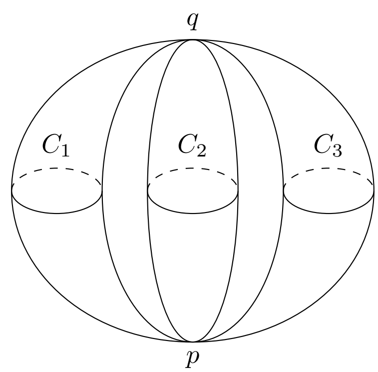
Here $\mathrm{Hilb}^{n,\beta}(X)$ is the Hilbert scheme of subschemes of X supported in the class $\beta \in H_2(X)$ and having holomorphic Euler characteristic n, and $e(-,\nu)$ denotes topological Euler characteristic weighted by Behrend's constructible function ν [3].

For a basis of divisor classes $D_1, \ldots, D_r$, we define the *DT partition function* by

$$Z_X(p, q_1, \ldots, q_r) = \sum_{n,\beta} \mathsf{DT}_{n,\beta}(X)\,(-p)^n\, q_1^{\beta\cdot D_1} \cdots q_r^{\beta\cdot D_r}\,.$$

The nano-manifolds $\widetilde{X}_N$ are of *banana type* which makes it possible to compute the DT partition function of $\widetilde{X}_N$ for all fiber curve classes. To be of banana type means that $\widetilde{X}_N \to \mathbb{P}^1$ is the compactification of a group scheme $\widetilde{X}_N^{\circ} \to \mathbb{P}^1$ by banana configurations.

A *banana configuration* in a CY3 is a union of three smoothly embedded $(-1,-1)$ rational curves C_1, C_2, C_3 whose intersection consists of two points $\{p,q\}$. Moreover, the group of the singular fiber where a banana configuration occurs is $\mathbb{C}^* \times \mathbb{C}^*$ and it acts on the banana configuration and its formal neighborhood (see [4]).



There is a basis of divisor classes for $\widetilde{X}_N$ given by $\widetilde{F}, \widetilde{\Delta}, \widetilde{S}', \widetilde{S}$ where $\widetilde{F}$ is the class of the fiber, and $\widetilde{\Delta}, \widetilde{S}', \widetilde{S}$ are classes which, when restricted to a generic fiber, span the Picard group and have intersection form $\left(\begin{smallmatrix} -2N^2 & 0 & 0 \\ 0 & 0 & N \\ 0 & N & 0 \end{smallmatrix}\right)$. We assign variables z, y, q, Q to the divisors $\widetilde{F}, \widetilde{\Delta}, \widetilde{S}', \widetilde{S}$ respectively and we compute the DT partition function for fiber curve classes, i.e. we compute the $z \to 0$ limit of $Z_{\widetilde{X}_N}(p, z, y, q, Q)$.

Theorem 2. *Define positive integers $c(a,m)$ as the Fourier coefficients of the ratio of theta functions $\theta_4(q^2,y)/\theta_1(q^4,y)^2$, namely:*

$$\sum_{a=-1}^{\infty} \sum_{m\in\mathbb{Z}} c(a,m) q^a y^m = \frac{\sum_{m\in\mathbb{Z}} q^{m^2}(-y)^m}{\left(\sum_{m\in\mathbb{Z}+\frac{1}{2}} q^{2m^2}(-y)^m\right)^2}. \tag{1}$$

Then the DT partition function of $\widetilde{X}_N$, for fiber curve classes is given by

$$Z_{\widetilde{X}_N}(p, y, q, Q) = \prod_{k\in\Theta_N} \prod_{m,r,s,t} \left(1 - p^m q^{Nr/k} Q^{ks} y^{Nt}\right)^{-c(4rs-t^2,m)}$$

where in the sum m, r, s, t are integers with $r, s, r+s+t \geq 0$ and if $r = s = t = 0$ then $m > 0$, and Θ_N is the 4-tuple given by

$$\Theta_5 = (5,5,1,1), \quad \Theta_6 = (6,3,2,1), \quad \Theta_8 = (4,4,2,2), \quad \Theta_9 = (3,3,3,3).$$

Remark 3. The methods developed in [4] to compute the fiber curve class DT partition function of the ordinary banana manifold applies equally well to the banana nano-manifolds $\widetilde{X}_N$. The primary difference is that in the ordinary banana manifold, the three banana curves of any singular fiber span the fiber curve classes whereas the classes of the banana curves in the singular fibers of $\widetilde{X}_N$ are more complicated and vary from singular fiber to singular fiber. For this reason, it is more convenient to express the paritition function in terms of the divisor classes and it accounts for the differences between the formula in this paper versus the one in [4].

1.2.2. *GW theory.* We define the genus g GW potential for fiber curve classes by

$$F_g^{\widetilde{X}_N}(Q,q,y) = \sum_{\substack{\beta\in H_2(\widetilde{X}_N)\\ \pi_*\beta=0}} \langle\ \rangle_{g,\beta}^{\widetilde{X}_N}\ Q^{\beta\cdot\widetilde{S}} q^{\beta\cdot\widetilde{S}'} y^{\beta\cdot\widetilde{\Delta}}$$

where $\langle\ \rangle_{g,\beta}^{\widetilde{X}_N}$ is the genus g GW invariant of $\widetilde{X}_N$ in the class β.

The following is a consequence of Theorem 2:

Corollary 4. *The genus g GW potential of $\widetilde{X}_N$ is given by*

$$F_g^{\widetilde{X}_N}(Q,q,y) = \sum_{k\in\Theta_N} F_g^{\mathsf{ban}}(Q^k, q^{N/k}, y^N) \tag{2}$$

where for $g \geq 2$, $F_g^{\mathsf{ban}}(Q,q,y)$ is a meromorphic Siegel modular form of weight $2g-2$ with

$$Q = e^{2\pi i\sigma}, q = e^{2\pi i\tau}, y = e^{2\pi i z}, \text{ for } \left(\begin{smallmatrix}\tau & z\\ z & \sigma\end{smallmatrix}\right) \in \mathbb{H}_2$$

where $\mathbb{H}_2$ is the genus 2 Siegel upper half space. Namely, F_g^{ban} is the Maass lift of the index 1, weight $2g-2$ Jacobi form

$$\alpha_g \cdot \phi_{-2,1}(q,y) \cdot E_{2g}(q)$$

where $E_{2g}(q)$ is the weight $2g$ Eisenstein series, $\phi_{-2,1}(q,y)$ is the unique weak Jacobi form of weight -2 and index 1, and $\alpha_g = \frac{|B_{2g}|}{2g(2g-2)!}$ (c.f. [4, § A.4] *for notation).*

We show that (up to a change of variables) the GW potentials $F_g^{\widetilde{X}_N}$ are Siegel modular forms for a certain subgroup of $\mathrm{Sp}_4(\mathbb{R})$ as follows. For $N \in \{5,6,8,9\}$ consider the group

$$P_N = \mathrm{Sp}_4(\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & \frac{N}{d_N}\mathbb{Z} & \frac{1}{d_N}\mathbb{Z} & \frac{1}{N}\mathbb{Z} \\ \frac{1}{d_N}\mathbb{Z} & \mathbb{Z} & \frac{1}{N}\mathbb{Z} & \frac{1}{Nd_N}\mathbb{Z} \\ \frac{N}{d_N}\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \frac{1}{d_N}\mathbb{Z} \\ N\mathbb{Z} & \frac{N^2}{d_N}\mathbb{Z} & \frac{N}{d_N}\mathbb{Z} & \mathbb{Z} \end{pmatrix} \tag{3}$$

where $d_N = \gcd\Theta_N$. Explicitly, $d_5 = 1$, $d_6 = 1$, $d_8 = 2$, and $d_9 = 3$. We additionally consider the involution

$$\iota_N = \begin{pmatrix} 0 & \sqrt{N} & 0 & 0 \\ \frac{1}{\sqrt{N}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{N}} \\ 0 & 0 & \sqrt{N} & 0 \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{R}) \tag{4}$$

and observe that conjugation by ι_N preserves P_N. We can therefore produce an index 2 normal extension of P_N

$$P_N^* = P_N \cup \iota_N P_N$$

and prove the following

Corollary 5. *Up to a change of variables (see Section 5.3), the GW potentials $F_g^{\widetilde{X}_N}$ for $g \geq 2$ are meromorphic Siegel modular forms of weight $2g-2$ for the discrete subgroup $P_N^* \subset \mathrm{Sp}_4(\mathbb{R})$.*

The involution ι_N induces the transformation $(Q,q,y) \mapsto (q^{\frac{1}{N}}, Q^N, y)$, which is a symmetry of $F_g^{\widetilde{X}_N}$ after the change of variables. We note that this is reminiscent of the behaviour of *Siegel paramodular forms* where in particular, invariance under $(Q,q,y) \mapsto (q^{\frac{1}{N}}, Q^N, y)$ is a key property [11, § 3.1].

1.2.3. *GV theory.* The genus g, curve class β GV invariant of a CY3 X is an integer valued curve counting invariant $n^g_\beta(X)$ which has been given a geometric definition by Maulik and Toda [17]. The GV invariants are conjecturally equivalent to the DT/GW invariants by the Gopakumar-Vafa formula which can alternatively be used to give a (non-geometric) definition. Either by assuming the Gopakumar-Vafa formula holds for $\widetilde{X}_N$, or by using the non-geometric definition, we may use our computation of the DT partition function to compute the (fiber class) GV invariants of $\widetilde{X}_N$.

The curve classes of a smooth fiber of $\widetilde{X}_N \to \mathbb{P}^1$ generate (over $\mathbb{Q}$) the fiber curve classes of $\widetilde{X}_N$. Consequently, the fiber curve classes inherit a quadratic form $||\cdot||$ coming from the intersection pairing on a smooth fiber.

Proposition 6. *Assuming that the Gopakumar-Vafa formula holds for $\widetilde{X}_N$, the Gopakumar-Vafa invariants of $\widetilde{X}_N$ in an effective fiber class β with $2||\beta|| = a$, are given by*

$$n^g_\beta(\widetilde{X}_N) = \epsilon_N(\beta)\, n^g_a$$

where

$$\epsilon_N(\beta) = \sum_{k\in\Theta_N} \epsilon_{N,k}(\beta), \qquad \epsilon_{N,k}(\beta) = \begin{cases} 1 & \text{if } k \,\Big|\, (\beta\cdot\widetilde{S}) \text{ and } \frac{N}{k} \,\Big|\, (\beta\cdot\widetilde{S}') \\ 0 & \text{otherwise,} \end{cases}$$

and the integers n^g_a are given by

$$\sum_{a=-1}^{\infty}\sum_{g=0}^{\infty} n^g_a\,(y^{\frac12}+y^{-\frac12})^{2g} q^{a+1} = \prod_{n=1}^{\infty} \frac{(1+yq^{2n-1})(1+y^{-1}q^{2n-1})(1-q^{2n})}{(1+yq^{4n})^2(1+y^{-1}q^{4n})^2(1-q^{4n})^2}.$$

We remark that in the usual banana manifold, the GV invariants in an effective fiber class β only depend on the quadratic form induced by the generic fiber whereas for $\widetilde{X}_N$, they also depend mildly on divisibility conditions through the number $\epsilon_N(\beta) \in \{1,2,3,4\}$.

1.3. **Arithmetic.** Let X be a rigid CY3 defined over $\mathbb{Q}$. Then there is a continuous system of two dimensional representations of $G_\mathbb{Q} = \mathrm{Gal}(\overline{\mathbb{Q}},\mathbb{Q})$ on $H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathbb{Q}_l)$. It was shown by Gouvêa-Yui [10] and Dieulefait [9] using the proof of Serre's conjectures by Khare-Wintenberger [13, 14] that this system of Galois representations is *modular*. In particular, there exists a weight 4 modular cusp form

$$f_X(q) = \sum_{n=1}^{\infty} a_n q^n$$

uniquely characterized by the condition that

$$a_p = \mathrm{tr}(\mathrm{Frob}_p \,\Big|\, H^3_{\text{ét}}(X_{\overline{\mathbb{F}}_p},\mathbb{Q}_l)) \tag{5}$$

for almost all primes p. Moreover, $\mathrm{tr}(\mathrm{Frob}_p)$ can be determined by the Lefschetz fixed point formula in terms of $\#X_{\mathbb{F}_p}$ and so the arithmetic of counting points in X over $\mathbb{F}_p$ gives rise to a modular form $f_X(q)$[1].

Theorem 7. *The rigid CY3 $\widetilde{X}_N$ is defined over $\mathbb{Q}$. The corresponding modular form $f_{\widetilde{X}_N}(q)$ is the unique weight 4 cusp form on the congruence subgroup $\Gamma_0(N)$. It can be expressed as the eta product*

$$f_{\widetilde{X}_N}(q) = \prod_{k\in\Theta_N} \eta(q^k)^2$$

where $\eta(q) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dedekind eta function, and Θ_N is as in Theorem 2.

The proof of this theorem, given in Appendix A, reduces to our case to a previously known case. Namely, we show that

$$f_{\widetilde{X}_N}(q) = f_{X_N^{\mathsf{Ver}}}(q)$$

where X_N^{Ver} are the rigid CY3s studied by Verrill [23, Appendix] who carried out the requisite point counts to determine $f_{X_N^{\mathsf{Ver}}}(q)$.

[1]Modularity for rigid CY3s defined over $\mathbb{Q}$ is analogous to the famous modularity theorem of Wiles-Taylor for elliptic curves over $\mathbb{Q}$. In that case the associated modular form $f_E(q)$ is weight 2.

