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G-bundles on Abelian surfaces, hyperkähler manifolds, and stringy Hodge numbers.

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ABSTRACT. We study the moduli space $M_G(A)$ of flat G-bundles on an Abelian surface A, where G is a compact, simple, simply connected, connected Lie group. Equivalently, $M_G(A)$ is the (coarse) moduli space of s-equivalence classes of holomorphic semi-stable $G^{\mathbf{C}}$-bundles with trivial Chern classes.

$M_G(A)$ has the structure of a hyperkähler orbifold. We show that when G is $Sp(n)$ or $SU(n)$, $M_G(A)$ has a natural hyperkähler desingularization which we exhibit as a moduli space of $G^{\mathbf{C}}$-bundles with an altered stability condition. In this way, we obtain the two known families of hyperkähler manifolds, the Hilbert scheme of points on a $K3$ surface and the generalized Kummer varieties. We show that for G not $Sp(n)$ or $SU(n)$, the moduli space $M_G(A)$ does *not* admit a hyperkähler resolution.

Inspired by the physicists Vafa and Zaslow, Batyrev and Dais define "stringy Hodge numbers" for certain orbifolds. These numbers have been proven to agree with the Hodge numbers of a crepant resolution (when it exists). We directly compute the stringy Hodge numbers of $M_{SU(n)}(A)$ and $M_{Sp(n)}(A)$, thus deriving formulas (originally due to Göttsche and Göttsche-Soergel) for the Hodge numbers of the Hilbert schemes of points on $K3$ surfaces and generalized Kummer varieties.

1. Results and the motivating examples.

Recent advances in certain string theories have inspired a resurgence of interest in the moduli space of G-bundles on elliptic curves [14] [15] [16] [17] [18] [36]. In these studies, care has been taken to develop methods that apply to arbitrary G and that are well suited to families of elliptic curves—the situation of physical interest is principal bundles on elliptic fibrations with structure group contained in $E_8 \times E_8$.

In this paper we study flat G-bundles on an Abelian surface A. We are primarily interested in the geometry of $M_G(A)$, the coarse moduli space, and so we will not address the existence of a universal family or the variation of $M_G(A)$ in families. This affords us the opportunity to keep the discussion of $M_G(A)$ very concrete and elementary; we have strived to give the paper some expository value in addition to reporting our findings.

Before we begin, we summarize our results in the following theorem, deferring definitions, explanations, and details to the rest of the paper.

Theorem 1.1. *Let G be a compact, simple, simply connected Lie group. Let $M_G(A)$ be the moduli space of flat G bundles on an Abelian surface A. Then*

1. *$M_G(A)$ has a hyperkähler resolution if and only if G is $SU(n)$ or $Sp(n)$ (Theorem 3.5);*
2. *In these cases, the resolution is realized as a certain moduli space of G-bundles, namely the moduli space of Mukai-stable (see Definition 5.2) $G^{\mathbf{C}}$-bundles (Theorems 5.3 and 5.4).*
3. *The stringy Hodge numbers of $M_{SU(n)}(A)$ and $M_{Sp(n)}(A)$ coincide with the ordinary Hodge numbers of their corresponding hyperkähler resolutions (Theorems 4.2 and 4.3).*

1.1. Notation

Fix A to be a principally polarized Abelian surface (we do this primarily for convenience—our results can be adapted to any complex torus of dimension 2 without much trouble) and fix E to be an elliptic curve. We choose origins $p_0 \in A$ and $p_0 \in E$. We will freely identify A and E with their duals $A^\vee$ and $E^\vee$ (using the polarizations). Let G be a compact, simple, simply connected, connected Lie group (e.g. $SU(n)$ or $Sp(n)$). Let $G^{\mathbf{C}}$ be the complexification of G (e.g. $SL(n, \mathbf{C})$ or $Sp(n, \mathbf{C})$), let r be the rank of G, and let W be its Weyl group.

1.2. Questions.

Let $M_G(A)$ (respectively $M_G(E)$) denote the moduli space of flat G connections on A (respectively E). Equivalently, $M_G(A)$ (respectively $M_G(E)$) is the (coarse) moduli space of s-equivalence classes of semi-stable holomorphic $G^{\mathbf{C}}$-bundles on A (respectively E) with trivial Chern classes.

In contrast to $M_G(E)$, $M_G(A)$ is not in general connected. We denote by $M^0_G(A)$ the component containing the trivial connection. This component can be described as a quotient of an r-fold product of A by an action of the Weyl group

$$M^0_G(A) \cong A^r/W.$$

The action of W preserves the natural holomorphic symplectic form on A^r and so $M^0_G(A)$ has a holomorphic symplectic form on the open dense locus of W orbits with trivial stabilizer (see Sections 2 and 3 for the details of these assertions).

The questions that motivated this work are:

Question 1.1. Does $M^0_G(A)$ have a smooth resolution $\widetilde{M}^0_G(A)$ to which the holomorphic symplectic form extends? Such a resolution would admit a hyperkähler metric.

Question 1.2. If $\widetilde{M}^0_G(A)$ exists, can it be realized as a moduli space for some moduli problem related to $G^{\mathbf{C}}$-bundles on A?

Question 1.3. How are the Hodge numbers of the desingularization $\tilde{M}^0_G(A)$ (if it exists) encoded in the action of W on A^r?

The answer to the first two questions is "yes" in the case when G is $Sp(n)$ or $SU(n)$. In these cases, $M_G^0(A) = M_G(A)$ and the hyperkähler manifolds obtained are exactly the two known families of irreducible hyperkähler manifolds. $\widetilde{M}_{Sp(n)}(A)$ is $\mathrm{Hilb}^n(X)$, the Hilbert scheme of n points on X, the Kummer $K3$ surface associated to A. $\widetilde{M}_{SU(n)}(A)$ is KA_{n-1}, the so called generalized Kummer variety which is the fiber of the map $\mathrm{Hilb}^n(A) \to A$ given by summing the points using the group law of A. We realize these resolutions as the moduli spaces of "Mukai-stable" $G^{\mathbf{C}}$-bundles (see Definition 5.2).

A framework for answering the third question is nicely provided by the "stringy Hodge numbers". These can be computed purely from the group theory that defines the action of W on A^r. In the case of $Sp(n)$ and $SU(n)$ we prove that they give exactly the Hodge numbers of the resolution[1].

1.3. The case of $SU(n)$

Since the case of $SU(n)$ was one of the motivating examples, we describe it in more detail. The rank of $SU(n)$ is $n-1$ and we have

$$A^{n-1} \hookrightarrow A^n$$

as the set of points $(x_1, \ldots, x_n)$ with $\sum x_i = 0$. The Weyl group W is the symmetric group S_n and its action on A^{n-1} is the restriction of the natural action on A^n. The identification

$$M_{SU(n)}(A) \cong A^{n-1}/S_n$$

is easy to understand in concrete terms. Points of $M_{SU(n)}(A)$ naturally correspond to s-equivalence classes of holomorphic semi-stable SL_n bundles with trivial Chern classes, that is bundles $\mathcal{E} \to A$ with $c_1(\mathcal{E}) = c_2(\mathcal{E}) = 0$ and an isomorphism $\det \mathcal{E} \cong \mathcal{O}_A$. In this case, every semi-stable bundle is strictly semi-stable and $\mathcal{E}$ can be decomposed (up to s-equivalence) into a sum of flat line bundles:

$$\mathcal{E} \cong L_{x_1} \oplus \cdots \oplus L_{x_n}$$

where L_x is the line bundle corresponding to $x \in A \cong \mathrm{Pic}^0 A$. This decomposition is unique up to s-equivalence and reordering the factors. The condition that $\det \mathcal{E} \cong \mathcal{O}$ imposes the condition $\sum x_i = 0$.

The singular points of $M_{SU(n)}(A)$ occur on the S_n-orbits with a non-trivial stabilizer. This occurs when two or more of the line bundles in the above description coinciding. When this happens, s-equivalence is rather brutal. It identifies many non-isomorphic bundles to a single moduli point. To illustrate, consider $SL(2, \mathbf{C})$ bundles on A. The moduli space is

$$M_{SU(2)}(A) \cong A/\pm 1$$

[1]After this paper was written, we learned that this result (and, in fact, Conjecture 4.1) follows from general results of Batyrev [2] and Denef-Loeser [11]. In light of this, one can view our computation, combined with the results in [2] or [11], as giving a new computation of the Hodge numbers of the Hilbert scheme of points on $K3$. The first computation of these numbers is due to Göttsche [22].

where the orbit $\{x, -x\}$ corresponds to the bundle $L_x \oplus L_{-x} = L_x \oplus L_x^{-1}$. The singular points occur for the sixteen two torsion points of A where $x = -x$. For a two torsion point τ, the moduli point $\{\tau, \tau\} \in A/\pm 1$ corresponds to the s-equivalence class of

$$L_\tau \otimes (\mathcal{O} \oplus \mathcal{O}).$$

For any non-trivial extension

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O} \to 0$$

the bundle $L_\tau \otimes \mathcal{E}$ is s-equivalent to $L_\tau \otimes (\mathcal{O} \oplus \mathcal{O})$. The natural parameter space for isomorphism classes of non-trivial extensions of $\mathcal{O}$ by $\mathcal{O}$ is

$$\mathbf{P}(\mathrm{Ext}^1(\mathcal{O}, \mathcal{O})) \cong \mathbf{P}(H^1(A, \mathcal{O})) \cong \mathbf{P}^1.$$

This suggests that if one could find a way to "destabilize" $L_\tau \otimes (\mathcal{O} \oplus \mathcal{O})$ and remove it from the moduli problem, then the corresponding moduli space should replace each of the sixteen double points of $M_{SU(2)}(A)$ with $\mathbf{P}^1$'s. Of course, if we blow up $A/\pm 1$ at the sixteen double points, we obtain X, the Kummer $K3$ surface which is a solution to Question 1.1 in this case. The above discussion also suggests a strategy for constructing S as a moduli space in order to answer Question 1.2. Following a suggestion of Aaron Bertram (that goes back to ideas of Mukai), we carry this out for G equal to $SU(n)$ or $Sp(n)$ in Section 5. In that section we define a new notion of stability (Mukai stability). The moduli space of Mukai stable bundles is then related to the Hilbert scheme of points on the dual Abelian surface via the Fourier-Mukai transform. Functorial properties of the Fourier-Mukai transform allow us to carefully analyze the condition that a bundle has a symplectic structure (the $Sp(n)$ case) where many subtleties occur.

1.4. The general case

Unfortunately, this program does not succeed in producing new examples of compact hyperkähler manifolds. We prove that $M_G(A)$ admits *no* hyperkähler resolution (in fact, no crepant resolution) for G not $SU(n)$ or $Sp(n)$ (Theorem 3.5). This situation has an analogue for the moduli space $M_G(E)$ of bundles on the elliptic curve E. In [37], Looijenga proves that $M_G(E)$ is a weighted projective space. The weighted projective space is smooth if and only if G is $SU(n)$ or $Sp(n)$. As we will explain, the same mechanism that causes $M_G(E)$ to fail smoothness, causes $M_G(A)$ to not admit a crepant resolution. This analogy continues to hold when we replace E with $\mathbf{C}$ and A with $\mathbf{C}^2$: Chevalley's theorem asserts that $\mathbf{C}^r/W$ is always smooth; a recently announced result of Bezrukavnikov-Ginzburg claims that $\mathbf{C}^{2r}/W$ always admits a holomorphic symplectic resolution. Thus the failure of $M_G(A)$ ($G \neq SU(n)$ or $Sp(n)$) to admit a holomorphic symplectic resolution has to do with global properties of A (like torsion points). We discuss this analogy and these results further in Section 3.

