Curve counting invariants for crepant resolutions

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Abstract

We construct curve counting invariants for a Calabi-Yau threefold $Y$ equipped with a dominant birational morphism $\pi : Y \rightarrow X$. Our invariants generalize the stable pair invariants of Pandharipande and Thomas which occur for the case when $\pi : Y \rightarrow Y$ is the identity. Our main result is a PT/DT-type formula relating the partition function of our invariants to the Donaldson-Thomas partition function in the case when $Y$ is a crepant resolution of $X$, the coarse space of a Calabi-Yau orbifold $\mathcal{X}$ satisfying the hard Lefschetz condition. In this case, our partition function is equal to the Pandharipande-Thomas partition function of the orbifold $\mathcal{X}$. Our methods include defining a new notion of stability for sheaves which depends on the morphism $\pi$. Our notion generalizes slope stability which is recovered in the case where $\pi$ is the identity on $Y$.

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1 Introduction

Donaldson-Thomas (DT) theory of a Calabi-Yau threefold $X$ gives rise to subtle deformation invariants. They are considered to be the mathematical counterparts of BPS state counts in topological string theory compactified on $X$. Principles of physics (see [47], [51]) indicate that the
string theory of an orbifold Calabi-Yau threefold and that of its crepant
resolution ought to be equivalent, so one expects that the DT theories of
an orbifold and its crepant resolution to be equivalent in some way. In the
case where the orbifold satisfies the hard Lefschetz condition, the crepant
resolution conjecture of [14] gives a formula determining the DT invariants
of the orbifold in terms of the DT invariants of the crepant resolution.

In this article, we begin a program to prove the crepant resolution con-
jecture using Hall algebra techniques inspired by those of Bridgeland [12].
In the process, we construct curve counting invariants for a Calabi-Yau
threefold \(Y\) equipped with a birational morphism \(\pi : Y \to X\). Our invari-
ants generalize the stable pair invariants of Pandharipande and Thomas
which occur for the case when \(\pi : Y \to Y\) is the identity. Our main result
is a PT/DT-type formula relating the partition function of our invari-
ants to the Donaldson-Thomas partition function in the case when \(Y\) is a
crepant resolution of \(X\), the coarse space of a Calabi-Yau orbifold \(X\) sat-
sifying the hard Lefschetz condition. In this case, our partition function
is equal to the Pandharipande-Thomas partition function of the orbifold
\(X\).

**Donaldson-Thomas theory**

Let \(Y\) be a smooth projective Calabi-Yau threefold. Let \(K(Y)\) be the
numerical K-theory of \(Y\), i.e. the quotient of the K-group of coh(\(Y\)) by
the kernel of the Chern character map to cohomology. The Hilbert scheme
of \(Y\), Hilb\(^a\)(\(Y\)), parametrizes quotients \(O_Y \to O_Z\), such that the class of
\(O_Z\) in \(K(Y)\) is \(a\). The group \(K(Y)\) is filtered by the dimension of the
support:

\[
F_0 K(Y) \subset F_1 K(Y) \subset F_2 K(Y) \subset F_3 K(Y) = K(Y).
\]

In this article, we will focus on curves, i.e., \(a \in F_1 K(Y)\), with \(\text{ch}(a) = (0, 0, \beta, n)\), where \(\beta \in H^4(Y, \mathbb{Z})\) is a curve class, and \(n \in H^6(Y, \mathbb{Z}) \cong \mathbb{Z}\) is
the holomorphic Euler characteristic. In [44], an obstruction theory for
this moduli space is constructed, which produces (by [5]) a virtual funda-
damental cycle. Donaldson-Thomas invariants are defined by integrating
over the zero-dimensional virtual fundamental class:

\[
DT^a(Y) = \int_{[\text{Hilb}^a(Y)]^{vir}} 1.
\]

Since the obstruction theory is symmetric, we may also express the
invariants as the Euler characteristic of Hilb\(^a\)(\(Y\)) weighted by Behrend’s
microlocal function [3]:

\[
DT^a(Y) = \sum_{n \in \mathbb{Z}} n \chi(\nu^{-1}(n)),
\]

where \(\nu : \text{Hilb}^a(Y) \to \mathbb{Z}\) is Behrend’s function.

Following [36], we assemble the invariants into a partition function

\[
\text{DT}(Y) = \sum_{a \in F_1 K(Y)} DT^a(Y) q^a.
\]
Remark 1. In [30], Donaldson-Thomas invariants are greatly generalized, from the case of structure sheaves of curves to that of arbitrary sheaves. The price of admission to this generality is the formidable machinery of Joyce [25, 26, 27, 28, 29]. An even more ambitious program of generalization is being lead by Kontsevich and Soibelman [32].

The Donaldson-Thomas crepant resolution conjecture

We follow [14] in our treatment of the crepant resolution conjecture.

An orbifold CY3 is defined to be a smooth, quasi-projective, Deligne-Mumford stack $X$ over $\mathbb{C}$ of dimension three having generically trivial stabilizers and trivial canonical bundle,

$$K_X \cong O_X.$$ 

The definition implies that the local model for $X$ at a point $p$ is $[\mathbb{C}^3/G_p]$ where $G_p \subset SL(3, \mathbb{C})$ is the (finite) group of automorphisms of $p$. The orbifold CY3s that appear in this article will all be projective and satisfy the hard Lefschetz condition [16, definition 1.1], which in this case is equivalent [15, lemma 24] to the condition that all $G_p$ are finite subgroups of $SO(3) \subset SU(3)$ or $SU(2) \subset SU(3)$.

Let $X$ denote the coarse space of $X$. A crepant resolution of $X$ is a resolution of singularities $\pi: Y \rightarrow X$ such that $\pi^* K_X \cong K_Y$. Lemma (1) and Proposition (1) of [49] prove that

$$R^* \pi_* O_Y \cong O_X. \quad (2)$$

The results of [13] and [19] prove that one distinguished crepant resolution of $X$ is

$$Y = \text{Hilb}^{[\mathcal{O}_p]}(\mathcal{X}), \quad (3)$$

the Hilbert scheme parametrizing substacks in the class $[\mathcal{O}_p] \in F_0 K(\mathcal{X})$. The hard Lefschetz condition implies that the resolution is semi-small (i.e., that the fibres of $\pi$ are zero- or one-dimensional), and that the singular locus of $X$ is one-dimensional; see [8, 15]. Furthermore, [13] and [19] prove that there is a Fourier-Mukai isomorphism

$$\Psi : D^b(Y) \rightarrow D^b(\mathcal{X})$$

defined by

$$E \mapsto Rq_* p^* E$$

where

$$p : Z \rightarrow Y, \quad q : Z \rightarrow \mathcal{X}$$

are the projections from the universal substack $Z \subset \mathcal{X} \times Y$ onto each factor. This isomorphism descends to an isomorphism of $K$-theory also denoted $\Psi : K(Y) \rightarrow K(\mathcal{X})$. It does not respect the filtration by dimension. However, the hard Lefschetz condition implies that the image of $F_0 K(\mathcal{X})$ is contained in $F_1 K(Y)$, under the inverse $\Phi$ of $\Psi$. We call the image $F_{exc} K(Y)$; its elements can be represented by formal differences of sheaves supported on the exceptional fibres of $\pi : Y \rightarrow X$. We define the
multi-regular part of $K$-theory, $F_{mr}(\mathcal{X})$, to be the preimage of $F_1K(Y)$ under $\Psi$. Its elements can be represented by formal differences of sheaves supported in dimension one where at the generic point of each curve in the support, the associated representation of the stabilizer group of that point is a multiple of the regular representation. The following filtrations are respected by $\Psi$:

$$F_{exc}K(Y) \subset F_1K(Y) \subset K(Y)$$

$$F_0K(\mathcal{X}) \subset F_{mr}K(\mathcal{X}) \subset K(\mathcal{X}).$$

Define the exceptional DT generating series of $Y$, the multi-regular generating series, and degree zero generating series of $\mathcal{X}$ to be:

$$DT_{exc}(Y) = \sum_{\alpha \in F_{exc}K(Y)} DT^\alpha(Y)q^\alpha,$$

$$DT_{mr}(\mathcal{X}) = \sum_{\alpha \in F_{mr}K(\mathcal{X})} DT^\alpha(\mathcal{X})q^\alpha,$$

$$DT_0(\mathcal{X}) = \sum_{\alpha \in F_0K(\mathcal{X})} DT^\alpha(\mathcal{X})q^\alpha.$$

We state the crepant resolution conjecture of [14, conjecture 1]:

**Conjecture 4.** Let $\mathcal{X}$ be an orbifold CY3 satisfying the hard Lefschetz condition. Let $Y$ be the Calabi-Yau resolution of $\mathcal{X}$ given by equation 3. Then using $\Psi$ to identify the variables, we have an equality

$$\frac{DT_{mr}(\mathcal{X})}{DT_0(\mathcal{X})} = \frac{DT(Y)}{DT_{exc}(Y)}.$$

This article makes progress towards proving this conjecture.

In his recent article [17], John Calabrese proves a relationship between the DT invariants of a Calabi-Yau threefold and its flop. This problem is similar in many respects to the crepant resolution conjecture studied in this thesis, and Calabrese uses many similar techniques. He constructs a torsion pair and new counting invariants which he relates to invariants on the flop via equations in the Hall algebra and the integration map. While this is very similar to our approach in outline, the actual torsion pair and counting invariants that Calabrese considers (even when adapted to the orbifold setting) are quite different from ours. It would be very interesting to find the precise relationship between the two approaches. An even more recent preprint [18] of Calabrese proves the DT crepant resolution conjecture, utilizing his earlier paper [17].

**$\pi$-stable pairs**

Objects of the Hilbert scheme may be viewed as two-term complexes,

$$\mathcal{O}_Y \xrightarrow{\gamma} G,$$

where the cokernel of $\gamma$ must be zero, and where $G$ may be any sheaf admitting such a map $\gamma$. The new invariants introduced in this article, $\pi$-stable pairs, are a modification of this idea. They have been constructed
with a view towards proving the crepant resolution conjecture, and as such, they depend on a crepant resolution \( Y \xrightarrow{\pi} X \) as described in the previous section. The objects of our moduli space allow more variation in our cokernels, but less in the sheaf \( G \). In particular, a two-term complex

\[ \mathcal{O}_Y \rightarrow G \]

is a \( \pi \)-stable pair (c.f. definition 14) if:

1. \( R^\bullet \pi_* \text{coker}(\gamma) \) is a zero-dimensional sheaf on \( X \), and
2. \( G \) admits only the zero map from any sheaf \( P \) with the property above, namely that \( R^\bullet \pi_* P \) is a zero-dimensional sheaf.

**Remark 5.** These pairs were inspired by, and are a generalization of, the stable pairs of Pandharipande and Thomas [40]. In fact, when \( X = Y \) and \( \pi \) is the identity map, the above definition reduces to their definition of stable pairs.

Below, we prove that there is a finite-type constructible space, \( \pi\text{-Hilb}^\alpha \) parametrizing these objects with \( [G] = \alpha \in K(Y) \). We may then define invariants

\[ \pi\text{-PT}^\alpha(Y) = \sum_{n \in \mathbb{Z}} n \chi(\nu^{-1}(n)), \]

where \( \nu : \pi\text{-Hilb}^\alpha \rightarrow \mathbb{Z} \) is Behrend’s microlocal function. Note that if \( \pi : Y \rightarrow X \) is the identity, then \( \pi\text{-PT}^\alpha(Y) = \text{PT}^\alpha(Y) \), the usual Pandharipande-Thomas invariants of \( Y \). As with Donaldson-Thomas theory, we collect the invariants into a generating series,

\[ \pi\text{-PT}(Y) = \sum_{\alpha \in F_1 K(Y)} \pi\text{-PT}^\alpha(Y)q^\alpha. \]

**Main result**

The following theorem rests the work of Bridgeland [12] and Joyce–Song [30], and we therefore require our Calabi-Yau threefold \( Y \) to satisfy

\[ H^1(Y, \mathcal{O}_Y) = 0. \]

**Theorem 6.** Let \( X \) be a projective Calabi-Yau threefold that is the coarse space of an orbifold CY3 \( X \) that satisfies the hard Lefschetz condition. Let \( \pi : Y \rightarrow X \) be the resolution given by equation 3. Then the generating series for the \( \pi \)-stable pair invariants and the DT invariants are related by the equation

\[ \pi\text{-PT}(Y) = \frac{\text{DT}(Y)}{\text{DT}_{\text{exc}}(Y)}. \]

The aim of this article is to prove this theorem. We summarize the chapters below. In chapter 2, we describe a torsion pair \( (\mathcal{P}_\pi, \mathcal{Q}_\pi) \) that is crucial to our definition of \( \pi \)-stable pairs. We explain the similarities between \( \pi \)-stable pairs and PT stable pairs and objects of the Hilbert scheme. The chapter ends by establishing results about the moduli space of \( \pi \)-stable pairs.

In chapter 3, we recall the concept of a stability condition in the sense of Joyce. We then define the stability condition that we will use through
out. The rest of the chapter is dedicated to proving that we may apply Joyce’s powerful machinery.

In chapter 4, we introduce the Harder-Narasimhan filtration for our stability condition, which will be our main tool to prove the relationship between the stability condition and the torsion pair from chapter 2.

In chapter 5, we introduce the motivic Hall algebra.

In chapter 6, we introduce the infinite-type Hall algebra as a purely pedagogical tool. It helps us to give the essence of the idea of many results, without having to concern ourselves with convergence issues, which are handled in the next chapter.

In chapter 7, we introduce the Laurent Hall algebra, address the convergence issues alluded to in the previous chapter, and prove theorem 6.

Remark 7. To prove the crepant resolution conjecture, we need to prove that \( \pi\text{-PT}(Y) = \text{PT}(X) \) and then use (2) Bayer’s proof of the PT/DT correspondence on \( X \),

\[
\text{PT}(X) = \frac{\text{DT}_{m_R}(X)}{\text{DT}_0(X)}.
\]

The hope is that the Fourier-Mukai isomorphism \( \Psi \) takes \( \pi\)-stable pairs (as an object in \( D^b(Y) \)) to a PT pair on \( X \).

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\section{\( \pi \)-stable pairs}

In this section, we define \( \pi \)-stable pairs, and prove some basic results.

\subsection{Categorical constructions}

Let \( \mathcal{A} \) be an abelian category. Here we recall the notion of torsion pairs.

Definition 8. Let \((\mathcal{P}, \mathcal{Q})\) be a pair of full subcategories of \( \mathcal{A} \). We say \((\mathcal{P}, \mathcal{Q})\) is a torsion pair if the following conditions hold.

- \( \text{Hom}(T, F) = 0 \) for any \( T \in \mathcal{P} \) and \( F \in \mathcal{Q} \).
- Any object \( E \in \mathcal{A} \) fits into a unique exact sequence,

\[
0 \to T \to E \to F \to 0,
\]

\[ (9) \]

with \( T \in \mathcal{P} \) and \( F \in \mathcal{Q} \).

