MULTIPLE COVERS AND THE INTEGRALITY CONJECTURE FOR RATIONAL CURVES IN CALABI-YAU THREEFOLDS

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Abstract. We study the contribution of multiple covers of an irreducible rational curve C in a Calabi-Yau threefold Y to the genus 0 Gromov-Witten invariants in the following cases.

1. If the curve C has one node and satisfies a certain genericity condition, we prove that the contribution of multiple covers of degree d is given by
$$\sum_{n|d}\frac{1}{n^3}.$$
2. For a smoothly embedded contractable curve $C \subset Y$ we define schemes C_i for $1 \leq i \leq l$ where C_i is supported on C and has multiplicity i, the number $l \in \{1, \ldots, 6\}$ being Kollár's invariant "length". We prove that the contribution of multiple covers of C of degree d is given by
$$\sum_{n|d}\frac{k_{d/n}}{n^3}$$
where k_i is the multiplicity of C_i in its Hilbert scheme (and $k_i = 0$ if $i > l$).

In the latter case we also get a formula for arbitrary genus (Theorem 1.7).

These results show that the curve C contributes an integer amount to the so-called instanton numbers that are defined recursively in terms of the Gromov-Witten invariants and are conjectured to be integers.

1. Motivation and Results

From string theory and M-theory, physicists insist on the existence of *instanton numbers*. Let Y be a Calabi-Yau threefold, and $\beta \in H_2(Y, \mathbf{Z})$. Then there is supposed to be an integer invariant n_β, an instanton number, so that the genus 0 Gromov-Witten invariant is given by

$$N_\beta = \sum_{n|\beta}\frac{n_{\beta/n}}{n^3}. \tag{1}$$

A slightly more precise version of this conjecture is stated as Conjecture 7.4.5 in [11]. Since Equation (1) determines the n_β's recursively in terms of the N_β's, one can regard the equation as the definition of n_β. The integrality of n_β has been verified empirically for Y where the Gromov-Witten invariants are known. However, there is no Y for which the integrality of n_β has been proven to hold for all β. M-theory

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suggests an approach to an intrinsic definition of the n_β's in terms of sheaves [14], but this has not yet been made precise mathematically. Methods to compute the n_β's directly from the physical definition have been developed in [17].

For smoothly embedded $\mathbf{P}^1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, it has been proven by Aspinwall-Morrison, Voisin, Kontsevich, Manin, Pandharipande, and Lian-Liu-Yau (see [11] [1] [22] [19] [21] [28])[1] that the contribution to the genus 0 Gromov-Witten invariant of degree d multiple covers is d^{-3}. Thus if we imagine that all the rational curves in Y are smoothly embedded with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then n_β is precisely the number of rational curves in the class β. However this assumption is too optimistic. For example, even if we assume Clemens' conjecture, a generic quintic 3-fold always has nodal rational curves in degree 5 by Vainsencher [27]. Therefore, to understand the physicists' instanton numbers and/or to extract more precise enumerative information from the Gromov-Witten invariants, we need to understand how non-generic curves in Y contribute to the invariants.

This problem is open in almost any reasonable situation other than $(-1,-1)$ curves. For example: isolated smooth rational curves with a non-generic normal bundles, and generic nodal curves. In this paper we treat the case of any contractable smooth rational curve (which can have normal bundle $(-1,-1)$, $(0,-2)$, or $(1,-3)$) and the case of a generic irreducible rational curve with one node.

To state our results we make the following definitions.

A $(-1,-1)$ curve has no infinitesimal deformations so the curve is rigid; in fact, no multiple of the curve has infinitesimal deformations so we say the curve is super-rigid. For nodal curves, this is the notion of rigidity that we need. It is an adaptation of a concept due to Pandharipande:

Definition 1.1 (c.f. Pandharipande [23]). *A curve C in a projective variety Y is called g-*super-rigid *if every genus g stable map $f : C' \to Y$ such that f is a local immersion and $f(C') \subset C$ has no infinitesimal deformations as a stable map. A curve that is g-super-rigid for all g is simply called* super-rigid.

Super-rigidity is a genericity condition for nodal curves in the following sense. One can weaken the super-rigidity condition by only requiring that the defining property holds for maps f with degree at most N onto C. Note that for each fixed N, this is an open condition in any family of pairs (Y, C) and so super-rigidity is an intersection of a countable number of open conditions. In practice, we will only need that the condition holds for fixed N. Our result for nodal curves is the following.

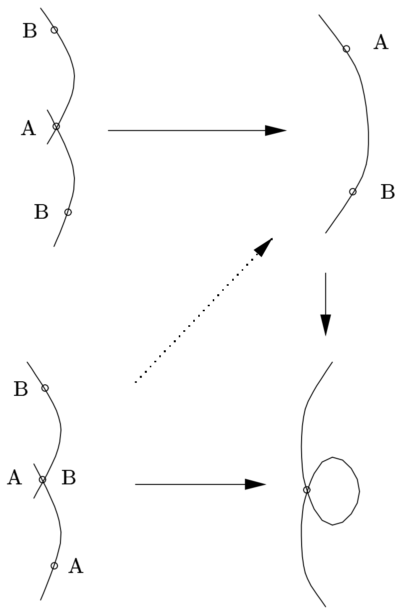
Theorem 1.2. *Let $C \subset Y$ be an irreducible 0-super-rigid rational curve with exactly one node in a 3-fold Y with K_Y trivial in a neighborhood of C. Then the*

[1] Aspinwall and Morrison's calculation predates the definition of Gromov-Witten invariants and so their argument must be regarded as incomplete in the present context. It would be interesting to find a direct comparison between their calculation and the calculations in Gromov-Witten theory. We should also warn the reader that the theorem in Voisin [28] only computes the part of the contribution of multiple covers of immersed $\mathbf{P}^1$'s coming from maps that factor through the normalization (and gets d^{-3}). As our results show, the contribution of the image curve to the Gromov-Witten invariant need not be d^{-3} if the $\mathbf{P}^1$ is not embedded (due to maps that do not factor through the normalization). Voisin's computation thus does *not* necessarily compute the full contribution to the Gromov-Witten invariant, as asserted in her abstract, unless the $\mathbf{P}^1$ is embedded.

contribution of degree d multiple covers of C to the genus 0 Gromov-Witten invariant is

$$\sum_{n|d} \frac{1}{n^3}.$$

Remark 1.3. Unlike the case of a smooth curve, the moduli space of stable maps that multiply-cover a nodal curve has a number of different path components arising from the possible "jumping" behavior of the map at the node. The basic phenomenon is illustrated below for degree two maps. The moduli space has different components depending on whether the map factors through the normalization or not. In the figure below, points labeled by A and B are mapped to corresponding points labeled A and B and the normalization map identifies A and B to the nodal point. Notice that the bottom map cannot be factored through the normalization (the dashed map doesn't exist) even though the map to the nodal curve is well defined.



Thus we see that the moduli space of genus 0 stable maps that double cover C consists of two connected components (of different dimensions) depending on whether the map factors through the normalization or not. See Proposition 3.2 for a description of the components in the case of arbitrary degree.

Remark 1.4. A necessary condition for the super-rigidity of C in this case is that the pullback of $TY|_C$ to the normalization $\pi : \tilde{C} \to C$ is generic; *i.e.* if we let $E = \pi^*(TY|_C)$, then

$$0 \to T\tilde{C} \to E \to N \to 0$$

where $N \cong \mathcal{O}_{\tilde{C}}(-1) \oplus \mathcal{O}_{\tilde{C}}(-1)$.