1.5. Stringy Hodge numbers

When a Calabi-Yau manifold X is acted on by a finite group H preserving the holomorphic volume form, Batyrev and Dais (based on ideas of the physicists Vafa [50] and

Zaslow [53]) define "stringy Hodge numbers" $h^{p,q}_{st}(X,H)$ [3]. In particular, if X is holomorphic symplectic (e.g. A^r) and the action of H preserves the symplectic form (e.g. W acting on A^r), then the numbers $h^{p,q}_{st}(X,H)$ are well defined. The stringy Hodge numbers are conjectured[2] to coincide with the ordinary Hodge numbers of a crepant resolution of X/H, if it exists (see Conjecture 4.1). This conjecture is part of the generalized McKay correspondence.

The situations where this conjecture has been tested are somewhat limited. It has been verified for $\dim X \leq 3$, and for H Abelian. Since the (ordinary) Hodge numbers of the resolutions of $M_{SU(n)}(A)$ and $M_{Sp(n)}(A)$ are known, the pairs (A^r, W) provide higher dimensional examples with non-Abelian group actions where the conjecture can be tested. This was done for $G = SU(n)$ by Göttsche (see Theorem 4.2); we verify the conjecture for $Sp(n)$ (Theorem 4.3). To our knowledge, there are no other higher dimensional, non-Abelian examples where this conjecture has been verified.

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2. G-bundles on elliptic curves and Abelian surfaces.

Much has been written recently concerning flat G-bundles/holomorphic $G^{\mathbf{C}}$-bundles on elliptic curves (for example [14] [15] [16] [17] [18] [36]). In this section we follow a "standard" approach to the construction of $M_G(E)$ and we develop the theory for Abelian surfaces in parallel. Since we are mainly interested in the geometry of the coarse moduli space $M_G(A)$, we take an elementary approach to its construction and ignore the issues of the existence of a universal bundle and the variation of $M_G(E)$ in a family.

Definition 2.1. Let $M_G(X)$ denote the moduli space of flat G-bundles on a path connected space X. It is given by

$$M_G(X) = \mathrm{Hom}(\pi_1(X), G)/G$$

where G acts on a representation by conjugation.

When X is Kähler, there is a correspondence between flat G-bundles and certain holomorphic $G^{\mathbf{C}}$-bundles. In the case of E and A it is a special case of the famous theorems of Narasimhan-Seshadri and Donaldson (generalized by Uhlenbeck and Yau [49]):

[2] This conjecture has, in fact, been verified. See the earlier footnote in section 1.2.

Theorem 2.1 (Narasimhan-Seshadri, Donaldson). *$M_G(E)$ (respectively $M_G(A)$) is isomorphic to the coarse moduli space of s-equivalence classes of semi-stable holomorphic $G^{\mathbf{C}}$-bundles on E (respectively A) with vanishing Chern classes. In particular, $M_G(E)$ and $M_G(A)$ are projective varieties.*

See [32] or Section 5 for the definitions of semi-stable and s-equivalence. For the most part we will work with the topological description of these moduli spaces, but we will identify and use the holomorphic structure coming from the above theorem.

2.1. Reduction to a finite quotient

Choose a maximal torus $T \subset G$. By a classical result of Borel [6], any pair of commuting elements in a compact, simply connected Lie group lie in the same maximal torus and thus can be simultaneously conjugated to the fixed torus T. Noting that $W = N(T)/T$ is the normalizer of T quotiented by T, we have

$$\begin{aligned} M_G(E) &\cong \mathrm{Hom}(\pi_1(E), G)/G \\ &\cong \mathrm{Hom}(\pi_1(E), T)/W \\ &\cong (T \times T)/W. \end{aligned}$$

In general, three or more commuting elements *do not* all lie in the same maximal torus (although it is true for $SU(n)$ and $Sp(n)$), so the above analysis for $M_G(A)$ does not apply. However, the condition that commuting elements all lie in the same maximal torus is both open and closed in $M_G(A)$ so if we restrict our attention to the connected component containing the trivial connection, the above argument will apply.

Definition 2.2. Let $M^0_G(A) \subset M_G(A)$ be the connected component containing the trivial connection.

By the previous argument, we then have

$$\begin{aligned} M^0_G(A) &\cong \mathrm{Hom}(\pi_1(A), T)/W \\ &\cong (T \times T \times T \times T)/W. \end{aligned}$$

The above description does not make the complex structure of $M^0_G(A)$ apparent. To do this we define the *coroot lattice* Λ by the kernel of the exponential map to T:

$$0 \to \Lambda \to \mathfrak{t} \to T \to 0.$$

An element $\pi_1(A) \to T$ of $\mathrm{Hom}(\pi_1(A), T)$ is dual to a homomorphism

$$\mathrm{Hom}(T, S^1) \to \mathrm{Hom}(\pi_1(A), S^1) \cong A^\vee \cong A.$$

The first group is just $\Lambda^\vee$ and so the above homomorphism is an element of $\Lambda \otimes A$. In this way we have a natural isomorphism

$$\mathrm{Hom}(\pi_1(A), T) \cong \Lambda \otimes A.$$

The action of W on Λ induces an action on $\Lambda \otimes A$ and the complex structure of A induces a holomorphic structure on the quotient. The same discussion applies to E and so we have

$$\begin{aligned} M_G(E) &\cong (\Lambda \otimes E)/W \qquad (1)\\ M_G^0(A) &\cong (\Lambda \otimes A)/W. \end{aligned}$$

Although we will not prove it, this holomorphic structure is the same as the one determined by Theorem 2.1. Since Λ is a rank r lattice, we may choose a $\mathbf{Z}$-basis and write $\Lambda \otimes A \cong A^r$ as we did in the first section.

2.2. An example of an unusual commuting triple

Before we continue our study of $M_G^0(A)$, we give an example (adapted from a talk of Witten) showing that there are commuting elements in a simply connected Lie group that do not all lie in the same maximal torus T.

Proposition 2.2. *Consider the following commuting matrices in $SO(8)$:*

$$\begin{aligned} a &= \mathrm{Diag}(+1,-1,+1,-1,+1,-1,+1,-1)\\ b &= \mathrm{Diag}(+1,+1,-1,-1,+1,+1,-1,-1)\\ c &= \mathrm{Diag}(+1,+1,+1,+1,-1,-1,-1,-1). \end{aligned}$$

Choose lifts $\widetilde{a}, \widetilde{b}, \widetilde{c} \in Spin(8)$. Then $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ are mutually commuting elements of $Spin(8)$ that do not all lie in a single maximal torus.

Remark 2.1. In a non-simply connected group, it is easy to find even just two commuting elements that do not lie in a single maximal torus. For example, $\mathrm{Diag}(-1,-1,+1)$ and $\mathrm{Diag}(+1,-1,-1)$ are a commuting pair of $SO(3)$ matrices that are in different maximal tori (they have different axis of rotation), however any lifts of these elements to the simply connected cover $SU(2)$ will not commute—their commutator is $-Id$. For an extensive study of commuting pairs and triples see [7] and also [46] or [33].

Proof of Proposition 2.2: We first show that a, b, and c do not lie in the same maximal torus in $SO(8)$. We then show that the lifts $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ mutually commute. The result will then follow since if $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ were contained in the same maximal torus in $Spin(8)$, then a, b, and c would be contained in the image torus in $SO(8)$.

Let T^3 be the three torus. The elements a, b, and c determine a representation $\pi_1(T^3) \to SO(8)$. *i.e.* a flat $SO(8)$ connection. If a, b, and c were contained in the same maximal torus, then they could be simultaneously conjugated to T, and the associated flat bundle would correspond to a moduli point in $(T \times T \times T)/W$. This bundle would hence have deformations as a flat bundle. We will show that it does not have deformations.

Real line bundles with a flat connection are parameterized by $H^1(T^3, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^3$. Let $\{R_\alpha\}$ be the eight flat line bundles corresponding to elements α of $H^1(T^3, \mathbf{Z}/2)$. The holonomy of the direct sum connection on the rank eight bundle

$$E = \bigoplus_{\alpha \in H^1(T^3, \mathbf{Z}/2)} R_\alpha$$

around the generators of $\pi_1(T^3)$ is given by the matrices a, b, and c. To show that E has no deformations as a flat $SO(8)$ bundle we compute the deformation space

$$H^1(T^3, \mathfrak{so}(E)).$$

The bundle $\mathfrak{so}(E)$ can be described as the skew symmetric endomorphisms:

$$\begin{aligned} \mathfrak{so}(E) &\subset \mathrm{End}(E) \\ &= \oplus_{\alpha,\beta} R_\alpha \otimes R_\beta^\vee \\ &\cong \oplus_{\alpha,\beta} R_\alpha \otimes R_\beta \end{aligned}$$

so that, $\mathfrak{so}(E) = \oplus M_{\alpha\beta}$ where the sum is over unordered pairs (α, β) with $\alpha \neq \beta$ and $M_{\alpha\beta}$ is by definition the rank 1 subbundle of $(R_\alpha \otimes R_\beta) \oplus (R_\beta \otimes R_\alpha)$ with local sections $(s_\alpha \otimes s_\beta, -s_\beta \otimes s_\alpha)$. Note that $M_{\alpha\beta} \cong R_\alpha \otimes R_\beta$ as a flat line bundle and $R_\alpha \otimes R_\beta \cong R_{\alpha+\beta}$ is the trivial bundle if and only if $\alpha = \beta$. Thus $\mathfrak{so}(E)$ is a sum of flat line bundles with no trivial factors. Our claim will then follow when we show that $H^1(T^3, R_\alpha) = 0$ if $\alpha \neq 0$.

Viewing T^3 as $S^1 \times S^1 \times S^1$ we can decompose α as $(\alpha_1, \alpha_2, \alpha_3)$ by the Kunneth theorem. Then

$$R_\alpha \cong \pi_1^*(R_{\alpha_1}) \otimes \pi_2^*(R_{\alpha_2}) \otimes \pi_3^*(R_{\alpha_3})$$

where π_i is the projection on to the ith factor and R_{α_i} is the line bundle corresponding to $\alpha_i \in H^1(S^1, \mathbf{Z}/2)$. We then have (again by the Kunneth theorem)

$$\begin{aligned} H^1(T^3, R_\alpha) &\cong H^1(S^1, R_{\alpha_1}) \otimes H^0(S^1, R_{\alpha_2}) \otimes H^0(S^1, R_{\alpha_3}) \\ &\oplus H^0(S^1, R_{\alpha_1}) \otimes H^1(S^1, R_{\alpha_2}) \otimes H^0(S^1, R_{\alpha_3}) \\ &\oplus H^0(S^1, R_{\alpha_1}) \otimes H^0(S^1, R_{\alpha_2}) \otimes H^1(S^1, R_{\alpha_3}). \end{aligned}$$

Now $H^0(S^1, R_{\alpha_i}) = 0$ for R_{α_i} non-trivial and $\dim H^0(S^1, R_{\alpha_i}) = \dim H^1(S^1, R_{\alpha_i})$ by the index theorem and so $H^1(S^1, R_{\alpha_i}) = 0$ for R_{α_i} non-trivial. Thus $H^1(T^3, R_\alpha) = 0$ unless $\alpha = (\alpha_1, \alpha_2, \alpha_3) = 0$ and so we conclude that $H^1(T^3, \mathfrak{so}(E)) = 0$.