We borrow the following lemma from Toda [46].
Lemma 10. Suppose that $A$ is a noetherian abelian category.

(i) Let $P \subset A$ be a full subcategory which is closed under extensions and quotients in $A$. Then for $Q = \{E \in A : \text{Hom}(P, E) = 0\}$, the pair $(P, Q)$ is a torsion pair on $A$.

(ii) Let $Q \subset A$ be a full subcategory which is closed under extensions and subobjects in $A$. Then for $P = \{E \in A : \text{Hom}(E, Q) = 0\}$, the pair $(P, Q)$ is a torsion pair on $A$.

Proof: We only show (i), as the proof of (ii) is similar. Take $E \in A$ with $E \not\in Q$. Then there is $T \in P$ and a non-zero morphism $T \to E$. Since $P$ is closed under quotients, we may assume that $T \to E$ is a monomorphism in $A$. Take an exact sequence in $A$,

$$0 \to T \to E \to F \to 0. \quad (11)$$

By the noetherian property of $A$ and the assumption that $P$ is closed under extensions, we may assume that there is no $T \subseteq T' \subset E$ with $T' \in P$. Then we have $F \in Q$ and (11) gives the desired sequence. □

Example 12. Let

$P = \{0\text{-dimensional sheaves on } Y\}$,

and let

$Q = \{E \in \text{coh}(Y) : \text{Hom}(P, E) = 0 \text{ for all } P \in P\}.$

Lemma 10 easily proves that the pair $(P, Q)$ is a torsion pair.

Let $C = \text{coh}_{\leq 1}(Y)$

denote the full subcategory of coherent sheaves on $Y$ whose support is of dimension no more than one. We make the following definitions:

$P_\pi = \{P \in C | R^1\pi_*P \text{ is a zero-dimensional sheaf on } X\}$,

and

$Q_\pi = \{F \in C | \text{for all } P \in P_\pi, \text{Hom}(P, F) = 0\} = P_\pi^\perp.$

Lemma 13. The pair $(P_\pi, Q_\pi)$ is a torsion pair in $C$.

Proof: By lemma 10, it suffices to prove that $P_\pi$ is closed under extensions and quotients.

Let $P', P'' \in P_\pi$, and consider the short exact sequence

$$0 \to P' \to P \to P'' \to 0.$$

We are to show that such a $P$ must live in $P_\pi$. Consider now the long exact sequence,

$$0 \to \pi_*P' \to \pi_*P \to \pi_*P'' \to R^1\pi_*P' \to R^1\pi_*P \to R^1\pi_*P'' \to 0.$$ 

Since $P', P'' \in P_\pi$, we know that $R^1\pi_*P' = 0$ and $R^1\pi_*P'' = 0$, so $R^1\pi_*P = 0$. We also know that $\pi_*P'$ and $\pi_*P''$ are zero-dimensional
sheaves, and so it is clear then that $\pi_*P$ must be so as well. This proves that $\mathcal{P}_\pi$ is closed under extensions.

Let $P \in \mathcal{P}_\pi$, and consider a quotient $P \twoheadrightarrow B \to 0$. Denote the kernel of this map by $K$. As before, we get a long exact sequence,

$$0 \to \pi_*K \to \pi_*P \to \pi_*B \to R^1\pi_*K \to R^1\pi_*P \to R^1\pi_*B \to 0.$$ 

Since $P \in \mathcal{P}_\pi$, $R^1\pi_*P = 0$, and so $R^1\pi_*B = 0$. It remains to show that $\pi_*B$ is zero-dimensional. We know that $\pi_*K$ is zero dimensional, since it is a subsheaf of (the zero-dimensional sheaf) $\pi_*P$. The support of $R^1\pi_*K$ is contained in the singular locus. Suppose $\dim \operatorname{supp}(R^1\pi_*K) = 1$. Then, $K$ must have been supported in dimension two, however this contradicts the fact that $K \in \operatorname{coh}_{\leq 1}(Y)$.

Hence $R^1\pi_*K$ is zero dimensional. Now, $\pi_*B$ is the extension of zero-dimensional sheaves, so it too is zero-dimensional. This completes the proof that $B \in \mathcal{P}_\pi$, and that $(\mathcal{P}_\pi, \mathcal{Q}_\pi)$ is a torsion pair.

\[\blacksquare\]

**Definition 14.** A map $\gamma : \mathcal{O}_Y \to G$ is a $\pi$-stable pair if $G \in \mathcal{Q}_\pi$ and $\operatorname{coker}(\gamma) \in \mathcal{P}_\pi$.

**Remark 15.** Our notion of $\pi$-stable pair is a generalization of the stable pairs of Pandharipande and Thomas [40]. In the trivial case when $X = Y$ and $\pi = \text{id}$, we have that $(\mathcal{P}_\pi, \mathcal{Q}_\pi) = (\mathcal{P}, \mathcal{Q})$ of example 12, and the $\pi$-stable pairs are exactly PT stable pairs.

**Definition 16.** Two $\pi$-stable pairs $\gamma_1 : \mathcal{O}_Y \to G_1$ and $\gamma_2 : \mathcal{O}_Y \to G_2$ are isomorphic if there exists an isomorphism of sheaves $\theta : G_1 \to G_2$ making the following diagram commute:

$$
\begin{array}{ccc}
\mathcal{O}_Y & \xrightarrow{\gamma_1} & G_1 \\
\gamma_2 \downarrow & & \downarrow \theta \\
& G_2 & \\
\end{array}
$$

A family of $\pi$-stable pairs on $Y$ over a scheme $T$ is a coherent sheaf $G$ on $Y \times T$, flat over $T$ and a morphism $\gamma : \mathcal{O}_{Y \times T} \to G$ such that for all closed points $t \in T$, the restriction $\gamma_t : \mathcal{O}_Y \to G_t$ is a $\pi$-stable pair.

**Remark 17.** In [12], the tilt of $\mathcal{A}$ with respect to the torsion pair $(\mathcal{P}, \mathcal{Q})$ of example 12 is denoted $\mathcal{A}^\#$, and lemma 2.3 of [12] proves that $\mathcal{A}^\#$-epimorphisms of the form $\mathcal{O}_Y \to F$ are precisely stable pairs. The abelian category generated by $\mathcal{O}_Y$ and $\mathcal{C}$ has a tilt whose epimorphisms of the form $\mathcal{O}_Y \to G$ are precisely $\pi$-stable pairs. This is analogous to the tilt used in [12]. However, since it is not strictly necessary for any of our arguments, we will not present a proof here.

We associate to every $\pi$-stable pair $\gamma : \mathcal{O}_Y \to G$ a short exact sequence

$$0 \to \mathcal{O}_C \to G \to P \to 0,$$

where $P = \operatorname{coker}(\gamma) \in \mathcal{P}_\pi$ and $\mathcal{O}_C = \mathcal{O}_Y / \ker(\gamma)$.
Proposition 18. Let $G$ be a non-zero sheaf. If $O_Y \to G$ is a $\pi$-stable pair, then $G$ is not supported exclusively on exceptional curves, and $R^1\pi_*G = 0$.

Proof: Let

$$0 \to O_C \to G \to P \to 0$$

be the associated short exact sequence. We will first show that $R^1\pi_*O_C = 0$. Consider the short exact sequence

$$0 \to I_C \to O_Y \to O_C \to 0.$$ 

Pushing forward yields

$$0 \to \pi_*I_C \to \pi_*O_Y \to \pi_*O_C \to R^1\pi_*I_C \to R^1\pi_*O_Y \to R^1\pi_*O_C \to 0$$

which is exact since the dimension of the fibres of $Y \to X$ is at most one. Thus the vanishing of $R^1\pi_*O_Y$ by equation 2 implies that of $R^1\pi_*O_C$.

Now consider the following long exact sequence.

$$0 \to \pi_*O_C \to \pi_*G \to \pi_*P \to R^1\pi_*O_C \to R^1\pi_*G \to R^1\pi_*P \to 0.$$

From above, we know $R^1\pi_*O_C = 0$. As well, $P \in P_\pi$ implies $R^1\pi_*P = 0$. Thus $R^1\pi_*G = 0$.

Now, if $C$ consists of only exceptional curves, then $\pi_*O_C$ is zero-dimensional. This implies that $\pi_*G$ is the extension of zero-dimensional sheaves, and therefore zero-dimensional. This means that $G \in P_\pi$. By definition of $\pi$-stable pair, $G \in Q_\pi$. By definition of $Q_\pi$ the only map from an object of $P_\pi$ to an object of $Q_\pi$ is the zero map, hence the identity of $G$ is the zero map, and $G$ is the zero object. $\blacksquare$

Let us introduce some terminology and results taken from [12] (modified for our purposes, since we are only interested in sheaves supported in dimension no more than one). Let $M$ denote the stack of objects of $\text{coh}_{\leq 1}(Y)$. It is an algebraic stack, locally of finite type over $\mathbb{C}$. Let $M(O)$ denote the stack of framed sheaves, that is, the stack whose objects over a scheme $S$ are pairs $(E, \gamma)$ where $E$ is a $S$-flat coherent sheaf on $S \times Y$, of relative dimension no more than one, together with a map $O_{S \times Y} \to E$. Given a morphism of schemes $f : T \to S$, and an object $(F, \delta)$ over $T$, a morphism in $M(O)$ lying over $f$ is an isomorphism

$$\theta : f^*(E) \to F$$

such that the following diagram commutes

$$\begin{array}{ccc}
    f^*(O_{S \times Y}) & \xrightarrow{f^*(\gamma)} & f^*(E) \\
    \downarrow \text{can} & & \downarrow \theta \\
    O_{T \times Y} & \xrightarrow{\delta} & F.
\end{array}$$

(19)

The symbol “can” denotes the canonical isomorphism of pullbacks.

There is a natural map

$$M(O) \xrightarrow{q} M$$

(20)

sending a sheaf with a section to the underlying sheaf.

The following lemmas are 2.4 and 2.5 of [12].
Lemma 21. The stack $M(\mathcal{O})$ is algebraic and the morphism $q$ is representable and of finite type.

Lemma 22. There is a stratification of $M$ by locally-closed substacks $M_r \subset M$ such that objects $F$ of $M_r(\mathbb{C})$ are coherent sheaves satisfying
\[
\dim_{\mathbb{C}} H^0(Y, F) = r.
\]
Furthermore, the pullback of the morphism $q$ to $M_r$ is a Zariski fibration with fibre $\mathbb{C}^r$.

Both of these are proven in [12].

Let
\[
\pi\text{-Hilb}^{(\beta,n)} \subset M(\mathcal{O})
\]
denote the subcategory of $M(\mathcal{O})$ consisting of families of $\pi$-stable pairs on $Y$ whose sheaf $G$ has chern character $(0,0,\beta,n)$.

Lemma 23. $\pi\text{-Hilb}^{(\beta,n)}$ is a constructible set, that is, it has a finite decomposition into subcategories which are each represented by schemes.

Remark 24. We expect that $\pi\text{-Hilb}^{(\beta,n)}$ is in fact represented by a projective scheme, but we do not pursue that in this paper. The above lemma suffices for our purposes: the use of $\pi\text{-Hilb}^{(\beta,n)}$ in the Hall algebra.

Proof: Since $M(\mathcal{O})$ is a locally constructible stack, the content of the lemma is (1) the subcategory $\pi\text{-Hilb}^{(\beta,n)}$ is bounded (see definition 41) and (2) the automorphism group of an object in $\pi\text{-Hilb}^{(\beta,n)}$ is trivial. We will prove (1) in Lemma 46 and we prove (2) below.

Let $\mathcal{O}_Y \to G$ be a $\pi$-stable pair. We will show that it has only the trivial automorphism. Consider the associated short exact sequence,
\[
0 \to \mathcal{O}_C \to G \to P \to 0.
\]
An automorphism of this $\pi$-stable pair leads to a diagram of the form,
\[
\begin{array}{ccc}
0 & \to & \mathcal{O}_C \\
\gamma & \to & G \\
\downarrow{\text{id}} & & \downarrow{g} \\
0 & \to & \mathcal{O}_C
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
P & \to & 0 \\
\delta & \to & \mathcal{O}_C \\
\downarrow{h} & & \downarrow{\gamma} \\
0 & \to & P
\end{array}
\]
\[
\begin{array}{ccc}
0 & \to & \mathcal{O}_C \\
\gamma & \to & G \\
\downarrow{\text{id}} & & \downarrow{\gamma} \\
0 & \to & \mathcal{O}_C
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
P & \to & 0 \\
\delta & \to & \mathcal{O}_C \\
\downarrow{h} & & \downarrow{\gamma} \\
0 & \to & P
\end{array}
\]

We will show that $g$ is the identity map. Consider the following diagram, obtained by subtracting the identity from the diagram above,
\[
\begin{array}{ccc}
0 & \to & \mathcal{O}_C \\
\gamma & \to & G \\
\downarrow{\text{id}} & & \downarrow{g} \\
0 & \to & \mathcal{O}_C
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
P & \to & 0 \\
\delta & \to & \mathcal{O}_C \\
\downarrow{h} & & \downarrow{\gamma} \\
0 & \to & P
\end{array}
\]
\[
\begin{array}{ccc}
0 & \to & \mathcal{O}_C \\
\gamma & \to & G \\
\downarrow{\text{id}} & & \downarrow{\gamma} \\
0 & \to & \mathcal{O}_C
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
P & \to & 0 \\
\delta & \to & \mathcal{O}_C \\
\downarrow{h} & & \downarrow{\gamma} \\
0 & \to & P
\end{array}
\]

Since the left-most vertical arrow is zero, a diagram chase proves that the dotted morphism $\delta$ exists and commutes with the diagram. However, $P \in \mathcal{P}_s$ and $G \in \mathcal{Q}_s$, so $\delta$ must be zero, and consequently, $\gamma - \text{id} = 0$. 

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Standard homological algebra then implies that the morphism \( g - \text{id} \) must be of the form \( \gamma \circ \epsilon \circ h \) for some \( \epsilon \in \text{Hom}(P, \mathcal{O}_C) \). However, any non-zero \( \epsilon \) would give rise to a non-zero map \( \gamma \circ \epsilon : P \to G \) which contradicts \( P \in \mathcal{P}_\pi \) and \( G \in \mathcal{Q}_\pi \). Thus \( g - \text{id} = 0 \) and so \( g = \text{id} \).  

\[ \blacksquare \]

The Behrend function identity

We state and prove a variation of [12, theorem 3.1] of Bridgeland.

**Lemma 26.** Let \( \gamma : \mathcal{O}_Y \to G \) be a \( \pi \)-stable pair. Then there is an equality of Behrend’s microlocal functions

\[ \nu_{\mathcal{M}(O)}(\gamma) = (-1)^{\chi(G)}\nu_{\mathcal{M}}(G). \]

**Proof:** The case when \( G \) is a stable pair is taken care of by theorem 3.1 of [12]. Thus, we may assume that the cokernel \( P \) of \( \gamma : \mathcal{O}_Y \to G \) has one-dimensional support.