The banana nano-manifold $\widetilde{X}_N \to \mathbb{P}^1$ has four singular fibers over points $p_1, \ldots, p_4 \in \mathbb{P}^1$ and thus there is an associated elliptic curve E_N given by the double branched cover of $\mathbb{P}^1$ ramified over $p_1, \ldots, p_4$. We have observed the following curious proposition which we will prove in Appendix A.

Proposition 8. *There is a model for E_N defined over $\mathbb{Q}$ such that the associated weight 2 modular form $f_{E_N}(q)$ is given by*

$$f_{E_N}(q) = \prod_{k \in \Theta_N} \eta(q^{2k})$$

so that $f_{E_N}(q)^2 = f_{\widetilde{X}_N}(q^2)$.

There is also elliptic curve associated to any rigid CY3 X, namely the intermediate Jacobian $J(X)$ which is given by

$$0 \to H_3(X, \mathbb{Z}) \xrightarrow{i} H^{3,0}(X)^* \to J(X) \to 0$$

where the embedding i is given by integrating the Calabi-Yau form over 3-cycles. This raises the natural

Question 8.1. *Is $E_N \cong J(\widetilde{X}_N)$?*

More generally, we ask the following

Question 8.2. *Let X be a rigid CY3 defined over $\mathbb{Q}$ and let $E = J(X)$ be its intermediate Jacobian. When does there exist a model for E defined over $\mathbb{Q}$ such that the equation*

$$f_X(q^2) = f_E(q)^2$$

holds?

1.4. **Plan of the paper.** In Section 2 we give our construction of $\widetilde{X}_N$ which is given as a free quotient of a crepant resolution of a certain fiber product of extremal rational elliptic surfaces. In Section 3 we use toric geometry to prove some key facts about our construction. In Section 4, we find a basis for curve classes and for divisors on $\widetilde{X}_N$ and compute their intersection numbers. In Section 5, we use the techniques of [4] to compute the DT invariants of $\widetilde{X}_N$ and we use that computation to get the GW and GV invariants. In Appendix A, we prove Theorem 7 by use étale cohomology techniques to reduce our computation to a related computation done by Verrill [23].

2. THE CONSTRUCTION OF $\widetilde{X}_N$

Let $S_N \to \mathbb{P}^1$ be one of the four unique rational elliptic surfaces having four singular fibers of type $(I_{k_1}, I_{k_2}, I_{k_3}, I_{k_4})$ where $\Theta_N = (k_1, k_2, k_3, k_4)$ is given below:

N	Singularity Type Θ_N	Mordell-Weil Group G_N	Modular group Γ_N
5	$(5, 5, 1, 1)$	$\mathbb{Z}_5$	$\Gamma_1(5)$
6	$(6, 3, 2, 1)$	$\mathbb{Z}_6$	$\Gamma_1(6)$
8	$(4, 4, 2, 2)$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\Gamma_1(4) \cap \Gamma(2)$
9	$(3, 3, 3, 3)$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\Gamma(3)$

The surface S_N is a compactification of the universal elliptic curve over the modular curve

$$\mathbb{P}^1 - \{4 \text{ points}\} \cong \mathbb{H}/\Gamma_N$$

where $\mathbb{H}$ is the upper half-plane and $\Gamma_N \subset SL_2(\mathbb{Z})$ is the congruence subgroup listed above. S_N admits an action of G_N, the associated Mordell-Weil group of sections listed above. The modular curve can be regarded as parameterizing elliptic curves E equipped with an injective homomorphism $G_N \to E$. Note that the order of G_N is N. S_N is an *extremal elliptic surface*: it admits no deformations preserving the singularity type [18, 2].

We will henceforth often suppress the N subscript of S_N and G_N. Let

$$S'_{\mathsf{sing}} = S/G, \quad S' \to S'_{\mathsf{sing}}$$

be the quotient and its minimal resolution. The action of G on S is free away from the nodes of the singular fibers and it is easy to see that the stabilizer of the nodes in an I_k fiber is $\mathbb{Z}_{N/k}$. It then follows that S' is isomorphic to S where the isomorphism is induced from an automorphism of the base $\mathbb{P}^1$ which reverses the order of the singular fibers. Indeed, the quotient of an I_k fiber in S is an irreducible nodal rational curve in S'_{sing} with a $\mathbb{Z}_{N/k}$ quotient singularity at the node. We denote this type of fiber by $I_1^{\mathbb{Z}_{N/k}}$.

Let

$$X_{\mathsf{sing}} = S \times_{\mathbb{P}^1} S'_{\mathsf{sing}},$$
$$X_{\mathsf{con}} = S \times_{\mathbb{P}^1} S'.$$

It follows from the previous analysis that

$$X_{\mathsf{sing}} \to \mathbb{P}^1 \text{ and } X_{\mathsf{con}} \to \mathbb{P}^1$$

are Abelian surface fibrations with four singular fibers of type

$$I_k \times I_1^{\mathbb{Z}_{N/k}} \text{ and } I_k \times I_{N/k}$$

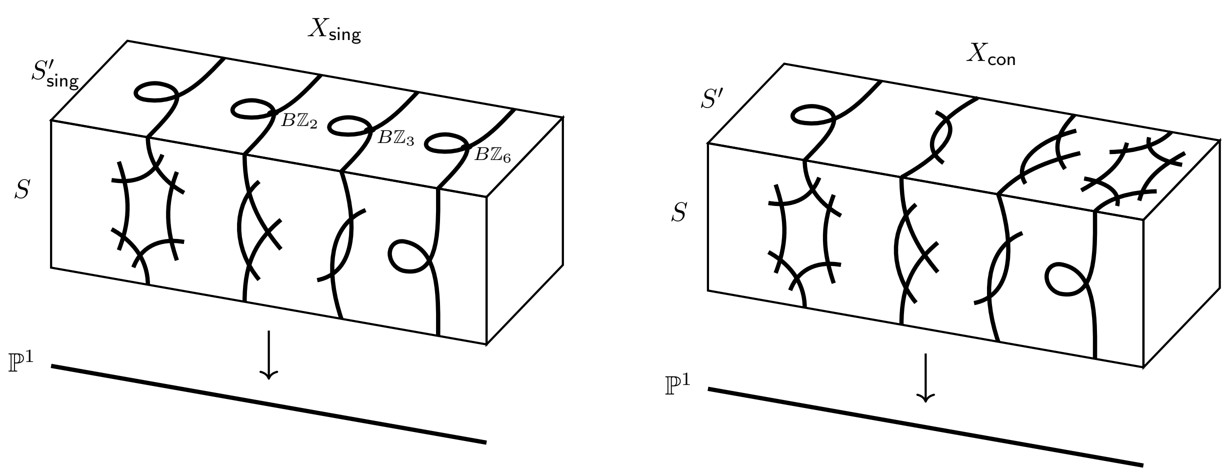
respectively for $k \in \Theta_N$.

The threefolds X_{sing} and X_{con} are singular CY3s. X_{con} has $4N$ singularities occurring at the points

$$n_i \times n'_j \in I_k \times I_{N/k}$$

where $n_i \in I_k$ and $n'_j \in I_{N/k}$ are nodes. The singularities of X_{con} are all conifold singularities whereas the singularities of X_{sing} are more complicated but are locally hypersurface singularities given by the equation $\{ab = x^l y^l\}$ where $l = N/k$.

We illustrate X_{sing} and X_{con} below in the case of $N = 6$:



It follows from the previous analysis of the stabilizers of G that the diagonal copy of G in $G \times G$ acts freely on $S \times_{\mathbb{P}^1} S'$, transitively permuting the N conifold points in each singular fiber.

One of our main technical results is that there is a projective conifold resolution of X_{con} given by blowing up an explicit (non-Cartier) Weil divisor passing through all the conifold points. Namely, let

$$\Gamma \subset X_{\mathsf{con}}$$

be the proper transform of $\gamma \subset X_{\mathsf{sing}}$ where γ is the graph of the quotient map

$$f : S \to S'_{\mathsf{sing}}.$$

Theorem 9. *Let $\Gamma \subset X_{\mathsf{con}}$ be as above and let*

$$X_N = \mathrm{Bl}_\Gamma(X_{\mathsf{con}})$$

be the blow-up of X_{con} along Γ. Then X_N is a non-singular CY3 with $h^{2,1}(X_N) = 0$ and $h^{1,1}(X_N) = 4N$. Moreover the quotient of X_N by the diagonal copy of G_N

$$\widetilde{X}_N = X_N / G_N$$

is a CY3 with $h^{2,1}(\widetilde{X}_N) = 0$ and $h^{1,1}(\widetilde{X}_N) = 4$.

Proof. The most difficult part of the proof of the above theorem is showing that blowing up Γ yields a conifold resolution. We defer that to the next section.

Assuming the conifold resolution exists, we compute the Hodge numbers as follows. First we show that X_N is rigid. For any CY3 given as a conifold resolution $Z \to Z_{\mathsf{con}}$, the deformations of Z are given by the deformations of Z_{con} which do not smooth any of the singularities [20, § 3.1]. In the case of $X_{\mathsf{con}} = S \times_{\mathbb{P}^1} S'$, all deformations arise from deformations of S, S', or from composing the map $S \to \mathbb{P}^1$ with a Mobiüs transformation of the base [22]. But since S and S' are extremal elliptic surfaces, any such deformation results in smoothing one or more of the conifold singularities. Therefore, X_N is rigid and $h^{2,1}(X_N) = 0$.

The topological Euler characteristic of X_{con} can be computed as follows. Since the generic fibers of $X_{\mathsf{con}} \to \mathbb{P}^1$ are Abelian surfaces and have Euler characteristic 0, the Euler characteristic of X_{con} is equal to the sum of the Euler characteristics of the singular fibers. Then since the singular fibers admit $\mathbb{C}^* \times \mathbb{C}^*$ action whose only fixed points are the singularities, we find

$$\begin{aligned} e(X_{\mathsf{con}}) &= \text{number of conifold singularities} \\ &= 4N. \end{aligned}$$

The conifold resolution $X_N \to X_{\mathsf{con}}$ replaces each conifold singularity with a $\mathbb{P}^1$ and so we find that $e(X_N) = 8N$. Then since for any CY3 X, $e(X) = 2h^{1,1}(X) - 2h^{2,1}(X)$ we see that $h^{1,1}(X_N) = 4N$ as was asserted. Finally since G_N acts freely, the Euler characteristic of $\widetilde{X}_N$ is 8 and subsequently, $h^{1,1}(\widetilde{X}_N) = 4$. □

2.1. **Schoen nano-manifolds.** We briefly digress to discuss some nano-manifolds closely related to $\widetilde{X}_N$. Consider

$$X_N^{\mathsf{Sch}} = S_N \times_\phi S'_N$$

where the notation means that we take the fiber product of $S_N \to \mathbb{P}^1$ with the composition $S'_N \to \mathbb{P}^1 \xrightarrow{\phi} \mathbb{P}^1$ where $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is a generic Möbius transformation.

The fiber product of any two rational elliptic surfaces such that the singular fibers of each surface do not coincide (such as X_N^{Sch}) is called a Schoen manifold and is a smooth CY3 with $h^{1,1} = h^{2,1} = 19$. The 19 deformations are given by the 8 deformations of each rational elliptic surface along with the 3 dimensional family of Möbius transformations of the base.

Consider the quotient of X_N^{Sch} by the free action of the diagonal $G_N \subset G_N \times G_N$:

$$\widetilde{X}_N^{\mathsf{Sch}} = X_N^{\mathsf{Sch}} / G_N.$$

Proposition 10. *$\widetilde{X}_N^{\mathsf{Sch}}$ is a nano-manifold of height 6 with*

$$h^{1,1}(\widetilde{X}_N^{\mathsf{Sch}}) = h^{2,1}(\widetilde{X}_N^{\mathsf{Sch}}) = 3.$$

Moreover, $\widetilde{X}_N^{\mathsf{Sch}}$ and $\widetilde{X}_N$ are related by a conifold transition.

Proof. Since S and S' are extremal, the only deformations of X_N^{Sch} which preserve the G_N action are the deformations of ϕ. Therefore $h^{2,1}(\widetilde{X}_N^{\mathsf{Sch}}) = 3$ and since

$$e(X_N^{\mathsf{Sch}}) = e(\widetilde{X}_N^{\mathsf{Sch}}) = 0$$

we have $h^{1,1}(\widetilde{X}_N^{\mathsf{Sch}}) = 3$. Under the deformation taking a generic ϕ to the identity, $\widetilde{X}_N^{\mathsf{Sch}}$ deforms to X_{con}/G_N which has $\widetilde{X}_N$ as a conifold resolution. □

The enumerative geometry of the Schoen nano-manifolds $\widetilde{X}_N^{\mathsf{Sch}}$ is expected to be interesting. On one hand, it is related to the enumerative geometry of $\widetilde{X}_N$ by the usual conifold transition formulas. On the other hand, the enumerative geometry of $\widetilde{X}_N^{\mathsf{Sch}}$ should be related to the enumerative geometry of the CHL model

$$(K3\times E)/G_N$$

where G_N acts symplectically on $K3$ and by translation on the elliptic curve E. The DT invariants of these models were studied in [5].[2]

3. Local Toric Geometry, Resolutions, and Intersections

In this section, we use toric geometry to analyze $\Gamma \subset X_{\mathsf{con}}$ in a formal neighborhood of the singular points and prove (among other things) that $X_N = \mathrm{Bl}_\Gamma X_{\mathsf{con}}$ is a conifold resolution of X_{con}.