Finally, the lifts $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ mutually commute if and only if the bundle E is spin. We compute $w_2(E)$ by the Whitney product formula:

$$w(E) = \prod_{\alpha \in H^1(T^3; \mathbf{Z}/2)} (1 + \alpha)$$

and so

$$w_2(E) = \sum_{\alpha, \beta \in H^1(T^3; \mathbf{Z}/2)} \alpha \cup \beta = 0$$

by the skew-symmetry of the cup product on $H^1(T^3; \mathbf{Z}/2)$. Thus E is spin and the proposition is proved. $\square$

G	$(g_1, \dots, g_r)$
$SU(n)$	$(1, \dots, 1)$
$Sp(n)$	$(1, \dots, 1)$
$Spin(2n)$	$(1, 1, 1, 2, \dots, 2)$
$Spin(2n+1)$	$(1, 1, 2, \dots, 2)$
G_2	$(1, 2)$
F_4	$(1, 2, 2, 3)$
E_6	$(1, 1, 2, 2, 2, 3)$
E_7	$(1, 2, 2, 2, 3, 3, 4)$
E_8	$(2, 2, 3, 3, 4, 4, 5, 6)$

TABLE 1. Coefficients of the highest coroots in terms of the simple coroots.

We leave it as an exercise to the reader to translate the above argument into a purely algebraic proof.

2.3. Looijenga's theorem

We return to our study of $M_G(E)$ and $M_G^0(A)$. The geometry of $M_G(E)$ is completely determined by Looijenga's theorem:

Theorem 2.3 (Looijenga [37]). *$M_G(E) \cong (\Lambda \otimes E)/W$ is isomorphic to a weighted projective space $\mathbf{P}(1, g_1, \dots, g_r)$ where the weights g_i are the coefficients of the highest coroot expressed in terms of the simple coroots (see Table 1). In particular, $(\Lambda \otimes E)/W$ is a smooth projective space $\mathbf{CP}^r$ if and only if G is $SU(n)$ or $Sp(n)$.*

Notice that the theorem fails for G not simple. For example, if $G = U(n)$, then $\Lambda_{U(n)} \cong \mathbf{Z}^n$ and $W_{U(n)} \cong S_n$ acting on $\mathbf{Z}^n$ by permuting the factors. Thus

$$\begin{aligned} M_{U(n)}(E) &\cong E^n/S_n \\ &= \mathrm{Sym}^n(E) \end{aligned}$$

is the nth symmetric product of E. However, we have an inclusion $\Lambda_{SU(n)} \subset \Lambda_{U(n)}$ as the rank $n-1$ sublattice of points $e_1, \dots, e_n$ with $\sum e_i = 0$ and the action of $W_{SU(n)} = W_{U(n)}$ on $\Lambda_{SU(n)}$ is the restriction of the action on $\Lambda_{U(n)}$. Thus we see that $M_{SU(n)}(E)$ is the fiber over the origin p_0 of the sum map

$$\begin{tikzcd} M_{SU(n)}(E) \arrow[r] & \mathrm{Sym}^n(E) \arrow[d, "\mathrm{sum}"] \\ & E \end{tikzcd}$$

By viewing $\mathrm{Sym}^n(E)$ as the space of effective degree n divisors on E and using the canonical isomorphism $E \cong \mathrm{Pic}^n E$, the above sum map is identified with the Abel-Jacobi map. Then the fiber $M_{SU(n)}(E)$ gets identified with the linear system $|\mathcal{O}(np_0)|$ which is indeed a projective space of dimension $n-1$ as predicted by Looijenga's theorem.

The predicted isomorphism $M_{Sp(n)}(E) \cong \mathbf{P}^n$ arises in a slightly different way. In this case $\Lambda_{Sp(n)} \cong \mathbf{Z}^n$ and $W_{Sp(n)}$ is a semi-direct product of S_n and $\{\pm 1\}^n$. W acts on Λ by permuting the factors and multiplying each factor by ± 1 (see the proof of Theorem A.2 for a detailed discussion of the coroot lattice and W action in this case). Thus

$$\begin{aligned} M_{Sp(n)}(E) &\cong E^n/(S_n \ltimes \{\pm 1\}^n) \\ &\cong (E/\pm 1)^n/S_n \\ &\cong \mathrm{Sym}^n(\mathbf{P}^1) \\ &\cong \mathbf{P}^n. \end{aligned}$$

2.4. Examples: $M_{SU(n)}(A)$ and $M_{Sp(n)}(A)$

We can apply the previous analysis in the case of the Abelian surface A. For these cases ($Sp(n)$ and $SU(n)$), any collection of commuting matrices is contained in the same maximal torus. Thus $M^0_{Sp(n)}(A) = M_{Sp(n)}(A)$ and $M^0_{SU(n)}(A) = M_{SU(n)}(A)$. As in the elliptic curve case, we find that $M_{SU(n)}(A)$ is the fiber of the sum map

$$\begin{tikzcd} M_{SU(n)}(A) \arrow[r] & \mathrm{Sym}^n(A) \arrow[d, "\mathrm{sum}"] \\ & A \end{tikzcd}$$

and $M_{Sp(n)}(A)$ is a symmetric product

$$M_{Sp(n)}(A) \cong \mathrm{Sym}^n(A/\pm 1).$$

Note that unlike for curves, the symmetric product of a surface is singular. Similarly, while $E/\pm 1 \cong \mathbf{P}^1$ is smooth, $A/\pm 1$ is singular. However, these spaces have natural desingularizations. In general for a surface S, the Hilbert scheme of n points on S together with the Hilbert-Chow map is a desingularization of $\mathrm{Sym}^n(S)$:

$$\mathrm{Hilb}^n(S) \to \mathrm{Sym}^n(S)$$

(we discuss the Hilbert scheme of points in more detail in Section 3). Likewise, $A/\pm 1$ has a natural desingularization which is the Kummer $K3$ surface X associated to A. Thus we can construct *ad hoc* desingularizations of $M_{SU(n)}(A)$ and $M_{Sp(n)}(A)$ as follows. Define $\widetilde{M}_{SU(n)}(A)$ to be the fiber over p_0 of the composition $\mathrm{Hilb}^n(A) \to \mathrm{Sym}^n(A) \to A$ so that we have:

$$\begin{tikzcd}
\widetilde{M}_{SU(n)}(A) \arrow[r] \arrow[d] & M_{SU(n)}(A) \arrow[d] \\
\mathrm{Hilb}^n(A) \arrow[r] \arrow[d] & \mathrm{Sym}^n(A) \arrow[d, "\mathrm{sum}"] \\
A \arrow[r, equal] & A.
\end{tikzcd}$$

Define $\widetilde{M}_{Sp(n)}(A)$ to be $\mathrm{Hilb}^n(X)$ so that we get the desingularization:

$$\widetilde{M}_{Sp(n)}(A) = \mathrm{Hilb}^n(X) \to \mathrm{Sym}^n(X) \to \mathrm{Sym}^n(A/\pm 1) \cong M_{Sp(n)}(A).$$

In Section 3 we will see that these *ad hoc* desingularizations are exactly the two known families (up to deformation) of compact irreducible hyperkähler manifolds. These give an affirmative answer to Question 1.1 for $Sp(n)$ and $SU(n)$.

In Section 5 we will realize these desingularizations as moduli spaces giving an affirmative answer to Question 1.2 for these cases.

3. Hyperkähler manifolds, holomorphic symplectic manifolds, and crepant resolutions.

In this section we first give brief expositions of hyperkähler manifolds and holomorphic symplectic manifolds. A general source for this material is [41] and the references therein. We then use some basic facts about crepant resolutions to determine which $M_G(A)$ have hyperkähler resolutions. The main result of this section is that $M_G(A)$ does *not* admit a hyperkäher resolution unless G is $SU(n)$ or $Sp(n)$.

3.1. Hyperkähler manifolds.

A $4n$ dimensional Riemannian manifold (X, g) is called *hyperkähler* if the holonomy group of the Levi-Civita connection is contained in $Sp(n)$. It is called *irreducible hyperkähler* if the holonomy group is exactly $Sp(n)$. It is well known that up to finite covers, every hyperkäher manifold is a product of irreducible hyperkähler manifolds and flat tori (e.g. [4]). The hyperkähler condition is equivalent to the existence of a triple of almost complex structures (I, J, K) such that each is integrable and the metric g is Kähler with respect to any of these structures, and (I, J, K) satisfy the algebra of the quaternions; that is

$$\nabla I = \nabla J = \nabla K = 0,$$

and

$$I^2 = J^2 = K^2 = IJK = -1.$$

In fact, there is a whole 2-sphere of Kähler structures: for each (a,b,c) with $a^2+b^2+c^2=1$, the almost complex structure $\lambda = aI+bJ+cK$ is integrable and g is Kähler with respect to λ. This family of Kähler structures is called the twistor family (c.f. [9]).

The holonomy condition imposes very restrictive conditions on the Hodge theory of a compact hyperkähler manifold X. Since $Sp(n) \subset SU(2n)$, hyperkähler manifolds are Ricci flat and so $h^{0,2n}(X) = 1$. In fact, the whole Hodge diamond is "mirror symmetric"; that is,

$$H^{p,q}(X) \cong H^{2n-p,q}(X).$$

This isomorphism is obtained by wedging a harmonic (p,q) form with a holomorphic symplectic form (see below) $n-p$ times [31].

Examples of hyperkähler manifolds can be obtained from other hyperkähler manifolds by a process analogous to symplectic reduction. Suppose a hyperkähler manifold admits an action of a compact Lie group G preserving (g,I,J,K), then Hitchin et. al. [29] introduced the notion of a hyperkähler moment map

$$\mu : X \to \mathbf{R}^3 \otimes \mathfrak{g}^*$$

and under suitable conditions, they show that the quotient $\mu^{-1}(\zeta)/G$ has a natural induced hyperkähler structure (for example, see [35]). However, no known, non-trivial examples of this type are compact unless the original hyperkähler manifold and group are both infinite dimensional.

3.2. Holomorphic symplectic manifolds

A Kähler manifold X of complex dimension $2n$ is a *holomorphic symplectic manifold* if there exists a closed, non-degenerate holomorphic 2-form $\sigma \in H^0(X, \Omega_X^2)$. Non-degenerate means that σ^n is a non-vanishing section of $\Omega_X^{2n} = K_X$. A holomorphic symplectic manifold is called *irreducible* if $h^0(X, \Omega_X^2) = 1$. The following is due to Beauville [4]:

Theorem 3.1. *A compact manifold X has an (irreducible) hyperkähler metric if and only if it has a metric such that it is an (irreducible) holomorphic symplectic manifold.*

Remark 3.1. If one removes the Kähler condition in the definition of holomorphic symplectic, then this theorem no longer holds. Examples of compact (non-Kähler) complex manifolds with holomorphic symplectic forms and no hyperkähler structure were constructed by Guan [25][26].

Sketch of proof of Theorem 3.1: Suppose that (X,g) is hyperkähler and let (ω_I, I), (ω_J, J), and (ω_K, K) be the defining Kähler structures. Then with respect to the Kähler structure (ω_I, I), it is easy to check that the form $\sigma = \omega_J + i\omega_K$ is a holomorphic symplectic form. Conversely, suppose (X,g) is a holomorphic symplectic manifold with

Kähler form ω and holomorphic symplectic form σ. Then σ^n defines a trivialization of K_X and so by Yau's solution to the Calabi conjecture [52], there is a unique Ricci flat metric for which ω is Kähler. This gives a reduction of the holonomy group from $U(2n)$ to $SU(2n)$. Since the Ricci curvature is zero, the standard Bôchner argument using the Weitzenböck formula shows that $\nabla\sigma \equiv 0$. Thus the holonomy group is contained in $SU(2n) \cap Sp(n, \mathbf{C}) = Sp(n)$. One can sharpen this argument to conclude that the two notions of irreducibility coincide. □

3.3. Examples

Theorem 3.1 can be used to construct examples of compact hyperkähler manifolds using the Hilbert scheme of points on a surface.