An additional necessary condition can be described as follows. Let $A, B \in \tilde{C}$ be the preimages of the nodal point $p \in C$. Since E_A and E_B are both canonically isomorphic to T_pY, we get a canonical isomorphism

$$\phi_1 : E_A \to E_B.$$

Now the image of $T_A\tilde{C}$ under ϕ_1 is a line that projects to a line in N_B determining a point $l_B \in \mathbf{P}(N_B)$. Similarly, $\phi_1^{-1}(T_B\tilde{C})$ determines a point $l_A \in \mathbf{P}(N_A)$. On the other hand, since N is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, there is a canonical identification

$$\phi_2 : \mathbf{P}(N_A) \to \mathbf{P}(N_B).$$

For super-rigidity to hold, we must have $\phi_2(l_A) \neq l_B$. This geometric condition can be seen to be equivalent to the surjectivity of (9.25) in [11], where super-rigidity for a nodal curve is explored from a different viewpoint. Whether this necessary condition is also a sufficient condition is a topic for further study.

An isolated smoothly embedded rational curve C in a Calabi-Yau 3-fold Y can have a normal bundle other than $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. In [25], Reid proves that if the normal bundle is $\mathcal{O} \oplus \mathcal{O}(-2)$, then the curve is isolated if and only if the curve contracts; that is, there exists a birational morphism $f : Y \to X$ with $f(C) = p \in X$ that is an isomorphism of $Y \backslash C$ with $X \backslash p$. If a rational curve contracts, it must be isolated, the normal bundle must be $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $\mathcal{O} \oplus \mathcal{O}(-2)$, or $\mathcal{O}(1) \oplus \mathcal{O}(-3)$ and p is a compound DuVal (cDV) singularity[2] [24]. The singularity types of the generic hyperplane section through p were classified by Katz-Morrison [18] who found that they are determined by Kollár's invariant "length".

Definition 1.5 (Kollár [10] page 95)**.** *Let $C \subset Y$ be a smooth rational curve in a 3-fold Y with K_Y trivial in a neighborhood of C. Suppose that there exists a contraction $\pi : Y \to X$ with $\pi(C) = p \in X$ and π is an isomorphism of $Y \backslash C$ onto $X \backslash p$. Define the* length *of p to be the length (at the generic point of C) of the scheme supported on C with structure sheaf $\mathcal{O}_Y/\pi^{-1}(\mathfrak{m}_{X,p})$ where $\mathfrak{m}_{X,p}$ is the maximal ideal of the point p.*

The length of p is a number $l \in \{1, \ldots, 6\}$ and if $l = 1$ then C has normal bundle $N_{C/Y}$ isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}(-2)$ and if $l > 1$ then $N_{C/Y} \cong \mathcal{O}(1) \oplus \mathcal{O}(-3)$. When $l > 1$ we can define a sequence of higher order neighborhoods of C:

Definition 1.6. *Let $C \subset Y$ be as in Definition 1.5 and let l be the length of p. Let $X_0 \subset X$ be a generic hyperplane section through p and let Y_0 be the proper transform of X_0. For $i = 1, \ldots, l$ define C_i to be the subscheme of Y_0 defined by the saturation $\mathcal{I}^{(i)}$ of the i^{th} power of the ideal sheaf $\mathcal{I}$ defining C in Y_0.*

Note that $\mathcal{I}^{(i)}$ is the sheaf of functions in $\mathcal{O}_{Y_0}$ which are in the i^{th} power of the ideal $\mathcal{I}$ away from the singular points of Y_0. For example, if Y_0 has an A_1 singularity $xy = z^2$ and C is the curve locally defined by $x = z = 0$, then $x \in \mathcal{I}^{(2)}$ even though $x \notin \mathcal{I}^2$.

Note that $C_1 = C$ and that C_l is the scheme used in the definition of length. The crucial property of C_i is that under a generic deformation of Y, C_i deforms to k_i smoothly embedded $(-1,-1)$ curves in the homology class $i[C]$ where k_i is the multiplicity of C_i in its Hilbert scheme (see Proposition 2.3 and Lemma 2.8).

[2] One might have hoped that all isolated curves contract, but this is false: Clemens has an example of an isolated $(2,-4)$ curve where no multiple of the curve deforms [9] and Jiménez has an example of a non-contractable, isolated $(1,-3)$-curve [15].

Our main result for embedded smooth curves is the following.

Theorem 1.7. *Let $C \subset Y$ be a smoothly embedded contractable rational curve in a 3-fold Y with K_Y trivial in a neighborhood of C. Then the contribution of degree d multiple covers of C to the genus g Gromov-Witten invariants is*

$$\sum_{n|d} k_{d/n} \frac{|B_{2g}| n^{2g-3}}{2g \cdot (2g-2)!}$$

where B_{2g} is the $2g$-th Bernoulli number (c.f. [12] Theorem 3) and k_i is the multiplicity of C_i in its Hilbert scheme if $i \leq l$ and $k_i = 0$ otherwise. In particular, the genus 0 formula is

$$\sum_{n|d} \frac{k_{d/n}}{n^3}.$$

For example, if $l = 1$ then C is a $(-1,-1)$ or $(0,-2)$ curve and k_1 coincides with Reid's invariant "width" (Definition 5.3 of [25]).

Our technique for proving Theorem 1.2 is to relate the invariants of a super-rigid 1-nodal curve to the invariants of a contractable chain of rational curves via a method similar to that used by Bryan-Leung to study contributions of multiple covers of nodal curves in $K3$ surfaces (Section 5 of [7]). The invariants of a contractable chain of curves (as well as the contractable curve in Theorem 1.7) are then computed by deforming a neighborhood of the curve. The deformation is constructed and analyzed by employing the beautiful subject of deformations of DuVal singularities and their simultaneous (partial) resolutions. The subject goes back to Brieskorn [4][5][6] and has been used to study 3-fold singularities with small resolutions by Pinkham [24] and others (c.f. [13][18][25]).

It is crucial that the deformations we construct in this way can be blown down to an open subset (in the analytic topology) of an affine variety. This enables us to conclude that all the complete curves in the deformed 3-fold lie in the exceptional set of the blowdown.

Unfortunately, we do not know how to generalize these techniques to curves with more than one node; in fact, we do not even have a reasonable conjectural formula (see Remark 3.4). There are also problems generalizing the proof of Theorem 1.2 to higher genus (see Remark 3.3). However, considerations of M-theory [14] lead us to the following conjecture:

Conjecture 1.8. *Let $C \subset Y$ be an irreducible super-rigid rational curve with exactly one node in a 3-fold Y with K_Y trivial in a neighborhood of C. Then the contribution of degree d multiple covers of C to the genus g Gromov-Witten invariants is*

$$\sum_{n|d} \left(\frac{|B_{2g}| n^{2g-3}}{2g(2g-2)!} + \frac{\delta_{1,g}}{n} \right)$$

where $\delta_{1,g} = 1$ if $g = 1$ and $\delta_{1,g} = 0$ if $g \neq 1$.

Via M-theory, genus g invariants n^g_β can be assigned for all $g \geq 0$ and all homology classes β, from which the 0-point Gromov-Witten invariants can be conjecturally computed by an explicit formula generalizing equation (1), with $n^0_\beta = n_\beta$ (see [14]). Let us consider these invariants (if they exist) to be *defined* by this formula (c.f. [23]). We can then discuss the contributions of a curve C to $n^g_{d[C]}$. Theorem 1.2

proves that the contribution of C to $n_{d[C]}$ is 1 for all $d \geq 1$, and Conjecture 1.8 asserts that the contribution to $n^1_{d[C]}$ is also 1 for all $d \geq 1$ while the contribution to $n^g_{d[C]}$ is 0 for $g \geq 2$.