Since $\Gamma\subset X_{\mathsf{con}}$ is a divisor, it is Cartier away from the $4N$ conifold singularities and hence the blow-up does nothing away from the singularities. Thus understanding $X_N = \mathrm{Bl}_\Gamma(X_{\mathsf{con}})$ reduces to the local problem of understanding Γ in a neighborhood of each singular point. This is still difficult since Γ is defined as the proper transform of $\gamma\subset X_{\mathsf{sing}}$ and is hence given by a closure (for example, it turns out that Γ is necessarily singular and non-normal although we will not need to prove that here).

For each $k\in\Theta_N$ let

$$l = N/k$$

so that the corresponding singular fibers in X_{sing} and X_{con} are of type $I_k\times I_1^{\mathbb{Z}_l}$ and $I_k\times I_l$ respectively. At the node in $I_1^{\mathbb{Z}_l}$, the surface S'_{sing} is formally locally modelled on

$$\begin{aligned} S^{\mathsf{loc}}_{\mathsf{sing}} &\cong \mathbb{A}^2/\mathbb{Z}_l\\ &\cong \operatorname{Spec}\left(\mathbb{C}[x,y]^{\mathbb{Z}_l}\right)\\ &\cong \operatorname{Spec}\left(\mathbb{C}[x^l,y^l,xy]\right)\\ &\cong \operatorname{Spec}\left(\mathbb{C}[a,b,c]/(ab-c^l)\right) \end{aligned}$$

and the map to the base is locally modelled on the map

$$S^{\mathsf{loc}}_{\mathsf{sing}}\to\mathbb{A}^1$$

given by $(x,y)\mapsto xy$, i.e. $(a,b,c)\mapsto c$.

Let

$$S^{\mathsf{loc}}\to S^{\mathsf{loc}}_{\mathsf{sing}}$$

be the minimal resolution. Note that the exceptional fiber is a chain of $(l-1)$ $\mathbb{P}^1$s.

The local model of X_{sing} at a singular point in the $I_k\times I_1^{\mathbb{Z}_l}$ fiber is then

$$\begin{aligned} X^{\mathsf{loc}}_{\mathsf{sing}} &= \mathbb{A}^2\times_{\mathbb{A}^1} S^{\mathsf{loc}}_{\mathsf{sing}}\\ &= \operatorname{Spec}\left(\mathbb{C}[x,y,a,b,c]/(ab-c^l, xy-c)\right)\\ &= \operatorname{Spec}\left(\mathbb{C}[x,y,a,b,]/(ab-x^ly^l)\right) \end{aligned}$$

and $X_{\mathsf{con}}\to X_{\mathsf{sing}}$ is locally modelled on

$$X^{\mathsf{loc}}_{\mathsf{con}}\to X^{\mathsf{loc}}_{\mathsf{sing}}\quad\text{where}\quad X^{\mathsf{loc}}_{\mathsf{con}} = \mathbb{A}^2\times_{\mathbb{A}^1} S^{\mathsf{loc}}.$$

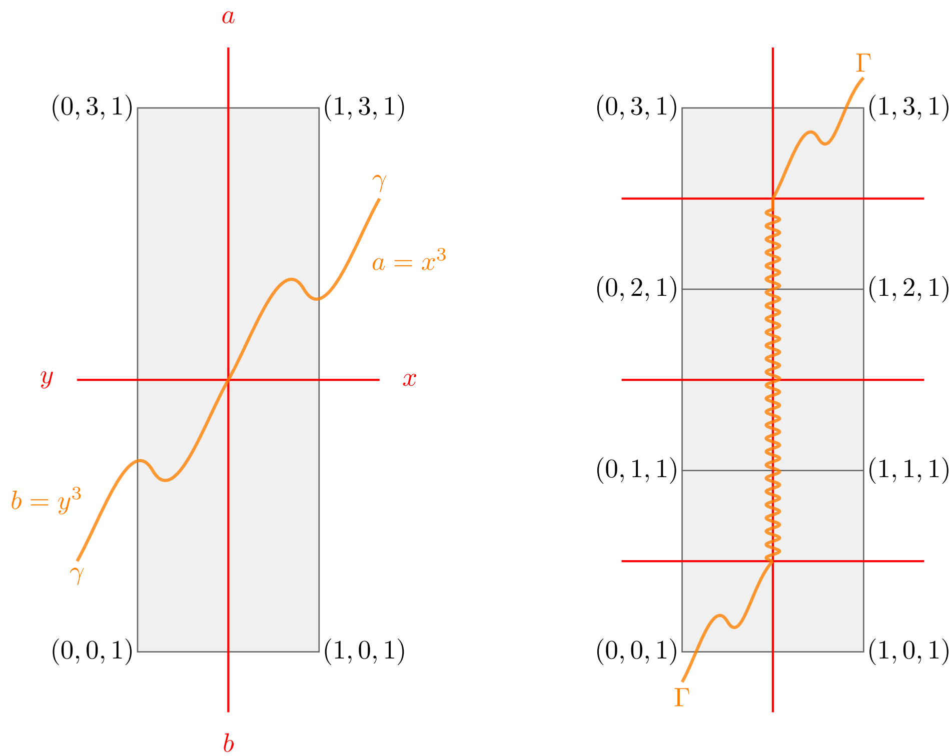
We note that graph of the quotient map $\mathbb{A}^2\to S^{\mathsf{loc}}_{\mathsf{sing}}$ is the Weil divisor $\gamma\subset X^{\mathsf{loc}}_{\mathsf{sing}}$ with ideal

$$(a-x^l, b-y^l)\subset \mathbb{C}[x,y,a,b]/(ab-x^ly^l).$$

The threefolds $X^{\mathsf{loc}}_{\mathsf{sing}}$ and $X^{\mathsf{loc}}_{\mathsf{con}}$ are both toric, the former is an affine toric variety with 1 singularity and the latter is a quasi-projective toric variety with l conifold singularities.

[2]The argument connecting these two models is a folklore degeneration argument that we learned from G. Oberdieck.

The toric threefolds $X^{\mathsf{loc}}_{\mathsf{sing}}$ and $X^{\mathsf{loc}}_{\mathsf{con}}$ determine fans (F_{sing} and F_{con} respectively) in $\mathbb{R}^3$ whose cones are generated by vertices in the $z=1$ plane in $\mathbb{R}^3$. For $X^{\mathsf{loc}}_{\mathsf{sing}}$, the fan is the cone over the rectangle with vertices $(0,0,1)$, $(0,l,1)$, $(1,0,1)$, and $(1,l,1)$. For $X^{\mathsf{loc}}_{\mathsf{con}}$, the fan is the union of the cones over the l squares with vertices $(0,i-1,1)$, $(0,i,1)$, $(1,i-1,1)$, and $(1,i,1)$ for $i=1,\dots,l$. We illustrate this below for the case of $l=3$:



The fans F_{sing} and F_{con} with the divisors γ and Γ.

In the above pictures (which live in the $z=1$ plane), the cones of the fans are given by the cones over the grey polygons. The dual polytope is depicted in red and corresponds to the torus invariant points, curves, and divisors[3]. For example, the plane perpendicular to the ray $(1,3,1)$ in the left picture corresponds to the torus invariant (Weil) divisor given by the ideal

$$(y,b)\subset\mathbb{C}[x,y,a,b]/(ab-x^3y^3).$$

This divisor is a copy of $\mathbb{A}^2$ with coordinates x and a and the intersection of γ with it is given by the curve $a=x^3$ which we draw schematically in orange. The proper transform of $\gamma\subset X^{\mathsf{loc}}_{\mathsf{sing}}$, namely $\Gamma\subset X^{\mathsf{loc}}_{\mathsf{con}}$, can intersect the exceptional curves of $X^{\mathsf{loc}}_{\mathsf{con}}\to X^{\mathsf{loc}}_{\mathsf{sing}}$ in complicated ways, which we depict in the right hand picture with a squiggly orange curve.

To deal with the potential complications of Γ and its blowup, we find a family of divisors that interpolates between Γ and a toric divisor. Let $D_\epsilon\subset X^{\mathsf{loc}}_{\mathsf{sing}}$ be the Weil divisor given by the ideal

$$(y^l-\epsilon b, a-\epsilon x^l)\subset\mathbb{C}[x,y,a,b]/(ab-x^ly^l).$$

Then $D_1=\gamma$ and D_0 is the torus invariant divisor with ideal (y^l,a). D_0 has support on the torus invariant divisor with ideal (y,a) which corresponds to the ray generated by $(1,0,1)$ in the fan. Note that D_0 has multiplicity l since on the interior of D_0 where $b,x\neq 0$, we have $a=x^ly^lb^{-1}=\text{unit}\cdot y^l$.

Let $D'_\epsilon\subset X^{\mathsf{loc}}_{\mathsf{con}}$ be the proper transform of $D_\epsilon\subset X^{\mathsf{loc}}_{\mathsf{sing}}$. For the toric case of $\epsilon=0$, D'_0 can be determined by the combinatorics of the fan. Indeed, as with any torus invariant divisor, $D_0\subset X^{\mathsf{loc}}_{\mathsf{sing}}$ is determined by an integer valued function on the vectors generating the rays of the fan. In this case, the function takes the values $(0,0,0,l)$ at the

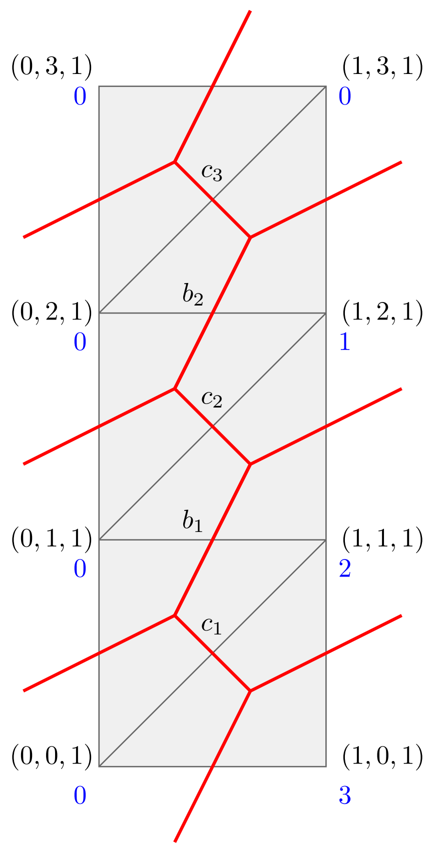
[3]The faces of the dual polytope should be perpendicular to the rays in the fan, so the red polytope does not lie in the $z=1$ plane. We depict a projection of the dual polytope to the plane.

generators $((0,0,1),(0,l,1),(1,l,1),(1,0,1))$ respectively. D_0 is not Cartier: if it were, its associated values on the vectors generating the fan would be the restriction of a linear function on the cone. It is however Cartier away from the singular point. This is reflected in the fact that the values are the restrictions of linear functions on the 2-dimensional faces of the cone. Since F_{con} is obtained from F_{sing} by adding various new generators to the 2-dimensional faces of F_{sing}, the proper transform D_0' is given by the function on the generators obtained by restricting the linear function on the faces of F_{sing} to the new generators. Namely, D_0' is given by the function taking values 0 on the generators $(0,j,1)$ and $l-j$ on the generators $(1,j,1)$ for $j=0,\dots,l$.

Note that $D_0' \subset X^{\mathsf{loc}}_{\mathsf{con}}$ is still non-Cartier: on the cone in F_{con} generated by $((0,i-1,1),(0,i,1),(1,i-1,1),(1,i,1))$, D_0' take values $(0,0,l-i+1,l-i)$. On this cone, which corresponds to the T-invariant affine neighborhood of the ith conifold point in $X^{\mathsf{loc}}_{\mathsf{con}}$, D_0' is the sum of the divisors taking values $(0,0,1,0)$ and $(0,0,l-i,l-i)$ on the generators. The latter is principal and hence doesn't affect the blowup. The blowup of the former gives the conifold resolution obtained by adding the 2-dimensional face spanned by $(0,i-1,1)$ and $(1,i,1)$ to the fan. Thus

$$X^{\mathsf{loc}} = \mathrm{Bl}_{D_0'}(X^{\mathsf{loc}}_{\mathsf{con}})$$

is smooth and is given by the fan F colored grey in the picture below:



The fan F of X^{loc}, the dual polytope, and the multiplicities of D_0''.

Let $D_0'' \subset X^{\mathsf{loc}}$ be the proper transform of D_0'. The values of D_0'' on the generators of the fan F are depicted in blue in the above picture and the dual polytope is depicted in red.

The exceptional locus of $X^{\mathsf{loc}} \to X^{\mathsf{loc}}_{\mathsf{sing}}$ is given by $c_1 \cup b_1 \cup \cdots \cup b_{l-1} \cup c_l$, a chain of $2l-1$ $\mathbb{P}^1$s. The l curves $c_1,\dots,c_l$ are the expectional curves of the conifold resolution $X^{\mathsf{loc}} \to X^{\mathsf{loc}}_{\mathsf{con}}$ and the $l-1$ curves $b_1,\dots,b_{l-1}$ are the proper transforms of the exceptional curves of $X^{\mathsf{loc}}_{\mathsf{con}} \to X^{\mathsf{loc}}_{\mathsf{sing}}$.