Let $\mathrm{Hilb}^n(X)$ denote the Hilbert scheme parameterizing 0 dimensional subschemes of length n in a smooth projective surface X (a.k.a. the Hilbert scheme of n points). This turns out to be a smooth projective variety of dimension $2n$ with many beautiful properties (see the book by Göttsche [22]). There is a proper morphism (the Hilbert-Chow morphism) from the Hilbert scheme to the symmetric product

$$\mathrm{Hilb}^n(X) \to \mathrm{Sym}^n(X)$$

that sends a subscheme $Z \subset X$ to its support (with multiplicities). Via this map, $\mathrm{Hilb}^n(X)$ is a smooth resolution of $\mathrm{Sym}^n(X)$.

The exceptional strata of $\mathrm{Hilb}^n(X)$ are in general very complicated, but over the locus in $\mathrm{Sym}^n(X)$ where no more than two points coincide, $\mathrm{Hilb}^n(X)$ can be described explicitly: The Hilbert-Chow morphism is an isomorphism on the locus of configurations of n distinct points; over configurations with exactly two points coinciding at x the fiber is a $\mathbf{CP}^1$ parameterizing the lines in T_xX. Geometrically, $\mathrm{Hilb}^n(X)$ records in which direction the two points come together. A local model[3] for a configuration in $\mathrm{Sym}^n(X)$ with exactly 2 points coinciding is

$$\mathrm{Sym}^2(\mathbf{C}^2) \times \mathbf{C}^{2n-4}$$

and the Hilbert-Chow morphism is locally a product:

$$\mathrm{Hilb}^2(\mathbf{C}^2) \times \mathbf{C}^{2n-4} \to \mathrm{Sym}^2(\mathbf{C}^2) \times \mathbf{C}^{2n-4}.$$

Now $\mathrm{Sym}^2(\mathbf{C}^2) = (\mathbf{C}^2 \times \mathbf{C}^2)/S_2$ which, after a linear change of variables, is just $(\mathbf{C}^2/\pm 1) \times \mathbf{C}^2$. The rational double point in $\mathbf{C}^2/\pm 1$ can be resolved by blowing up. The resulting space is the total space of the cotangent bundle of $\mathbf{CP}^1$ and the map

$$T^*\mathbf{CP}^1 \to \mathbf{C}^2/\pm 1$$

[3]We say that $A \subset X$ *has a local model* or *is locally modeled on* $A \subset Y$ if there is an analytic neighborhood of A in X that is complex analytically isomorphic to a neighborhood of A in Y. Note that A could just be a point and if the subspace or the ambient space is clear from the context, then we will drop them from the terminology (e.g. "the subset B is locally modeled on Y").

contracts the zero section to the double point. The resolution $\mathrm{Hilb}^n(X) \to \mathrm{Sym}^n(X)$ near the locus where 2 points coincide is locally modeled on

$$T^*\mathbf{CP}^1 \times \mathbf{C}^{2n-2} \to (\mathbf{C}^2/\pm 1) \times \mathbf{C}^{2n-2}. \tag{2}$$

Note that this local model has a holomorphic symplectic form since for any complex manifold M, T^*M has a canonical holomorphic symplectic form (a fact analogous to the corresponding fact for real manifolds and real symplectic forms).

This description enabled Fujiki [19] and Beauville [4] to construct examples of compact holomorphic symplectic manifolds (and hence compact hyperkähler manifolds) from Hilbert schemes of points.

Theorem 3.2. *If X is an algebraic surface that is holomorphic symplectic, then $\mathrm{Hilb}^n(X)$, the Hilbert scheme of n points on X, is a holomorphic symplectic manifold (of complex dimension $2n$).*

SKETCH OF PROOF: Recall that $\mathrm{Hilb}^n(X)$ is a smooth resolution of $\mathrm{Sym}^n(X) = X^n/S_n$. If X has a holomorphic symplectic form, then X^n has a natural holomorphic symplectic form that is invariant under the action of S_n. Thus $\mathrm{Sym}^n(X)$ has a holomorphic symplectic form on the open set of S_n-orbits with trivial stabilizer. The map $\mathrm{Hilb}^n(X) \to \mathrm{Sym}^n(X)$ restricts to an isomorphism on this set, so we get a holomorphic symplectic form on $\mathrm{Hilb}^n(X)$ defined on the complement of the exceptional set. We need to show that this form extends to a non-degenerate form on all of $\mathrm{Hilb}^n(X)$. This form can be extended to the complement of the codimension 2 set where 3 or more points come together using the canonical symplectic form on the local model (Equation 2) on this locus. The form then automatically extends across the codimension 2 strata (by Hartog's theorem) to a form σ. The form is non-degenerate since if σ^n had a non-empty zero set, it would have codimension one, but σ is non-degenerate in codimension two by construction. □

Remark 3.2. The restriction to algebraic surfaces is not necessary. The same argument applies when X is a non-algebraic, holomorphic symplectic surface if we replace the Hilbert scheme with the corresponding Douady space. We restrict to algebraic surfaces for convenience only.

From the Kodaira-Enriques classification of compact complex surfaces, we know that if a compact algebraic surface X is holomorphic symplectic, then X must be either a $K3$ or an Abelian surface. If X is a $K3$ surface, then $h^0(\mathrm{Hilb}^n(X), \Omega^2) = 1$ (Göttsche [21]) so $\mathrm{Hilb}^n(X)$ is an irreducible hyperkähler manifold. Since any two $K3$ surfaces are deformation equivalent, all the examples produced in this way are deformation equivalent.

For an Abelian surface A, $\mathrm{Hilb}^n(A)$ is not irreducible. However, one can easily see that the holomorphic symplectic form is non-degenerate on the fibers of the map $\mathrm{Hilb}^n(A) \to A$ given by the composition of the Hilbert-Chow map and the sum map $\mathrm{Sym}^n(A) \to A$. Thus the fibers of $\mathrm{Hilb}^n(A) \to A$, which are, by definition, the generalized Kummer varieties KA_{n-1}, are holomorphic symplectic. One can also check that KA_{n-1} are irreducible.

Until recently, the only known examples of compact irreducible hyperkähler manifolds were deformation or birationally equivalent to $\mathrm{Hilb}^n(X)$ for a $K3$ surface X or KA_n for some Abelian surface A. In particular, all the known examples had the same Betti numbers[4] as $\mathrm{Hilb}^n(X)$ or KA_n. However, O'Grady has recently constructed an isolated example in dimension 10 that does not have the same Betti numbers as $\mathrm{Hilb}^5 X$ or KA_5 [42].

3.4. Crepant resolutions

The hyperkähler manifolds KA_n and $\mathrm{Hilb}^n(X)$ appear as resolutions of orbifolds. As we showed in Section 2, these orbifolds are $M^0_G(A) = A^n/W$ where G is $SU(n+1)$ and $Sp(n)$ respectively. The orbifolds of the form A^n/W are holomorphic symplectic in the sense that A^n has a holomorphic symplectic form preserved by W. The resolution of $M^0_G(A)$ that we seek (and have for $SU(n+1)$ and $Sp(n)$) should have a holomorphic symplectic form that agrees with the holomorphic symplectic form on the smooth locus of $M^0_G(A)$. We call such a resolution a holomorphic symplectic resolution and since A^n/W is projective, such a resolution is a hyperkähler manifold.

Definition 3.1. Let M be a quasi-projective variety non-singular in codimension 1 with a holomorphic symplectic form defined on the smooth locus of M. We say that a smooth resolution $\widetilde{M} \to M$ is a *holomorphic symplectic resolution* if $\widetilde{M}$ has a global holomorphic symplectic form that agrees with the form pulled back from M on the corresponding locus. If M is projective, we will also call such a resolution a *hyperkähler resolution.*

Hyperkähler resolutions are special cases of crepant resolutions:

Definition 3.2. Let M be a quasi-projective variety, non-singular in codimension one, with a holomorphic volume form defined on the smooth locus of M (equivalently, M has a trivial canonical class $K_M \cong \mathcal{O}_M$). We say a smooth resolution $\widetilde{M} \to M$ is *crepant* if $\widetilde{M}$ has a global holomorphic volume form that agrees with the form pulled back from M on the corresponding locus.

Remark 3.3. We've restricted our definition of crepant to the case where M has trivial canonical class to emphasize the analogy with holomorphic symplectic resolutions. In general, an arbitrary proper, birational morphism $\phi : Y \to X$ has a *discrepancy divisor* $\Delta = K_Y - \phi^* K_X$ and ϕ is *crepant* if $\Delta = 0$.

Crepant resolutions may not exist in general. Locally, at an isolated orbifold point, the issue is:

Question 3.1. Let $H \subset SL(n, \mathbf{C})$ be a finite group. When does $\mathbf{C}^n/H$ admit a crepant resolution? What can one say about the geometry of a resolution if it exists?

[4] Birationally equivalent hyperkähler manifolds have the same Betti numbers [1]. In fact, it is believed (but not proven) that birationally equivalent hyperkähler manifolds are actually deformation equivalent (and hence diffeomorphic). c.f. [31]

These questions are the subject of study of the so-called generalized McKay correspondence, and to some extent they motivated the definition of stringy Hodge numbers (see [3] [43] [44]). We will return to this topic in more detail in Section 4.

For this section, we quote some existence and non-existence results for holomorphic symplectic and crepant resolutions.

Consider $\mathbf{C}^{2r} = \mathbf{C}^2 \otimes \Lambda$ with its W action. The orbifold $\mathbf{C}^{2r}/W$ is the affine analogue of our moduli space $M^0_G(A) \cong A^r/W$. In a recent announcement of Bezrukavnikov and Ginzburg [5], they construct a holomorphic symplectic resolution of this space.

Theorem 3.3 (Bezrukavnikov-Ginzburg). *$\mathbf{C}^{2r}/W = (\mathbf{C} \otimes \Lambda)/W$ admits a holomorphic symplectic resolution.*

At first glance, this (announced) theorem suggests that A^r/W should also have a holomorphic symplectic (and hence hyperkähler) resolution. However, *not all the orbifold points of A^r/W have local models of the type $\mathbf{C}^{2r}/W$*; there are additional possibilities arising from the presence of torsion in A.

The only non-existence result we need is a very simple one that we have borrowed from the McKay correspondence literature (it is implied by Theorem 5.4 of [3] or see [44] example 5.4).

Theorem 3.4. *Let $\mathbf{Z}/2 = \{\pm 1\}$ act by -1 on all the factors of $\mathbf{C}^{2d}$, $d > 1$. Then $\mathbf{C}^{2d}/\pm 1$ does not admit a crepant resolution.*

Our main result of this section is the following:

Theorem 3.5. *Let G be a compact, simple, simply connected Lie group. Let $M_G(A)$ be the moduli space of flat G bundles on an Abelian surface A. Then $M_G(A)$ admits a crepant resolution if and only if G is $SU(n)$ or $Sp(n)$; in particular, $M_G(A)$ has a hyperkähler resolution if and only if G is $SU(n)$ or $Sp(n)$.*

We devote the rest of this section to the proof. To prove the theorem as stated, it obviously suffices to prove it for $M^0_G(A)$ since when G is $Sp(n)$ or $SU(n)$, $M^0_G(A) = M_G(A)$.

3.5. The basic examples: G_2, B_3, and D_4.

To prove Theorem 3.5, we first prove it in the cases when G is G_2, $Spin(7)$, and $Spin(8)$, which in Cartan's classification, corresponds to the Dynkin diagrams G_2, B_3, and D_4. We will later show how the basic examples can be propagated to every other G not equal to $SU(n)$ or $Sp(n)$.

Theorem 3.5 for $G = G_2$ follows from Theorem 3.4 and the following:

Theorem 3.6. *Let W and Λ be the Weyl group and coroot lattice for G_2. There exists a point of $(A \otimes \Lambda)/W$ locally modeled on $\mathbf{C}^4/\pm 1$.*

PROOF: Λ is the rank two sublattice of $\mathbf{Z}^3$ consisting of those elements summing to zero:

$$\Lambda = \{(a_1, a_2, a_3) \in \mathbf{Z}^3 : a_1 + a_2 + a_3 = 0\}.$$

The Weyl group W is the dihedral group of order 12. W is a $\mathbf{Z}/2$ extension of the symmetric group S_3 and the S_3 action on Λ is given by permuting the a_i's and the $\mathbf{Z}/2 = \{\pm 1\}$ action is given by $(a_1, a_2, a_3) \mapsto (-a_1, -a_2, -a_3)$.