In this language, [14] also gives a conjectural formula for these invariants in terms of moduli spaces of $U(d)$ bundles on C. These moduli spaces have been computed in [26], and we have checked that this computation leads to C contributing 1 to $n_{d[C]}$ for all d, as predicted.

Our paper is organized as follows. In Section 2 we study the neighborhood of a contractable curve in a Calabi-Yau 3-fold. We explicitly construct deformations of the neighborhood in order to compute its contribution to the Gromov-Witten invariants. In Section 3, we relate the Gromov-Witten contribution of a 0-super-rigid nodal curve to the contribution of contractable chains of curves. As an example of Theorem 1.7, we work out the length 2 case in detail in Section 4, computing k_1 and k_2 directly from explicit equations for the 3-fold.

2. The neighborhood of a contractable curve and its deformations

In this section we use the deformation invariance property of Gromov-Witten invariants to compute them for a neighborhood of a contractable curve in a local Calabi-Yau 3-fold. This is achieved by deforming the neighborhood so that all the curves in the neighborhood are smooth $(-1,-1)$ curves (c.f. Friedman [13] pg. 678–679). We focus on the two cases of interest: a generic linear chain of curves that contracts to a cA_n singularity and a single contractable smooth curve which necessarily must contract to a cA_1, cD_4, cE_6, cE_7, or cE_8 singularity. We will relate the former case to the invariants of a nodal curve in Section 3.

We begin by outlining the theory of deformations of DuVal singularities and their simultaneous resolutions. This is due to Brieskorn [4][5][6] and, in a refined form that we need, Katz-Morrison (see Section 3, especially Theorem 1 of [18]). The application of these ideas to Gorenstein 3-fold singularities with small resolutions was pioneered by Pinkham [24] and Reid [25] (see also [8]).

Let $C \subset Y$ be a rational curve (not necessarily irreducible) in a 3-fold Y with K_Y trivial in a neighborhood of C and $\pi : Y \to X$ is a birational morphism such that $\pi(C) = p \in X$ and $\pi|_{Y\backslash C}$ is an isomorphism onto $X\backslash p$. We consider an analytic neighborhood of p (still denoted X) and its inverse image under π (still denoted Y). By a lemma of Reid [25] (1.1,1.14), the generic hyperplane section through p is a surface X_0 with an isolated rational double point, and the proper transform of X_0 is a partial resolution $Y_0 \to X_0$ (*i.e.* the minimal resolution $Z_0 \to X_0$ factors through $Y_0 \to X_0$).

The partial resolution $Y_0 \to X_0$ determines combinatorial data $\Gamma_0 \subset \Gamma$ consisting of an ADE Dynkin diagram Γ (the type of the singularity p) and a subgraph Γ_0 (the dual graph of the exceptional set of Y_0).

Let $\mathcal{Z} \to \mathrm{Def}(Z_0)$, $\mathcal{Y} \to \mathrm{Def}(Y_0)$, and $\mathcal{X} \to \mathrm{Def}(X_0)$ be semi-universal deformations of Z_0, Y_0, and X_0. Following [18], there are identifications

$$\mathrm{Def}(Z_0) \cong U =: \mathrm{Res}(\Gamma)$$
$$\mathrm{Def}(Y_0) \cong U/W_0 =: \mathrm{PRes}(\Gamma, \Gamma_0)$$
$$\mathrm{Def}(X_0) \cong U/W =: \mathrm{Def}(\Gamma)$$

where U is the complex root space associated to Γ and W is its Weyl group. $W_0 \subset W$ is the subgroup generated by reflections of the simple roots corresponding to

$\Gamma - \Gamma_0$. Deformations of Z_0 or Y_0 can be blown down to give deformations of X_0 ([29] Theorem 1.4) and the induced classifying maps are given by the natural maps $U \to U/W$ and $U/W_0 \to U/W$ under the above identifications.

By choosing a one parameter family $\{X_t\}_{t\in\Delta}$ of hyperplane sections of X with X_0 a generic hyperplane section passing through p, we can view X as the total space of the family X_t defined by a classifying map

$$g : \Delta \to \mathrm{Def}(\Gamma).$$

Similarly, we get a compatible family Y_t given by a map

$$f : \Delta \to \mathrm{PRes}(\Gamma, \Gamma_0).$$

That is, we get the diagram

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "\tilde{\sigma}"] \arrow[d] & \mathcal{Y} \arrow[r, "\tilde{\rho}"] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathrm{Res}(\Gamma) \arrow[r, "\sigma"] & \mathrm{PRes}(\Gamma, \Gamma_0) \arrow[r, "\rho"] & \mathrm{Def}(\Gamma) \\
& \Delta \arrow[u, "f"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

where Y is the pullback of $\mathcal{Y}$ by f and X is the pullback of $\mathcal{X}$ by g.

Our aim is to deform Y by deforming the map f and to describe the curves in the resulting space. To do this we need to understand the discriminant loci of $\mathrm{PRes}(\Gamma, \Gamma_0)$ and the curves in the corresponding surfaces. We first remark that if $\tilde{f} : \Delta \to \mathrm{PRes}(\Gamma, \Gamma_0)$ is a generic deformation of f (so in particular $\tilde{f}$ is transverse to the discriminant locus), then $\tilde{Y} := \tilde{f}^{-1}(\mathcal{Y})$ is smooth. This is true because the singularities of $\mathcal{Y}$ all lie over a set of at least codimension 2 in $\mathrm{PRes}(\Gamma, \Gamma_0)$. Ultimately, we show that all the curves in $\tilde{Y}$ are smooth $(-1,-1)$ curves. If we then know the number of such curves and their homology class, then the Gromov-Witten invariants of $\tilde{Y}$ can be easily computed from the following multiple cover formula of Faber and Pandharipande.

Theorem 2.1 (Faber-Pandharipande [12]). *If C is a smoothly embedded $(-1,-1)$ curve in a Calabi-Yau 3-fold, then the contribution of degree d multiple covers of C to the genus g Gromov-Witten invariants is given by*

$$\frac{|B_{2g}|d^{2g-3}}{2g\cdot(2g-2)!}.$$

The exceptional curve of the minimal resolution $Z_0 \to X_0$ is a rational curve whose dual graph is Γ. Thus the components of the exceptional divisor are rational curves C_{e_i} which are in one to one correspondence with the simple roots $e_1, \ldots, e_n$ (n is the rank of the root system). A generic deformation of Z_0 has no complete curves, but the curves C_{e_i} and various combinations of C_{e_i} will deform in certain codimension one families of deformations. To make this precise, we define the *discriminant locus* $\mathfrak{D} \subset \mathrm{Res}(\Gamma)$ to be those $t \in \mathrm{Res}(\Gamma)$ such that the corresponding surface $\mathcal{Z}_t$ has a complete curve. The following proposition is essentially part 3 of Theorem 1 in [18].

Proposition 2.2. *The irreducible components of the discriminant divisor $\mathfrak{D} \subset$ Res(Γ) are in one to one correspondence with the positive roots of Γ. Under the identification of* Res(Γ) *with the complex root space U, the component $\mathfrak{D}_v$ corresponding to the positive root $v = \sum_{i=1}^{n} a_i e_i$ is $v^{\perp} \subset U$,* i.e. *the hyperplane perpendicular to v.*

Moreover, $\mathfrak{D}_v$ corresponds exactly to those deformations of Z_0 in $\mathcal{Z}$ to which the curve

$$C_v := \bigcup_{i=1}^{n} a_i C_{e_i}$$

lifts. For a generic point $t \in \mathfrak{D}_v$, the corresponding surface $\mathcal{Z}_t$ has a single smooth -2 curve in the class $\sum_{i=1}^{n} a_i [C_{e_i}]$ thus there is a small neighborhood B of t such that the restriction of $\mathcal{Z}$ to B is isomorphic to a product of $\mathbf{C}^{n-1}$ with the semi-universal family over Res(A_1).