Let \( \mathcal{O}_C \subset G \) be the image of \( \gamma \). It is the structure sheaf of a subscheme \( C \subset Y \) of dimension 1. There is a line bundle \( L \) on \( Y \) such that \( H^i(Y, G \otimes L) = 0 \) (27) for all \( i > 0 \), and there is a divisor \( H \in |L| \) such that \( H \) meets \( C \) at finitely many points, none of which are in the support of \( \text{coker}(\gamma) \). This claim is verified in lemma 30. From here, the proof is identical to Bridgeland’s, but we include a portion of it to illustrate his ideas.

There is a short exact sequence

\[ 0 \to \mathcal{O}_Y \xrightarrow{s} L \to \mathcal{O}_H(H) \to 0 \]

where \( s \) is the section of \( L \) corresponding to the divisor \( H \). Tensoring it with \( G \), and using the above assumptions yields a diagram of sheaves

\[ \begin{array}{cccccc}
0 & \to & \mathcal{O}_Y & \xrightarrow{\gamma} & G & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & K & \to & 0 \\
& & \downarrow{\delta} & & \downarrow{\gamma} & & \downarrow{\delta} & & \downarrow{\beta} & & \downarrow{\gamma} & \\
& & 0 & & G & & F & & K & & 0
\end{array} \] (28)

where \( F = G \otimes L \). The support of the sheaf \( K \) is zero-dimensional, and disjoint from the support of \( \text{coker}(\gamma) \). In particular,

\[ \text{Hom}_Y(K, F) = 0. \] (29)

Consider two points of the stack \( \mathcal{M}(O) \) corresponding to the maps

\[ \gamma : \mathcal{O}_Y \to G \text{ and } \delta : \mathcal{O}_Y \to F. \]

The statement of lemma 26 holds for the map \( \delta \) because lemma 22 together with equation 27 implies that

\[ q : \mathcal{M}(O) \to \mathcal{M} \]
is smooth of relative dimension $\chi(F) = H^0(Y, F)$ over an open neighbourhood of the point $F \in \mathcal{M}(\mathbb{C})$. On the other hand, tensoring sheaves with $L$ defines an automorphism of $\mathcal{M}$, so the microlocal function of $\mathcal{M}$ at the points corresponding to $G$ and $F$ are equal. To prove the lemma, it suffices to show that

$(-1)^{\chi(G)} \cdot \nu_{\mathcal{M}(\mathcal{O})}(\gamma) = (-1)^{\chi(F)} \cdot \nu_{\mathcal{M}(\mathcal{O})}(\delta)$.

Consider the stack $W$ whose $S$-valued points are diagrams of $S$-flat sheaves on $S \times Y$ of the form

$$
\begin{array}{c}
\mathcal{O}_{S \times Y} \\
\gamma_S \\
\delta_S \\
\downarrow \\
0 \\
\alpha_S \\
G_S \\
\alpha_S \\
F_S \\
\beta_S \\
K_S \\
\downarrow \\
0
\end{array}
$$

There are two morphisms

$p : W \rightarrow \mathcal{M}(\mathcal{O}), \quad q : W \rightarrow \mathcal{M}(\mathcal{O}),$

taking such a diagram to the maps $\gamma_S$ and $\delta_S$ respectively. By passing to an open substack of $W$, we may assume that equation 29 holds for all $\mathbb{C}$-valued points of $W$. It follows that $p$ and $q$ induce injective maps on stabilizer groups of $\mathbb{C}$-valued points, and hence are representable.

Recall that Behrend’s microlocal function satisfies the property that when $f : T \rightarrow S$ is a smooth morphism of relative dimension $d$, there is an identity [3, proposition 1.5]

$$\nu_T = (-1)^d f^*(\nu_S).$$

Using this identity, it will be enough to show that at the point $w \in W(\mathbb{C})$ corresponding to the diagram 28, the morphisms $p$ and $q$ are smooth of relative dimension $\chi(K)$ and 0, respectively. For the proof of these facts, see [12, pages 11–13].

**Lemma 30.** Given a $\pi$-stable pair $\gamma : \mathcal{O}_Y \rightarrow G$, we may choose a very ample divisor $H$ on $X$ such that its pull-back is equal to its proper transform (we denote both by $\bar{H}$), it satisfies $\text{supp}(\text{coker} \, \gamma) \cap H = \emptyset$, $H \cap \text{supp} \, G$ is 0-dimensional, and $H^1(Y, G(\bar{H})) = 0$.

**Proof:** First we collect a little notation. Let $E$ be the exceptional locus of $Y$, let $E'$ be the image of $E$ in $X$. Define a subset $Z$ as follows

$$Z = \{p \in X : G|_{\pi^{-1}(p)} \text{ is one dimensional} \} \subset E'.$$

Notice that $Z$ is a finite collection of points, namely it is the image under $\pi$ of the exceptional components in the support of $G$.

Since the cokernel of $\gamma$ is supported in dimension one and it lies in $\mathcal{P}_e$, it is supported on points and exceptional curves. Hence $\pi(\text{supp}(\text{coker} \, \gamma))$ is zero dimensional. Moreover, $\pi(\text{supp} \, G)$ is one dimensional. Thus we may choose an ample divisor $H$ on $X$ so that

$$H \cap \pi(\text{supp}(\text{coker} \, \gamma)) = \emptyset$$

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and $H \cap \pi(\text{supp} G)$ is zero dimensional and does not contain any of the points in $Z$. It follows that $\tilde{H} \cap \text{supp}(\text{coker} \gamma)$ is empty, and $\tilde{H} \cap \text{supp}(G)$ is zero dimensional. Moreover, by Serre vanishing, we may assume that $H$ is sufficiently ample on $X$ so that

$$H^1(X, (\pi_*G)(H)) = 0.$$ (31)

We now show that $H^1(Y, G(eH)) = 0$. By proposition 18, we know that $R^1\pi_*G = 0$, since $\mathcal{O}_Y \to G$ is a $\pi$-stable pair. The sequence

$$0 \to G \to G(\tilde{H}) \to G(eH)|_{\tilde{H}} \to 0$$

gives

$$\ldots \to R^1\pi_*G \to R^1\pi_*G(\tilde{H}) \to R^1\pi_*G(eH)|_{\tilde{H}} \to 0.$$

However, we know $R^1\pi_*G = 0$ and $G(\tilde{H})|_{\tilde{H}}$ is supported on points so $R^1\pi_*G(eH)|_{\tilde{H}} = 0$, so $R^1\pi_*G(\tilde{H}) = 0$. Now by the Leray spectral sequence,

$$H^1(Y, G(\tilde{H})) = H^1(X, \pi_*G(\tilde{H})) = H^1(X, \pi_*G \otimes \pi^*O_X(H)) = H^1(X, (\pi_*G)(H)) = 0,$$

where the last equality comes from equation 31.

3 Stability conditions

In this section, we define a stability condition on $\mathcal{C} = \text{coh}_{\leq 1}Y$. We follow Joyce’s treatment of stability conditions as found in section 4 of [27], though not in as great generality.

Let $N_1(Y)$ denote the abelian group of cycles of dimension one modulo numerical equivalence. We begin by quoting lemmas 2.1 and 2.2 of [12].

**Lemma 32.** An element $\beta \in N_1(Y)$ has only finitely many decompositions of the form $\beta = \beta_1 + \beta_2$ with $\beta_i$ effective.

**Lemma 33.** The Chern character map induces an isomorphism

$$ch = (ch_2, ch_3) : F_1K(Y) \to N_1(Y) \oplus \mathbb{Z}.$$

Define

$$\Delta = \{[E] \in F_1K(Y) : E \in \mathcal{C}\}$$

to be the positive or effective cone of $F_1K(Y)$.

**Definition 34.** A stability condition on $\mathcal{C}$ is a triple $(T, \tau, \leq)$ where $(T, \leq)$ is a set $T$ with a total ordering $\leq$, and $\tau$ is a map $\Delta \to T$ from the effective cone to $T$, satisfying the following condition: whenever $\alpha + \beta = \gamma$ in $\Delta$, then

$$\tau(\alpha) < \tau(\gamma) < \tau(\beta),$$

or

$$\tau(\beta) < \tau(\gamma) < \tau(\alpha),$$

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or
\[ \tau(\alpha) = \tau(\gamma) = \tau(\beta). \]

A triple \((T, \tau, \leq)\) is called a weak stability condition if it satisfies the weaker condition that whenever \(\alpha + \beta = \gamma\) in \(\Delta\), \(\tau(\alpha) \leq \tau(\gamma) \leq \tau(\beta)\) or \(\tau(\beta) \leq \tau(\gamma) \leq \tau(\alpha)\).

**Definition 35.** A non-zero sheaf \(G\) is
1. \(\tau\)-semistable if for all \(S \subset G\), such that \(S \not\cong 0\), we have that \(\tau(S) \leq \tau(G/S)\);
2. \(\tau\)-stable if for all \(S \subset G\), such that \(S \not\cong 0\), we have that \(\tau(S) < \tau(G/S)\);
3. \(\tau\)-unstable if it is not \(\tau\)-semistable.

**Lemma 36.** Let \(F\) and \(G\) be \(\tau\)-semistable sheaves, and let \(F \rightarrow G\) be a map of sheaves. Then either \(\tau(F) \leq \tau(G)\) or \(f = 0\).

**Proof:** Consider the inclusion map \(\iota : \text{im}(f) \rightarrow G\). Since \(G\) is semistable, we know either \(\text{im}(f) = 0\) or \(\tau(\text{im}(f)) \leq \tau(G)\). Consider now the corestriction of \(f\), \(\text{cor}(f) : F \rightarrow \text{im}(f)\). Since \(F\) is semistable, we know that either \(\text{im}(f) = 0\) or \(\tau(F) \leq \tau(\text{im}(f))\). This implies either \(\text{im}(f) = 0\) or \(\tau(F) \leq \tau(G)\). □

Now we define a stability condition on \(C\).

**Definition 37.** Choose an ample divisor \(H\) on \(X\), let \(e_H\) denote the total transform of \(H\) in \(Y\). Let \(A\) be an ample line bundle on \(Y\) and we let \(L = e_H + A\). Note that \(L\) is ample and that \(L \cdot C > e_H \cdot C\) for any curve class \(C\). Given a sheaf \(G\) in \(C\), define the \(\pi\)-slope of \(G\) to be
\[ \mu_{\pi}(G) = \left( \frac{\chi(G)}{\beta \cdot H}, \frac{\chi(G)}{\beta \cdot L} \right) \in (-\infty, +\infty] \times (-\infty, +\infty], \]
where by convention \(\chi/0 = +\infty\) for any \(\chi \in \mathbb{Z}\), \((-\infty, +\infty] \times (-\infty, +\infty]\) is ordered lexicographically, and \(\beta = \beta_G\) the homology class associated to the support of \(G\).

To make Joyce’s Hall algebra machinery work, he introduces the additional notion of permissibility for a (weak) stability condition [27, Def. 4.7]. The main result of this section is the following:

**Theorem 38.** The map
\[ \mu_{\pi} : \Delta \rightarrow (-\infty, +\infty] \times (-\infty, +\infty] \]
defines a weak permissible stability condition.

Following definitions 4.1 and 4.7 of [27], we see that we must prove the following three properties:

1. (weak seesaw property) for any short exact sequence
\[ 0 \rightarrow A \rightarrow G \rightarrow B \rightarrow 0, \]
either \(\mu_{\pi}(A) \leq \mu_{\pi}(G) \leq \mu_{\pi}(B)\) or \(\mu_{\pi}(A) \geq \mu_{\pi}(G) \geq \mu_{\pi}(B)\),
2. \( C \) is \( \mu_\pi \)-artinian, i.e. there exists no infinite descending chain \( \cdots A_2 \subset A_1 \subset A \in C \) such that \( A_i \neq A_{i+1} \), and \( \mu_\pi(A_{i+1}) \geq \mu_\pi(A_i/A_{i+1}) \) for all \( i \); and

3. the substack of \( \mu_\pi \)-semistable objects of a fixed Chern character of the stack parametrizing objects of \( C \) is a constructible substack of \( \mathcal{M} \).

We will prove the first property in lemma 39 and the second in lemma 40. The third property amounts to showing that the family of \( \mu_\pi \)-semistable sheaves of a fixed chern class is bounded (see the proof of theorem 4.20 in [27]) which we prove in lemma 47.

**Lemma 39.** The function \( \mu_\pi \) satisfies the weak seesaw property.

**Proof:** Let 0 \( \rightarrow A \rightarrow G \rightarrow B \rightarrow 0 \) be a short exact sequence of sheaves, and suppose

\[
\left( \frac{\chi(A)}{\beta_A \cdot H}, \frac{\chi(A)}{\beta_A \cdot L} \right) \leq \left( \frac{\chi(G)}{\beta_G \cdot H}, \frac{\chi(G)}{\beta_G \cdot L} \right),
\]

from which we are to deduce that \( \mu_\pi(G) \leq \mu_\pi(B) \). Before we start a case-by-case analysis, notice that \( \chi(G) = \chi(A) + \chi(B) \) and \( \beta_G = \beta_A + \beta_B \).

**case 1:** \( \frac{\chi(A)}{\beta_A \cdot H} < \frac{\chi(G)}{\beta_G \cdot H} \) and no denominator is zero.

Then this follows from the observation

\[
\frac{a}{b} < \frac{a+c}{b+d} \quad \Rightarrow \quad \frac{a+c}{b+d} < \frac{c}{d}
\]

provided \( b, d > 0 \). In particular, we assume

\[
\frac{\chi(A)}{\beta_A \cdot H} \leq \frac{\chi(G)}{\beta_G \cdot H}.
\]

Rewriting the second term yields

\[
\frac{\chi(A)}{\beta_A \cdot H} < \frac{\chi(A) + \chi(B)}{(\beta_A + \beta_B) \cdot H}.
\]

The observation above then proves that

\[
\frac{\chi(A) + \chi(B)}{(\beta_A + \beta_B) \cdot H} < \frac{\chi(B)}{\beta_B \cdot H},
\]

as desired.

**case 2:** \( \frac{\chi(A)}{\beta_A \cdot H} = \frac{\chi(G)}{\beta_G \cdot H}, \frac{\chi(A)}{\beta_A \cdot L} \leq \frac{\chi(G)}{\beta_G \cdot L} \) and no denominator is zero.

We are given that \( \frac{\chi(A)}{\beta_A \cdot H} = \frac{\chi(G)}{\beta_G \cdot H} \), so

\[
\chi(A)(\beta_G \cdot \tilde{H}) = (\beta_A \cdot \tilde{H})\chi(G).
\]

Writing everything in terms of \( A \) and \( B \),

\[
\chi(A)((\beta_A + \beta_B) \cdot \tilde{H}) = \beta_A \cdot \tilde{H}(\chi(A) + \chi(B)),
\]

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which implies 
\[ \chi(A) (\beta_B \cdot \tilde{H}) = (\beta_A \cdot \tilde{H}) \chi(B). \]

Since we assume that all denominators are non-zero, we have
\[ \frac{\chi(A)}{\beta_A \cdot \tilde{H}} = \frac{\chi(B)}{\beta_B \cdot \tilde{H}}. \]

So we must show that
\[ \frac{\chi(G)}{\beta_G \cdot L} < \frac{\chi(B)}{\beta_B \cdot L}. \]

This follows from the same observation made in case 1.