Choose a triple of distinct 2-torsion points in A that sum to zero; *i.e.* let $p = (\tau_1, \tau_2, \tau_3) \in A^3$ such that $2\tau_i = 0$, $\tau_i \neq \tau_j \neq 0$ for all $i \neq j$, and $\tau_1 + \tau_2 + \tau_3 = 0$. Note that $p \in A \otimes \Lambda$ since

$$A \otimes \Lambda = \{(x_1, x_2, x_3) \in A^3 : x_1 + x_2 + x_3 = 0\}.$$

Since the τ_i are distinct, no non-trivial permutation fixes p; on the other hand, since $\tau_i = -\tau_i$, we have that $p = -p$. Thus the stabilizer of p is $\mathbf{Z}/2 = \{\pm 1\} \subset W$.

Therefore a neighborhood of the image of p in $(A \otimes \Lambda)/W$ is modeled on $\mathbf{C}^4/\pm 1$, where ± 1 actions non-trivially on all factors. $\square$

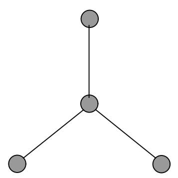
Remark 3.4. If we replace A with E in the above discussion, we see that in $M_{G_2}(E) = (E \otimes \Lambda)/W$ there is a point modeled on $\mathbf{C}^2/\pm 1$. Looijenga's theorem (Theorem 2.3) tells us that $M_{G_2}(E)$ is in fact $\mathbf{CP}(1,1,2)$ which has a unique singular point (modeled on $\mathbf{C}^2/\pm 1$). In an elliptic curve E, there are exactly 3 non-zero 2-torsion points and so the choice of the τ_i is unique (up to permutation). Thus the orbit of p is the unique singular point in $\mathbf{CP}(1,1,2)$. In A, there are many choices for the τ_i's and so there are multiple points in $M_{G_2}(A)$ where a crepant resolution does not exist locally.

The basic examples for B_3 and D_4 ($Spin(7)$ and $Spin(8)$) are variations on the same theme that are slightly more involved. See the appendix for their construction (Theorems A.1 and A.2).

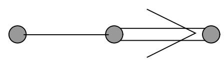
Remark 3.5. It is no accident that the examples we have for D_4, B_3, and G_2 are all very similar. The D_4 Dynkin diagram has an action of the symmetric group S_3 and the "quotient" of the D_4 diagram by S_3 "is" the G_2 diagram, while the "quotient" of the D_4 diagram by $\mathbf{Z}/2 \subset S_3$ "is" the B_3 diagram. What this really means is that there is an S_3 action on Λ_{D_4} so that Λ_{G_2} and Λ_{B_3} are respectively the S_3 and $\mathbf{Z}/2$ invariant sublattices. In this way, we get an S_3 action on $M^0_{Spin(8)}(A)$ so that the S_3 and $\mathbf{Z}/2$ fixed point sets give inclusions $M^0_{G_2}(A) \subset M^0_{Spin(8)}(A)$ and $M^0_{Spin(7)}(A) \subset M^0_{Spin(8)}(A)$. The basic examples for G_2 and B_3 are just the restriction of the basic example for D_4 under the above inclusions.

3.6. Propagating the basic examples.

The Dynkin diagrams of D_4 and B_3 (corresponding to $Spin(8)$ and $Spin(7)$ respectively) are



D_4



B_3

The D_4 diagram is a sub-diagram of the diagrams of E_6, E_7, E_8, and D_n, $n \geq 4$. The Dynkin diagram of B_3 is a sub-diagram of the diagrams of F_4 and B_n, $n \geq 3$. We can thus get from the diagrams of G_2, B_3, and D_4 to any other Dynkin diagram not in the A_n or C_n series by inclusion.

The following lemma will show that the operation of inclusion allows us to "propagate" our basic examples to find points in $M^0_G(A)$ ($G \neq Sp(n)$ or $SU(n)$) with local models of the form $(\mathbf{C}^{2l}/\pm 1) \times \mathbf{C}^{2k}$, $l > 1$, which then by Theorem 3.4 do not admit crepant resolutions. This will complete the proof of Theorem 3.5.

Lemma 3.7. *Suppose that $\Lambda \subset \Lambda'$ and $W \subset W'$ are the inclusions of a coroot lattice and its Weyl group into another coroot lattice and Weyl group that are induced by an inclusion of a rank l Dynkin diagram into a rank $l+k$ diagram. Let $p \in A \otimes \Lambda$ be a point and let $W_p \subset W$ denote the W-stabilizer of p. Then there exists a point $p' \in A \otimes \Lambda'$ such that its W'-stabilizer is isomorphic to W_p. Moreover, the point $[p'] \in (A \otimes \Lambda')/W'$ is locally modeled on $(\mathbf{C}^{2l}/W_p) \times \mathbf{C}^{2k}$.*

Proof: Since Λ'/Λ is torsion free, the induced map $A \otimes \Lambda \to A \otimes \Lambda'$ is injective. Via this inclusion, the W'-stabilizer of p (denoted W'_p) contains the W-stabilizer, *i.e.* $W_p \subset W'_p$.

Using translation by p and the exponential map, we W'_p-equivariantly identify a small neighborhood of $p \in A \otimes \Lambda \subset A \otimes \Lambda'$ with a small neighborhood of $0 \in \mathbf{C}^2 \otimes \Lambda \subset \mathbf{C}^2 \otimes \Lambda'$. Let N be the orthogonal complement of $\mathbf{C}^2 \otimes \Lambda$ in $\mathbf{C}^2 \otimes \Lambda'$. Let q be a small, generic, non-zero element of N which, after exponentiation and translation, gives us an element $p' \in A \otimes \Lambda'$ lying in a small neighborhood of p. We need to show that $W'_{p'} = W_p$; equivalently we need to show that the W'_p-stabilizer of $q \in N \subset \mathbf{C}^2 \otimes \Lambda'$ (denoted $(W'_p)_q$) is W_p.

Since W is generated by reflections through planes perpendicular to vectors in Λ, elements of $W_p \subset W$ fix $N = (\mathbf{C}^2 \otimes \Lambda)^{\perp}$ and so $(W'_p)_q$ contains W_p. To prove the converse, let $g \in (W'_p)_q \subset W'_p \subset W'$. Since q was chosen generically, g must fix all of N. We claim that any element of W' fixing $(\mathbf{C}^2 \otimes \Lambda)^{\perp}$ must in fact be an element of W. This claim implies that $g \in W'_p \cap W = W_p$ and so $(W'_p)_q = W_p$ as asserted.

To prove the claim, it is enough to prove the claim with $\mathbf{C}^2$ replaced by $\mathbf{R}$. In other words, suppose $g \in W'$ acts on $\mathfrak{t}' = \mathbf{R} \otimes \Lambda'$ preserving $\mathfrak{t}^{\perp} = (\mathbf{R} \otimes \Lambda)^{\perp}$; we wish to show that

$g \in W$. The set of Weyl chambers in $\mathfrak{t}'$ (respectively $\mathfrak{t}$) forms a W'-torsor (respectively W-torsor). By choosing a fundamental chamber $C' \subset \mathfrak{t}'$ for W' such that $C = C' \cap \mathfrak{t}$ is a fundamental chamber for W, we get a bijective correspondence between elements of W' (respectively W) and Weyl chambers in $\mathfrak{t}'$ (respectively $\mathfrak{t}$) with the following property. Those elements of W' that lie in W correspond to exactly those Weyl chambers in $\mathfrak{t}'$ whose intersection with $\mathfrak{t}$ is non-trivial. If $g \in W'$ preserves $\mathfrak{t}^{\perp}$ then it must be an orthogonal transformation of $\mathfrak{t}$ and so $g(C') \cap \mathfrak{t} = g(C)$ is a Weyl chamber in $\mathfrak{t}$ and hence $g \in W$ which proves the claim.

Finally, to finish the proof of the Lemma, we observe that via translation and exponentiation, the decomposition $\mathbf{C}^2 \otimes \Lambda' = (\mathbf{C}^2 \otimes \Lambda) \oplus N$ provides the local model whose existence is asserted by the Lemma. □

4. The stringy Hodge numbers of $M_{SU(n)}(A)$ and $M_{Sp(n)}(A)$.

The Calabi-Yau spaces that appear in the physics of string theory often have orbifold singularities. Based on mirror symmetry considerations, physicists have suggested a novel way to extend the definition of Hodge numbers to Calabi-Yau varieties with orbifold singularities [50][53].

These so called "stringy Hodge numbers" have been extensively studied by mathematicians recently, especially in the context of the generalized McKay correspondence (see [3][44][43]). The stringy Hodge numbers are conjectured to coincide with the ordinary Hodge numbers of any Crepant resolution, provided it exists[5].

In this section we compute the stringy Hodge numbers of $M_{Sp(n)}(A)$ and $M_{SU(n)}(A)$ and show that they coincide with the ordinary Hodge numbers of their resolutions $\widetilde{M}_{Sp(n)}(A)$ and $\widetilde{M}_{SU(n)}(A)$.

4.1. Stringy Euler numbers

Historically, the stringy Euler number was defined first [12][13]. Let X be a smooth Calabi-Yau manifold and let H be a finite group acting on X preserving K_X

The ordinary Euler number of the quotient $Y = X/H$ can be expressed as

$$e(X/H) = \frac{1}{|H|} \sum_g e(X^g)$$

where X^g is the fixed point set of an element $g \in H$.

In contrast, the stringy Euler number is *defined* by

$$e_{st}(X,H) = \frac{1}{|H|} \sum_{gh=hg} e(X^g \cap X^h)$$

where the sum is over pairs of commuting elements in H.

[5] See the footnote in section 1.2.

Note that this sum can be rearranged as a sum over conjugacy classes in H. We let $\{g\}$ denote the conjugacy class represented by an element g. Let $C(g)$ denote the centralizer of g. Then $C(g)$ acts on X^g and it is easy to see that $e_{st}(X,H)$ can be rewritten as

$$e_{st}(X,H) = \sum_{\{g\}\in \mathrm{Conj}(H)} e(X^g/C(g)).$$

Via localization, it can be shown that $e(X/H)$ is the Euler characteristic of the H-equivariant cohomology of X while $e_{st}(X,H)$ turns out to be equal to the Euler characteristic of the H-equivariant K-theory of X (see [28], [8]).

4.2. Stringy Hodge numbers

The stringy Euler number was generalized to Hodge numbers by Zaslow [53]. We follow the definition as given by Batyrev and Dais [3].

Let X be a smooth Calabi-Yau manifold with an action of a finite group H that preserves the holomorphic volume form. Let $Y = X/H$ be the quotient. We will define the *stringy Hodge numbers* $h^{p,q}_{st}(X,H)$, which we will just write as $h^{p,q}_{st}(Y)$ when the orbifold structure of Y is clear from the context.

Definition 4.1. Let X, H, and Y be as above. For each $g \in H$, let

$$X^g = X^g_1 \cup \cdots \cup X^g_{r_g}$$

denote a decomposition of the fixed locus of g into groups of components which are the orbits of the connected components under the centralizer $C(g)$ of g. For any $x \in X^g_i$, g acts on T_xX with eigenvalues $e^{2\pi i\alpha_1},\dots,e^{2\pi i\alpha_n}$ where we choose the *weights* α_j so that $0 \le \alpha_j < 1$ and we define the *Fermion shift number* F^g_i of the component X^g_i as the sum of the corresponding weights; *i.e.*

$$F^g_i = \sum_{j=1}^n \alpha_j .$$

Note that the F^g_i's are integers since the action of g preserves the holomorphic volume form.