Katz and Morrison prove this (among other things) by constructing explicit equations for $\mathcal{Z}$. Although we do not need the explicit form of these equations for the proofs of our theorems, they are useful for computing the values of k_i in concrete cases (see Section 4).

Since the Weyl group acts transitively on the positive roots, we see that the discriminant locus in Def(Γ), which is the image of $\mathfrak{D}$, is irreducible. It corresponds to those deformations of X_0 which are singular.

The two cases relevant to this paper are when $\Gamma = \Gamma_0 = A_n$ (so PRes$(\Gamma, \Gamma_0) =$ Res(Γ)) and when Γ_0 is a single vertex. In this latter case, Γ must be one of A_1, D_4, E_6, E_7, or E_8 and Γ_0 must be the central vertex or, in the case of E_8, it can also be the vertex one away from the center on the long branch. These six possibilities correspond to the six possible values of the length (this is the main theorem of [18]).

2.1. **The case of a contractable smooth $\mathbf{P}^1$.** Assume then that Γ_0 is a single vertex so that (Γ, Γ_0) is one of the above six possibilities. Let $D \subset$ PRes(Γ, Γ_0) denote the discriminant locus, *i.e.* the image of $\mathfrak{D}$ under the quotient of W_0. The corresponding surfaces are those which are singular or contain complete curves (or both). We write $D = D^{sing} \cup D^{curv}$ where the surfaces corresponding to the points in D^{curv} contain a complete curve and D^{sing} is the union of the remaining components. A generic point of D^{curv} corresponds to a smooth surface with a smooth -2 curve and a generic point of D^{sing} corresponds to a surface with no curves and a single A_1 singularity.

For the purposes of computing the Gromov-Witten invariants of Y we need to understand the components of D^{curv}. Since the family $\mathcal{Y}$ is obtained from $\mathcal{Z}$ by simultaneously blowing down the curves in $\mathcal{Z}_t$ corresponding to $\Gamma - \Gamma_0$ followed by taking the quotient by W_0, we can study D^{curv} by studying the orbit structure of W_0 acting on $\mathfrak{D} = \cup \mathfrak{D}_v$.

Proposition 2.3. *The divisor $D^{curv} \subset$ PRes(Γ, Γ_0) consists of l irreducible components $D_1, \ldots, D_l$. Moreover, if t is a generic point in D_i then the surface $\mathcal{Y}_t$ has exactly one smooth -2 curve in the class $i[C]$. These curves form the fibers of a divisor $\mathcal{C}_i$ inside the restriction of $\mathcal{Y}$ over D_i. The scheme-theoretic fiber of $\mathcal{C}_i$ over the origin is precisely the subscheme $C_i \subset Y_0$.*

PROOF: Consider the action of W_0 on the roots $\{v\}$ and the induced action on the hyperplanes $\{\mathfrak{D}_v\}$. Let e_1 denote the simple root corresponding to Γ_0. The coefficient α_1 of any root $v=\sum_{i=1}^n \alpha_i e_i$ is preserved by the generators of W_0 and so it is an invariant of the orbits of W_0 acting on $\{\mathfrak{D}_v\}$. For orbits with $\alpha_1 \neq 0$ it turns out that α_1 is a complete invariant:

Lemma 2.4. *If $v=\sum_{i=1}^n \alpha_i e_i$ is a root with $\alpha_1 \neq 0$, then $v'=\sum_{i=1}^n \alpha_i' e_i$ is in the W_0-orbit of v if and only if $\alpha_1'=\alpha_1$.*

Since the groups and root systems are finite and there are only a finite number of cases, this can easily be checked by hand. It is a fun exercise (really!) which we encourage the reader to carry out.

The W_0 orbits of $\{\mathfrak{D}_v\}$ with $\alpha_1 \neq 0$ are exactly the components of D^{curv} and the W_0 orbits of $\{\mathfrak{D}_v\}$ with $\alpha_1=0$ are the components of D^{sing}. By inspection, the possible non-zero values of α_1 are $1,\ldots,l$, and so define D_k to be the image of $\{\mathfrak{D}_v : v=\sum \alpha_i e_i \text{ has } \alpha_1 = k\}$ in $\mathrm{PRes}(\Gamma,\Gamma_1)$. Since the surface $\mathcal{Z}_t$ for $t\in\mathfrak{D}_v$ has a curve in the class $\sum_{i=1}^n \alpha_i[C_{e_i}]$, the surface $\mathcal{Y}_{t'}$ (where $t\mapsto t'$) has a curve in the class $\alpha_1[C]$. For generic $t\in\mathfrak{D}_v$ this curve is a single smooth $(-1,-1)$ curve and $\mathcal{Z}_t \cong \mathcal{Y}_{t'}$.

For the claim about C_i, note that after pulling back to $\mathrm{Res}(\Gamma)$, we can pick a discriminant component $\mathfrak{D}_v$ mapping to D_i, and a corresponding (possibly reducible) divisor $\mathcal{C}_v$ in the restriction of $\mathcal{Z}$ to $\mathfrak{D}_v$ whose fibers are the (possibly reducible) curves corresponding to the discriminant component $\mathfrak{D}_v$. The divisor $\mathcal{C}_i$ is the scheme-theoretic image of $\mathcal{C}_v$ under the restriction of $\tilde{\sigma}$ over the relevant discriminant components. The fiber C_v of $\mathcal{C}_v$ over the origin is itself an exceptional ADE configuration, and its scheme structure is defined by the pullback of the maximal ideal of the singularity being resolved. Writing its cycle class as $C_v=\sum_{j\geq 1}\alpha_j C_{e_j}$, we have that $\alpha_1 = i$. We have that C_v is scheme-theoretically defined by the ideal of functions vanishing to order α_j at each C_{e_j} [2]. We now look at the scheme-theoretic image of C_v after contracting the C_{e_j} for $j>1$ via $\tilde{\sigma}$. Its ideal consists of those functions f such that $\mathrm{ord}_{C_{e_j}}\tilde{\sigma}^*(f)\geq \alpha_j$, where $\mathrm{ord}_{C_{e_j}}$ denotes the order of vanishing along C_{e_j}. It remains to show that if $f\in\mathcal{I}^{(i)}$, then $\mathrm{ord}_{C_{e_j}}\tilde{\sigma}^*(f)\geq\alpha_j$ for all j. To expedite this task we use the following lemma.

Lemma 2.5. *Let f be a function on a neighborhood of C in Y_0. Put $m_k = \mathrm{ord}_{C_{e_j}}\tilde{\sigma}^*(f)$. Then for each $\tilde{\sigma}$-exceptional curve C_{e_k} we have $2m_k \geq \sum_j m_j$.*

The lemma follows immediately from the obvious assertion: for all $\tilde{\sigma}$-exceptional curves D, we have $\tilde{\sigma}^*((f))\cdot D\leq 0$.

It is now a simple matter to check in each case that the inequalities of Lemma 2.5 together with $m_1\geq\alpha_1=i$ imply that $m_k\geq\alpha_k$ for all k. □

Remark 2.6. *The proof of Proposition 2.3 shows that the scheme theoretic exceptional set of $Y_0\to X_0$ is defined by the ideal $\mathcal{I}^{(l)}$.*

We now wish to compute the Gromov-Witten invariants of Y in the class $d[C]$. Let $\tilde{f}:\Delta\to\mathrm{PRes}(\Gamma,\Gamma_0)$ be a small generic deformation of f and define $\tilde{Y}$ to be the total space of the induced one parameter family of surfaces, $\tilde{f}^{-1}(\mathcal{Y})$.