**case 3:** \( \beta_A \cdot \tilde{H} = 0. \) Then \( +\infty = \frac{\chi(A)}{\beta_A \cdot \tilde{H}} \leq \frac{\chi(A) + \chi(B)}{(\beta_A + \beta_B) \cdot \tilde{H}}. \) This implies that \( \chi(A) = \chi(B) \) and \( \beta_A \cdot \tilde{H} = 0, \) hence \( \beta_B \cdot \tilde{H} = 0. \) This reduces us to Gieseker stability on \( Y, \) which we know satisfies the weak seesaw property.

**case 4:** \( \beta_G \cdot \tilde{H} = 0. \) We know \( \beta \cdot \pi^* H = \pi^*(\beta \cdot \pi^* H) = \pi_*(\beta \cdot H) \geq 0, \) hence \( \beta \cdot H \geq 0 \) for any effective curve class \( \beta, \) so we must have \( \beta_A \cdot \tilde{H} = 0 \) and \( \beta_B \cdot \tilde{H} = 0. \) This lands us back in the case of Gieseker stability on \( Y, \) and lemma 39 is proven.

\[ \blacksquare \]

**Lemma 40.** The category \( \mathcal{C} \) is \( \mu_\pi \)-artinian.

**Proof:** Joyce proves that a weak stability condition is Artinian if it is dominated by an Artinian weak stability condition [27, 4.10, 4.11]. Recall that a weak stability condition \( \tilde{\tau} \) is said to dominate \( \tau \) if for any \( A, B \) in \( \mathcal{C} \) with \( \tau(A) \leq \tau(B) \) then \( \tilde{\tau}(A) \leq \tilde{\tau}(B). \) Let
\[ \delta(G) = - \dim \text{supp} \, G \in \mathbb{Z}, \]

then \( \delta \) is an Artinian, weak stability condition [27, 4.19]. Thus to prove the lemma, it suffices to show that \( \mu_\pi \) is dominated by \( \delta. \)

Let \( \mu_\pi(A) \leq \mu_\pi(B). \) We need to show that this implies that \( \delta(A) \leq \delta(B). \) Expanding, we have that
\[ \left( \frac{\chi(A)}{\beta_A \cdot \tilde{H}}, \frac{\chi(A)}{\beta_A \cdot L} \right) \leq \left( \frac{\chi(B)}{\beta_B \cdot \tilde{H}}, \frac{\chi(B)}{\beta_B \cdot L} \right). \]

We proceed with a case-by-case analysis.

**case 1:** the denominators are non-zero. Since we have restricted our attention to sheaves supported in dimension \( \leq 1, \) it follows that neither \( A \) nor \( B \) is 0-dimensional. Thus, they are both one dimensional, and \( \delta(A) = \delta(B). \) In particular, \( \delta(A) \leq \delta(B). \)

**case 2:** \( \beta_B \cdot \tilde{H} = 0 \) and \( \beta_A \cdot \tilde{H} \neq 0. \) Then \( \dim \text{supp} \, A \geq 1 \geq \dim \text{supp} \, B. \) So \( \delta(A) \leq \delta(B). \)

**case 3:** Both \( \beta_B \cdot \tilde{H} = 0 \) and \( \beta_A \cdot \tilde{H} = 0. \) Then \( \mu_\pi(A) \leq \mu_\pi(B) \) amounts to regular Geiseker stability, which is dominated by \( \delta \) as demonstrated by Joyce [27, §4.4].
To finish the proof that \( \mu_\pi \) is a permissible weak stability condition, it remains only to prove the family of all \( \mu_\pi \)-semistable sheaves of a fixed Chern class is bounded. This is proven in lemma 47 in the next section.

Boundedness

In this section, we prove that the family of \( \pi \)-stable pairs with fixed chern classes is bounded (lemma 46) and we prove that the family of \( \mu_\pi \)-semistable sheaves with fixed chern classes is bounded (lemma 47).

We begin by recalling some basic results concerning boundedness (cf. [24]).

**Definition 41.** A subcategory \( \mathcal{U} \) of \( \text{coh}(Y) \) is bounded if there exists a scheme \( S \) of finite type and a sheaf \( U \) on \( X \times S \) such that for every object \( U_i \) of \( \mathcal{U} \), there exists a closed point \( s_i \in S \) such that \( U_i \cong U|_{X \times \{s_i\}} \).

Notice that this definition still makes sense if we have a set of isomorphism classes of sheaves instead of a category.

**Definition 42.** Let \( Y \) be a scheme, let \( \mathcal{O}(1) \) be an ample line bundle, and let \( m \) be an integer. A sheaf \( F \) on \( Y \) is \( m \)-regular if, for all \( i \geq 0 \),

\[
H^i(Y, F(m - i)) = 0.
\]

A proof for the following may be found in [31], as well as in [38].

**Lemma 43.** If \( F \) is \( m \)-regular, then the following statements are true:

1. \( F \) is \( m' \)-regular for all \( m' \geq m \).
2. \( F(m) \) is globally generated.
3. For all \( n \geq 0 \), the natural map \( H^0(Y, F(m)) \otimes H^0(Y, \mathcal{O}(n)) \to H^0(Y, F(n + m)) \) is surjective.

**Definition 44.** The Mumford-Castelnuovo regularity of a sheaf \( F \) is the number \( \text{reg}(F) = \inf\{m \in \mathbb{Z} : F \text{ is } m\text{-regular}\} \).

**Lemma 45.** Let \( \mathcal{U} \) be a category of sheaves on \( Y \). The following statements are equivalent.

1. \( \mathcal{U} \) is bounded.
2. The set of Hilbert polynomials of objects \( U_i \) of \( \mathcal{U} \) is finite, and there is an integer \( N \) such that for all objects \( U_i \) of \( \mathcal{U} \), \( \text{reg} U_i < N \).
3. The set of Hilbert polynomials of objects \( U_i \) of \( \mathcal{U} \) is finite, and there exists a sheaf \( F \) such that each object of \( \mathcal{U} \) is isomorphic to a quotient of \( F \).

The proof of this lemma may be found in [21].

**Lemma 46.** The family of \( \pi \)-stable pairs \( O_Y \xrightarrow{\gamma} G \) with a fixed Chern class is bounded.
**Proof:** To each such $\pi$-stable pair there is an associated a short exact sequence,
$$0 \to \mathcal{O}_C \to G \to P \to 0,$$
where $\mathcal{O}_C$ is the image of the map $\gamma$, and $P$ is the cokernel. We will show that the family of possibilities for $\mathcal{O}_C$ and the family of possible $P$s are both bounded families. Once this is established, it is clear that the family of sheaves underlying a $\pi$-stable pair is a bounded family of sheaves.

First we will consider the family of possibilities for $\mathcal{O}_C$. To show that this family is bounded, we will show:

1. the Hilbert polynomials of this family take only a finite number of values; and
2. there exists a single sheaf that surjects onto each member of this family.

The second requirement is trivially satisfied, since each member of this family is the structure sheaf of a subscheme of $Y$, and hence, admits a surjective map from $\mathcal{O}_Y$. It remains to find upper and lower bounds for the coefficients of the Hilbert polynomial of a general element from this family.

In contrast to $PT$-theory, the support of $\mathcal{O}_C$ is not equal to the support of $G$, since $P$ is not necessarily zero-dimensional. However, we still have $\beta_G = \beta_C + \beta_P$, where all $\beta$ are effective. We know that there are only finitely many decompositions of $\beta_G$ into the sum of two effective curve classes. This forces an upper and lower bound on the linear coefficient of the Hilbert polynomial of $\mathcal{O}_C$. It remains to find upper and lower bounds for the Euler characteristic of $\mathcal{O}_C$ (the constant coefficient of the Hilbert polynomial).

The Leray spectral sequence proves that $\chi(P) = \chi(R^*_\pi P) \geq 0$, the inequality following from the fact that $R^*_\pi P$ is zero dimensional. Now, $\chi(G) = \chi(P) + \chi(\mathcal{O}_C)$, and $\chi(P) \geq 0$ implies $\chi(\mathcal{O}_C) \leq \chi(G)$. This gives us an upper bound, since the Chern character, and hence, the Euler characteristic, of $G$ is fixed. For the lower bound, let $\alpha_G = (\beta_G, n_G)$ be the Chern character of $G$, and let $\alpha_C = (\beta_C, n_C)$ be the Chern character of $C$.

In general, if $\text{Hilb}^{(\beta, n)}$ is non-empty (say one of its points represents a curve $J$), then $\dim \text{Hilb}^{(\beta, n+k)} \geq 3k$ since we get a $3k$-dimensional space of curves coming from the curve $J$ with $k$ "wandering points." This line of reasoning tells us that
$$\dim \text{Hilb}^{(\beta_C, n_G)} \geq 3(n_G - n_C).$$

Rearranging this yields
$$n_C \geq n_G - \frac{1}{3} \dim \text{Hilb}^{(\beta_C, n_G)}.$$

This gives us a lower bound for $n_C$, which completes the proof that the corresponding family is bounded.

Now to show that the family of cokernels is bounded, we will show:

1. the Hilbert polynomials of this family take on only a finite number of values; and
2. there is a common upper-bound to the index of regularity.

Using the Leray spectral sequence again, we note that for all $P \in \mathcal{P},$
\[ H^1(Y, P) = H^1(X, \pi_* P) = 0, \]
thus, all $P \in \mathcal{P}$ are 1-regular. To show this family is bounded, it remains to find upper and lower bounds for the coefficients of the Hilbert polynomial of a general object.

As above, there are only a finite number of options for the support curve of $P$. This yields upper and lower bounds on the linear coefficient of the Hilbert polynomial.

We know that $\chi(G) = \chi(P) + \chi(O_C)$. Since $\chi(O_C)$ is bounded, and $\chi(G)$ is fixed, so too must $\chi(P)$ be bounded.

This completes the proof that the sheaves underlying a $\pi$-stable pair of fixed $K$-class forms a bounded family of sheaves. ■

**Lemma 47.** The family of $\mu_\pi$-semistable sheaves with fixed chern character $(0, 0, \beta, n)$ is bounded.

**Proof:** For a sheaf $G$ of dimension one, we use the notation $\beta_G$ to denote the corresponding curve class and we let
\[ \mu_N(G) = \frac{\chi(G)}{N \cdot \beta_G} \in (\infty, \infty) \]
be the $N$-slope, for any $\mathbb{Q}$-divisor $N$. Note that $\mu_\pi$ still denotes the $\pi$-slope so that in this notation
\[ \mu_\pi(G) = (\mu_{\bar{H}}(G), \mu_L(G)) \in (\infty, \infty] \times (\infty, \infty]. \]

We will construct an ample divisor $A_\epsilon$ such that every $\pi$-semistable sheaf $F$ of chern character $(0, 0, \beta, n)$ is either $\mu_L$-semistable or $\mu_{A_\epsilon}$-semistable. The lemma will then follow since for any ample divisor $N$, the family of $\mu_N$-semistable sheaves of fixed chern classes form a bounded family [24, Thm 3.3.7].

Let $F$ be a $\mu_\pi$-semistable sheaf with $ch(F) = (0, 0, \beta, n)$. We may assume that $\bar{H} \cdot \beta > 0$ since if $\bar{H} \cdot \beta = 0$, then the $\mu_\pi$-semistability of $F$ implies $\mu_L$-semistability and we are done.

We construct our ample $A_\epsilon$ as follows. Let $A$ be an ample $\mathbb{Q}$-divisor with $A \cdot \beta = \bar{H} \cdot \beta$ and let
\[ A_\epsilon = (1 - \epsilon)\bar{H} + \epsilon A. \]

Since $\bar{H} = \pi^*(H)$ is in the boundary of the nef cone and $A$ is ample, $A_\epsilon$ is ample for any $\epsilon \in \mathbb{Q} \cap (0, 1)$. We note that $A_\epsilon \cdot \beta = \bar{H} \cdot \beta$ for all $\epsilon$. We will choose an appropriate $\epsilon$ below.

Since there are a finite number of decompositions $\beta = \beta_1 + \beta_2$ with $\beta_i$ effective [12, Lemma 2.1], the set
\[ \{\bar{H} \cdot \beta_K\}_{K \in F} \]
is finite. By $\mu_\pi$-semistability, we know that
\[ \mu_{\bar{H}}(K) \leq \mu_{\bar{H}}(F) \]
for all $K \subset F$. Since the set $\{H \cdot \beta_K\}_{K \subset F}$ is finite, there exists some $\delta > 0$ such that

$$\mu_{\tilde{H}}(K) + \delta < \mu_{\tilde{H}}(F)$$

for all $K \subset F$ such that $\mu_{\tilde{H}}(K) < \mu_{\tilde{H}}(F)$. In other words, if the $\tilde{H}$-slope of a subsheaf $K \subset F$ is strictly less than the $\tilde{H}$-slope of $F$, then it is bounded away from the $\tilde{H}$-slope of $F$ by $\delta$, a number independent of $K$ and $F$ (but depending on $\beta$ and $n$). If this were not the case, there would have to be an infinite number of possible denominators in $\mu_{\tilde{H}}(K) = \chi(K)/(H \cdot \beta_K)$ which is not true.

We now choose $\epsilon > 0$ small enough so that

$$\epsilon \cdot \mu_{\tilde{H}}(F) \cdot \left(1 - \frac{A \cdot \beta_K}{H \cdot \beta_K}\right) < \delta$$

for all $K \subset F$ with $\mu_{\tilde{H}}(K) < \mu_{\tilde{H}}(F)$. Then for all such $K \subset F$ we get

$$\epsilon \cdot \mu_{\tilde{H}}(F) \cdot \left(1 - \frac{A \cdot \beta_K}{H \cdot \beta_K}\right) + \mu_{\tilde{H}}(K) < \delta + \mu_{\tilde{H}}(K) < \mu_{\tilde{H}}(F)$$

which implies

$$\frac{\epsilon \chi(F)}{H \cdot \beta} = \frac{\epsilon \chi(F) (A \cdot \beta_K)}{(H \cdot \beta)(H \cdot \beta_K)} + \frac{\chi(K)}{H \cdot \beta_K} \chi(F) \cdot \frac{A \cdot \beta_K}{H \cdot \beta_K}.$$

Clearing denominators and rearranging, we get

$$\chi(K) \cdot \tilde{H} \cdot \beta < \chi(F) (\epsilon A \cdot \beta_K + (1 - \epsilon) \tilde{H} \cdot \beta_K) = \chi(F) (A \cdot \beta_K).$$

Using the fact that $A \cdot \beta = \tilde{H} \cdot \beta$ the above implies

$$\frac{\chi(K)}{A \cdot \beta_K} < \frac{\chi(F)}{A \cdot \beta}$$

So we’ve proved that

$$\mu_{A_e}(K) < \mu_{A_e}(F)$$

for all $K \subset F$ with $\mu_{\tilde{H}}(K) < \mu_{\tilde{H}}(F)$.