We then define the *stringy Hodge numbers* of Y by

$$h^{p,q}_{st}(Y) = \sum_{\{g\}\in \mathrm{Conj}(H)} h^{p,q}_g(X,H)$$

where

$$h^{p,q}_g(X,H) = \sum_{i=1}^{r_g} h^{p-F^g_i,q-F^g_i}(X^g_i/C(g)).$$

Note that X^g_i is smooth and has an action of $C(g)$. The notation on the right in the above equation is as follows: for any finite group K acting on a smooth complex manifold V, let $h^{p,q}(V/K)$ denote the dimension of the K-invariant part of $H^{p,q}(V)$.

It is easy to see that the stringy Euler characteristic is then given by

$$e_{st}(Y) = \sum_{p,q} (-1)^{p+q} h^{p,q}_{st}(Y).$$

The main conjecture concerning the stringy Hodge numbers is the following.

Conjecture 4.1 (Zaslow). *If $Z \to X/H$ is any crepant resolution of X/H then*

$$h^{p,q}(Z) = h^{p,q}_{st}(X/H).$$

Remark 4.1. We learned after the first version of this paper was written that this conjecture is now proved. It follows from general results of Batyrev [2] and Denef-Loeser [11]. See the discussion in the footnote in section 1.2.

Remark 4.2. The definition of stringy Hodge numbers and the above conjecture are often both extended to the case where we do not assume that X is Calabi-Yau, but we merely assume that the singularities of X/H are Gorenstein.

4.3. The Fermionic shifts in the holomorphic symplectic case

For a finite group acting on a holomorphic symplectic manifold preserving the holomorphic symplectic form, the Fermionic shifts F^g_i become very simple.

Recall that the Hodge diamond of a holomorphic symplectic manifold is mirror symmetric; in other words, if $\dim X = 2n$, then the Hodge diamond is completely symmetric about (n, n) meaning that

$$h^{n+p,n+q}(X) = h^{n-p,n-q}(X) = h^{n-p,n+q}(X) = h^{n+p,n-q}(X).$$

The shifts F^g_i in the formula for $h^{p,q}_{st}(Y)$ are such that each individual contribution is completely symmetric (in the above sense) about (n, n). This can be seen as follows. Since H preserves the symplectic form, the fixed point sets X^g_i are each smooth and holomorphic symplectic. Thus their Hodge diamonds are completely symmetric about $(n - \frac{1}{2}\operatorname{codim}(X^g_i), n - \frac{1}{2}\operatorname{codim}(X^g_i))$. Thus we just want to show that

$$F^g_i = \frac{1}{2}\operatorname{codim}(X^g_i). \tag{3}$$

This follows from the fact that g acts on $T_xX^g_i$ symplectically: The eigenvalues of any symplectic transformation come in pairs λ, λ^{-1} so the weights α_j come in pairs of the form $(\alpha, 1-\alpha)$ or $(0, 0)$. The number of non-zero eigenvalues is exactly the codimension of X^g_i and so we have that

$$F^g_i = \sum_{j=1}^{2n} \alpha_j = \frac{1}{2}\operatorname{codim}(X^g_i)$$

as asserted.

4.4. The computation of $h_{st}^{p,q}(\mathrm{Sym}^n(X))$

In [23], Göttsche showed that for an algebraic surface X, the stringy Hodge numbers of the symmetric product $\mathrm{Sym}^n(X) = X^n/S_n$ coincide with the ordinary Hodge numbers of the resolution $\mathrm{Hilb}^n(X)$, verifying Conjecture 4.1 in this case (c.f. Remark 4.2).

In order to state his formula and to facilitate our computations in this section, we introduce some of Göttsche's notation:

Definition 4.2. Let $P(n)$ be the set of partitions of n. We write $\alpha \in P(n)$ as $(1^{\alpha_1}, 2^{\alpha_2}, \dots, n^{\alpha_n})$ so that α_i is the number of i's in the partition. Thus $n = \sum_i i\alpha_i$ and we put $|\alpha| = \sum_i \alpha_i$. We will use the following shorthand:

$$\begin{aligned} X^{(n)} &:= \mathrm{Sym}^n(X), \\ X^{[n]} &:= \mathrm{Hilb}^n(X), \\ X^{\alpha} &:= X^{\alpha_1} \times \cdots \times X^{\alpha_n}, \text{ and} \\ X^{(\alpha)} &:= X^{(\alpha_1)} \times \cdots \times X^{(\alpha_n)}, \end{aligned}$$

where by convention, $X^{(0)}$ or X^0 is just a single point. For an element of the symmetric group $g \in S_n$, let $\alpha(g)$ denote its cycle type and note that $g \mapsto \alpha(g)$ defines a bijection between conjugacy classes of S_n and $P(n)$. Let X be smooth with an action of a finite group H. Define the Hodge and stringy Hodge polynomials by

$$\begin{aligned} h(X, x, y) &:= \sum_{p,q} h^{p,q}(X) x^p y^q \\ h_{st}(X/H, x, y) &:= \sum_{p,q} h_{st}^{p,q}(X/H) x^p y^q. \end{aligned}$$

Recall that when we use the ordinary Hodge number notation for an orbifold $h^{p,q}(X/H)$ we mean the dimension of the H-invariant part of $H^{p,q}(X)$. We will then also have the corresponding polynomial:

$$h(X/H, x, y) := \sum_{p,q} h^{p,q}(X/H) x^p y^q.$$

Note that the Hodge polynomial is multiplicative:

$$h(Y \times Z) = h(Y) h(Z).$$

If it happens that the Fermionic shift numbers for all the different components of X^g agree (*i.e.* $F_1^g = \cdots = F_{r_g}^g$) for all g, then the stringy Hodge polynomial can be written as follows:

$$\begin{aligned} h_{st}(X/H) &= \sum_{p,q} h_{st}^{p,q}(X/H)x^p y^q \\ &= \sum_{p,q}\sum_{\{g\}} h^{p-F^g,q-F^g}(X^g/C(g))x^p y^q \\ &= \sum_{\{g\}} (xy)^{F^g} h(X^g/C(g)). \end{aligned} \tag{4}$$

In [23], Göttsche computes the stringy Hodge numbers of $S^{(n)}$ and A^{n-1}/S_n (a.k.a. $M_{SU(n)}(A)$). Having computed the ordinary Hodge numbers of $S^{[n]}$ and KA_{n-1} in previous works ([22][24]), he then verifies Conjecture 4.1 in these cases.

Theorem 4.2 (Göttsche). *For any projective surface S we have*

$$h(S^{[n]},x,y) = h_{st}(S^{(n)},x,y) = \sum_{\alpha\in P(n)} (xy)^{n-|\alpha|} h(S^{(\alpha)},x,y).$$

In particular, in the notation of Section 2,

$$h(\widetilde{M}_{U(n)}(A)) = h_{st}(M_{U(n)}(A)).$$

Moreover,

$$h(KA_{n-1}) = h_{st}(A^{n-1}/S_n),$$

or, in the notation of Section 2,

$$h(\widetilde{M}_{SU(n)}(A)) = h_{st}(M_{SU(n)}(A)).$$

4.5. The stringy Hodge number of $M_{Sp(n)}(A)$

Göttsche's theorem verifies Conjecture 4.1 for $M_{SU(n)}(A)$ (with its resolution $\widetilde{M}_{SU(n)}(A)$). We wish to do the same for $M_{Sp(n)}(A)$, *i.e.* we need to compute the stringy Hodge polynomial of $M_{Sp(n)}(A)$ and compare it to the ordinary Hodge polynomial of $\widetilde{M}_{Sp(n)}(A)$. The result is the following:

Theorem 4.3. *Let $M_{Sp(n)}(A)$ and $\widetilde{M}_{Sp(n)}(A)$ be as in Section 2. Then*

$$h(\widetilde{M}_{Sp(n)}(A)) = h_{st}(M_{Sp(n)}(A)).$$

In particular, Conjecture 4.1 holds for $M_{Sp(n)}(A)$ (with its resolution $\widetilde{M}_{Sp(n)}(A)$).

Proof: Recall that $M_{Sp(n)}(A) = A^n/S_n \ltimes \{\pm 1\}^n$. To compute $h_{st}^{p,q}(M_{Sp(n)}(A))$ we first need to identify the conjugacy classes of $W = S_n \ltimes \{\pm 1\}^n$. An element of W consists of a permutation along with n signs. We give an overall sign to each cycle in the permutation by multiplying all of the signs in the cycle together. In this way, each element determines a splitting of a partition $\alpha \in P(n)$ into two partitions $\alpha^+ + \alpha^- = \alpha$, where α_i^+ and α_i^- are the number of cycles of length i of positive and negative type respectively.

The conjugacy classes of W are in bijective correspondence with the data (α^+,α^-) (see Carter [10]). For each (α^+,α^-), we choose the representative group element to be such that the positive cycles are first, arranged in order of increasing length, followed by the negative cycles, also arranged in order of increasing length. Furthermore, each positive cycle should have all positive signs and each negative cycle should have exactly one minus sign at the beginning of each cycle. This determines a unique representative $g(\alpha^\pm)$ of each conjugacy class (α^+,α^-).

We next determine the fixed point set of $g(\alpha^\pm)$ acting on A^n. The fixed point locus of each positive cycle fixes is the diagonal in the product of factors that the cycle acts on. The fixed point locus of each negative cycle is the 2-torsion points in the diagonal in the product of the factors that the cycle acts on. Thus the fixed point set of $g(\alpha^\pm)$ is isomorphic to a product of copies of A (one for each positive cycle) and a product of copies of the set of 2-torsion points of A, denoted A_2, one for each negative cycle. In other words,

$$\begin{aligned}(A^n)^{g(\alpha^\pm)} &= \prod_{i=1}^n A^{\alpha_i^+}\times A_2^{\alpha_i^-}\\ &= A^{\alpha^+}\times A_2^{\alpha^-}.\end{aligned}$$

We next need to determine the action of the centralizer $C(g(\alpha^\pm))$ on the fixed set $(A^n)^{g(\alpha^\pm)}$. Elements that commute with $g(\alpha^\pm)$ permute the cycles of $g(\alpha^\pm)$ of the same length and type. It is easy to see that all such permutations can be realized (possibly with signs). We only need to understand the signs of the elements of the centralizer acting on the $A^{\alpha_i^+}$ factors since the $A_2^{\alpha_i^-}$ factors are not changed by multiplication by -1 (recall that A_2 is the group of 2-torsion points in A). Assume then that all the $\alpha_i^\pm$ are zero except for a single $\alpha_l^+ = n/l = m$; that is $g = g(\alpha^\pm)$ is a permutation consisting of m cycles all of length l and no minus signs. We claim that $C(g)/\langle g\rangle$ is then $S_m \ltimes \{\pm 1\}^m$ acting on $(A^n)^g \cong A^m$ in the standard way. The $S_m \subset C(g)$ is generated by elements that exchange two of the l cycles in g; they are a product of l disjoint transpositions with all positive signs. If each of the transpositions in this product is given a single minus sign at the beginning of each transposition, then this element is also in $C(g)$. The effect of the action of this element on $(A^n)^g \cong A^m$ is to permute two of the factors and then multiply one of them by -1. One can directly check that elements of this form generate $C(g)/\langle g\rangle$.