It makes sense to speak of the Gromov-Witten invariants of a local manifold like Y since for any closed 3-fold $\overline{Y}$ containing Y, the moduli space of (non-constant) stable maps has a connected component consisting of maps whose image lies entirely in Y. This is because a curve that lies in Y must be in the exceptional set of $Y \to X$ since X is an open set in an affine variety. We understand the "Gromov-Witten invariants of Y" to mean the virtual fundamental cycle restricted to that component. Alternatively, one can apply the analytic definition of the Gromov-Witten invariants directly to Y since the above argument shows that the moduli space of stable maps is compact, the (analytic) virtual class constructions of Li and Tian apply [20].

Lemma 2.7. *The Gromov-Witten invariants of $\tilde{Y}$ are well defined and equal to the Gromov-Witten invariants of Y.*

By the above argument, all of the complete curves of our deformations of Y lie in the exceptional set of $\mathcal{Y} \to \mathcal{X}$ which in turn lies over $D^{curv} \subset \mathrm{PRes}(\Gamma, \Gamma_0)$. Let

$$F : \Delta \times [0, \epsilon] \to \mathrm{PRes}(\Gamma, \Gamma_0)$$

be a smooth map so that $f_s := F|_{\Delta \times \{s\}}$ is analytic, $f_0 = f$, and F is transverse to D^{curv}. Furthermore, we may assume, by making ϵ smaller if necessary, that $F^{-1}(D^{curv})$ is bounded away from $\partial\Delta$. Let $\tilde{f} = f_\epsilon$. The parameterized moduli spaces are then compact and so the deformation invariance arguments in [20] apply. □

Since $\tilde{f}$ is transverse to D^{curv}, by Proposition 2.3, we see that the complete curves of $\tilde{Y}$ are all smooth $(-1,-1)$ (disjoint) curves in the classes $[C]$, $2[C]$, ... , $l[C]$. The number of irreducible curves in the class $i[C]$ is the intersection number of Δ with D_i, *i.e.* the cardinality of $\tilde{f}^{-1}(D_i)$. Thus Theorem 1.7 follows easily from Theorem 2.1 once we show that the intersection numbers of Δ with D_i are equal to the Hilbert scheme multiplicities.

Lemma 2.8. *The intersection number $\#\{\tilde{f}^{-1}(D_i)\}$ is equal to k_i, the multiplicity of C_i (Definition 1.6) in its Hilbert scheme.*

Proof: For each point in D_i, the corresponding surface has a unique irreducible complete curve. Let $\mathcal{C}_i \to D_i$ be the corresponding family of curves. We claim this family is a flat family of rational curves $\mathcal{C}_i \subset \mathcal{Y}$.

From the explicit construction in [18], we see that $\mathcal{C}_i \to D_i$ is projective so for flatness, it suffices to check that the genus and degree of the fibers are constant where any convenient polarization can be chosen for the purpose of defining the degree. The fibers are all rational, since they are the exceptional set of the map of a surface to a surface with a rational singularity. To see that the degree is constant, consider the pullback of $\mathcal{C}_i \to D_i$ to $\mathrm{Res}(\Gamma)$. We get a family $\sigma^{-1}(\mathcal{C}_i) \to \mathfrak{D}_i$ where $\mathfrak{D}_i = \sigma^{-1}(D_i)$ is the union of the components $\mathfrak{D}_v$ mapping to D_i. It is enough to show that $\sigma^{-1}(C_i) \to \mathfrak{D}_i$ has constant fiberwise degree. By restricting $(\tilde{\rho} \circ \tilde{\sigma})^{-1}(\mathcal{X})$, $\tilde{\sigma}^{-1}(\mathcal{Y})$, and $\mathcal{Z}$ to $\mathfrak{D}_i$, we get the families of singular, partially resolved, and minimally resolved surfaces over $\mathfrak{D}_i$ which we denote by $\mathcal{X}'$, $\mathcal{Y}'$, and $\mathcal{Z}'$ respectively. From [18], these surfaces all have projective compactifications and hence polarizations $L_{\mathcal{X}'}$, $L_{\mathcal{Y}'}$, and $L_{\mathcal{Z}'}$. We need to show that $\sigma^{-1}(\mathcal{C}_i) \subset \mathcal{Y}'$ has constant degree, *i.e.* the fibers of $\sigma^{-1}(\mathcal{C}_i)$ have constant intersection with $L_{\mathcal{Y}'}$, which we will refer to as " $\sigma^{-1}(\mathcal{C}_i) \cdot L_{\mathcal{Y}'}$ is constant on the fibers". Let $\mathcal{C} \subset \mathcal{Z}'$ be the exceptional set for the blowdown $\mathcal{Z}' \to \mathcal{X}'$ and let $\mathcal{C} = \mathcal{C}' + \mathcal{C}''$ where $\mathcal{C}''$ is the exceptional set for $b : \mathcal{Z}' \to \mathcal{Y}'$. Then the exceptional set $\mathcal{C} \to \mathfrak{D}_i$ is flat. This was written explicitly for all v in [18] in the cases of A_n (page 469) and D_n (page 473). This also holds true for E_n as can be checked directly case by case. It follows that $\mathcal{C} \cdot L_{\mathcal{Z}'}$ is constant on fibers. Furthermore, $L'_{\mathcal{Z}}$ can be chosen to be the pullback of $L'_{\mathcal{Y}}$ plus an appropriate multiple m of $\mathcal{C}''$. Since $b(\mathcal{C}'')$ is of lower dimension, $b_*(\mathcal{C}'') = 0$, and we have as an identity on the fibers

$$\sigma^{-1}(\mathcal{C}_i) \cdot L_{\mathcal{Y}'} = b_*(\mathcal{C}') \cdot L_{\mathcal{Y}'} = b_*(\mathcal{C}) \cdot L_{\mathcal{Y}'} = \mathcal{C} \cdot (L_{\mathcal{Z}'} - m\mathcal{C}''),$$

which is constant since the intersection number of the fibers of $\mathcal{C}$ and $\mathcal{C}''$ is constant. Thus $\mathcal{C}_i \to D_i$ is flat.

Now consider an analytic family f_ϵ of deformations of f parameterized by a disk B, *i.e.* an analytic map

$$F : \Delta \times B \to \mathrm{PRes}(\Gamma, \Gamma_0)$$

where $F|_{\Delta\times\epsilon}$ is denoted f_ϵ. Assume that $\tilde{f}$ is given by f_1. We get a finite map $\tilde{B} \to B$ whose degree is the intersection number $\#\{\tilde{f}^{-1}(D_i)\}$ by projecting $F^{-1}(D_i)$ to B. By the flatness of $\mathcal{C}_i \to D_i$, we get a flat family of curves $F^{-1}(\mathcal{C}_i) \to B$. This family sit inside the family of 3-folds $F^{-1}(\mathcal{Y}) \to B$ (whose fiber over ϵ is the 3-fold $f_\epsilon^{-1}(\mathcal{Y})$) and so determines a subscheme of the relative Hilbert scheme (or Douady space) of $F^{-1}(\mathcal{Y})/B$ that is finite over B. This family is isomorphic to $\tilde{B} \to B$ and since its degree is exactly the multiplicity of C_i in $Y = f_0^{-1}(\mathcal{Y})$, the lemma is proved.