This is now enough to prove our claim: if $F$ is $\mu_e$-semistable with $\text{ch}(F) = (0, 0, \beta, n)$, then either $F$ is $\mu_{A_e}$-semistable or $\mu_L$-semistable, for if not, then there exists $K \subset F$ such that $\mu_{A_e}(K) > \mu_{A_e}(F)$ and $\mu_L(K) > \mu_L(F)$. But then $\mu_e(K) \leq \mu_e(F)$ implies $\mu_{\tilde{H}}(K) < \mu_{\tilde{H}}(F)$ which then by construction implies $\mu_{A_e}(K) < \mu_{A_e}(F)$ which is a contradiction.

Thus the family of $\mu_e$-semistable sheaves of Chern character $(0, 0, \beta, n)$ is contained in the union of the families of $\mu_{A_e}$-semistable and $\mu_L$-semistable sheaves of Chern character $(0, 0, \beta, n)$ and is thus bounded.
4 The torsion pair and the stability condition

In this section, we show that \( P_\pi \) may be conveniently expressed in terms of the stability condition, and similarly for \( Q_\pi \). First we give a rapid introduction to the modern Harder-Narasimhan property, a generalization of the Harder-Narasimhan filtration of [23].

Definition 48. A weak stability condition \((T, \tau, \leq)\) on \( \mathcal{C} \) is said to have the Harder-Narasimhan property if for every sheaf \( G \), there exists a unique filtration of \( G 

\begin{align*}
0 = HN_\tau(G)_0 &\subset HN_\tau(G)_1 \subset \cdots \subset HN_\tau(G)_{N-1} \subset HN_\tau(G)_{N} = G
\end{align*}

(where the inclusions are strict) such that the quotients

\[ Q_i = HN_\tau(G)_i/HN_\tau(G)_{i-1} \]

are \( \tau \)-semistable and

\[ \tau(Q_i) > \tau(Q_{i+1}) \]

for all \( i > 0 \). When it is clear from the context, most of the notation will be suppressed, and we will denote the Harder-Narasimhan filtration of \( G \) with respect to \( \tau \) by \( 0 \subset G_1 \subset G_2 \subset \cdots \subset G_{N-1} \subset G_N = G \). The \( G_i \) are called the filtered objects of the Harder-Narasimhan filtration, and the \( Q_i \) are called the quotient objects.

We borrow the following definition and theorem from Joyce [27]. In [25, §9], Joyce proves that the category of coherent sheaves satisfies assumptions 3.7 of [27]. This is enough for us to conclude that the assumptions are also true of \( \mathcal{C} \) the category of coherent sheaves supported in dimension one or less.

Theorem 49 ([27], Theorem 4.4). Let \((T, \tau, \leq)\) be a weak stability condition on an abelian category \( \mathcal{A} \). If \( \mathcal{A} \) is Noetherian and \( \tau \)-artinian, then \((T, \tau, \leq)\) has the Harder-Narasimhan property.

Corollary 50. The weak stability condition \( \mu_\pi \) on \( \mathcal{C} \) has the Harder-Narasimhan property.

Proof: The category \( \mathcal{C} \) is Noetherian because it is a subcategory of the category of coherent sheaves, which is Noetherian. Corollary 40 proves that \( \mathcal{C} \) is \( \mu_\pi \)-artinian.

When we refer to the Harder-Narasimhan filtration in what follows, we will always be referring to the filtration with respect to the stability condition \( \mu_\pi \).

We present some notation before we state and prove the main result of this section. Recall that our slope function \( \mu_\pi \) takes values in the lexicographically ordered set \(( -\infty, +\infty ] \times ( -\infty, +\infty ] \). To avoid awkwardly writing the ordered pairs \(( +\infty, +\infty )\) and \(( +\infty, 0 )\) through-out, let us denote

\[ \infty : = ( +\infty, +\infty ) , \]

and

\[ \frac{\infty}{2} : = ( +\infty, 0 ) . \]
Given an interval $I \subset (-\infty, +\infty] \times (-\infty, +\infty]$, we define $SS(I) \subset C$ to be the full subcategory of zero objects together with those one-dimensional sheaves whose Harder-Narasimhan quotients have $\mu_\pi$-value in the interval $I$. If $a, b \in (-\infty, +\infty] \times (-\infty, +\infty]$ such that $a < b$, then we denote the closed interval between $a$ and $b$ by $a \leq \square \leq b$, and similarly for open, half-open, etc. intervals.

**Lemma 51.**

$$P_\pi = SS(\square \geq \frac{\infty}{2}),$$

and

$$Q_\pi = SS(\square < \frac{\infty}{2}).$$

**Proof:** First we will show that $P_\pi \subset SS(\square \geq \frac{\infty}{2})$.

Case 1: Let $P \in P_\pi$ be semi-stable. We will show that $P \in SS(\square \geq \frac{\infty}{2})$. Since $P$ is semi-stable, it suffices to show that $\mu_\pi(P) \geq \frac{\infty}{2}$. Now $P \in P_\pi$ implies that $\chi(P) \geq 0$. By ampleness of $L$, we know $\beta_P \cdot L \geq 0$. Hence $\mu_\pi(P) \geq (+\infty, 0)$.

Case 2: Let $P \in \mathcal{P}_s$ be general, let the following be its Harder-Narasimhan filtration,

$$0 = P_0 \subset P_1 \subset \cdots \subset P_{N-1} \subset P_N = P,$$

and let $Q_i = \frac{P_{i-1}}{P_{i-2}}$ be the $i$th quotient; we must show that $\mu_\pi(Q_i) \geq \frac{\infty}{2}$. Since $P_N = P' \in \mathcal{P}_s$ and $\mathcal{P}_s$ is closed under quotients, it follows that $Q_N \in \mathcal{P}_s$. By definition of the HN filtration, $Q_N$ is semi-stable, hence $\mu_\pi(Q_N) \geq \frac{\infty}{2}$ and $Q_N \in SS(\square \geq \frac{\infty}{2})$ by the previous case. Another defining property is that $\mu_\pi(Q_1) > \mu_\pi(Q_2) > \cdots > \mu_\pi(Q_N)$. Hence, $\mu_\pi(Q_i) \geq \frac{\infty}{2}$ for all $i$, in other words, $P \in SS(\square \geq \frac{\infty}{2})$.

Now we will show that $SS(\square \geq \frac{\infty}{2}) \subset \mathcal{P}_s$.

Case 1: Let $G \in SS(\square \geq \frac{\infty}{2})$ be semi-stable. In this case, $G \in SS(\square \geq \frac{\infty}{2})$ implies that $\mu_\pi(G) \geq \frac{\infty}{2}$. Since $(\mathcal{P}_s, \mathcal{Q}_s)$ is a torsion pair, for every sheaf there exists a uniquely associated short exact sequence,

$$0 \rightarrow A \rightarrow G \rightarrow B \rightarrow 0$$

where $A \in \mathcal{P}_s$ and $B \in \mathcal{Q}_s$. By the above, we know that $A \in SS(\square \geq \frac{\infty}{2})$. By the semi-stability of $G$, $\mu_\pi(B) \geq \mu_\pi(G) \geq \frac{\infty}{2}$ and hence we have $\chi(B) \geq 0$.

Notice that $G$ must be supported on a fibre of $\pi$ because $\mu_\pi(G) \in \{+\infty\} \times (0, +\infty]$. Hence $B$ is also supported on a fibre. We claim that this forces $H^0(B) = 0$. For suppose there was a non-zero map $O_Y \rightarrow B$. This would yield non-trivial $0 \rightarrow O_C \rightarrow B$ where $C$ is the support of the map $O_Y \rightarrow B$. From the proof of proposition 18, we know that $R^1\pi_*O_C = 0$, which implies that $O_C \in \mathcal{P}_s$ which contradicts the definition of $Q_s$. Hence $H^0(B) = 0$.

However, $\chi(B) \geq 0$ so $\dim H^1(B) \leq 0$. This implies that $H^1(B) = 0$, which implies that $R^1\pi_*B = 0$ (by the theorem of cohomology and base-change) and hence $B \in \mathcal{P}_s$. Since $B \in \mathcal{Q}_s$ we conclude that $B = 0$ and $G = A \in \mathcal{P}_s$.

Case 2: Let $G \in SS(\square \geq \frac{\infty}{2})$ be general. We need to show that $G \in \mathcal{P}_s$. Let

$$0 = G_0 \subset G_1 \subset \cdots \subset G_{N-1} \subset G_N = G$$
be the HN filtration of \( G \), and let \( Q_i = \frac{G_i}{G_{i-1}} \) denote the corresponding semistable quotients. By assumption, \( \mu_\pi(Q_i) \geq \frac{\infty}{2} \); notice that \( G_1 = Q_1 \), so we have that \( G_1 \) is semistable and \( \mu_\pi(G_1) \geq \frac{\infty}{2} \). The previous case then proves that for all \( i \), \( Q_i \in P_\pi \), and since \( P_\pi \) is closed under extensions, we see that \( G \in P_\pi \).

This completes the proof that \( P_\pi = \text{SS}(\square \geq \frac{\infty}{2}) \).

The pairs \( (P_\pi, Q_\pi) \) and \( (\text{SS}(\square \geq \frac{\infty}{2}), \text{SS}(\square < \frac{\infty}{2})) \) both form torsion pairs; the former we proved in Lemma 13, the later because \( \mu_\pi \) is a stability condition. Since any torsion pair is completely determined by its torsion part, \( Q_\pi = \text{SS}(\square < \frac{\infty}{2}) \) follows from \( P_\pi = \text{SS}(\square \geq \frac{\infty}{2}) \) and the lemma is proved.

\[ \blacksquare \]

5 The motivic Hall algebra

Here we provide a quick summary of the constructions and results of Bridgeland’s papers [11], [12] (which came into existence as a gentle introduction to part of Joyce’s theory of motivic Hall algebras [25], [26], [27], [28]).

Let \( S \) be a stack, locally of finite type over \( \mathbb{C} \) and with affine stabilizers.

**Definition 52.** The relative Grothendieck group \( K(\text{St}/S) \) of stacks over \( S \) is the \( \mathbb{Q} \)-vector space spanned by symbols

\[
\left[ T \overset{m}{\rightarrow} S \right]
\]

(where \( T \) is a finite-type stack and \( m \) is a morphism), subject to the following relations.

a) \( [T \rightarrow S] = [U \rightarrow S] + [F \rightarrow S] \) where \( U \) is an open substack of \( T \) and \( F \) is the corresponding closed complement.

b) \( [T_1 \overset{\alpha}{\rightarrow} S] = [T_2 \overset{\alpha}{\rightarrow} S] \), if \( T_1 \overset{f}{\rightarrow} B \) and \( T_2 \overset{g}{\rightarrow} B \) are Zariski fibrations over \( B \) with identical fibres, and \( B \overset{\alpha}{\rightarrow} S \) is a morphism of stacks.

c) \( [T \overset{\alpha}{\rightarrow} S] = [T' \overset{\beta}{\rightarrow} S] \) if there exists a commutative diagram

\[
\begin{array}{c}
T \\
\alpha \downarrow \quad \beta \\
S \\
\end{array} \quad \begin{array}{c} \text{s} \end{array} \quad \begin{array}{c} \text{such that the associated map on C-points T(\mathbb{C}) \overset{\text{\circ}}{\rightarrow} T'(\mathbb{C}) is an equivalence of categories.} \end{array}
\]

\( ^1 \text{a Zariski-local product space} \)
The vector space $K(\text{St} / S)$ is a $K(\text{St} / \text{Spec} \mathbb{C})$-module, whose action we now describe. Let $[A \to \text{Spec} \mathbb{C}] = [A] \in K(\text{St} / \text{Spec} \mathbb{C})$, and let $[T_m \to S] \in K(\text{St} / S)$. Then we define $[A] \cdot [T_m \to S] = [A \times_{\text{Spec} \mathbb{C}} T \xrightarrow{f} S]$, where $f$ is the composition of the projection $A \times_{\text{Spec} \mathbb{C}} T \to T$ and the map $T \to S$.

We are most interested in the case where $S$ is the stack of objects in $\mathcal{C}$, i.e. coherent sheaves on $Y$ supported in dimension one or less. We denote this stack by $\mathcal{M}$, and we denote $K(\text{St} / \mathcal{M})$ by $H(\mathbb{C})$. The vector space $H(\mathbb{C})$ is the motivic Hall algebra; let us justify the name by endowing it with the structure of an algebra. First, we define $\mathcal{M}^{(2)}$ to be the stack of short-exact sequence of sheaves on $\mathcal{M}$. Now, given $[A \to \mathcal{M}]$ and $[B \to \mathcal{M}]$ we define the convolution product $[A \to \mathcal{M}] \ast [B \to \mathcal{M}]$ to be $[Z \to \mathcal{M}]$ where $Z \xrightarrow{c} \mathcal{M}$ is defined by the following Cartesian diagram:

$$
\begin{array}{ccc}
Z & \xrightarrow{g} & \mathcal{M}^{(2)} & \xrightarrow{c} & \mathcal{M} \\
\downarrow & & \downarrow^{(l,r)} & & \\
A \times B & \longrightarrow & \mathcal{M} \times \mathcal{M}.
\end{array}
$$

The morphisms $l, r$ are the “left hand” and “right hand” morphisms, which project a short exact sequence to its left-most (resp. right-most) non-zero entry. The morphism $c$ is the “centre” morphism. Intuitively, given families of sheaves $A \to \mathcal{M}$ and $B \to \mathcal{M}$ their product in the Hall algebra is the family $Z \to \mathcal{M}$ parametrizing extensions of objects of $B$ by objects of $A$.

The motivic Hall algebra is useful tool. It holds enough information to allow us to retrieve Euler characteristics, yet is flexible enough to produce decompositions of elements in terms of extensions. We will describe an “integration” map on $H(\mathbb{C})$ taking values in the ring of polynomials. Equations among elements of $H(\mathbb{C})$ will be integrated to yield equations of polynomials. This entire framework will then be souped-up to incorporate Laurent series, and our theorem will be the result of applying the souped-up integration map to equations in the souped-up Hall algebra.