The above discussion easily generalizes to an arbitrary conjugacy class (α^+,α^-). The centralizer $C(g(\alpha^\pm))$ acts independently on each $A^{\alpha_i^+}$ and $A_2^{\alpha_i^-}$ factor, acting by $S_{\alpha_i^-}$ on $A_2^{\alpha_i^-}$ and by $S_{\alpha_i^+}\ltimes\{\pm 1\}^{\alpha_i^+}$ on $A^{\alpha_i^+}$. Thus we have

$$\begin{aligned}(A^n)^{g(\alpha^\pm)}/C(g(\alpha^\pm)) &= \prod_{i=1}^n K^{(\alpha_i^+)}\times A_2^{(\alpha_i^-)}\\ &= K^{(\alpha^+)}\times A_2^{(\alpha^-)}\end{aligned}$$

where $K = A/\pm 1$ is the (singular) Kummer surface associated to A.

Since the codimension of $(A^n)^{g(\alpha^\pm)}$ is $2n - 2|\alpha^+|$ and the action is symplectic, the Fermionic shift number is just $n - |\alpha^+|$ (see Equation 3).

From Equation 4 and the multiplicative properties of the Hodge polynomial, we then have

$$\begin{aligned} h_{st}(M_{Sp(n)}(A)) &= \sum_{\alpha^\pm} (xy)^{n-|\alpha^+|} h(K^{(\alpha^+)}) h(A_2^{(\alpha^-)}) \\ &= \sum_{\alpha^\pm} (xy)^{n-|\alpha|} \prod_{i=1}^{n} h(K^{(\alpha_i^+)}) h(A_2^{(\alpha_i^-)})(xy)^{\alpha_i^-} \\ &= \sum_{\alpha \in P(n)} (xy)^{n-|\alpha|} \prod_{i=1}^{n} \left\{ \sum_{\alpha_i^+ + \alpha_i^- = \alpha_i} h(K^{(\alpha_i^+)}) h(A_2^{(\alpha_i^-)})(xy)^{\alpha_i^-} \right\}. \end{aligned} \tag{5}$$

Let X be the smooth Kummer $K3$ surface associated to A, *i.e.* the blowup of K at the sixteen double points. Note that

$$h(X) = h(K) + h(A_2)xy.$$

To each polynomial $h(x, y)$ with positive, integral coefficients, we can assign a bigraded vector space where the dimension of the (p, q) graded piece is the coefficient of $x^p y^q$ in $h(x, y)$. The nth symmetric power of this vector space is also a bigraded vector space and so gives rise to a polynomial which we denote $\mathrm{Sym}^n(h(x, y))$. By what is essentially a tautology of the definitions, we have

$$h(X^{(l)}) = \mathrm{Sym}^l(h(X)),$$

and from well known properties of the symmetric tensor products we also have

$$\mathrm{Sym}^l(f + g) = \sum_{l^+ + l^- = l} \mathrm{Sym}^{l^+}(f)\,\mathrm{Sym}^{l^-}(g).$$

Thus we get

$$\begin{aligned} h(X^{(\alpha_i)}) &= \mathrm{Sym}^{\alpha_i}(h(X)) \\ &= \mathrm{Sym}^{\alpha_i}(h(K) + h(A_2)xy) \\ &= \sum_{\alpha_i^+ + \alpha_i^- = \alpha_i} \mathrm{Sym}^{\alpha_i^+}(h(K))\,\mathrm{Sym}^{\alpha_i^-}(h(A_2)xy) \\ &= \sum_{\alpha_i^+ + \alpha_i^- = \alpha_i} h(K^{(\alpha_i^+)}) h(A_2^{(\alpha_i^-)})(xy)^{\alpha_i^-} \end{aligned}$$

and so substituting into Equation 5 and using Theorem 4.2 we get

$$\begin{aligned} h_{st}(M_{Sp(n)}(A)) &= \sum_{\alpha\in P(n)} (xy)^{n-|\alpha|} \prod_{i=1}^{n} h(X^{(\alpha_i)}) \\ &= \sum_{\alpha\in P(n)} (xy)^{n-|\alpha|} h(X^{(\alpha)}) \\ &= h(X^{[n]}) \\ &= h(\widetilde{M}_{Sp(n)}(A)) \end{aligned}$$

completing the proof of Theorem 4.3. □

4.6. Stringy speculations.

It turns out that the series

$$\sum_{n=0}^{\infty} h_{st}(M_{Sp(n)}(A), x, y) q^n$$

and

$$\sum_{n=0}^{\infty} h_{st}(M_{SU(n)}(A), x, y) q^n$$

have some interesting arithmetic properties. Certain series obtained by setting x and y to special values can be expressed in terms of modular and quasi-modular forms. For example, by setting $(x, y) = (-1, 1)$ one obtains the generating series for the signature, while setting $(x, y) = (-1, -1)$ one obtains the generating series for the Euler characteristic. Göttsche gives expressions for these series as the Fourier expansions of certain quasi-modular forms ([22] pages 37–39, 51–53, and 57).

Regardless of the existence of crepant resolutions, the stringy Hodge numbers $h^{p,q}_{st}(M_G(A))$ are well defined for any G. It would be interesting to compute the generating series for the B_n and D_n series (*i.e.* Spin(odd) and Spin(even)) and determine if they also have nice expressions. This calculation is straight forward, although rather complicated. We conjecture that the generating series for the stringy signature and stringy Euler characteristic have closed expressions in terms of quasi-modular forms.

5. Realization of the desingularizations as moduli spaces

The previous desingularizations of $M_{SU(n)}(A)$ and $M_{Sp(n)}(A)$ defined in Section 2 were *ad hoc.* In this section, we expand on a suggestion of Aaron Bertram (that also is implicitly contained in the work of Mukai), to realize these desingularizations as moduli spaces of holomorphic $G^{\mathbf{C}}$-bundles satisfying a stability condition that we call Mukai-stability. This will give an affirmative answer to Question 1.2 (for the $SU(n)$ and $Sp(n)$ cases).

The notion of Mukai-stability refines ordinary semi-stability in the sense that Mukai stable bundles are semi-stable. Furthermore, the generic semi-stable bundle is also Mukai stable. Consequently, the map from the moduli space of Mukai stable bundles to the moduli space of semi-stable bundles is generically an isomorphism. However, non-generically, the stability notions differ which results in a different equivalence relation for the corresponding moduli problems. In this way, the moduli space for Mukai stable bundles actually becomes a resolution of singularities for the ordinary moduli space.

To define Mukai-stability we use a "framing" of the fiber over p_0 of the holomorphic vector bundle associated to a principal $G^{\mathbf{C}}$ bundle. That is, if $\mathcal{E}$ is a holomorphic bundle, a *framing*[6] is a surjective sheaf map $\tau : \mathcal{E} \to \mathcal{O}_{p_0}$ where $\mathcal{O}_{p_0}$ is the one dimensional skyscraper sheaf at the origin p_0. We say that the bundle $\mathcal{E}$ is *Mukai-stable* if it is semi-stable and $\mathrm{Ker}(\tau)$ is simple, *i.e.* the automorphism group of $\mathrm{Ker}(\tau)$ is $\mathbf{C}^*$. It turns out that $\mathrm{Ker}(\tau)$ is independent of τ (as long as $\mathrm{Ker}(\tau)$ is simple).

We motivate how this sort of condition arises by considering how the desingularization $\mathrm{Hilb}^n(A) \to \mathrm{Sym}^n(A)$ occurs in the context of moduli spaces and then interpreting this in the context of bundles via the isomorphism $M_{U(n)}(A) \cong \mathrm{Sym}^n(A)$. The main tool is the Fourier-Mukai transform. A detailed study of this case allows us to generalize to the cases of $SU(n)$ and $Sp(n)$.

5.1. Sym, Hilb, and the moduli of sheaves

We begin by discussing the isomorphism $M_{U(n)}(A) \cong \mathrm{Sym}^n(A)$ from the point of view of holomorphic bundles rather than flat connections. Recall that we have been viewing $M_{U(n)}(A)$ as parameterizing flat $U(n)$ connections so that the isomorphism with $\mathrm{Sym}^n(A)$ was obtained by "diagonalizing" the representation $\pi_1(A) \to U(n)$ (see Section 2). By Donaldson's theorem (Theorem 2.1), we may also view $M_{U(n)}(A)$ as parameterizing s-equivalence classes of semi-stable holomorphic $Gl(n, \mathbf{C})$ bundles, or, equivalently, $M_{U(n)}(A)$ parameterizes s-equivalence classes of semi-stable, rank n, holomorphic bundles $\mathcal{E}$ with

$$ch(\mathcal{E}) = (n, 0, 0) \in H^0(A, \mathbf{Z}) \oplus H^2(A, \mathbf{Z}) \oplus H^4(A, \mathbf{Z}).$$

More generally, if $v = (n, c, \chi) \in H^0(A, \mathbf{Z}) \oplus H^2(A, \mathbf{Z}) \oplus H^4(A, \mathbf{Z})$ is any vector, there exists (Simpson [47], c.f. [32]) a (coarse) moduli space, which we denote $M(v)$, of s-equivalence classes of semi-stable, coherent, pure-dimensional sheaves $\mathcal{F}$ with $ch(\mathcal{F}) = (n, c, \chi)$.

Using this notation for this section, we write $M(n, 0, 0)$ instead of $M_{U(n)}(A)$. The isomorphism $M(n, 0, 0) \cong \mathrm{Sym}^n(A)$ is obtained by arguing that every bundle $\mathcal{E}$ with $ch(\mathcal{E}) = (n, 0, 0)$ is s-equivalent to a direct sum of degree 0 line bundles, *i.e.* $L_{x_1} \oplus \cdots \oplus L_{x_n}$ where $x_i \in \mathrm{Pic}^0(A) \cong A$. This follows from basic facts concerning s-equivalence and Jordan-Hölder filtrations: in general, every semi-stable bundle $\mathcal{E}$ admits a filtration $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$ where the sheaves $\mathcal{E}_{i+1}/\mathcal{E}_i$ are stable and have constant slope. Let $Gr(\mathcal{E}) = \oplus_i \mathcal{E}_{i+1}/\mathcal{E}_i$, then $\mathcal{F}$ is s-equivalent to $\mathcal{E}$ if and only if $Gr(\mathcal{E}) \cong Gr(\mathcal{F})$. In the

[6] We use the terminology "framing" following Huybrechts and Lehn [30] who more generally define a "framed module" as a pair $(\mathcal{E}, \tau)$ where $\tau : \mathcal{E} \to \mathcal{F}$ is a sheaf map to some fixed sheaf $\mathcal{F}$.

case at hand, the semi-stability of $\mathcal{E}$ and the stability and constant slope of the $\mathcal{E}_{i+1}/\mathcal{E}_i$'s imply that all the factors of $Gr(\mathcal{E})$ must be degree 0 line bundles.

We see that under s-equivalence, all the possible different extensions

$$0 \to \mathcal{E}_i \to \mathcal{E}_{i+1} \to \mathcal{E}_{i+1}/\mathcal{E}_i \to 0$$

get identified in the same s-equivalence class. In order to find a desingularization, we would like to find a different notion of equivalence that remembers such information.

To motivate how one should go about doing this, we first consider how the desingularization

$$\mathrm{Hilb}^n(A) \to \mathrm{Sym}^n(A)$$

naturally occurs in the context of moduli spaces of sheaves. $\mathrm{Hilb}^n(A)$, by definition, parameterizes length n, 0-dimensional subschemes of A. Such a subscheme Z is determined by its ideal sheaf $\mathcal{I}_Z$ which can be considered as a point in $M(1,0,n)$ (the moduli space of rank 1 semi-stable sheaves $\mathcal{F}$ with $c_1(\mathcal{F}) = 0$ and $c_2(\mathcal{F}) = -n$). In fact, there is an isomorphism

$$\mathrm{Hilb}^n(A) \times A \cong M(1,0,n)$$

given by $(Z,x) \mapsto \mathcal{I}_Z \otimes L_x$.