2.2. **The case of a generic contractable A_n curve.** We now assume that Y is a neighborhood of a generic, contractable chain of n rational curves $C = C_{e_1} \cup \cdots \cup C_{e_n}$. Thus we are in the case where $\Gamma_0 = \Gamma = A_n$ and $Y = f^{-1}(\mathcal{Y})$ where $f : \Delta \to \mathrm{Res}(A_n)$. Our genericity assumption is that f is transverse to the components $\mathfrak{D}_{ij} \subset \mathrm{Res}(A_n)$ of the discriminant locus. That is, although f intersects the discriminant locus only at the origin, it meets each hyperplane $\mathfrak{D}_{ij}$ transversely. Here the hyperplanes $\{\mathfrak{D}_{ij} : 1 \le i \le j \le n\}$ are perpendicular to the roots $v_{ij} = \sum_{k=i}^{j} e_k$. This condition is related to the non-existence of infinitesimal deformations:

Lemma 2.9. *Let $f : \Delta \to \mathrm{Res}(A_n)$ be as above so that f intersects each hyperplane $\mathfrak{D}_{ij}$ transversely exactly once at $0 \in \mathrm{Res}(A_n)$. Let $Y = f^{-1}(\mathcal{Y})$ and let $C_{e_1} \cup \cdots \cup C_{e_n} = C \subset Y$ be the corresponding contractable curve. For any $1 \le i \le j \le n$, let $C_{ij} = C_{e_i} \cup \cdots \cup C_{e_j}$ and let N_{ij} be the normal bundle of C_{ij} in Y. Then $H^0(C_{ij}, N_{ij}) = 0$ for all i and j;* i.e. *C_{ij} has no infinitesimal deformations. Conversely, if $H^0(C_{ij}, N_{ij}) \neq 0$, then f meets $\mathfrak{D}_{ij}$ non-transversely.*

Proof: This follows directly from the description of deformations of the curves C_{ij} in Chapter 6 of [30], or from the characterization of $\mathfrak{D}_v$ in terms of deformations of C_v that was given in Proposition 2.2. Suppose that for some i and j, C_{ij} has a non-trivial infinitesimal deformation. It defines a morphism $h : \mathcal{C} \to Y$ where $\pi : \mathcal{C} \to \mathbf{I}$ is a curve over $\mathbf{I} = \mathrm{Spec}(\mathbf{C}[\epsilon]/\epsilon^2)$ and $h|_{\pi^{-1}(0)} : C_{ij} \hookrightarrow Y$. Since $\mathfrak{D}_{ij} \subset \mathrm{Res}(A_n)$ classifies those deformations to which C_{ij} lifts, the map $\mathcal{C} \to Y$ is induced by a classifying map $\mathbf{I} \to \mathfrak{D}_{ij}$ so we get a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] & \mathcal{Y}|_{\mathfrak{D}_{ij}} \arrow[l, hook'] \arrow[d] & \mathcal{C} \arrow[l] \arrow[d] \\
\mathbf{I} \arrow[r] & \Delta \arrow[r] & \mathrm{Res}(A_n) & \mathfrak{D}_{ij} \arrow[l, hook'] & \mathbf{I}. \arrow[l]
\end{tikzcd}$$

and so Δ lies in $\mathfrak{D}_{ij}$ to first order and we conclude that f is not transverse to $\mathfrak{D}_{ij}$. Conversely, if f is not transverse to $\mathfrak{D}_{ij}$, then the reverse of the above argument gives an infinitesimal deformation of C_{ij}. □

We can now compute the Gromov-Witten invariants of Y as in the previous case by deforming $Y = f^{-1}(\mathcal{Y})$ to $\tilde{Y} = \tilde{f}^{-1}(\mathcal{Y})$. The Gromov-Witten invariants of Y and $\tilde{Y}$ are well defined and equal by the same argument as Lemma 2.7 and all the curves of $\tilde{Y}$ are smooth $(-1,-1)$ curves arising from the transverse intersection of $\tilde{f}$ with $\mathfrak{D}_{ij}$ at generic points. Thus we see that there is exactly one $(-1,-1)$ curve in the class $\sum_{k=i}^{j}[C_{e_k}]$ for each $1 \leq i \leq j \leq n$ and these are all of the complete curves in $\tilde{Y}$. Combining this observation with Theorem 2.1 we get the following proposition:

Proposition 2.10. *Assume that $d_i > 0$ for $i = 1, \ldots, n$. Let $N_g(d_1, \ldots, d_n)$ denote the contribution of the genus g Gromov-Witten invariant of Y in the class $\sum_{i=1}^{n} d_i[C_{e_i}]$. Then*

$$N_g(d_1, \ldots, d_n) = \begin{cases} \frac{|B_{2g}|}{2g(2g-2)!} d^{2g-3} & \text{if } d_k = d \text{ for all } k \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to formulate and prove a generalization of Proposition 2.10 to the non-generic case, where each C_{ij} is assumed isolated but not necessarily infinitesimally isolated. The multiplicity of each C_{ij} in its Hilbert scheme appears in the formula.

We will also need the following result in the sequel.

Corollary 2.11. *Let Y' be an analytic 3-fold containing a projective curve C' and suppose that the image of any non-constant stable map to Y' lies in C'. Suppose that the formal neighborhood of C' in Y' is isomorphic to the formal neighborhood of C in Y where $C \subset Y$ is a generic, contractable chain of n curves in a local Calabi-Yau 3-fold Y (as above). Then the Gromov-Witten invariants (of non-zero degree) of Y' are given by the formula in Proposition 2.10.*

Proof: Since the moduli spaces of non-constant stable maps for Y and Y' consist of maps whose images are in C and C', the moduli spaces are isomorphic. Moreover, their corresponding virtual classes only depend on the formal neighborhood of the image curve and so these virtual classes also coincide. □

3. Relating nodal curves to chains.

In this section we relate the Gromov-Witten invariants of a super-rigid nodal rational curve in a Calabi-Yau 3-fold to the Gromov-Witten invariants of a neighborhood of a contractable chain of curves in a Calabi-Yau 3-fold.

Let $C_0 \subset \overline{Y}$ be a super-rigid, irreducible, rational curve C_0 with exactly one node in a projective 3-fold $\overline{Y}$. We consider a neighborhood (in the analytic topology) of C_0, which we denote by Y, that is sufficiently small enough so that all non-constant stable maps to $\overline{Y}$ whose image is contained in Y have image C_0. Then the moduli space $\overline{M}_0(\overline{Y}, d[C_0])$ of genus 0 stable maps of degree $d[C_0]$ to $\overline{Y}$ contains the moduli space $\overline{M}_0(Y, d[C_0])$ as a connected component. The restriction of the virtual class $[\overline{M}_0(\overline{Y}, d[C_0])]^{vir}$ to $\overline{M}_0(Y, d[C_0])$ defines a rational number N_d which is the contribution of C_0 to the degree $d[C_0]$ genus 0 Gromov-Witten invariant of $\overline{Y}$.

We construct a local Calabi-Yau 3-fold Y_n with a chain of curves $C_n = C_{e_1} \cup \cdots \cup C_{e_n} \subset Y_n$ and a local isomorphism

$$g_n : Y_n \to Y$$

restricting to the local immersion

$$g_n : C_n \to C_0.$$

We will show that a formal neighborhood of C_n in Y_n is isomorphic to a formal neighborhood of a generic contractable chain of curves in a local Calabi-Yau 3-fold.