In [11], Bridgeland introduces regular elements. Let $K(\text{Var} / \mathbb{C})$ denote the relative Grothendieck group of varieties over $\mathbb{C}$ (cf. definition 52). Let $L$ denote the element $[A^1 \to \mathbb{C}]$, the Tate motive. Consider the maps of commutative rings,

$$K(\text{Var} / \mathbb{C}) \to K(\text{Var} / \mathbb{C})[L^{-1}] \to K(\text{St} / \mathbb{C}),$$

and recall from [11] that $H(\mathbb{C})$ is an algebra over $K(\text{St} / \mathbb{C})$. Define a $K(\text{Var} / \mathbb{C})[L^{-1}]$-module

$$H_{\text{reg}}(\mathbb{C}) \subset H(\mathbb{C})$$

to be the span of classes of maps $[V \xrightarrow{f} \mathcal{M}]$ with $V$ a variety. We call an element of $H(\mathbb{C})$ regular if it lies in this submodule. The following result is theorem 5.1 of [11].
**Theorem 53.** The submodule of regular elements is closed under the convolution product:

\[ H_{\text{reg}}(\mathcal{C}) \ast H_{\text{reg}}(\mathcal{C}) \subset H_{\text{reg}}(\mathcal{C}), \]

and is therefore a \( K(\text{Var}/\mathcal{C})[\mathbb{L}^{-1}] \)-algebra. Moreover the quotient

\[ H_n(\mathcal{C}) = H_{\text{reg}}(\mathcal{C})/(\mathbb{L} - 1) H_{\text{reg}}(\mathcal{C}), \]

is a commutative \( K(\text{Var}/\mathcal{C}) \)-algebra.

Bridgeland equips \( H_n(\mathcal{C}) \) with a Poisson bracket, defined by

\[ \{f, g\} = \frac{f \ast g - g \ast f}{\mathbb{L} - 1}. \]

The integration map \( I \) is defined on \( H_n(\mathcal{C}) \). Now we work toward the polynomial ring in which it takes values.

Recall that \( K(Y) \) is the numerical \( K \)-theory of \( Y \). Recall \( \Delta \subset F_1K(Y) \) is the effective cone of \( F_1K(Y) \), that is, the collection of elements of the form \( [F] \) where \( F \) is a one-dimensional sheaf. Define a ring \( \mathbb{C}[\Gamma] \) to be the vector space spanned by symbols \( x^\alpha \) for \( \alpha \in \Delta \) and defining the multiplication by

\[ x^\alpha \cdot x^\beta = x^{\alpha + \beta}. \]

We equip \( \mathbb{C}[\Delta] \) with the trivial Poisson bracket. We are now ready for the following theorem.

**Theorem 54 (5.1 of [12]).** There exists a Poisson algebra homomorphism

\[ I : H_n(\mathcal{C}) \to \mathbb{C}[\Delta] \]

such that

\[ I\left( [Z, f] \mapsto M_\alpha \right) = \chi(Z, f^*(\nu))x^\alpha, \]

where \( \nu : \mathcal{M} \to \mathbb{Z} \) is Behrend’s microlocal function of \( \mathcal{M} \), and \( M_\alpha \) denotes the component of \( \mathcal{M} \) with fixed Chern character \( \alpha \).

### 6 Equations in the infinite-type Hall algebra and the fake proof

For the sake of exposition only, we follow [12] and [17] by introducing an infinite-type version of the Hall algebra. This has the benefit of allowing non-finite-type stacks, but the devastating draw-back of not admitting an integration map. We use it because it will allow us to temporarily work without having to think about convergence of power series. Also, many of the arguments will be used again later. We end this chapter with a fake proof of our main result. It is our hope that this fake proof helps the reader to navigate the true one in the following chapter.

The **infinite-type Hall algebra** is defined by considering symbols as in definition 52, but with \( T \) assumed only to be locally of finite type over \( \mathbb{C} \), and use relations as before, except that we do not use relation (a). (Admitting relation (a) in this case would make every infinite-type Hall algebra trivial). We denote it by \( H_\infty(\mathcal{C}) \).
Given a substack $N \subset M$, we let

$$1_N = [N \xrightarrow{i} M]$$

denote the inclusion $i : N \to M$. Pulling back the morphism (20) to $N \subset M$ gives a stack denoted $N(O)$ with a morphism $N(O) \xrightarrow{s} N$, and hence an element

$$1_N^O = [N(O) \xrightarrow{s} M] \in H_\infty(C).$$

For example, $P_\pi$ and $Q_\pi$ are full subcategories of $C$, and define substacks of $M$, which we abusively denote with the same letters, $P_\pi, Q_\pi \subset M$. These substacks define elements of the infinite-type Hall algebra,

$$1_{P_\pi}, 1_{Q_\pi} \in H_\infty(C).$$

Other examples include

$$\mathcal{H} = [\text{Hilb}(Y) \to M],$$

$$\mathcal{H}_{exc} = [\text{Hilb}_{exc} \to M],$$

and

$$\mathcal{H}^\pi = [\pi\text{-Hilb}(Y) \to M] \in H_\infty(C),$$

where $\text{Hilb}_{exc}$ denotes the Hilbert scheme of curves supported on fibres of $\pi$, and the map to $M$ is given by taking $O_Y \to G$ to $G$. Note that all Hilbert schemes are restricted to the components parametrizing sheaves $G$ of dimension one.

**Lemma 55.**

$$1_C = 1_{P_\pi} \ast 1_{Q_\pi}$$

This lemma reflects the fact that $(P_\pi, Q_\pi)$ is a torsion pair.

**Proof:** Form the following Cartesian diagram:

$$
\begin{array}{ccc}
Z & \xrightarrow{f} & M^{(2)} & \xrightarrow{b} & M \\
\downarrow & & \downarrow \scriptstyle{(a_1,a_2)} & & \\
\mathcal{P}_\pi \times \mathcal{Q}_\pi & \xrightarrow{i} & M \times M.
\end{array}
$$

(56)

By lemma A.1 [11], the groupoid of $T$-valued points of $Z$ can be described as follows. The objects are short exact sequences of $T$-flat sheaves on $T \times Y$ of the form

$$0 \to A \to G \to B \to 0$$

such that $A$ and $B$ define families of sheaves on $Y$ lying in the subcategories $\mathcal{P}_\pi$ and $\mathcal{Q}_\pi$ respectively. The morphisms are isomorphisms of short exact sequences. The composition,

$$g = b \circ f : Z \to M$$

sends a short exact sequence to the object $G$. Since $\mathcal{P}_\pi$ and $\mathcal{Q}_\pi$ are subcategories of $C$, it follows that the composition factors through $C$. This morphism induces an equivalence on $C$-valued points because of the torsion pair property: every object $G$ of $C$ fits into a unique short exact
sequence of the form \((56)\). Thus, the identity follows from the relations in the infinite Hall algebra.

We will need a framed version of the previous lemma.

**Lemma 57.**

\[ 1^C_O = 1^C_{\mathcal{P}e} \ast 1^C_{\mathcal{Q}e}. \]

**Proof:** Form the following Cartesian diagram:

\[
\begin{array}{ccc}
U & \xrightarrow{p} & V \\
\downarrow & & \downarrow \\
\mathcal{P}_e(O) \times \mathcal{Q}_e(O) & \xrightarrow{p \times q} & \mathcal{M} \times \mathcal{M}.
\end{array}
\]

Then \(1^C_{\mathcal{P}e} \ast 1^C_{\mathcal{Q}e}\) is represented by the composite map \(b \circ j \circ p : U \to \mathcal{M}\). Since \(R^1 \pi_* P = 0\) for all \(P \in \mathcal{P}_e\), the argument of lemma 2.5 in [12] implies that the map \(\mathcal{P}_e(O) \to \mathcal{P}_e\) is a Zariski fibration with fibre \(H^0(P)\) over a point \(P \in \mathcal{P}_e\). By pullback, the same is true of the morphism \(p\).

The groupoid of \(T\)-valued points of \(V\) can be described as follows. The objects are short exact sequences of \(T\)-flat sheaves on \(T \times Y\) of the form

\[ 0 \to P \to G \to B \to 0 \]

such that \(P\) and \(B\) define flat families of objects in \(\mathcal{P}_e\) and \(\mathcal{Q}_e\) respectively, together with a map \(O_{T \times Y} \to B\). We represent the objects of \(V\) as diagrams of the form:

\[
\begin{array}{ccc}
O_{T \times Y} & \downarrow \\
0 & \xrightarrow{\delta} & P \xrightarrow{} G \xrightarrow{} B \xrightarrow{} 0.
\end{array}
\]

Consider the stack \(Z\) from lemma 55 with its map \(Z \to \mathcal{M}\). Form the diagram

\[
\begin{array}{ccc}
W \xrightarrow{h} & \mathcal{M}(O) \\
\downarrow & \downarrow \\
Z \xrightarrow{g} & \mathcal{M}.
\end{array}
\]

Since \(g\) induces an equivalence on \(\mathbb{C}\)-valued points, so does \(h\), so that the element \(1^C_O\) can be represented by the map \(g \circ h\).

We represent the objects of \(W\) as diagrams of the form:

\[
\begin{array}{ccc}
O_{T \times Y} & \downarrow \\
0 & \xrightarrow{\gamma} & P \xrightarrow{} G \xrightarrow{} B \xrightarrow{} 0.
\end{array}
\]

Setting \(\gamma = \beta \circ \delta\) defines a map of stacks \(W \to V\), which is a Zariski fibration with fibre an affine model of the vector space \(H^0(P)\) over a sheaf \(P\).

Now, since \(H^1(P) = 0\), \(U \to V\) is a Zariski fibration with fibre \(H^0(P)\), hence they represent the same element of the Hall algebra, namely \(1^C_O\).
Lemma 58.

\[ 1_C^O = H \ast 1_C \]

This is lemma 4.3 of [12]. Intuitively, this amounts to the fact that every map \( O \xrightarrow{\gamma} G \) factors uniquely into a surjection \( O \twoheadrightarrow \text{im}(\gamma) \) and an inclusion \( \text{im}(\gamma) \hookrightarrow G \). The following lemma is a restriction of the previous to the substack \( \mathcal{P}_\pi \).

Lemma 59.

\[ 1_{\mathcal{P}_\pi}^O = H_{\text{exc}} \ast 1_{\mathcal{P}_\pi} \]

**Proof:** Form the Cartesian diagram:

\[
\begin{array}{ccc}
Z & \longrightarrow & \mathcal{M}^{(2)} \\
\downarrow & & \downarrow^{(a_1,a_2)} \\
H_{\text{exc}} \times \mathcal{P}_\pi & \longrightarrow & \mathcal{M} \times \mathcal{M}.
\end{array}
\]

The groupoid of \( T \)-valued points of \( Z \) may be described as follows. The objects are short exact sequences of \( T \)-flat sheaves of \( T \times Y \):

\[ 0 \rightarrow A \rightarrow G \rightarrow B \rightarrow 0 \]

such that for all geometric points \( t \in T \), \( B_t \in \mathcal{P}_\pi \), and \( A_t \) is supported on exceptional fibres, together with an epimorphism \( \mathcal{O}_Y \rightarrow A_t \). We can represent these objects as diagrams of the form

\[
\begin{array}{ccc}
\mathcal{O}_{T \times Y} & \rightarrow & 0 \\
\downarrow_{\gamma} & & \downarrow \\
A & \longrightarrow & G \\
\alpha & \rightarrow & B.
\end{array}
\]

Let \( t \in T \) be an arbitrary geometric point. Since \( \mathcal{O}_Y \rightarrow A_t \) is an epimorphism, we know that \( A_t \) is of the form \( \mathcal{O}_{C_t} \) for some one-dimensional subscheme \( C_t \) of \( Y \). By the proof of proposition 18, we know that \( R\pi_* \mathcal{O}_{C_t} = 0 \). Since \( A_t = \mathcal{O}_{C_t} \) has exceptional support, it follows that \( \pi_* A_t \) is a zero-dimensional sheaf, hence \( A_t \in \mathcal{P}_\pi \). Since \( B_t \in \mathcal{P}_\pi \) by design, and since \( \mathcal{P}_\pi \) is closed under taking extensions, we conclude that in any such short exact sequence, \( G_t \in \mathcal{P}_\pi \). There is a map \( h : Z \rightarrow \mathcal{P}_\pi(\mathcal{O}) \) sending the above diagram to the composite map

\[ \delta = \alpha \circ \gamma : \mathcal{O}_{T \times Y} \rightarrow G. \]

This morphism \( h \) fits into a commuting diagram of stacks

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & \mathcal{P}_\pi(\mathcal{O}) \\
\downarrow^{b \circ f} & & \downarrow^{q} \\
\mathcal{M} & & \mathcal{M}
\end{array}
\]

We argue that the map \( h \) then induces an equivalence on \( \mathbb{C} \)-valued points. Suppose \( \mathcal{O}_{T \times Y} \xrightarrow{\delta} G \) is an arbitrary map of sheaves, with \( G \) defining a family of sheaves in \( \mathcal{P}_\pi \). Then we get the following diagram:
\[\mathcal{O}_{T \times Y} \xrightarrow{\gamma} 0 \rightarrow \text{im}(\delta) \rightarrow G \rightarrow \text{coker}(\delta) \rightarrow 0.\]

Since \(G_t \in \mathcal{P}_\pi\) we know that the one-dimensional component of its support is exceptional, hence \(\text{im}(\delta)|_t\) is also exceptional, so that \(\mathcal{O}_{T \times Y} \rightarrow \text{im}(\delta)\) defines a family of objects in \(\text{Hilb}_{\text{exc}}\). As well, we know that \(\mathcal{P}_\pi\) is closed under taking quotients, so \(\text{coker}(\delta)\) is in \(\mathcal{P}_\pi\). This completes the proof. 

Morally, the next lemma is similar to lemma 58 since \(\mathcal{H}^r\) may be thought of as the surjections \(\mathcal{O}_Y \rightarrow G\) in a tilt of the abelian category generated by \(\mathcal{O}\) and \(\mathcal{C}\). We provide a direct proof since we have not constructed this tilt.

**Lemma 60.**

\[1^O_{Q_\pi} = \mathcal{H}^r * 1_{Q_\pi}.\]

**Proof:** Form the following Cartesian diagram:

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & \mathcal{M}^{(2)} \xrightarrow{h} \mathcal{M} \\
\downarrow & & \downarrow \begin{pmatrix} \alpha_1 \circ \alpha_2 \end{pmatrix} \\
\mathcal{H}^r \times Q_\pi & \xrightarrow{i} & \mathcal{M} \times \mathcal{M}
\end{array}
\]

The groupoid of \(T\)-valued points of \(Z\) is described as follows. The objects are short exact sequences of \(T\)-flat sheaves on \(T \times Y\)

\[0 \rightarrow A \rightarrow G \rightarrow B \rightarrow 0\]

with the property that \(B \in Q_\pi\), together with a map \(\mathcal{O}_{T \times Y} \rightarrow A\) that pulls back to a \(\pi\)-stable pair \(\mathcal{O}_Y \rightarrow A_t\) for every \(t \in T\). We can represent these objects as diagrams of the form:

\[\mathcal{O}_{T \times Y} \xrightarrow{\gamma} 0 \rightarrow A \xrightarrow{\alpha} G \xrightarrow{\beta} B \rightarrow 0.\]  

(61)

Since \(A\) and \(B\) are objects of \(Q_\pi\), and \(Q_\pi\) is closed under extensions, we conclude \(G \in Q_\pi\). Thus, there is a map \(h: Z \rightarrow Q_\pi(\mathcal{O})\) sending the above diagram to the composite map

\[\sigma = \alpha \circ \gamma: \mathcal{O}_{T \times Y} \rightarrow G.\]

This map \(h\) fits into a commuting diagram of stacks

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & Q_\pi(\mathcal{O}) \\
\downarrow b \circ f & & \downarrow q \\
\mathcal{M} & \xrightarrow{q} & 
\end{array}
\]

(62)

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The map \( h \) then induces an equivalence on \( \mathbb{C} \)-valued points because of the following argument. Let \( \mathcal{O}_Y \to G \) be an arbitrary map, with \( G \in \mathcal{Q}_s \). We need to produce a diagram,

\[
\begin{align*}
\mathcal{O}_Y \\
\downarrow \gamma \\
0 & \longrightarrow A \quad \alpha \longrightarrow G \quad \beta \longrightarrow B \longrightarrow 0,
\end{align*}
\]

with \( \mathcal{O}_Y \to A \) a \( \pi \)-stable pair, \( B \in \mathcal{Q}_s \), and \( \alpha \circ \gamma = \sigma \).