We can now compute the intersection numbers $D_0'' \cdot c_i$ and $D_0'' \cdot b_i$. Let $D[r, s, 1]$ denote the torus invariant divisor corresponding to the ray generated by $(r, s, 1)$. So then

$$D_0'' = \sum_{k=0}^{l} (l-k) D[1, k, 1].$$

Now the intersection of the proper curves b_i and c_i with $D[r, s, 1]$ can be computed using standard toric geometry. The intersection number is 0 if the curve and divisor are disjoint and 1 if they meet in a single point. If the curve is contained in the divisor, one can find a linearly equivalent divisor (by adding a suitable global linear function on the fan) such that no component of the new divisor contains the curve and then easily compute the intersection number. The results are

$$b_i \cdot D[1, k, 1] = \begin{cases} 1 & \text{if } k = i+1 \\ -1 & \text{if } k = i \\ 0 & \text{otherwise.} \end{cases}$$

$$c_i \cdot D[1, k, 1] = \begin{cases} 1 & \text{if } k = i-1 \\ -1 & \text{if } k = i \\ 0 & \text{otherwise.} \end{cases}$$

We then get

$$\begin{aligned} b_i \cdot D_0'' &= l - (i+1) - (l-i) \\ &= -1 \\ c_i \cdot D_0'' &= l - (i-1) - (l-i) \\ &= 1. \end{aligned}$$

We summarize the above discussion in the following lemma.

Lemma 11. *Let $X^{\mathsf{loc}} = \mathrm{Bl}_{D_0'}(X^{\mathsf{loc}}_{\mathsf{con}})$. Then X^{loc} is a smooth toric CY3. The exceptional set of the resolution $X^{\mathsf{loc}} \to X^{\mathsf{loc}}_{\mathsf{sing}}$ is a chain of $2l-1$ curves $c_1 \cup b_1 \cup \cdots \cup b_{l-1} \cup c_l$ where the c_i's are the exceptional curves of $X^{\mathsf{loc}} \to X^{\mathsf{loc}}_{\mathsf{con}}$ and the b_i's are the proper transforms of the exceptional curves of $X^{\mathsf{loc}}_{\mathsf{con}} \to X^{\mathsf{loc}}_{\mathsf{sing}}$. Let $D_0'' \subset X^{\mathsf{loc}}$ be the proper transform of D_0'. Then the intersection numbers of D_0'' with the exceptional curves are given by*

$$D_0'' \cdot c_i = +1, \text{ for } i = 1, \ldots, l, \tag{6}$$

$$D_0'' \cdot b_i = -1, \text{ for } i = 1, \ldots, l-1. \tag{7}$$

We next show that the above lemma holds for the whole family of divisors parameterized by ϵ. This will allow us to convert the toric result about X^{loc} into a global result for X since the divisor $\Gamma = D'_{\epsilon=1}$ is well defined globally.

Let

$$\mathbb{D} \subset \mathbb{A}^1 \times X^{\mathsf{loc}}_{\mathsf{sing}}$$

be the Weil divisor with ideal

$$(y^l - \epsilon b, a - \epsilon x^l) \subset \mathbb{C}[x, y, a, b, \epsilon]/(ab - x^l y^l).$$

Let $\mathbb{D}'$ be the proper transform of $\mathbb{D}$ under $\mathbb{A}^1 \times X^{\mathsf{loc}}_{\mathsf{con}} \to \mathbb{A}^1 \times X^{\mathsf{loc}}_{\mathsf{sing}}$ and consider

$$\mathbb{X}^{\mathsf{loc}} = \mathrm{Bl}_{\mathbb{D}'}(\mathbb{A}^1 \times X^{\mathsf{loc}}_{\mathsf{con}}).$$

By the functoriality of blowups, the fiber of $\mathbb{X}^{\mathsf{loc}} \to \mathbb{A}^1$ over ϵ is given by $\mathrm{Bl}_{D'_\epsilon}(X^{\mathsf{loc}}_{\mathsf{con}})$. Then since we've shown that $\mathrm{Bl}_{D_0'}(X^{\mathsf{loc}}_{\mathsf{con}})$ is non-singular, it follows that $\mathrm{Bl}_{D'_\epsilon}(X^{\mathsf{loc}}_{\mathsf{con}})$ is non-singular for generic ϵ and hence for all ϵ. In fact, since the singularities are all conifolds and the resolutions are conifold resolutions, we have $\mathbb{X}^{\mathsf{loc}} = X^{\mathsf{loc}} \times \mathbb{A}^1$. Let $\mathbb{D}'' \subset \mathbb{X}^{\mathsf{loc}}$ be the proper transform of $\mathbb{D}'$. Then $D''_\epsilon \cdot c_i$ and $D''_\epsilon \cdot b_i$ are independent of ϵ since they are given by $\deg(\mathcal{O}(\mathbb{D}'')|_{\epsilon \times c_i})$ and $\deg(\mathcal{O}(\mathbb{D}'')|_{\epsilon \times b_i})$.

In particular, when $\epsilon = 1$, the divisor $D'_{\epsilon=1} \subset X^{\mathsf{loc}}_{\mathsf{con}}$ is a local model for $\Gamma \subset X_{\mathsf{con}}$ and then the local results for $D'_{\epsilon=1}$ imply the following global results:

Proposition 12. *Let $\Gamma \subset X_{\mathsf{con}}$ be as defined in § 2. Let $X_N = \mathrm{Bl}_\Gamma(X_{\mathsf{con}})$ and let us also denote by $\Gamma \subset X_N$ the proper transform of $\Gamma \subset X_{\mathsf{con}}$. Then X_N is a smooth CY3 and*

$$\Gamma \cdot c = 1 \tag{8}$$

for any exceptional curve c of $X \to X_{\mathsf{con}}$ and

$$\Gamma \cdot b = -1 \tag{9}$$

for any curve b given by the proper transform of an exceptional curve of $X_{\mathsf{con}} \to X_{\mathsf{sing}}$.

4. Divisors, curves, and intersection numbers on X_N

We now study the cohomology classes of X_N and $\widetilde{X}_N$. In particular we find a $\mathbb{Q}$ basis for curve classes and divisor classes and we compute their intersections numbers.

The divisor classes of X_N necessarily consist of the divisor classes of the generic fiber and the fiber classes. The generic fiber is spanned by the three divisors

$$S = S \times_{\mathbb{P}^1} \{0\}, \quad S' = \{0\} \times_{\mathbb{P}^1} S', \quad \text{and} \quad \Gamma.$$

The fiber divisor classes are given by the generic fiber F and the irreducible components of the singular fibers.

We now develop notation for the curves and divisors in the singular fibers of $X_N \to \mathbb{P}^1$. As discussed in Section 2, for each $k \in \Theta_N$ there is a corresponding singular fiber of $X_N \to \mathbb{P}^1$ which is the proper transform of $I_k \times I_l \subset X_{\mathsf{con}}$ where we recall that

$$l = \frac{N}{k}.$$

Fibers of this type are called multi-banana fibers and were studied by Kanazawa-Lau [12] and Morishige [19]. We label[4] such a fiber by $F(k)$.

The singular fiber $F(k) \subset X_N$ is a non-normal toric surface and its formal neighborhood $\widehat{F}(k)$ has a universal cover which is a formal toric Calabi-Yau threefold $\widehat{U}$ modelled on the toric CY3 whose fan in $\mathbb{R}^3$ consists of the cones generated by

$$(i,j,1),\ (i+1,j,1),\ (i+1,j+1,1), \quad (i,j) \in \mathbb{Z}\times\mathbb{Z},$$

and

$$(i,j,1),\ (i,j+1,1),\ (i+1,j+1,1), \quad (i,j) \in \mathbb{Z}\times\mathbb{Z}.$$

An element $(s,t) \in \mathbb{Z}\times\mathbb{Z}$ in the group of deck transformations of $\widehat{U} \to \widehat{F}(k)$ acts on the generators of the cones by translation by $(sk, tl, 0)$ (see [19, § 2.2] for more details). We then may choose a fundamental domain for the $\mathbb{Z}\times\mathbb{Z}$ action on the cones, namely the cones given above with $i \in \{0,\dots,k-1\}$ and $j \in \{0,\dots,l-1\}$. The quotient of $\widehat{F}(k)$ by the action of G_N is then given by the quotient of $\widehat{U}$ by $\mathbb{Z}\times\mathbb{Z}$ where now (s,t) acts by translation by $(s,t,0)$ on the generators of the cones. This quotient is the formal neighborhood of a banana fiber, $\widehat{F}_{\mathsf{ban}}$ in the notation of [4]:

$$\widehat{F}(k)/G_N \cong \widehat{F}_{\mathsf{ban}}.$$

We label the torus invariant divisors in $F(k)$ by $D_{ij}(k)$, $i = 1,\dots,k$, $j = 1,\dots,l$ and the torus invariant curves by $a_{ij}(k)$, $b_{ij}(k)$, and $c_{ij}(k)$. We do this such that the zero section meets $D_{kl}(k)$ and the b and c curves coincide with the b and c curves in the local model. In particular, $c_{ij}(k)$ are the exceptional curves of the conifold resolution $X_N \to X_{\mathsf{con}}$, $b_{ij}(k)$ for $j \neq l$ are the proper transforms of the exceptional curves of $X_{\mathsf{con}} \to X_{\mathsf{sing}}$, and $b_{il}(k)$ is the proper transform of the curve

$$n_i \times I_1^{\mathbb{Z}_l} \subset I_k \times I_1^{\mathbb{Z}_l} \subset X_{\mathsf{sing}}.$$

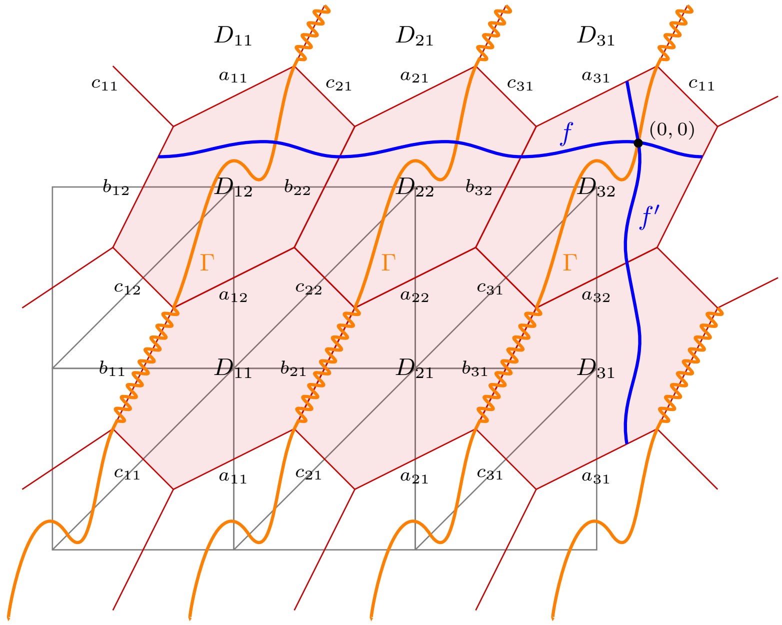
Finally, $a_{ij}(k)$ is the proper transform of

$$a_i \times n_j \subset I_i \times I_l \subset X_{\mathsf{con}}$$

where a_i is the ith curve in I_i and n_j is the jth node in I_l.

We illustrate this below for $N = 6$, $k = 3$, and $l = 2$ (we suppress the k from the notation in the diagram):

[4] If k is repeated in Θ_N we will use primes to further distinguish so that the labels will be in $(1,2,3,6)$, $(1,1',5,5')$, $(2,2',4,4')$, or $(3,3',3'',3''')$.



A fundamental domain for the toric diagram (gray) and dual polytope (red) for $\widehat{F}(3)$, the formal neighborhood of the singular fiber $F(3) \subset X_6$.

Note that in the above example, the fan and dual polytope of the local model appears $k = 3$ times, each with $l = 2$. The intersection of Γ with $F(3)$ is depicted in orange.

The $4N + 4$ divisor classes given by

$$\{F, S, S', \Gamma, D_{ij}(k)\}_{k\in\Theta_N,\, i=1,\dots,k,\, j=1,\dots,l}$$

admit four obvious relations, namely

$$\sum_{i,j} D_{ij}(k) = F$$

for each $k \in \Theta_N$. Since the Picard number of X_N is 4N, there are no further relations.

We obtain the following intersection numbers easily since in each case, the curve and divisor are either disjoint or intersect transversely in a single point:

$$S \cdot a_{ij}(k) = S' \cdot b_{ij}(k) = S \cdot c_{ij}(k) = S' \cdot c_{ij}(k) = 0.$$

and

$$S' \cdot a_{ij}(k) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases}$$

$$S \cdot b_{ij}(k) = \begin{cases} 1 & \text{if } j = l, \\ 0 & \text{if } j \neq l. \end{cases}$$

By Equations (8) and (9) in Proposition 12, we have

$$\begin{aligned} &\Gamma \cdot c_{ij}(k) = 1, \\ &\Gamma \cdot b_{ij}(k) = -1 \quad \text{ for } j \neq l. \end{aligned}$$

Thus to get all the intersection numbers of the fiber curves with $\{S, S', \Gamma\}$, it remains to compute $\Gamma \cdot a_{ij}(k)$ and $\Gamma \cdot b_{il}(k)$. To determine these, we use the curve classes

$$f = S \cdot F, \quad f' = S' \cdot F$$

which are the fiber curve classes of the elliptic fibrations $S \to \mathbb{P}^1$ and $S' \to \mathbb{P}^1$. We may write these classes in terms of $a_{ij}(k), b_{ij}(k), c_{ij}(k)$ since they are given as the total transform of the curves $I_k \times \{0\}$ and $\{0\} \times I_l$ respectively in the fiber $F(k) \subset X_N$. These curves are illustrated in the above diagram in blue.