To construct Y_n, let $g_\infty : Y_\infty \to Y$ be the universal cover of Y. We may assume that Y is a sufficiently small enough neighborhood of C_0 so that $\pi_1(Y) \cong \pi_1(C_0) \cong \mathbf{Z}$ and so Y_∞ is an open neighborhood of C_∞, a linear chain of a countable number of rational curves. Fix a subchain C_n of length n and let $Y_n \subset Y_\infty$ be an open neighborhood of C_n and let g_n be the restriction of g_∞ to Y_n. Since $K_{Y_\infty} = g_\infty^* K_Y$, we see that K_{Y_n} is trivial.

Lemma 3.1. *Super-rigidity of $C_0 \subset Y$ implies that a formal neighborhood of $C_n \subset Y_n$ is isomorphic to the formal neighborhood of a generic contractable curve (in the sense of subsection 2.2).*

PROOF: Let $N_{C_0/Y}$ be the normal bundle of C_0 in Y (note that C_0 and C_n are all local complete intersections). By super-rigidity,

$$H^0(C_n, g_n^*(N_{C_0/Y})) = 0.$$

Since g_n is a local isomorphism, $g_n^*(N_{C_0/Y}) \cong N_{C_n/Y_n}$. For any subchain $C_{ij} \subset C_n$, we have a injective sheaf map

$$0 \to N_{C_{ij}/Y_n} \to N_{C_n/Y_n}|_{C_{ij}}$$

and so $H^0(C_{ij}, N_{C_{ij}/Y_n}) = 0$ as well.

In particular, each component of $C_n \subset Y_n$ is a $(-1,-1)$-curve.

In his Ph.D. thesis [30], Zerger proves a contraction criterion for chains of rational curves in (formal) Calabi-Yau threefolds. It is an immediate consequence of his Theorems 6.1 and 6.2 that a chain of rational curves in a smooth Calabi-Yau threefold contracts in its formal neighborhood if no subchain of curves has infinitesimal deformations[3].

Thus C_n contracts in its formal neighborhood. Moreover, by Lemma 2.9, we have that this formal neighborhood of $C_n \subset Y_n$ is generic. $\square$

The connected components of $\overline{M}_0(Y, d[C_0])$ are identified by the following proposition:

[3]It is probably the case that this holds for an analytic neighborhood as well, but there is no proof in the literature.

Proposition 3.2. *The connected components of $\overline{M}_0(Y, d[C_0])$ are indexed by n-tuples $(d_1, \ldots, d_n)$ of positive integers with $\sum d_i = d$ and they are isomorphic to $\overline{M}_0(Y_n, (d_1, \ldots, d_n))$ where $(d_1, \ldots, d_n) \in H_2(Y_n, \mathbf{Z}) \cong H_2(C_n, \mathbf{Z})$ via the natural basis of $H_2(C_n, \mathbf{Z})$ indexed by the components of C_n. Furthermore, the virtual class on $\overline{M}_0(Y, d[C_0])$ agrees with the class induced by Y_n on $\overline{M}_0(Y_n, (d_1, \ldots, d_n))$.*

It follows immediately from this proposition and the genus 0 case of Corollary 2.11 (c.f. Proposition 2.10) that

$$N_d = \sum_{n|d} \frac{1}{n^3}$$

which proves Theorem 1.2.

PROOF: Let $f : \Sigma \to Y$ be a genus 0 stable map. Since $\pi_1(\Sigma) = 0$, f lifts to a map to the universal cover

$$\tilde{f} : \Sigma \to Y_\infty.$$

This map is unique up to deck transformations. Since Σ is compact, the image of $\tilde{f}$ is supported on some finite chain. After a deck transformation, we can assume that the image of $\tilde{f}$ is exactly C_n for some n. Therefore, each f factors uniquely as $g_n \circ \tilde{f}$ where $\tilde{f} \in \overline{M}_0(Y_n, (d_1, \ldots, d_n))$ for some n-tuple $(d_1, \ldots, d_n)$ of positive integers with $\sum d_i = d$. The same argument works equally well for families of stable maps and so we get an isomorphism of moduli functors

$$\coprod_{(d_1,\ldots,d_n), \sum d_i = d} \overline{M}_0(Y_n, (d_1, \ldots, d_n)) \to \overline{M}_0(Y, d[C_0])$$

where the isomorphism is given by composition with the appropriate g_n.

The virtual class $[\overline{M}_0(Y_n, (d_1, \ldots, d_n))]^{vir}$ is determined (as in [3]) by the perfect relative obstruction theory

$$[R^\bullet \pi_* \tilde{f}^*(T_{Y_n})]^\vee \to L^\bullet_{\overline{M}_0(Y_n,(d_1,\ldots,d_n))/\mathfrak{M}_0}$$

where $\pi : \mathcal{C} \to \overline{M}_0(Y_n, (d_1, \ldots, d_n))$ is the universal curve and $\tilde{f} : \mathcal{C} \to Y_n$ is the universal map. Since g_n is a local isomorphism, $T_{Y_n} \cong g_n^* T_Y$ and so $\tilde{f}^*(T_{Y_n}) = \tilde{f}^* g_n^*(T_Y)$. Since $g_n \circ \tilde{f}$ is the universal map for $\overline{M}_0(Y, d[C_0])$ under the above isomorphism, the two obstruction theories are isomorphic and thus the virtual classes agree. $\square$

Remark 3.3. The difficulty in generalizing this proof to higher genus Gromov-Witten invariants is that when Σ is no longer simply connected, we must consider maps that factor through the finite covers of Y as well. The two terms in Conjecture 1.8 correspond to the contributions of (1) maps factoring through the universal cover (which we know), and (2) maps that don't factor where the conjecture predicts the contribution is the same as if C_0 were a smooth super-rigid elliptic curve. Note that for the genus 0 invariants, we only need to assume 0-super-rigidity whereas to extend the argument to higher genus we would need 1-super-rigidity as well.

Remark 3.4. The difficulty in generalizing this proof to a rational curve with more nodes is that the universal cover is a curve whose dual graph is a complicated tree. One thus has to deal with curves not necessarily of ADE type and so our deformation technique does not apply. However, for degree five or less, the maps will factor through a map to the universal cover that either has an ADE image or is degree one on each component. Thus under the appropriate genericity conditions we can derive a formula for multiple covers of n-nodal irreducible rational curves for multiple covers of degree 5 or less.

4. Explicit computation in the length two (cD_4) case

In this section, we show how to compute the multiplicities k_1 and k_2 (see Theorem 1.7) directly from explicit equations for the 3-fold in the case of a curve contracting to a cD_4 singularity.

Recall that in the case of contractable a smooth rational curve C inside Y , the length of the singular point $p = \pi(C)$ inside X is at most six. When the length equals one, p is a cA_1 singularity and the divisor D^{curv} inside $\mathrm{PRes}(\Gamma, \Gamma_0)$ is just the origin of the affine line.

In this section we discuss in detail the length two case which corresponds to p being a cD_4 singularity. First we want to describe the divisor $D^{curv} = D_1 \cup D_2$ in a suitable coordinate system of $\mathrm{PRes}(\Gamma, \Gamma_0)$. To start, let x_1, x_2, x_3 and x_4 be any coordinate system for $\mathrm{PRes}(\Gamma, \Gamma_0)$ and the threefold Y is given by a map f:

$$\begin{aligned} f \quad &: \quad \Delta \to \mathrm{PRes}(\Gamma, \Gamma_0) \\ f(t) \quad &= \quad (x_1(t), x_2(t), x_3(t), x_4(t)). \end{aligned}$$

Notice that $x_i(t)$ vanishes at $t = 0$ for any i. In fact, we can choose a good coordinate system for $\mathrm{PRes}(\Gamma, \Gamma_0)$ in such a way that X can be described in term of an explicit equation as follows (see [16] and [18])[4]:

$$X = \left\{ x^2 + \frac{1}{z}(yz + x_2x_4)^2 = \frac{1}{z}F(z,t) \right\} \subset \mathbf{C}^3 \times \Delta,$$

where $F(z,t) = \left((z - x_2)^2 + x_1^2 z\right)\left(z^2 + x_3 z + x_4^2\right)$ and (x, y, z, t) are coordinates on $\mathbf{C}^3 \times \Delta$.