Consider the cokernel \( K \) of \( \sigma \). Since \( (\mathcal{P}_s, \mathcal{Q}_s) \) is a torsion pair, we know that \( K \) fits into a short exact sequence

\[
0 \to P \to K \to Q \to 0
\]

where \( P \in \mathcal{P}_s \) and \( Q \in \mathcal{Q}_s \). Let \( G \to d \to K \) be the canonical map from \( G \) to the cokernel \( K \) of \( \sigma \), and let \( G \to d \to Q \) be the composition of \( G \to K \) and \( K \to Q \). Define \( A \) to be the kernel of \( d \). Consider the following diagram.

\[
\begin{align*}
0 & \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y \longrightarrow 0 \\
\downarrow \psi & \quad \downarrow \sigma \\
0 & \longrightarrow A \quad G \quad \downarrow c \quad Q \longrightarrow 0 \\
\downarrow \psi & \quad \downarrow d \\
0 & \longrightarrow P \quad K \quad \downarrow e \quad Q \longrightarrow 0 \\
\downarrow \psi & \quad \downarrow 0 \\
0 & \longrightarrow 0 \\
\end{align*}
\]

We know that \( Q \in \mathcal{Q}_s \), and a diagram chase proves that the dotted vertical morphisms exist and that \( P \) is the cokernel of \( \mathcal{O}_Y \to A \). The sheaf \( A \) is a subsheaf of \( G \in \mathcal{Q}_s \), and \( \mathcal{Q}_s \) is closed under taking subsheaves, so \( A \in \mathcal{Q}_s \). This proves that \( \mathcal{O}_Y \to A \) is a \( \pi \)-stable pair, and thus \( h \) is a surjection on \( \mathbb{C} \)-valued points. Moreover, the above diagram is uniquely determined up to isomorphism (since the exact sequence \( P \to K \to Q \) is unique up to isomorphism) and consequently the preimage of \( \mathcal{O}_Y \to G \) under \( h \) is unique up to isomorphism. Thus \( h \) is a geometric bijection and thus a (constructible) equivalence of stacks [11, Lemma 3.2].

We end this section by giving a fake proof of theorem 6 that depends on a fake integration map. In fact, no such integration map is known to exist, but if there was one, the proof of our theorem would be simpler. As it stands, we have a chapter dedicated to convergence issues to get around the fact that no such integration map exists on the infinite type Hall algebra. It is our hope that this fake proof will make the true one easier to follow.

**Fake proof:** Lemma 4.3 of [12] proves

\[
\mathcal{H} \ast 1_{\mathcal{C}} = 1_{\mathcal{C}}^0.
\]

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Using lemma 55, we may rewrite $\mathcal{H} \ast 1_C$:

$$\mathcal{H} \ast 1_C = \mathcal{H} \ast 1_{1_P} \ast 1_{Q_x}.$$ 

Lemma 57 allows us to write

$$1_C^O = 1_{P_x}^O \ast 1_{Q_x}^O.$$ 

Putting these together yields

$$\mathcal{H} \ast 1_{P_x} \ast 1_{Q_x} = 1_{P_x}^O \ast 1_{Q_x}^O.$$ 

Applying lemma 59,

$$\mathcal{H} \ast 1_{P_x} \ast 1_{Q_x} = \mathcal{H}_{exc} \ast 1_{P_x} \ast 1_{Q_x}^O.$$ 

From lemma 60, we have

$$1_{Q_x}^O = \mathcal{H}^{\pi} \ast 1_{Q_x}.$$ 

so we get

$$\mathcal{H} \ast 1_{P_x} \ast 1_{Q_x} = \mathcal{H}_{exc} \ast 1_{P_x} \ast \mathcal{H}^{\pi} \ast 1_{Q_x}.$$ 

Now, for reasons we will explain in the next section, $1_{P_x}$ and $1_{Q_x}$ are invertible in the Hall algebra. We may therefore cancel the copies of $1_{Q_x}$ and isolate $\mathcal{H}$.

$$\mathcal{H} = \mathcal{H}_{exc} \ast 1_{P_x} \ast \mathcal{H}^{\pi} \ast 1_{P_x}^{-1}.$$ 

The elements $\mathcal{H}, \mathcal{H}_{exc}, \mathcal{H}^{\pi}$ all lie in the subalgebra $\mathcal{H}_{reg}(C)$ since they are represented by (constructible) schemes. As we will see in the next section, conjugation by $1_{P_x}$ induces a Poisson homomorphism of $\mathcal{H}_{reg}(C)$ of the form: identity + terms expressed in the Poisson bracket. Since the Poisson bracket of the polynomial ring is trivial, these terms vanish when we apply the fake integration map, and we are left with

$$I(\mathcal{H}) = I(\mathcal{H}_{exc}) \cdot I(\mathcal{H}^{\pi})$$

or equivalently,

$$\frac{I(\mathcal{H})}{I(\mathcal{H}_{exc})} = I(\mathcal{H}^{\pi}).$$

Up to signs arising from lemma 26, the “polynomials” $I(\mathcal{H}), I(\mathcal{H}_{exc}),$ and $I(\mathcal{H}^{\pi})$ are the generating series of $DT(Y), DT_{exc}(Y),$ and $\pi$-PT$(Y)$, respectively, and we see that the above equation is the formula claimed in theorem 6.

The true proof will follow precisely these steps, fully justified, and with the appropriate convergence arguments. The next chapter describes the Laurent Hall algebra, which does have an integration map.
7 Equations in the Laurent Hall algebra and the true proof

Laurent subsets

In this section, we will formally modify the algebra $H(\mathbb{C})$ and its integration map so that the modified integration map on the modified algebra takes valued in power series. This section is a summary of sections 5.2 and 5.3 of [12].

**Definition 64.** A subset $S \subset \Delta$ is Laurent if for all $\beta \in N_1(Y)$, the collection of elements of the form $(\beta, n) \in S$ is such that $n$ is bounded below.

Let $\Phi$ denote the set of all Laurent subsets. It has the following properties:

1. if $S, T \in \Phi$ then so it $S + T = \{\alpha + \beta : \alpha \in S, \beta \in T\}$
2. if $S, T \in \Phi$, and $\alpha \in \Delta$ then there are only finitely many ways to write $\alpha = \beta + \gamma$ such that $\beta \in S$ and $\gamma \in T$.

Given a ring $A$ graded by $\Delta$, $A = \bigoplus_{\gamma \in \Delta} A_\gamma$, we can use the Laurent subset to define a new algebra, which we will denote $A_\Phi$. Elements of $A_\Phi$ are of the form

$$a = \sum_{\gamma \in S} a_\gamma$$

where $S \in \Phi$, and $a_\gamma \in A_\gamma \subset A$. Given an element $a \in A_\Phi$ as above, we define $\rho_\gamma(a) = a_\gamma \in A$. (Here, our notation differs from [12], since we are using the symbol $\pi$ for the map $\pi : Y \to X$.) The projection operator $\rho$ allows us to define a product $*$ on $A_\Phi$ by:

$$\rho_\gamma(a * b) = \sum_{\gamma = \alpha + \beta} \rho_\alpha(a) * \rho_\beta(b).$$

$A_\Phi$ admits a natural topology that may be identified by declaring a sequence $(a_j)_{j \in \mathbb{N}} \subset A_\Phi$ to be convergent if for any $(\beta, n) \in \Delta$, there exists an integer $K$ such that for all $m < n$

$$i, j > K \Rightarrow \rho_{(\beta, n)}(a_i) = \rho_{(\beta, n)}(a_j).$$

**Lemma 65.** If $A$ is a $\mathbb{C}$-algebra and $a \in A_\Phi$ satisfies $\rho_0(a) = 0$ then any series

$$\sum_{j \geq 1} c_j a^j$$

with coefficients $c_j \in \mathbb{C}$ is convergent in the topological ring $A_\Phi$.

See Lemma 5.3 of [12] for a proof.

Given two $\Delta$-graded algebras $A$ and $B$, and a morphism $f : A \to B$ that preserves the $\Delta$-grading, we get an induced continuous map

$$f_\Phi : A_\Phi \to B_\Phi$$

by defining

$$\rho_\gamma(f_\Phi(a)) = f(\rho_\gamma(a)).$$
Applying this process to the map of \( \Delta \)-graded algebras \( I : H_{sc}(\mathcal{C}) \to \mathbb{C}[\Delta] \) yields a continuous map \( I_{\Phi} : H_{sc}(\mathcal{C})_{\Phi} \to \mathbb{C}[\Delta]_{\Phi} \). We call \( H_{sc}(\mathcal{C})_{\Phi} \) the Laurent Hall algebra; it, too, is equipped with a Poisson bracket.

**Definition 66.** A morphism of stacks \( f : W \to \mathcal{C} \) is \( \Phi \)-finite if

a) \( W_\alpha = f^{-1}(C_\alpha) \) is of finite type for all \( \alpha \in \Delta \), and
b) there is a Laurent subset \( S \subset \Delta \) such that \( W_\alpha \) is empty unless \( \alpha \in S \).

A \( \Phi \)-finite morphism of stacks \( f : W \to \mathcal{C} \) defines an element of \( H_{sc}(\mathcal{C})_{\Phi} \) by the formal sum

\[
\sum_{\alpha \in S} [W_\alpha \to \mathcal{C}].
\]

In [12], it is shown that \( [\text{Hilb} \to \mathcal{M}] \) is \( \Phi \)-finite, and hence defines an element in \( H_{sc}(\mathcal{C})_{\Phi} \).

**Lemma 67.** The maps

\( \pi \text{-Hilb} \to \mathcal{C}, \quad \text{Hilb} \to \mathcal{C}, \quad \text{Hilb}_{exc} \to \mathcal{C}, \)

are \( \Phi \)-finite. The corresponding elements \( \mathcal{H}^\pi, \mathcal{H} \) and \( \mathcal{H}_{exc} \) of \( H_{sc}(\mathcal{C})_{\Phi} \) satisfy

\[
I_{\Phi}(\mathcal{H}^\pi) = \sum_{(\beta, n) \in \Delta} (-1)^n \pi \text{-PT}(\beta, n)x^\beta q^n = \pi \text{-PT}(Y)(x, -q),
\]

where we have written \( x^\beta = q^{(\beta, 0)} \) and \( q = q^{(0, 1)} \). Similarly,

\[
I_{\Phi}(\mathcal{H}) = \sum_{(\beta, n) \in \Delta} (-1)^n DT(\beta, n)x^\beta q^n = DT(Y)(x, -q)
\]

\[
I_{\Phi}(\mathcal{H}_{exc}) = \sum_{(\beta, n) \in \Delta, \pi, \beta = 0} (-1)^n DT(\beta, n)x^\beta q^n = DT_{exc}(Y)(x, -q).
\]

**Proof:** Lemma 23 proves that \( \pi \text{-Hilb} \) is constructible and locally of finite type, and is of finite type once the Chern character is fixed. As well, the set of elements \( \alpha \in \Delta \) for which \( \pi \text{-Hilb}^\alpha \) is non-empty is Laurent. To prove this, it suffices to show that for any curve class \( \beta \), there exists an integer \( N \) such that for any \( n < N \), the moduli space \( \pi \text{-Hilb}^{(\beta, n)} \) is empty. Fix a curve class \( \beta \), and consider all \( \pi \text{-stablepairs} \mathcal{O}_Y \to G \) in that class. There is the associated short exact sequence,

\[
0 \to \mathcal{O}_C \to G \to P \to 0
\]

and since there are only finitely many decompositions \( \beta = \beta_1 + \beta_2 \) of \( \beta \) into a sum of effective curve classes, we lose no generality in fixing the curve class of \( \mathcal{O}_C \) and \( P \). Now the structure sheaf \( \mathcal{O}_C \) lives in a Hilbert scheme, and the set of elements \( (\beta_1, n_1) \in \Delta \) for which \( \text{Hilb}^{(\beta_1, n_1)} \) is non-empty is Laurent, so there is a “minimal” Euler characteristic of \( \mathcal{O}_C \), which we denote by \( N_1 \). As for \( P \), the Leray spectral sequence shows that \( \chi(P) \geq 0 \) (see lemma 46). This proves that we may take \( N = N_1 \), and for any \( n < N, \pi \text{-Hilb}^{(\beta, n)} \) is empty.

The formulae then follow from lemma 26 and Behrend’s description of DT invariants as a weighted Euler characteristic.

\( \text{Hilb}_{exc} \) is a subscheme of \( \text{Hilb} \), so the desired properties follow from [12, lemma 5.5].

\[\square\]
Lemma 68. Let $I \subset (-\infty, +\infty] \times (-\infty, +\infty]$ be an interval bounded from below. Then
\[ 1_{SS(I)} \to \mathcal{C} \]
is $\Phi$-finite.

Proof: Since this holds for Gieseker stability ([24, theorem 3.3.7]), it suffices to prove that for any $b = (b_1, b_2) \in (-\infty, +\infty] \times (-\infty, +\infty]$ there exists a number $M$ such that the family of all $G$ with $\mu_\pi(G) \geq b$ satisfies $\mu(G) \geq M$. Here, $\mu$ stands for Gieseker slope stability, namely
\[ \mu(G) = \frac{\chi(G)}{\beta \cdot L}. \]

Case 1: $\chi(G) > 0$
Here, we have
\[ 0 < \frac{\chi(G)}{\beta \cdot L} = \mu(G), \]
so we may take $M = 0$ in this case.

Case 2: $\chi(G) < 0$
Now, since $\beta \cdot \tilde{H} \leq \beta \cdot L$ (c.f. definition 37) and $\chi(G) < 0$, we have
\[ b_1 \leq \frac{\chi(G)}{\beta \cdot H} \leq \frac{\chi(G)}{\beta \cdot L}. \]
In this case, we may take $M = b_1$.

Case 3: $\chi(G) = 0$
In this case, $\mu(G)$ is either $0$ or $+\infty$, so we may take $M = 0$ in this case.

Case-by-case analysis reveals that we may use $M = \min\{0, b_1\}$. ■

Equations in the Laurent Hall algebra
In this section, following [12] we establish equations in $H_{sc}(\mathcal{C})_\Phi$, and ultimately prove theorem 6.