Since $I_k \times \{0\}$ is homologous to $I_k \times n_j$ where n_j is any node, and the total transform of $I_k \times n_j$ is the union

$$\bigcup_i c_{ij}(k) \cup a_{ij}(k)$$

we get the following relation for any fixed k and j:

$$f = \sum_{i=1}^{k} c_{ij}(k) + a_{ij}(k).$$

Similarly, for any k and i we get

$$f' = \sum_{j=1}^{l} c_{ij}(k) + b_{ij}(k).$$

Taking a representative for f' and f in the generic fiber, we see that[5]

$$f' \cdot \Gamma = 1, \qquad f \cdot \Gamma = N.$$

Combining, we get

$$\begin{aligned} 1 &= f' \cdot \Gamma \\ &= \sum_{j=1}^{l} (c_{ij}(k) + b_{ij}(k)) \cdot \Gamma \\ &= l - (l-1) + b_{il}(k) \cdot \Gamma \end{aligned}$$

which implies

$$b_{il}(k) \cdot \Gamma = 0$$

for any i and k.

We note that $a_{ij}(k) \cdot \Gamma$ is independent of i since the action of the first factor of $G_N \times G_N$ preserves Γ and permutes the i indices of $a_{ij}(k)$. Then for any j and k we have

$$\begin{aligned} N &= f \cdot \Gamma \\ &= \sum_{i=1}^{k} (c_{ij}(k) + a_{ij}(k)) \cdot \Gamma \\ &= k + k a_{ij}(k) \cdot \Gamma \end{aligned}$$

where the last equality holds for all i, j, k. Thus we get

$$a_{ij}(k) \cdot \Gamma = l - 1.$$

We summarize these results in the following

Lemma 13. *The intersection numbers of the fiber curve classes $a_{ij}(k), b_{ij}(k), c_{ij}(k)$ with the divisors S, S', Γ are given by the following table:*

[5] We remark that the picture of the intersection of f and Γ in the diagram is misleading. The curve f should meet Γ twice in each of D_{12}, D_{22}, D_{32} for a total of $N = 6$.

ϵ	$S'\cdot\epsilon$	$S\cdot\epsilon$	$\Gamma\cdot\epsilon$
$a_{ij}(k)$	$\left\{\begin{smallmatrix} 1 \text{ if } i=k \\ 0 \text{ if } i\neq k \end{smallmatrix}\right\}$	0	$l-1$
$b_{ij}(k)$	0	$\left\{\begin{smallmatrix} 1 \text{ if } j=l \\ 0 \text{ if } j\neq l \end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix} 0 & \text{if } j=l \\ -1 & \text{if } j\neq l \end{smallmatrix}\right\}$
$c_{ij}(k)$	0	0	1

We are primarily interested in curve and divisor classes on $\widetilde{X}_N = X_N/G_N$. Let

$$\pi : X_N \to \widetilde{X}_N$$

be the quotient map. We note that for each k, G_N acts simply transitively on the sets $\{a_{ij}(k)\}$, $\{b_{ij}(k)\}$, $\{c_{ij}(k)\}$, and $\{D_{ij}(k)\}$. In particular the classes

$$\tilde{a}(k) = \pi_*(a_{ij}(k)), \quad \tilde{b}(k) = \pi_*(b_{ij}(k)), \quad \tilde{c}(k) = \pi_*(c_{ij}(k))$$

are independent of the choice of i and j. We also define

$$\widetilde{\Gamma} = \pi_*(\Gamma), \quad \widetilde{S} = \pi_*(S), \quad \widetilde{S}' = \pi_*(S')$$

and finally we define $\widetilde{F}$ to be the class of the fiber of $\widetilde{X}_N \to \mathbb{P}^1$ so that (in slight notational conflict with the other tilded classes)

$$\widetilde{F} = \tfrac{1}{N}\pi_*(F).$$

The divisor classes $\{\widetilde{F}, \widetilde{S}, \widetilde{S}', \widetilde{\Gamma}\}$ span $\operatorname{Pic}(\widetilde{X}_N)$ and to compute their intersection numbers with the fiber curve classes $\widetilde{a}(k)$, $\widetilde{b}(k)$, $\widetilde{c}(k)$ we use the following.

If D is any divisor class on X_N, $\widetilde{\epsilon}$ is a curve class on $\widetilde{X}_N$, and $\widetilde{D} = \pi_*(D)$ then

$$\begin{aligned} \widetilde{D}\cdot\widetilde{\epsilon} &= \pi_*(D)\cdot\widetilde{\epsilon} \\ &= \pi_*(D\cdot\pi^*\widetilde{\epsilon}) \\ &= D\cdot\pi^*\widetilde{\epsilon} \end{aligned}$$

so to compute $\widetilde{S}\cdot\widetilde{a}(k)$ for example, we need to compute

$$S\cdot\pi^*(\widetilde{a}(k)) = S\cdot\sum_{i,j} a_{ij}(k)$$

and this can be read off from the table in Lemma 13. Carrying this out we get the following table of intersections on $\widetilde{X}_N$.

$\widetilde{\epsilon}$	$\widetilde{S}'\cdot\widetilde{\epsilon}$	$\widetilde{S}\cdot\widetilde{\epsilon}$	$\widetilde{\Gamma}\cdot\widetilde{\epsilon}$
$\widetilde{a}(k)$	l	0	$N(l-1)$
$\widetilde{b}(k)$	0	k	$-k(l-1)$
$\widetilde{c}(k)$	0	0	N

Geometrically, the classes $\widetilde{a}(k), \widetilde{b}(k), \widetilde{c}(k)$ are represented by the three banana curves in the fiber $\widetilde{F}(k) = F(k)/G_N$. It will be convenient to consider a slightly different basis of curve and divisor classes. Namely, consider the curve classes

$$\widetilde{a}(k) + \widetilde{c}(k), \quad \widetilde{b}(k) + \widetilde{c}(k), \quad \widetilde{c}(k)$$

and the divisor class

$$\widetilde{\Delta} = \widetilde{\Gamma} - N\widetilde{S}' - \widetilde{S}.$$

Lemma 14. *The intersection numbers for the above classes are given in the following:*

$\widetilde{\epsilon}$	$\widetilde{S}'\cdot\widetilde{\epsilon}$	$\widetilde{S}\cdot\widetilde{\epsilon}$	$\widetilde{\Delta}\cdot\widetilde{\epsilon}$
$\widetilde{a}(k)+\widetilde{c}(k)$	l	0	0
$\widetilde{b}(k)+\widetilde{c}(k)$	0	k	0
$\widetilde{c}(k)$	0	0	N

4.1. **The intersection form of a smooth fiber.** The divisor classes

$$\Gamma, \quad S, \quad S', \quad \Delta = \Gamma - NS' - S$$

on X_N and the corresponding classes

$$\widetilde{\Gamma}, \quad \widetilde{S}, \quad \widetilde{S}', \quad \widetilde{\Delta}$$

on $\widetilde{X}_N$ restrict to a fiber to give curve classes

$$\gamma = \Gamma \cdot F, \quad f = S \cdot F, \quad f' = S' \cdot F, \quad \delta = \Delta \cdot F$$

on X_N, and correspondingly

$$\widetilde{\gamma} = \widetilde{\Gamma} \cdot \widetilde{F}, \quad \widetilde{f} = \widetilde{S} \cdot \widetilde{F}, \quad \widetilde{f}' = \widetilde{S}' \cdot \widetilde{F}, \quad \widetilde{\delta} = \widetilde{\Delta} \cdot \widetilde{F}$$

on $\widetilde{X}_N$.

Viewed as divisor classes on a smooth fiber $E \times E' \subset X_N$ or $(E \times E')/G_N \subset \widetilde{X}_N$, these classes are endowed with an intersection form. Geometrically, we may explicitly write these cycles:

$$f = \{(x, 0) \in E \times E'\}, \quad f' = \{(0, x') \in E \times E'\}, \quad \gamma = \{(x, g(x)) \in E \times E'\}$$

where $g : E \to E'$ is the quotient map.

The self-intersection of f, f', γ are all zero since translation by a generic element in $E \times E'$ creates a disjoint homologous cycle. The remaining intersections are transverse and easily counted using the above descriptions. They are given by

$$f \cdot f' = 1, \quad f \cdot \gamma = N, \quad f' \cdot \gamma = 1.$$

It follows then for $\delta = \gamma - Nf' - f$ that

$$\delta \cdot f = 0, \quad \delta \cdot f' = 0, \quad \delta \cdot \delta = -2N$$

so in the basis $\langle \delta, f, f' \rangle$ for $\mathrm{Pic}(E \times E')$ the intersection form is

$$\begin{pmatrix} -2N & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

All classes $\alpha \in \{f, f', \gamma\}$ have the property that

$$\begin{aligned} \pi^* \pi_* \alpha &= G_N \text{ orbit of } \alpha \\ &= N\alpha \end{aligned}$$

so for $\widetilde{\alpha}, \widetilde{\beta} \in \{\widetilde{f}, \widetilde{f}', \widetilde{\gamma}, \widetilde{\delta}\}$ we have

$$\widetilde{\alpha} \cdot \widetilde{\beta} = N\alpha \cdot \beta$$

since

$$\begin{aligned} \alpha \cdot \beta &= \pi_*(\alpha \cdot \beta) \\ &= \tfrac{1}{N} \pi_*(\pi^* \pi_* \alpha \cdot \beta) \\ &= \tfrac{1}{N} (\pi_* \alpha \cdot \pi_* \beta) \\ &= \tfrac{1}{N} \widetilde{\alpha} \cdot \widetilde{\beta}. \end{aligned}$$

Thus we've proved the following

Lemma 15. *The classes $\widetilde{\delta}, \widetilde{f}, \widetilde{f}'$ given by the restriction of the divisor classes $\widetilde{\Delta}, \widetilde{S}, \widetilde{S}'$ in $\widetilde{X}_N$ to a smooth fiber have intersection pairing given by the matrix*

$$\begin{pmatrix} -2N^2 & 0 & 0 \\ 0 & 0 & N \\ 0 & N & 0 \end{pmatrix}.$$

It is also useful to write the classes of the banana curves $\widetilde{a}(k), \widetilde{b}(k), \widetilde{c}(k) \subset \widetilde{F}(k)$ in terms of the curve classes $\widetilde{\delta}, \widetilde{f}, \widetilde{f}'$.

Lemma 16. *The following equations of curve classes hold*

$$\begin{aligned} \widetilde{a}(k) + \widetilde{c}(k) &= \frac{1}{k}\widetilde{f} \\ \widetilde{b}(k) + \widetilde{c}(k) &= \frac{1}{l}\widetilde{f}' \\ \widetilde{c}(k) &= -\frac{1}{2N}\widetilde{\delta}. \end{aligned}$$

Proof. It suffices to show that both sides of each equation have the same pairing with the divisors $\widetilde{S}', \widetilde{S}$, and $\widetilde{\Delta}$. The pairings of $\widetilde{f}, \widetilde{f}', \widetilde{\delta}$ with $\widetilde{S}, \widetilde{S}', \widetilde{\Delta}$ are given by the intersection pairing in Lemma 15. In particular, the only non-zero pairings are

$$\begin{aligned} \frac{1}{k}\widetilde{f} \cdot \widetilde{S}' &= \frac{N}{k} = l \\ \frac{1}{l}\widetilde{f}' \cdot \widetilde{S} &= \frac{N}{l} = k \\ -\frac{1}{2N}\widetilde{\delta} \cdot \widetilde{\Delta} &= -\frac{1}{2N}(-2N^2) = N. \end{aligned}$$

The pairing with the classes on the left are given by Lemma 14 which are the same. □

5. Computation of the enumerative invariants of $\widetilde{X}_N$

5.1. **Review of the partition function of Banana manifolds.** The fiber curve DT partition function of the generic banana manifold X_{ban} was computed in [4] with a combination of motivic and topological vertex methods. The method works for any other CY3 of banana type including $\widetilde{X}_N$. What is needed is a CY3 X with an Abelian surface fibration $X \to \mathbb{P}^1$ whose singular fibers $F_1, \dots, F_r$ are banana fibers, so that in particular the formal neighborhoods $\widehat{F}_i$ are isomorphic to $\widehat{F}_{\mathsf{ban}}$ see [4, § 4]. The result is that the fiber curve DT partition function is a product over the contributions from singular fibers

$$Z_X = \prod_{i=1}^{r} Z_{\widehat{F}_i}$$

where each $Z_{\widehat{F}_i}$ is, up to a change of variables, given by

$$\begin{aligned} Z_{\widehat{F}_{\mathsf{ban}}}(p, Q_1, Q_2, Q_3) &= \sum_{\vec{\mathbf{d}}, m} \mathsf{DT}_{m, \beta_{\vec{\mathbf{d}}}}(-p)^m Q_1^{d_1} Q_2^{d_2} Q_3^{d_3} \\ &= \prod_{\vec{\mathbf{d}}, m} (1 - p^m Q_1^{d_1} Q_2^{d_2} Q_3^{d_3})^{-c(\vec{\mathbf{d}}^2, m)} \end{aligned}$$

where if a, b, c are the banana curves in $\widehat{F}_{\mathsf{ban}}$,

$$\beta_{\vec{\mathbf{d}}} = d_1 a + d_2 b + d_3 c.$$

Moreover, the coefficients $c(\vec{\mathbf{d}}^2, m)$ are defined as in Equation (1) and $\vec{\mathbf{d}}^2$ is by definition

$$\vec{\mathbf{d}}^2 = 2d_1 d_2 + 2d_2 d_3 + 2d_3 d_1 - d_1^2 - d_2^2 - d_3^2.$$

Finally, in the above product we require $d_1, d_2, d_3 \geq 0$ and if $d_1 = d_2 = d_3 = 0$ we require $m > 0$.