By direct computations, $D^{curv} = D_1 \cup D_2$ can be described explicitly as follows:

$$\begin{aligned} D_1 \quad &= \quad \left\{ (x_2^2 + x_2x_3 + x_4^2)^2 + x_1^2(x_1^2x_4^2 - 4x_2x_4^2 - x_3x_4^2 - x_3x_2^2) = 0 \right\}, \\ D_2 \quad &= \quad \{x_1 = 0\}. \end{aligned}$$

From a local analysis near the singular points of the partial resolution of a D_4 singularity, it is shown in [16] that the smoothness of Y implies that the order of vanishing of $x_i(t)$ at $t = 0$ is exactly one for $i = 2, 3$ or 4. On the other hand, the order of vanishing of $x_1(t)$ at $t = 0$ is precisely k_2 via the above description of D_2.

To determine k_1 we must compute the intersection number of $f(\Delta)$ with the hypersurface D_1 at the origin. In fact, k_1 is at least four since D_1 is defined by a polynomial which vanishes at the origin to the fourth order. We expect that $k_1 = 4$ in generic situation. We can describe this generic condition rather easily in terms of either $F(z,t)$ or f. If we set $z = x_2(t)$ then we have

$$F(x_2(t), t) = x_1^2 x_2 (x_2^2 + x_2x_3 + x_4^2)$$

Since $x_1(t)$ vanishes at $t = 0$ to order k_2 and the order of vanishing of the other $x_i(t)$'s is one, the order of vanishing of $F(x_2(t), t)$ at $t = 0$ is at least $2k_2 + 3$. If it is exactly $2k_2 + 3$, then $f(\Delta)$ is transverse to the hypersurface $x_2^2 + x_2x_3 + x_4^2 = 0$. By the description of D_1, this implies that $k_1 = 4$. An alternative way to see this is as follows. Let us write $x_i(t) = c_i t + h_i(t)t^2$ for $i = 2, 3$ or 4 for nonzero c_i's.

[4] The coordinate x_1 (resp. x_2, x_3 and x_4) we use here corresponds to σ_1 (resp. σ_2, s_2 and σ_2') in [16].

Then $c = [c_2, c_3, c_4]$ determines a point in $\mathbf{P}^2$. Then we have $k_1 = 4$ as long as this point does not lie inside the conic defined by $x_2^2 + x_2x_3 + x_4^2 = 0$. Therefore in such generic situation, we have $k_1 = 4$ and k_2 equals of order of vanishing of $x_1(t)$ at $t = 0$. When c does lie inside the above conic, then, in order to determine k_1, we also need to consider the fifth order terms in the defining equation of D_1. We would still obtain an explicit description of k_1 but it will be not as simple as in the generic case.

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References

[1] Paul S. Aspinwall and David R. Morrison. Topological field theory and rational curves. *Comm. Math. Phys.*, 151(2):245–262, 1993.

[2] W. Barth, C. Peters, and A. Van de Ven. *Compact Complex Surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, 1984.

[3] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.

[4] E. Brieskorn. Singular elements of semi-simple algebraic groups. pages 279–284, 1971.

[5] Egbert Brieskorn. Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen. *Math. Ann.*, 166:76–102, 1966.

[6] Egbert Brieskorn. Die Auflösung der rationalen Singularitäten holomorpher Abbildungen. *Math. Ann.*, 178:255–270, 1968.

[7] Jim Bryan and Naichung Conan Leung. The enumerative geometry of $K3$ surfaces and modular forms. alg-geom/9711031, 1997.

[8] D. M. Burns, Jr. and Jonathan M. Wahl. Local contributions to global deformations of surfaces. *Invent. Math.*, 26:67–88, 1974.

[9] Herbert Clemens. The infinitesimal Abel-Jacobi mapping and moving the $\mathcal{O}(2)\oplus\mathcal{O}(-4)$ curve. *Duke Math. J.*, 59(1):233–240, 1989.

[10] Herbert Clemens, János Kollár, and Shigefumi Mori. Higher-dimensional complex geometry. *Astérisque*, (166):144 pp. (1989), 1988.

[11] David A. Cox and Sheldon Katz. *Mirror symmetry and algebraic geometry.* American Mathematical Society, Providence, RI, 1999.

[12] C. Faber and R. Pandharipande. Hodge integrals and Gromov-Witten theory, 1998. Preprint, math.AG/9810173.

[13] Robert Friedman. Simultaneous resolution of threefold double points. *Math. Ann.*, 274(4):671–689, 1986.

[14] Rajesh Gopakumar and Cumrun Vafa. M-theory and topological strings–II, 1998. Preprint, hep-th/9812127.

[15] Jesús Jiménez. Contraction of nonsingular curves. *Duke Math. J.*, 65(2):313–332, 1992.

[16] Sheldon Katz. Small resolutions of Gorenstein threefold singularities. In *Algebraic geometry: Sundance 1988*, pages 61–70. Amer. Math. Soc., Providence, RI, 1991.

[17] Sheldon Katz, Albrecht Klemm, and Cumrun Vafa. M-theory, topological strings and spinning black holes. Preprint: hep-th/9910181.

[18] Sheldon Katz and David R. Morrison. Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups. *J. Algebraic Geom.*, 1(3):449–530, 1992.

[19] Maxim Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.

[20] Jun Li and Gang Tian. Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds. In *Topics in symplectic 4-manifolds (Irvine, CA, 1996)*, pages 47–83. Internat. Press, Cambridge, MA, 1998.

[21] Bong H. Lian, Kefeng Liu, and Shing-Tung Yau. Mirror principle. I. *Asian J. Math.*, 1(4):729–763, 1997.

[22] Yu. I. Manin. Generating functions in algebraic geometry and sums over trees. In R. Dijkgraaf, C. Faber, and G. van der Geer, editors, *The moduli space of curves*, pages 401–417. Birkhauser, 1995.

[23] R. Pandharipande. Hodge integrals and degenerate contributions, 1998. Preprint, math.AG/9811140.

[24] Henry C. Pinkham. Factorization of birational maps in dimension 3. In *Singularities, Part 2 (Arcata, Calif., 1981)*, pages 343–371. Amer. Math. Soc., Providence, RI, 1983.

[25] Miles Reid. Minimal models of canonical 3-folds. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, pages 131–180. North-Holland, Amsterdam, 1983.

[26] Titus Teodorescu. *Semistable torsion-free sheaves over curves of arithmetic genus one.* PhD thesis, Columbia University, 1999.

[27] Israel Vainsencher. Enumeration of n-fold tangent hyperplanes to a surface. *J. Algebraic Geom.*, 4(3):503–526, 1995.

[28] Claire Voisin. A mathematical proof of a formula of Aspinwall and Morrison. *Compositio Math.*, 104(2):135–151, 1996.

[29] Jonathan M. Wahl. Equisingular deformations of normal surface singularities. I. *Ann. of Math. (2)*, 104(2):325–356, 1976.

[30] Thomas Zerger. Contracting rational curves on smooth complex threefolds, 1996. Thesis, Oklahoma State University.

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