Lemma 69. Let $\mu \in (-\infty, +\infty] \times (-\infty, +\infty]$ such that $\mu < \frac{\infty}{2}$. Then the following equality holds in $H_{sc}(\mathcal{C})_\Phi$:
\[ 1_{SS(\mu \leq \square < \frac{\infty}{2})} = 1_{P_\pi} \ast 1_{SS(\mu \leq \square < \frac{\infty}{2})}. \]

Proof: Form the following Cartesian diagram:
\[ Z \xrightarrow{f} \mathcal{M}(\mathcal{C}) \xrightarrow{c} \mathcal{M} \]
\[ \mathcal{P}_\pi \times SS(\mu \leq \square < \frac{\infty}{2}) \xrightarrow{\mathcal{M} \times \mathcal{M}} \]
$T$-valued points of $Z$ are short exact sequences $0 \to A \to G \to B \to 0$ of $T$-flat sheaves on $T \times Y$ such that $A$ defines a family of objects in $\mathcal{P}_\pi$ and
B a family in $\text{SS} (\mu \leq \Box < \frac{\infty}{2})$. By lemma 51, we know $\frac{\infty}{2} \leq \mu(A) \leq \infty$.

Now by [12, lemma 6.2], we know that $G$ defines a family of objects in $\text{SS} (\mu \leq \Box \leq \infty)$.

Now let $G \in \text{SS} (\mu \leq \Box \leq \infty)$. If $G \in \mathcal{P}_\pi$ or $\text{SS} (\mu \leq \Box < \frac{\infty}{2})$ then we are done, since then $G$ will be an extension where one term is zero (recall that all $\text{SS} (a < \Box < b)$ include the zero objects). Otherwise, let

$$0 = G_0 \subset G_1 \subset \ldots \subset G_{N-1} \subset G_N = G$$

be the Harder-Narasimhan filtration of $G$, let $Q_i$ be the associated quotients. Then there exists an index $j \in \mathbb{N}$, $1 < j < N$ such that for all $i < j$, $\mu_e(Q_i) \geq \frac{\infty}{2}$ and $\mu_e(Q_j) < \frac{\infty}{2}$. Finally, by the uniqueness of the Harder-Narasimhan filtration, we have that $G_j \in \mathcal{P}_\pi$ and $G/G_j \in \text{SS} (\mu \leq \Box < \frac{\infty}{2})$.

Remark 70. The proof of this lemma is strikingly similar to the proof of Lemma 55. This is no coincidence. The above is actually just a minor refinement of Lemma 55 which says that we may cut off the tail end of $Q_\pi$ and have the corresponding result still hold. As we go on, we will be less explicit about the proofs of lemmas when the argument has been already made in the infinite-type case.

Lemma 71. Let $\mu \in (-\infty, +\infty] \times (-\infty, +\infty]$. Then, as $\mu \to -\infty$, we have

$$\mathcal{H} \ast 1_{\text{SS} (\mu \leq \Box \leq \infty)} - 1_{\text{SS} (\mu \leq \Box \leq \infty)} \to 0.$$ 

Proof: Fix $(\beta, n) \in \Delta$. Then there are only finitely many decompositions $(\beta, n) = (\beta_1, n_1) + (\beta_2, n_2)$ such that both $\rho((\beta_1, n_1), (\mathcal{H}))$ and $\rho((\beta_2, n_2), (1_{\text{SS} (\mu \leq \Box \leq \infty)}))$ are non-zero. This follows from the fact that there are only finitely many decompositions $\beta = \beta_1 + \beta_2$ with both $\beta_i$ effective. Now for each fixed $\beta$, there exist finitely many $n$ such that both $\rho((\beta_1, n_1), (\mathcal{H}))$ and $\rho((\beta_2, n_2), (1_{\text{SS} (\mu \leq \Box \leq \infty)}))$ are non-zero.

By the boundedness of the Hilbert scheme, we may assume that $\mu$ is small enough so that for any of the decompositions, $\beta = \beta_1 + \beta_2$, all points $O_Y \to A$ of $\text{Hilb}((\beta_1, n_1))$ satisfy $A \in \text{SS} (\mu \leq \Box \leq \infty)$. Consider a diagram of sheaves,

$$\begin{array}{c}
\mathcal{O}_Y \\
\downarrow_f \\
0 \longrightarrow A \overset{\alpha}{\longrightarrow} G \overset{\beta}{\longrightarrow} B \longrightarrow 0
\end{array}$$

with $O_Y \to A$ in $\text{Hilb}((\beta_1, n_1))$ and $\text{ch}(\mathcal{O}) = (\beta, n)$.

Now $G \in \text{SS} (\mu \leq \Box \leq \infty)$ if and only if $B \in \text{SS} (\mu \leq \Box \leq \infty)$. Since Bridgeland proves [12, prop 6.5]

$$\rho((\beta, n), (\mathcal{H} \ast 1_{\text{SS} (\mu \leq \Box \leq \infty)})) = \rho((\beta, n), (1_{\text{SS} (\mu \leq \Box \leq \infty)})),$$

the claim is proven.  

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Lemma 73. Let $\mu \in (-\infty, +\infty) \times (-\infty, +\infty)$. Then, as $\mu \to -\infty$, we have

$$
\mathcal{H}^n \ast 1_{\SS(\mu \leq \Box < \frac{1}{2})} - 1^O_{\SS(\mu \leq \Box < \frac{1}{2})} \to 0.
$$

Proof: Fix $(\beta, n) \in \Delta$. Then there are only finitely many decompositions $(\beta, n) = (\beta_1, n_1) + (\beta_2, n_2)$ such that both $\rho(\beta_1, n_1)(\mathcal{H}^n)$ and $\rho(\beta_2, n_2)(1_{\SS(\mu \leq \Box < \frac{1}{2})})$ are non-zero.

By the boundedness of the moduli space of $\pi$-stable pairs, we may assume that $\mu$ is small enough so that for any decompositions, $\beta = \beta_1 + \beta_2$, all points $O_Y \to A$ of $\pi$-Hilb$^{(\beta_1, n_1)}$ satisfy $A \in \SS(\mu \leq \Box < \frac{1}{2})$. Consider a diagram of sheaves,

$$
\begin{array}{c}
\mathcal{O}_Y \\
\downarrow \gamma \\
0 \rightarrow A \overset{\alpha}{\rightarrow} G \overset{\beta}{\rightarrow} B \rightarrow 0
\end{array}
$$

with $O_Y \to A$ in $\pi$-Hilb$(\beta_1, n_1)$ and $[G] = (\beta, n)$. Using that $\SS(\beta_1) \ast \SS(\beta_2) \subset \SS(\beta_1)$, we see that $B \in \SS(\mu \leq \Box < \frac{1}{2})$ if and only if $G \in \SS(\mu \leq \Box < \frac{1}{2})$. Now since $A \in \mathcal{Q}_s$ and $B \in \mathcal{Q}_s$, we have that $G \in \mathcal{Q}_s$. Composing the map $O_Y \to A$ with the map $A \to G$, yields a map $O_Y \to G$; this represents an object of $1^O_{\mathcal{Q}_s}$. The proof of lemma 60, that

$$
1^O_{\mathcal{Q}_s} = \mathcal{H}^n \ast 1_{\mathcal{Q}_s},
$$

can be easily adapted to now prove that

$$
\rho(\beta, n)(\mathcal{H}^n \ast 1_{\SS(\mu \leq \Box < \frac{1}{2})}) = \rho(\beta, n)(1^O_{\SS(\mu \leq \Box < \frac{1}{2})})
$$

This completes the proof that

$$
\mathcal{H}^n \ast 1_{\SS(\mu \leq \Box < \frac{1}{2})} - 1^O_{\SS(\mu \leq \Box < \frac{1}{2})} \to 0
$$

as $\mu \to -\infty$. $\blacksquare$

Proposition 75. We have the following equality in the Laurent Hall algebra, $H_n(C)_{\mathsf{cf}}$:

$$
\mathcal{H} \ast 1_{P_n} = \mathcal{H}_{exc} \ast 1_{P_n} \ast \mathcal{H}^n.
$$

Proof: Using $1_{\SS(\mu \leq \Box \leq \infty)} = 1_{P_n} \ast 1_{\SS(\mu \leq \Box < \frac{1}{2})}$ and $1^O_{\SS(\mu \leq \Box \leq \infty)} = 1^O_{P_n} \ast 1^O_{\SS(\mu \leq \Box < \frac{1}{2})}$, we can rewrite

$$
\mathcal{H} \ast 1_{\SS(\mu \leq \Box \leq \infty)} - 1^O_{\SS(\mu \leq \Box \leq \infty)} \to 0
$$

as

$$
\mathcal{H} \ast 1_{P_n} + 1_{\SS(\mu \leq \Box < \frac{1}{2})} - 1^O_{P_n} + 1^O_{\SS(\mu \leq \Box < \frac{1}{2})} \to 0,
$$

as $\mu \to -\infty$.

Multiplying $\mathcal{H}^n \ast 1_{\SS(\mu \leq \Box < \frac{1}{2})} - 1^O_{\SS(\mu \leq \Box < \frac{1}{2})} \to 0$ on the left by $1^O_{P_n}$, and rewriting using $1^O_{P_n} = \mathcal{H}_{exc} \ast 1_{P_n}$ yields

$$
\mathcal{H}_{exc} \ast 1_{P_n} \ast \mathcal{H}^n \ast 1_{\SS(\mu \leq \Box < \frac{1}{2})} - 1^O_{P_n} \ast 1^O_{\SS(\mu \leq \Box < \frac{1}{2})} \to 0
$$
as $\mu \to -\infty$. Hence
\[ H_{\text{exc}} * 1_{P_\pi} * \mathcal{H}_\mu * 1_{\text{SS}(\mu \leq \square < \frac{\infty}{2})} - \mathcal{H} * 1_{P_\pi} * 1_{\text{SS}(\mu \leq \square < \frac{\infty}{2})} \to 0 \]
as $\mu \to -\infty$. Since $1_{\text{SS}(\mu \leq \square < \infty)}$ is invertible, we can cancel it from both sides:
\[ \mathcal{H} * 1_{P_\pi} = H_{\text{exc}} * 1_{P_\pi} * \mathcal{H}^{\pi}. \]

The proof of theorem 6

We first collect results. The next proposition is theorem 3.11 of [30], and is a very deep result whose proof depends on all the full power of the formalism of [25, 26, 27, 28, 29].

**Proposition 76.** For each slope $\mu \in ((-\infty, -\infty), (+\infty, +\infty)]$, we can write
\[ 1_{\text{SS}(\mu)} = \exp(\epsilon_\mu) \in H(C), \]
with $\nu_\mu = [C^\ast] \cdot \epsilon_\mu \in H_{\text{reg}}(C)$ a regular element.

**Proof:** The proof is identical to that of [12, theorem 6.3]. Bridgeland uses Joyce’s machinery, which applies in our case just as it does in his. ■

The following corollary corresponds to Bridgeland’s 6.4 [12].

**Corollary 77.** For any $\mu \in ((-\infty, -\infty), (+\infty, +\infty)]$, the element $1_{\text{SS}(\mu)} \in H(C)$ is invertible, and the automorphism
\[ \text{Ad}_{1_{\text{SS}(\mu)}} : H(C) \to H(C) \]
preserves the subring of regular elements. The induced Poisson automorphism of $H_{\text{reg}}(C)$ is given by
\[ \text{Ad}_{1_{\text{SS}(\mu)}} = \exp\{\eta_\mu, -\}. \]

**Proof:** The proof of this is identical to that of corollary 6.4 of [12]. ■

Now we can prove theorem 6. We have
\[ \mathcal{H} * 1_{P_\pi} = H_{\text{exc}} * 1_{P_\pi} * \mathcal{H}^{\pi}. \]

Rearranging yields
\[ \mathcal{H} = H_{\text{exc}} * 1_{P_\pi} * \mathcal{H}^{\pi} * (1_{P_\pi})^{-1}. \]

By lemma 51, we can write $1_{P_\pi} = \text{SS}(\frac{\infty}{2} \leq \square \leq \infty)$, and by lemma 6.2 of [12], we can write
\[ \text{SS}(\frac{\infty}{2} \leq \square \leq \infty) = \prod_{\frac{\infty}{2} \leq \mu \leq \infty} 1_{\text{SS}(\mu)}. \]

In [12, lemma 6.2], it is explained that given an interval $J \subset (-\infty, +\infty) \times (-\infty, +\infty]$ that is bounded below, and an increasing sequence of finite subsets
\[ V_1 \subset V_2 \subset \ldots \subset J \]
the sequence $1_{SS(V_j)}$ converges to $1_{SS(J)}$, where $1_{SS(V_j)}$ is defined to be

$$\prod_{v \in V_j} 1_{SS(v)},$$

where the product is taken in descending order of slope. So, letting $J$ denote the interval of slopes between $\frac{2}{3}$ and $\infty$, including $\frac{2}{3}$ and excluding $\infty$, we can write

$$H = H_{exc} \ast \lim_{\text{finite } V \subset J} 1_{SS(\mu_N)} \ast \cdots \ast 1_{SS(\mu_1)} \ast H^\pi \ast (1_{SS(\mu_N)})^{-1} \ast \cdots \ast (1_{SS(\mu_1)})^{-1},$$

where $\mu_i$ enumerate all the elements of $V$. Using proposition and corollary, we can rewrite

$$H = H_{exc} \ast \lim_{\text{finite } V \subset J} \exp\{\eta_{\mu_N}, \exp\{\eta_{\mu_{N-1}}, \cdots, \exp\{\eta_{\mu_1}, -\} \cdots\}(H^\pi).$$

Now hitting this equation with the integration map yields

$$I_\Phi(H) =$$

$$I_\Phi(H_{exc}) \cdot I_\Phi \left( \lim_{\text{finite } V \subset J} \exp\{\eta_{\mu_N}, \exp\{\eta_{\mu_{N-1}}, \cdots, \exp\{\eta_{\mu_1}, -\} \cdots\}(H^\pi) \right).$$

The integration map commutes with limits since it is continuous, thus

$$I_\Phi(H) =$$

$$I_\Phi(H_{exc}) \lim_{\text{finite } V \subset J} I_\Phi \left( \exp\{\eta_{\mu_N}, \exp\{\eta_{\mu_{N-1}}, \cdots, \exp\{\eta_{\mu_1}, -\} \cdots\}(H^\pi) \right).$$

Now, the Poisson bracket is a commutator which is trivial in the ring of Laurent series, so it vanishes after applying the integration map, and we are left with

$$I_\Phi(H) = I_\Phi(H_{exc}) \cdot I_\Phi(H^\pi).$$

Applying lemma 67, we get

$$DT(Y)(x, -q) = DT_{exc}(Y)(x, -q) \cdot \pi\text{-PT}(Y)(x, -q)$$

and substituting $q$ for $-q$ yields

$$DT(Y) = DT_{exc}(Y) \cdot \pi\text{-PT}(Y),$$

which is what we set out to prove.  

\textbf{References}