5.2. **The DT partition function of $\widetilde{X}_N$.** In the case of $\widetilde{X}_N$, we have four singular fibers, indexed by $k \in \Theta_N$ given by

$$\widehat{F}(k)/G_N \cong \widehat{F}_{\mathsf{ban}}$$

and containing banana curves $\widetilde{a}(k), \widetilde{b}(k), \widetilde{c}(k)$. Therefore

$$Z_{\widetilde{X}_N} = \prod_{k\in\Theta_N} Z_{\widehat{F}(k)/G_N}$$

and $Z_{\widehat{F}(k)/G_N} = Z_{\widehat{F}_{\mathsf{ban}}}$ up to a change of variables which we determine as follows.

The contribution of a class

$$\widetilde{\beta}_{\vec{\mathbf{d}}}(k) = d_1\widetilde{a}(k) + d_2\widetilde{b}(k) + d_3\widetilde{c}(k)$$

to the DT partition function contributes by definition to the coefficient of the monomial

$$y^{\widetilde{\beta}_{\vec{\mathbf{d}}}(k)\cdot\widetilde{\Delta}}\, q^{\widetilde{\beta}_{\vec{\mathbf{d}}}(k)\cdot\widetilde{S}'}\, Q^{\widetilde{\beta}_{\vec{\mathbf{d}}}(k)\cdot\widetilde{S}}.$$

By writing

$$\widetilde{\beta}_{\vec{\mathbf{d}}}(k) = d_1(\widetilde{a}(k)+\widetilde{c}(k)) + d_2(\widetilde{b}(k)+\widetilde{c}(k)) + (d_3-d_1-d_2)\widetilde{c}(k)$$

we can easily rewrite the above monomial using the table in Lemma 14. It is given by

$$\begin{aligned} y^{\widetilde{\beta}_{\vec{\mathbf{d}}}(k)\cdot\widetilde{\Delta}}\cdot q^{\widetilde{\beta}_{\vec{\mathbf{d}}}(k)\cdot\widetilde{S}'}\cdot Q^{\widetilde{\beta}_{\vec{\mathbf{d}}}(k)\cdot\widetilde{S}} &= y^{N(d_3-d_1-d_2)}\cdot q^{ld_1}\cdot Q^{kd_2} \\ &= \left(q^l y^{-N}\right)^{d_1}\left(Q^k y^{-N}\right)^{d_2}\left(y^N\right)^{d_3} \end{aligned}$$

where recall that $l = N/k$. So

$$Z_{\widehat{F}(k)/G_N}(p,y,q,Q) = Z_{\widehat{F}_{\mathsf{ban}}}(p,Q_1,Q_2,Q_3)$$

where

$$Q_1 = q^l y^{-N}, \quad Q_2 = Q^k y^{-N}, \quad Q_3 = y^N. \tag{10}$$

Combining, we've shown that

$$Z_{\widetilde{X}_N}(p,y,q,Q) = \prod_{k\in\Theta_N}\prod_{\vec{\mathbf{d}},m}\left(1-p^m y^{N(d_3-d_1-d_2)}q^{ld_1}Q^{kd_2}\right)^{-c(\vec{\mathbf{d}}^2,m)}.$$

Letting

$$r = d_1, \quad s = d_2, \quad t = d_3-d_1-d_2$$

and observing that

$$\vec{\mathbf{d}}^2 = 4rs - t^2$$

we then have the formula for $Z_{\widetilde{X}_N}$ as stated in Theorem 2 and thus have concluded its proof.

5.3. **GW potentials of $\widetilde{X}_N$.** In this section we prove Corollary 4. We begin by computing the genus g, fiber curve, GW potential of $\widetilde{X}_N$:

$$F_g^{\widetilde{X}_N}(Q,q,y) = \sum_{\substack{\beta\in H_2(\widetilde{X}_N)\\ \pi_*\beta=0}} \langle\,\rangle_{g,\beta}^{\widetilde{X}_N}\; Q^{\beta\cdot\widetilde{S}}q^{\beta\cdot\widetilde{S}'}y^{\beta\cdot\widetilde{\Delta}}.$$

Our calculation follows closely the computation in [4, App A]. The *reduced* GW potential F_g' is defined by removing the $\beta=0$ term from $F_g^{\widetilde{X}_N}$:

$$F_g'(Q,q,y) = F_g^{\widetilde{X}_N}(Q,q,y) - F_g^{\widetilde{X}_N}(0,0,0).$$

The GW/DT correspondence conjectured in [16] and recently proven by Pardon in [21], asserts that

$$\sum_{g=0}^{\infty} F_g'(Q,q,y)\lambda^{2g-2} = \log\left(\frac{Z_{\widetilde{X}_N}(p,y,q,Q)}{Z_{\widetilde{X}_N}(p,0,0,0)}\right)$$

under the change of variables $p = e^{i\lambda}$. Applying this to the DT partition function of $\widetilde{X}_N$ we get

$$\sum_{g=0}^{\infty} F'_g(Q,q,y)\lambda^{2g-2} = \log\left(\prod_{k\in\Theta_N}\prod_{m,r,s,t}(1-p^m q^{lr}Q^{ks}y^{Nt})^{-c(4rs-t^2,m)}\right) \tag{11}$$

$$= \sum_{k\in\Theta_N}\sum_{m,r,s,t} c(4rs-t^2,m)\sum_{n=1}^{\infty}\frac{1}{n}p^{nm}q^{nlr}Q^{nks}y^{nNt}$$

where the indices (m,r,s,t) in the product and sum are given by integers satisfying $r,s,r+s+t\geq 0$ and $(r,s,t)\neq(0,0,0)$. Now $c(d,m)=0$ if $d<-1$ [4, Prop 14] from which one can show that an equivalent indexing condition is given by $r,s\geq 0$ and $t>0$ if $r=s=0$.

In Appendix A of [4] it is shown that

$$\sum_{g=0}^{\infty} c_{2g-2}(d)\lambda^{2g-2} = \sum_{m\in\mathbb{Z}} c(d,m)e^{im\lambda} \tag{12}$$

where $c_{2g-2}(d)$ is defined by

$$\psi_{2g-2}(q,y) = \sum_{n=0}^{\infty}\sum_{t\in\mathbb{Z}} c_{2g-2}(4n-t^2)q^n y^t$$

and where $\psi_{2g-2}(q,y)$ is the weak Jacobi form of weight 2g-2 and index 1 given by

$$\psi_{2g-2}(q,y) = \phi_{-2,1}(q,y)\cdot\begin{cases}1 & g=0\\ \wp(q,y) & g=1\\ \frac{|B_{2g}|}{2g(2g-2)!}E_{2g}(q) & g>1\end{cases}$$

(see [4, App A] for further explination). Applying the substitution $p=e^{i\lambda}$ to Equation (11) and using Equation (12) we find

$$\sum_{g=0}^{\infty} F'_g(Q,q,y)\lambda^{2g-2} = \sum_{k\in\Theta_N}\sum_{r,s,t}\sum_{n=1}^{\infty}\frac{1}{n}q^{nlr}Q^{nks}y^{nNt}\sum_{g=0}^{\infty}c_{2g-2}(4rs-t^2)n^{2g-2}\lambda^{2g-2}$$

so that

$$F'_g(Q,q,y) = \sum_{k\in\Theta_N}\sum_{r,s,t} c_{2g-2}(4rs-t^2)\,\mathrm{Li}_{3-2g}(q^{lr}Q^{ks}y^{Nt}).$$

For $g>1$ we can add back in the constant terms using for example [16, § 2.1] to get

$$F_g^{\widetilde{X}_N}(Q,q,y) = \sum_{k\in\Theta_N}\left(c_{2g-2}(0)\cdot\frac{-B_{2g-2}}{4g-4} + \sum_{r,s,t}c_{2g-2}(4rs-t^2)\,\mathrm{Li}_{3-2g}(Q^{ks}q^{lr}y^{Nt})\right). \tag{13}$$

Let

$$F_g^{\mathsf{ban}}(Q,q,y) = c_{2g-2}(0)\cdot\frac{-B_{2g-2}}{4g-4} + \sum_{r,s\geq 0}\ \sum_{\substack{t\in\mathbb{Z}\\ t>0 \text{ if } r=s=0}} c_{2g-2}(4rs-t^2)\,\mathrm{Li}_{3-2g}(Q^s q^r y^t)$$

It is shown in Appendix A of [4] that $F_g^{\mathsf{ban}}(Q,q,y)$ is the genus 2 Siegel modular form given by the Maass lift of $\psi_{2g-2}(q,y)$. Then we can rewrite Equation (13) as

$$F_g^{\widetilde{X}_N}(Q,q,y) = \sum_{k\in\Theta_N} F_g^{\mathsf{ban}}(Q^k,q^l,y^N)$$

which completes the proof of Corollary 4.

Proof of Corollary 5. Let us define the function

$$\mathcal{F}_g^{\widetilde{X}_N}(Q,q,y) = \sum_{k\in\Theta_N} F_g^{\mathsf{ban}}(Q^{Nk},q^{\frac{N}{k}},y^N)$$

which is clearly related to the GW potential $F_g^{\widetilde{X}_N}$ through the simple change of variables $Q\mapsto Q^N$. We will prove Corollary 5 by establishing the automorphic properties of $\mathcal{F}_g^{\widetilde{X}_N}$.

Lemma 17. *If $F(Q,q,y)$ is a Siegel modular form on $\mathrm{Sp}_4(\mathbb{Z})$, then for N and k positive integers, $F(Q^{Nk},q^{\frac{N}{k}},y^N)$ is a Siegel modular form of the same weight for the subgroup*

$$L_{N,k}=\mathrm{Sp}_4(\mathbb{Q})\cap\begin{pmatrix}\mathbb{Z} & k\,\mathbb{Z} & \frac{k}{N}\,\mathbb{Z} & \frac{1}{N}\,\mathbb{Z}\\ \frac{1}{k}\,\mathbb{Z} & \mathbb{Z} & \frac{1}{N}\,\mathbb{Z} & \frac{1}{Nk}\,\mathbb{Z}\\ \frac{N}{k}\,\mathbb{Z} & N\,\mathbb{Z} & \mathbb{Z} & \frac{1}{k}\,\mathbb{Z}\\ N\,\mathbb{Z} & Nk\,\mathbb{Z} & k\,\mathbb{Z} & \mathbb{Z}\end{pmatrix}.$$

Proof. Recall that $Q=e^{2\pi i\sigma}$, $q=e^{2\pi i\tau}$, $y=e^{2\pi iz}$ where $\Omega=\left(\begin{smallmatrix}\tau & z\\ z & \sigma\end{smallmatrix}\right)$ is an element of the genus 2 Siegel upper half space $\mathbb{H}_2$. The diagonal matrix

$$g=\mathrm{diag}(N,Nk,k,1)$$

acts on $\mathbb{H}_2$ in the standard way by

$$\Omega\mapsto g\cdot\Omega=\begin{pmatrix}N & 0\\ 0 & Nk\end{pmatrix}\Omega\begin{pmatrix}k & 0\\ 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}\frac{N}{k}\tau & Nz\\ Nz & Nk\sigma\end{pmatrix}.$$

If $F(\Omega)=F(Q,q,y)$ is a Siegel modular form on $\mathrm{Sp}_4(\mathbb{Z})$, then $F(g\cdot\Omega)=F(Q^{Nk},q^{\frac{N}{k}},y^N)$ is a Siegel modular form of the same weight on $\mathrm{Sp}_4(\mathbb{Q})\cap(g^{-1}\,\mathrm{Sp}_4(\mathbb{Z})g)$. For the particular matrix g above, it is straightforward to verify that

$$L_{N,k}=\mathrm{Sp}_4(\mathbb{Q})\cap(g^{-1}\,\mathrm{Sp}_4(\mathbb{Z})g).$$

□

Recalling that $F_g^{\mathsf{ban}}(Q,q,y)$ is a Siegel modular form of weight $2g-2$ on $\mathrm{Sp}_4(\mathbb{Z})$, the above lemma implies that $\mathcal{F}_g^{\widetilde{X}_N}$ is a Siegel modular form of weight $2g-2$ for the group

$$\cap_{k\in\Theta_N}L_{N,k}$$

which one can easily show is exactly the subgroup $P_N\subset\mathrm{Sp}_4(\mathbb{Q})$ defined by Equation (3). Finally, we note that by the standard action of $\mathrm{Sp}_4(\mathbb{R})$ on $\mathbb{H}_2$, the involution ι_N (defined in equation (4)) induces the transformation

$$(Q,q,y)\mapsto(q^{\frac{1}{N}},Q^N,y)$$

under which $\mathcal{F}_g^{\widetilde{X}_N}$ is evidently invariant. It follows that $\mathcal{F}_g^{\widetilde{X}_N}$ is a Siegel modular form for the index 2 normal extension $P_N^*\subset\mathrm{Sp}_4(\mathbb{R})$, which completes the proof of Corollary 5.

□

5.4. **GV invariants of $\widetilde{X}_N$.** In [4] it is shown that if $a,b,c\subset\widehat{F}_{\mathsf{ban}}$ are banana curves the the GV invariants $n^g_\beta(\widehat{F}_{\mathsf{ban}})$ of an effective class

$$\beta=d_1a+d_2b+d_3c$$

only depend on g and the quantity

$$a=\vec{\mathbf{d}}^2=2d_1d_2+2d_2d_3+2d_3d_1-d_1^2-d_2^2-d_3^2$$

and then $n^g_\beta(\widehat{F}_{\mathsf{ban}})=n^g_a$ where the integers n^g_a are given by the formula

$$\sum_{a=-1}^{\infty}\sum_{g=0}^{\infty}n^g_a\,(y^{\frac{1}{2}}+y^{-\frac{1}{2}})^{2g}q^{a+1}=\prod_{n=1}^{\infty}\frac{(1+yq^{2n-1})(1+y^{-1}q^{2n-1})(1-q^{2n})}{(1+yq^{4n})^2(1+y^{-1}q^{4n})^2(1-q^{4n})^2}.$$

Now consider an effective curve class on $\widehat{F}(k)\subset\widetilde{X}_N$ given by

$$\beta=d_1\widetilde{a}(k)+d_2\widetilde{b}(k)+d_3\widetilde{c}(k).$$

Recall that we have a quadratic form $||\cdot||$ on fiber curve classes induced from the intersection form on a smooth fiber. We compute $||\beta||$ as follows:

$$
\begin{aligned}
||\beta|| &= ||d_1(\widetilde{a}(k)+\widetilde{c}(k))+d_2(\widetilde{b}(k)+\widetilde{c}(k))+(d_3-d_1-d_2)\widetilde{c}(k)|| \\
&= ||d_1\cdot\frac{1}{k}\cdot\widetilde{f}+d_2\cdot\frac{1}{l}\cdot\widetilde{f}'+(d_3-d_1-d_2)\cdot\frac{-1}{2N}\cdot\widetilde{\delta}|| \\
&= 2d_1d_2\cdot\frac{1}{kl}\cdot N+(d_3-d_1-d_2)^2\cdot\frac{1}{4N^2}\cdot(-2N^2) \\
&= \frac{1}{2}\left(4d_1d_2-(d_3-d_1-d_2)^2\right) \\
&= \frac{1}{2}\left(2d_1d_2+2d_2d_3+2d_3d_1-d_1^2-d_2^2-d_3^2\right).
\end{aligned}
$$

And thus if we let $a=2||\beta||$ then $n^g_\beta(\widehat{F}(k))=n^g_a$.

Then for a general effective fiber curve class β on $\widetilde{X}_N$ with $a=2||\beta||$ we have

$$
n^g_\beta(\widetilde{X}_N)=\epsilon_N(\beta)n^g_a \tag{14}
$$

where $\epsilon_N(\beta)$ is the number of singular fibers $F(k)\subset\widetilde{X}_N$ on which β is represented by an integral class.

For example the class

$$
\begin{aligned}
\widetilde{a}(N)+\widetilde{c}(N) &= \frac{1}{N}\cdot\widetilde{f} \\
&= \frac{k}{N}(\widetilde{a}(k)+\widetilde{c}(k))
\end{aligned}
$$

is represented by an effective curve in $F(N)$, but is not represented by a curve in $F(k)$ when $k\neq N$, so in this case, $\epsilon_N\left(\widetilde{a}(N)+\widetilde{c}(N)\right)=1$.

Lemma 18.

$$
\epsilon_N(\beta)=\sum_{k\in\Theta_N}\epsilon_{N,k}(\beta)
$$

where

$$
\epsilon_{N,k}(\beta)=\begin{cases}1 & \textit{if } k\,\Big|\,(\beta\cdot\widetilde{S}) \textit{ and } l\,\Big|\,(\beta\cdot\widetilde{S}') \\ 0 & \textit{otherwise.}\end{cases}
$$

Proof. An integral class on $F(k)$ given by

$$
\beta=d_1\widetilde{a}(k)+d_2\widetilde{b}(k)+d_3\widetilde{c}(k)
$$

can be written as

$$
\beta=d_1\cdot\frac{1}{k}\cdot\widetilde{f}+d_2\cdot\frac{1}{l}\cdot\widetilde{f}'+(d_3-d_1-d_2)\cdot\frac{-1}{2N}\cdot\widetilde{\delta}
$$

and thus satisfies

$$
\beta\cdot\widetilde{S}'=d_1l,\quad \beta\cdot\widetilde{S}=d_2k,\quad \beta\cdot\widetilde{\Delta}=(d_3-d_1-d_2)N
$$

and so in particular k divides $\beta\cdot\widetilde{S}$ and l divides $\beta\cdot\widetilde{S}'$. Moreover, N divides $\beta\cdot\widetilde{\Delta}$ for any effective fiber curve class.

Conversely, suppose that β is an effective fiber curve class satisfying $k\,\Big|\,(\beta\cdot\widetilde{S})$ and $l\,\Big|\,(\beta\cdot\widetilde{S}')$. Then β is represented by an integral curve class on $F(k)$ since we may define the integers

$$
d_1=\frac{1}{l}\beta\cdot\widetilde{S}',\quad d_2=\frac{1}{k}\beta\cdot\widetilde{S}
$$

and then

$$
d_3=\frac{1}{N}\beta\cdot\widetilde{\Delta}+d_1+d_2.
$$

□

Lemma 18 and Equation (14) then complete the proof of Proposition 6.

Appendix A. Rigid CY3s related by finite quotients and small resolutions (with Mike Roth)

Recall that for a rigid CY3 X defined over $\mathbb{Q}$ there exists a weight 4 modular cusp form

$$f_X(q) = \sum_{n=1}^{\infty} a_n q^n$$

uniquely characterized by the condition that

$$a_p = \operatorname{tr}(\operatorname{Frob}_p \Big| H^3_{\mathsf{\acute{e}t}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l))$$

for almost all primes p.

In [23, Appendix] Verrill considered rigid CY3s which are closely related to our banana nano-manifolds $\widetilde{X}_N$. Namely, let X_N^{Ver} be (any) projective conifold resolution of the fiber product $S_N \times_{\mathbb{P}^1} S_N$ (Schoen [22] proved that there exists projective resolutions of any self-fiber product of a rational surface). Then X_N^{Ver} is a rigid CY3 with $h^{1,1}(X_N^{\mathsf{Ver}}) = \sum_{k\in\Theta_N} k^2$. Verrill uses a particular model for S_N which is defined over $\mathbb{Q}$ and proves that

$$f_{X_N^{\mathsf{Ver}}}(q) = \prod_{k\in\Theta_N} \eta(q^k)^2$$

independent of the choice of the conifold resolution[6]

$$X_N^{\mathsf{Ver}} \xrightarrow{\pi_1} S_N \times_{\mathbb{P}^1} S_N.$$

We note that the quotient of $S_N \times_{\mathbb{P}^1} S_N$ by G_N acting on the second factor is

$$S_N \times_{\mathbb{P}^1} S_{\mathsf{sing}} = X_{\mathsf{sing}}.$$

Thus $\widetilde{X}_N$ and X_N^{Ver} are related by the following sequence of maps

$$X_N^{\mathsf{Ver}} \xrightarrow{\pi_1} S_N \times_{\mathbb{P}^1} S_N \xrightarrow{q_1} X_{\mathsf{sing}} \xleftarrow{\pi_2} X_N \xrightarrow{q_2} \widetilde{X}_N \tag{15}$$

where the maps π_1 and π_2 are crepant resolutions and q_1 and q_2 are both quotients by the action of the finite group G_N.

We will show that the above maps and varieties are defined over $\mathbb{Q}$ and that they induce an isomorphism

$$H^3_{\mathsf{\acute{e}t}}(X_N^{\mathsf{Ver}}, \mathbb{Q}_l) \cong H^3_{\mathsf{\acute{e}t}}(\widetilde{X}_N, \mathbb{Q}_l)$$

which is compatible with the action of Frobenius so that in particular

$$f_{X_N^{\mathsf{Ver}}}(q) = f_{\widetilde{X}_N}(q).$$

Let V be a variety defined over $\mathbb{C}$. We recall that the both the ordinary cohomology groups $H^i(V, \mathbb{Q})$, and the groups $H^i_c(V, \mathbb{Q})$ of cohomology with compact support carry weight filtrations, increasing filtrations $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ of $\mathbb{Q}$-subspaces. We denote by $\operatorname{Gr}_m H^i(V, \mathbb{Q})$ (or $\operatorname{Gr}_m H^i_c(V, \mathbb{Q})$) the quotient F_m/F_{m-1}. Given a map $\pi\colon V \longrightarrow V'$ of varieties, the pullback maps on cohomology are compatible with the weight filtrations, and so induce maps of the graded pieces.

Moreover, if one has an exact sequence of cohomology groups, for instance an excision sequence

$$\cdots \longrightarrow H^{i-1}(Z, \mathbb{Q}) \longrightarrow H^i_c(U, \mathbb{Q}) \longrightarrow H^i(V, \mathbb{Q}) \longrightarrow H^i(Z, \mathbb{Q}) \longrightarrow \cdots$$

then for any m the sequence of graded pieces

$$\cdots \longrightarrow \operatorname{Gr}_m H^{i-1}(Z, \mathbb{Q}) \longrightarrow \operatorname{Gr}_m H^i_c(U, \mathbb{Q}) \longrightarrow \operatorname{Gr}_m H^i(V, \mathbb{Q}) \longrightarrow \operatorname{Gr}_m H^i(Z, \mathbb{Q}) \longrightarrow \cdots$$

is again exact. This is a consequence of the fact that the maps on cohomology are not only compatible with the filtrations, but are strictly compatible [7, Proposition 1.1.11].

Lemma 19. *Let V and V' be projective threefolds defined over $\mathbb{C}$ and $\pi\colon V \longrightarrow V'$ a birational map. Let $Z \subset V$ be the exceptional locus of π and $Z' = \pi(Z)$.*

[6]The existence of a projective conifold resolution follows from a theorem of Schoen. However, Schoen's argument does not guarantee that the conifold resolution is defined over $\mathbb{Q}$. We will address this issue in Lemma 20.

(*a*) *Suppose that Z and Z' have the property that* $\mathrm{Gr}_3\, H^2(Z,\mathbb{Q}) = \mathrm{Gr}_3\, H^3(Z,\mathbb{Q}) = 0$ *and that* $\mathrm{Gr}_3\, H^2(Z',\mathbb{Q}) = \mathrm{Gr}_3\, H^3(Z',\mathbb{Q}) = 0$. *Then the pullback map* $\pi^*\colon H^3(V',\mathbb{Q}) \longrightarrow H^3(V,\mathbb{Q})$ *induces an isomorphism on* Gr_3.

(*b*) *The vanishing conditions in (a) hold in each of the following cases:*
- (*b1*) *For the morphisim* $\pi_2\colon X_N \longrightarrow X_{\mathsf{sing}}$ *above;*
- (*b2*) *When* $\pi\colon V \longrightarrow V'$ *is a conifold resolution (e.g.,* $\pi_1\colon X_N^{\mathsf{Ver}} \longrightarrow S_N \times_{\mathbb{P}^1} S_N$*).*

Proof. Let
$$U = V - Z, \quad U' = V' - Z',$$
and note that π induces an isomorphism $U \cong U'$.

The excision exact sequences for $U = V - Z$ and $U' = V' - Z'$ are compatible with the maps π^* on cohomology and lead to the following commutative diagram with exact rows:

$$\begin{tikzcd}
H^2(Z',\mathbb{Q}) \arrow[r] \arrow[d, "\pi^*"] & H^3_c(U',\mathbb{Q}) \arrow[r, "i"] \arrow[d, "\pi^*", "\wr"'] & H^3(V',\mathbb{Q}) \arrow[r] \arrow[d, "\pi^*"] & H^3(Z',\mathbb{Q}) \arrow[d, "\pi^*"] \\
H^2(Z,\mathbb{Q}) \arrow[r] & H^3_c(U,\mathbb{Q}) \arrow[r] & H^3(V,\mathbb{Q}) \arrow[r] & H^3(Z,\mathbb{Q})
\end{tikzcd}$$

Passing to Gr_3 and using the vanishing hypotheses, this diagram becomes

$$\begin{tikzcd}
0 \arrow[r] & H^3_c(U',\mathbb{Q}) \arrow[r, "i"] \arrow[d, "\pi^*", "\wr"'] & \mathrm{Gr}_3\, H^3(V',\mathbb{Q}) \arrow[r] \arrow[d, "\pi^*"] & 0 \\
0 \arrow[r] & \mathrm{Gr}_3\, H^3_c(U,\mathbb{Q}) \arrow[r] & \mathrm{Gr}_3\, H^3(V,\mathbb{Q}) \arrow[r] & 0
\end{tikzcd}$$

Consequently the map $\mathrm{Gr}_3\, H^3(V',\mathbb{Q}) \longrightarrow \mathrm{Gr}_3\, H^3(V,\mathbb{Q})$ is an isomorphism.

We next check the vanishing conditions for $\pi_2\colon X_N \longrightarrow X_{\mathsf{sing}}$. In this case $Z' \subset X_{\mathsf{sing}}$ is given by the union of the curves $I_k \times n'$ where $I_k \subset S$ are the singular fibres and $n' \in S'_{\mathsf{sing}}$ is the nodal point in the corresponding fibre. Since Z' is complex one-dimensional, $H^3(Z',\mathbb{Q}) = 0$. Since Z' is proper, all weights of $H^i(Z',\mathbb{Q})$ are $\leq i$, by [8, Théorème 8.2.4]. In particular all weights of $H^2(Z',\mathbb{Q})$ are ≤ 2 and so $\mathrm{Gr}_3\, H^2(Z',\mathbb{Q}) = 0$.

The exceptional locus $Z \to Z'$ is a union of components in the singular fibres of X_N which by the local toric description of Section 3 is a normal crossing divisor. The class of Z in the Grothendieck group of varieties is a polynomial in $\mathbb{L} = [\mathbb{A}^1]$, the class of the affine line. It follows that the weight polynomial of Z is supported in even degrees and in particular, we have that $\mathrm{Gr}_3\, H^3(Z) = 0$. Since Z is proper, $\mathrm{Gr}_3\, H^2(Z,\mathbb{Q}) = 0$ as above.

Finally, we check the vanishing conditions for a conifold resolution. Here Z' is a finite set of points and so has no cohomology above H^0, and Z is a curve, and so the degree 3 parts of H^2 and H^3 vanish as in the previous case. □

Lemma 20. *The morphisms and varieties given in equation* (15) *are all defined over* $\mathbb{Q}$.

Proof. As we make explicit in Appendix B, both Verrill's model for $S_N \to \mathbb{P}^1$ and the group action $G_N \times S_N \to S_N$ are defined over $\mathbb{Q}$. Consequently, $S_N \times_{\mathbb{P}^1} S_N$, $X_{\mathsf{sing}} = S_N \times_{\mathbb{P}^1} (S_N/G_N)$ and the morphism q_1 are all defined over $\mathbb{Q}$. The map q_1 is the composition of $X_N \to X_{\mathsf{con}}$ and $X_{\mathsf{con}} \to X_{\mathsf{sing}}$. This later map is induced by the minimal resolution $S'_N \to S_N/G_N$ on the second factor which is defined over $\mathbb{Q}$ because we may take S'_N to be the so-called G-Hilbert scheme which is a component of the Hilbert scheme of substacks of the stack quotient $[S_N/G_N]$ and the morphism to be the Hilbert-Chow morphism (the Hilbert scheme of substacks of a stack defined over $\mathbb{Q}$ is defined over $\mathbb{Q}$). The morphism $X_N \to X_{\mathsf{con}}$ is given by the blowup of Γ which is a divisor defined over $\mathbb{Q}$ since it is the proper transform of the graph of a morphism defined over $\mathbb{Q}$.

Finally, we need to see that Verrill's conifold resolution $X_N^{\mathsf{Ver}} \to S_N \times_{\mathbb{P}^1} S_N$ is defined over $\mathbb{Q}$ (this issue does not appear to be addressed in the original paper). Schoen proves the existence of a projective resolution of any $S \times_{\mathbb{P}^1} S$ self-product of a rational elliptic surface with singular fibers of I_n type and Verrill quotes this result. However, Schoen's argument (first blow up the diagonal and then successively blow up irreducible components of the singular fibers) does not guarentee that the result is defined over $\mathbb{Q}$ since components of the singular fibers may not be defined over $\mathbb{Q}$ (indeed they are not in general in our case). We may nevertheless find a conifold resolution defined over $\mathbb{Q}$ as follows. The diagonal, the G_N-orbits of the diagonal, and H-orbits of the diagonal for subgroups $H \subset G_N$ are all Weil divisors defined over $\mathbb{Q}$. We can obtain a projective conifold resolution of $S_N \times_{\mathbb{P}^1} S_N$ defined over $\mathbb{Q}$ by successively blowing up those Weil divisors and their proper transforms in various orders. The specifics of this process depend on N, which is not hard to determine with explicit analysis of the singular fibers. Explicitly, for $N = 5$ it suffices to first blowup the diagonal and then blowup the proper transform of the $\mathbb{Z}_5$ orbit of the diagonal. For $N = 6$, first blow up the diagonal, then blow up the proper transform of the $\mathbb{Z}_2$-orbit of the diagonal, then blowup the proper transform of the $\mathbb{Z}_3$-orbit of the diagonal. For $N = 8$, first blowup the $\mathbb{Z}_2$-orbit of the diagonal and then blowup the proper transform of the $\mathbb{Z}_4$-orbit of the diagonal. For $N = 9$, blowing up the full $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbit of the diagonal works. □

We will also use the following standard result which may be easily proved using the Leray-Serre spectral sequence.

Lemma 21. *Let G be a finite group acting on V. Then $H^i(V/G, \mathbb{Q}) \cong H^i(V, \mathbb{Q})^G$ where the inclusion $H^i(V/G, \mathbb{Q}) \cong H^i(V, \mathbb{Q})^G \hookrightarrow H^i(V, \mathbb{Q})$ is given by p^* where $p : V \to V/G$.*

We now complete the proof of Theorem 7. We examine the maps on cohomology induced by (15) :

$$H^3(X_N^{\mathsf{Ver}}, \mathbb{Q}) \xleftarrow{\pi_1^*} H^3(S_N \times_{\mathbb{P}^1} S_N, \mathbb{Q}) \xhookleftarrow{q_1^*} H^3(X_{\mathsf{sing}}, \mathbb{Q}) \xrightarrow{\pi_2^*} H^3(X_N, \mathbb{Q}) \xhookleftarrow{q_2^*} H^3(\widetilde{X}_N, \mathbb{Q})$$

Restricting to the weight 3 graded piece of the above and using the fact that X_N^{Ver}, X_N, and $\widetilde{X}_N$ are non-singular projective threefolds we get

$$H^3(X_N^{\mathsf{Ver}}, \mathbb{Q}) \xleftarrow{\pi_1^*} \mathrm{Gr}_3\, H^3(S_N \times_{\mathbb{P}^1} S_N, \mathbb{Q}) \xhookleftarrow{q_1^*} \mathrm{Gr}_3\, H^3(X_{\mathsf{sing}}, \mathbb{Q}) \xrightarrow{\pi_2^*} H^3(X_N, \mathbb{Q}) \xhookleftarrow{q_2^*} H^3(\widetilde{X}_N, \mathbb{Q})$$

By Lemma 19 π_2^* and π_1^* are isomorphisms on the degree 3 pieces. By Lemma 21, q_1^* and q_2^* are both injective. Since X_N^{Ver} and $\widetilde{X}_N$ are both rigid CY3s, $H^3(X_N^{\mathsf{Ver}}, \mathbb{Q})$ and $H^3(\widetilde{X}_N, \mathbb{Q})$ are both isomorphic to $\mathbb{Q}\oplus\mathbb{Q}$ and hence the injection $(\pi_1^*)^{-1} \circ q_1^* \circ (\pi_2^*)^{-1} \circ q_2^*$ is an isomorphism

$$H^3(\widetilde{X}_N, \mathbb{Q}) \cong H^3(X_N^{\mathsf{Ver}}, \mathbb{Q}).$$

Fix a prime l. Then by the comparison theorem [1, Lecture 11, Theorem 4.4] $H^3_{\text{ét}}(\tilde{X}_N, \mathbb{Q}_l) \cong H^3(\tilde{X}_N, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_l$ and $H^3_{\text{ét}}(X_N^{\mathsf{Ver}}, \mathbb{Q}_l) \cong H^3(X_N^{\mathsf{Ver}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_l$. The isomorphisms provided by the comparision theorem are compatible with pullbacks, and so the map $(\pi_1^*) \circ q_1^* \circ (\pi_2^*)^{-1} \circ q_2^*$ also induces an isomorphism

$$H^3_{\text{ét}}(\tilde{X}_N, \mathbb{Q}_l) \cong H^3_{\text{ét}}(X_N^{\mathsf{Ver}}, \mathbb{Q}_l).$$

As a consequence of Lemma 20, the maps π_1, π_2, q_1, and q_2 are all defined over $\mathbb{Q}$, and so this isomorphism on cohomology groups is also an isomorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ representations. Thus we have the equality

$$f_{\tilde{X}_N}(q) = f_{X_N^{\mathsf{Ver}}}(q).$$

This completes the proof of Theorem 7.

It remains to prove Propostion 8. The four singular fibers of $S_N \to \mathbb{P}^1$ occur at points $p_1, \ldots, p_4 \in \mathbb{P}^1$ which are given explicity in [23, Table 2]. In all cases, $p_1 = \infty$, and the cross-ratio of the four points is given by

$$\lambda = \frac{p_3 - p_2}{p_3 - p_4}$$

If E_N is the double cover of $\mathbb{P}^1$ branched at $\{p_1, \ldots, p_4\}$, then the j-invariant of E_N is given by

$$j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (1 - \lambda)^2}$$

Thus, one can compute $j(\lambda)$ in each case, and use the LMFDB [15] to find a suitable model of E_N over $\mathbb{Q}$ whose corresponding weight 2 cusp modular form $f_{E_N}(q)$ satisfies $f_{E_N}(q)^2 = f_{\widetilde{X}_N}(q^2)$. The data is presented in the following table, together with the appropriate LMFDB labels:

N	$j(\lambda)$	$\mathbb{Q}$-model of E_N (LMFDB)	Weierstrass form of $\mathbb{Q}$-model	$f_{E_N}(q)$
5	$\frac{488095744}{125}$	20.a1	$y^2 = x^3 + x^2 - 41x - 116$	$\eta(q^{10})^2\eta(q^2)^2$
6	$\frac{1556068}{81}$	24.a3	$y^2 = x^3 - x^2 - 24x - 36$	$\eta(q^{12})\eta(q^6)\eta(q^4)\eta(q^2)$
8	1728	32.a3	$y^2 = x^3 - x$	$\eta(q^8)^2\eta(q^4)^2$
9	0	36.a3	$y^2 = x^3 - 27$	$\eta(q^6)^4$

APPENDIX B. EXPLICIT $\mathbb{Q}$-MODELS FOR THE SURFACES S_N

For reference, we include some basic explicit data for the models over $\mathbb{Q}$ of the surfaces S_N studied by Beauville [2] and by Verrill [23]. In this model, the surface S_N is given by the minimal resolution of a hypersurface $\overline{S}_N \subset \mathbb{P}^2 \times \mathbb{P}^1$ with equation $f_N(x, y, z, \lambda, \mu) = 0$, which is homogeneous of degree 3 and 1 in the variables $(x, y, z) \in \mathbb{P}^2$ and $(\lambda, \mu) \in \mathbb{P}^1$ respectively.

In each case the Mordell-Weil group G_N is finite, and we include here explicit equations for the group action $G_N \times S_N \to S_N$ (which to our knowledge does not appear anywhere else in the literature). Departing notationally from Beauville and Verrill, we choose to index the cases by the order N of the Mordell-Weil group. Note that the $N = 3$ and $N = 4$ cases do not arise in the main body of the paper, as they do not lead to a construction of a banana nano-manifold.

N	f_N	G_N	Generator(s) for the G_N action
3	$\mu(x^2y + y^2z + z^2x) - \lambda xyz$	$\mathbb{Z}_3$	$(x, y, z) \mapsto (y, z, x)$
4	$\mu(x + y)(xy - z^2) - \lambda xyz$	$\mathbb{Z}_4$	$(x, y, z) \mapsto (xy, -z^2, xz)$
5	$\mu x(x - y)(y - z) - \lambda xyz$	$\mathbb{Z}_5$	$(x, y, z) \mapsto (y(x - z), -yz, z(x - y))$
6	$\mu(x + y + z)(xy + yz + zx) - \lambda xyz$	$\mathbb{Z}_6$	$(x, y, z) \mapsto (xy, yz, xz)$
8	$\mu x(x^2 + z^2 + 2zy) - \lambda z(x^2 - y^2)$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	(See below)
9	$\mu(x^3 + y^3 + z^3) - \lambda xyz$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$(x, y, z) \mapsto (y, z, x)$
			$(x, y, z) \mapsto (x, \omega y, \omega^2 z), \quad \omega^3 = 1$

In the case of $N = 8$, the following maps of order 4 and 2, respectively, generate the $\mathbb{Z}_4 \times \mathbb{Z}_2$ action

$$(x,y,z) \mapsto \Big((x-y)(x-z)^2(x^2+z(2y+z)),\ (x-y)(x-z)(x^3-x^2z+xz(2y+z)+z(2y^2+2yz+z^2),\ -(x+y)(x^2+z(2y+z))^2)\Big)$$

$$(x,y,z) \mapsto \Big(x(y+z)(x^2+yz)^2(x^2+z(2y+z)),\ (x^2+yz)(x^6+3x^4yz+y^3z^3+x^2z(y^3+6y^2z+3yz^2+z^3)),\ -x^2z(y+z)^3(x^2+z(2y+z))\Big)$$

In the case of $N \in \{3, 6, 9\}$, the generators of the action can be determined by inspection. The remaining cases require a straightforward calculation using the group law of a generic fiber of $\overline{S}_N \to \mathbb{P}^1$. This smooth cubic curve intersects its Hessian curve in 9 inflection points, one of which can be chosen as the origin. An analysis of the cubic pencil $f_N = 0$ determines the sections, and the translation morphism by a given section can thus be determined from the group law as a birational automorphism of $\mathbb{P}^2$.

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