Now the subscheme $Z \subset A$ is also determined by its structure sheaf $\mathcal{O}_Z$ which we can regard as a rank 0 sheaf on A where $ch(\mathcal{O}_Z) = (0,0,n)$. Thus $\mathcal{O}_Z$ determines a point in $M(0,0,n)$. However, $M(0,0,n)$ is isomorphic to $\mathrm{Sym}^n(A)$. The reason is that every sheaf in $M(0,0,n)$ is s-equivalent to a sheaf of the form $\mathcal{O}_{x_1} \oplus \cdots \oplus \mathcal{O}_{x_n}$. For example, if $Z \subset A$ is a subscheme of length 2 supported at $x \in A$ then $\mathcal{O}_Z$ can be written as a non-trivial extension

$$0 \to \mathcal{O}_x \to \mathcal{O}_Z \to \mathcal{O}_x \to 0$$

which is s-equivalent to the trivial extension $\mathcal{O}_x \oplus \mathcal{O}_x$.

Under these isomorphisms, the Hilbert-Chow morphism

$$\mathrm{Hilb}^n(A) \to \mathrm{Sym}^n(A)$$

is obtained by sending the moduli point of $\mathcal{I}_Z$ to the moduli point of $\mathcal{O}_Z$.

To translate this picture over to rank n bundles we need a correspondence between sheaves on A and sheaves on $\mathrm{Pic}^0(A)$ that generalizes the tautological correspondence between points in $\mathrm{Pic}^0(A)$ and line bundles on A. The Fourier-Mukai transform provides such a dictionary.

5.2. The Fourier-Mukai transform

Although we have been identifying A and $\mathrm{Pic}^0(A)$ throughout his paper via the polarization, for clarity in this section we write $\widehat{A}$ for $\mathrm{Pic}^0(A)$. The ideas of this subsection are due to Mukai; see the papers [39] and [40].

Let $P \to A \times \widehat{A}$ denote the normalized Poincare bundle, *i.e.* $P|_{A\times\{x\}} = L_x$ and $P|_{\{p_0\}\times\widehat{A}}$ is trivial. Let $\pi : A\times\widehat{A} \to A$ and $\widehat{\pi} : A\times\widehat{A} \to \widehat{A}$ denote the projections. Define the functors

$$S(?) = \widehat{\pi}_*(\pi^*(?) \otimes P)$$

and

$$\widehat{S}(?) = \pi_*(\widehat{\pi}^*(?) \otimes P).$$

If $\mathcal{F}$ is a sheaf on A (respectively, $\widehat{A}$), we obtain sheaves $R^iS(\mathcal{F})$ (respectively, $R^i\widehat{S}(\mathcal{F})$) on $\widehat{A}$ (respectively, A) via the right derived functors of S (respectively, $\widehat{S}$).

Definition 5.1. If $R^iS(\mathcal{F}) = 0$ for all i except some i_0, then we say that $\mathcal{F}$ satisfies the *Weak Index Theorem (W.I.T.)* with index $i(\mathcal{F}) = i_0$ and we call the sheaf $R^{i_0}S(\mathcal{F})$ the *Fourier-Mukai transform* of $\mathcal{F}$ and denote it $\widehat{\mathcal{F}}$.

We also have the analogous definition for sheaves on $\widehat{A}$ and furthermore, if $\mathcal{F}$ satisfies W.I.T., then $\widehat{\mathcal{F}}$ satisfies W.I.T. and $\widehat{\widehat{\mathcal{F}}} = (-1_A)^*\mathcal{F}$.

For example, any degree 0 line bundle $L_{\widehat{x}} \to A$ corresponding to the point $\widehat{x} \in \widehat{A}$ satisfies W.I.T. (with $i(L_{\widehat{x}}) = 2$) and $\widehat{L}_{\widehat{x}} = \mathcal{O}_{-\widehat{x}}$. More generally, every bundle $\mathcal{E} \to A$ with $ch(E) = (n, 0, 0)$ satisfies W.I.T. and the Fourier-Mukai transform induces an isomorphism

$$\widehat{\ }: M(n, 0, 0) \to M(0, 0, n).$$

Conversely, the structure sheaf $\mathcal{O}_Z$ of a length n, zero-dimensional subscheme $Z \subset A$, satisfies W.I.T. with $i(\mathcal{O}_Z) = 0$ and $\widehat{\mathcal{O}_Z}$ is a bundle $\mathcal{E}$ with $ch(\mathcal{E}) = (n, 0, 0)$.

Note that despite the fact that the moduli spaces $M(n, 0, 0)$ and $M(0, 0, n)$ (each isomorphic to $\mathrm{Sym}^n(\widehat{A})$) both suffer from an undiscriminating s-equivalence, the Fourier-Mukai transform itself does not lose information. For example, if $\mathcal{E}$ is a non-trivial extension

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O} \to 0$$

then $\widehat{\mathcal{E}} = \mathcal{O}_Z$ where Z is a length 2 subscheme supported at $p_0 \in \widehat{A}$ and so it is a non-trivial extension of $\widehat{\mathcal{O}} = \mathcal{O}_{p_0}$ by $\widehat{\mathcal{O}} = \mathcal{O}_{p_0}$:

$$0 \to \mathcal{O}_{p_0} \to \mathcal{O}_Z \to \mathcal{O}_{p_0} \to 0.$$

The $\mathbf{P}^1$'s worth of non-trivial extensions of $\mathcal{O}$ by $\mathcal{O}$ correspond to the $\mathbf{P}^1$'s worth of length 2 subschemes supported at a point.

It is not quite the case that the Fourier-Mukai transform is an equivalence of the category of coherent sheaves on A with the category of coherent sheaves on $\widehat{A}$. However, the Fourier-Mukai transform is an equivalence of the subcategories of sheaves satisfying W.I.T.. More generally, the derived functor of S defines an equivalence between the derived categories of coherent sheaves on A and $\widehat{A}$:

$$\mathbf{R}S : D(A) \to D(\widehat{A})$$

(see [48] for an introduction to the derived category or [51] for more detail). The inverse functor is easily determined because there is an isomorphism of functors, $\mathbf{R}S \circ \mathbf{R}\widehat{S} \cong (-1_A)^*[-2]$ where "$[-2]$" denotes "shift the complex 2 places to the right".

5.3. Translating points to bundles.

We now examine what happens to the sheaves $\mathcal{I}_Z$ and $\mathcal{O}_Z$, their corresponding moduli spaces, and the exact sequence $0 \to \mathcal{I}_Z \to \mathcal{O} \to \mathcal{O}_Z \to 0$ under the equivalence of categories $\mathbf{R}S$.

We saw in the last subsection that for any length n, zero dimensional subscheme Z, $\mathcal{O}_Z$ satisfies W.I.T. with $i(\mathcal{O}_Z) = 0$ and $\widehat{\mathcal{O}_Z}$ is a bundle $\mathcal{E}$ with $ch(\mathcal{E}) = (n, 0, 0)$. That is, $\mathbf{R}S(\mathcal{O}_Z)$ is represented by the complex of sheaves that in degree 0 is $\mathcal{E}$ and it is 0 in all other degrees. $\mathcal{O}$ also satisfies W.I.T. with $i(\mathcal{O}) = 2$ and $\widehat{\mathcal{O}} = \mathcal{O}_{p_0}$; that is, $\mathbf{R}S(\mathcal{O})$ is represented by the complex $\mathcal{O}_{p_0}[2]$ which is $\mathcal{O}_{p_0}$ in degree 2 and 0 otherwise.

The exact sequence of sheaves

$$0 \to \mathcal{I}_Z \to \mathcal{O} \to \mathcal{O}_Z \to 0$$

is an exact triangle in $D(A)$ when we regard the sheaves as complexes concentrated in degree 0. Since the functor $\mathbf{R}S$ is an equivalence of categories, it must take exact triangles to exact triangles and so

$$\mathbf{R}S(\mathcal{I}_Z) \to \mathcal{O}_{p_0}[2] \to \mathcal{E} \to \mathbf{R}S(\mathcal{I}_Z)[1]$$

is an exact triangle in $D(\widehat{A})$.

This triangle gives us a long exact sequence in cohomology from which we see immediately that $\mathbf{R}S(\mathcal{I}_Z)$ cannot be a sheaf; it is represented by a complex of sheaves whose cohomology is $\mathcal{O}_{p_0}$ in degree 2, $\mathcal{E}$ in degree 1, and 0 otherwise. The problem is that $\mathbf{R}S(\mathcal{O})$ and $\mathbf{R}S(\mathcal{O}_Z)$ are sheaves, but they are concentrated in different degrees. To rectify this problem, we employ another functor that is an equivalence of derived categories.

Let $\Delta(?) = \mathcal{H}om(?, \mathcal{O})$ be the dualizing functor and let

$$\mathbf{R}\Delta : D(\widehat{A}) \to D(\widehat{A})$$

be its derived functor (warning: our notation for Δ differs from Mukai's, his has the $\mathbf{R}$ built in and has an additional shift of the index by 2). Since $\mathcal{O}_A$ is the dualizing sheaf of A, $\mathbf{R}\Delta$ is an anti-equivalence of the category $D(\widehat{A})$ to itself. The composition

$$\mathbf{R}\Delta \circ \mathbf{R}S = \mathbf{R}(\Delta S) : D(A) \to D(\widehat{A})$$

is thus also an anti-equivalence. The derived dual of $\mathcal{E}$ is simply the ordinary dual, *i.e.* $\mathbf{R}\Delta(\mathcal{E}) = \mathcal{E}^\vee$. The derived dual of $\mathcal{O}_{p_0}[2]$ is also a sheaf concentrated in degree zero since

$$R^i\Delta(\mathcal{O}_{p_0}[2]) = \mathcal{E}xt^{2+i}(\mathcal{O}_{p_0}, \mathcal{O}) \cong \begin{cases} \mathcal{O}_{p_0} & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Thus if we apply $\mathbf{R}(\Delta S)$ to the sequence $\mathcal{I}_Z \to \mathcal{O} \to \mathcal{O}_Z$ (which reverses arrows), we get the exact triangle

$$\mathbf{R}(\Delta S)(\mathcal{I}_Z) \leftarrow \mathcal{O}_{p_0} \leftarrow \mathcal{E}^\vee \leftarrow \mathbf{R}(\Delta S)(\mathcal{I}_Z)[-1].$$

Since the map $\mathcal{O} \to \mathcal{O}_Z$ is non-zero, the map $\mathcal{E}^\vee \to \mathcal{O}_{p_0}$ must be non-zero and hence surjective. Thus we see that $\mathbf{R}(\Delta S)(\mathcal{I}_Z)$ is concentrated in degree -1 and is the sheaf $\mathrm{Ker}(\mathcal{E}^\vee \to \mathcal{O}_{p_0})$.

We have shown that the functor $\mathbf{R}(\Delta S)$, which is an (anti-)equivalence of categories, takes the ideal sheaves of zero-dimensional subschemes of A to sheaves on $\widehat{A}$ which are the kernel of a framing $\tau : \mathcal{E} \to \mathcal{O}_{p_0}$. It remains to see which bundles and framings arise in this way. This can be answered by reversing the question; when is $\mathbf{R}(\Delta S)(\mathrm{Ker}(\mathcal{E} \to \mathcal{O}_{p_0}))$ a sheaf of the form $\mathcal{I}_{\widehat{Z}}$ where $\mathcal{I}_{\widehat{Z}}$ is the ideal sheaf of a 0 dimensional subscheme $\widehat{Z}$ of $\widehat{A}$? We can reverse the question in this way because the functor $\mathbf{R}(\Delta\widehat{S})$ is the inverse of $\mathbf{R}(\Delta S)$, *i.e.* $\mathbf{R}(\Delta\widehat{S}) \circ \mathbf{R}(\Delta S)$ is isomorphic to the identity functor (Theorem 2.2. and Equation 3.8 of [39]). The answer to this question is given by the following theorem of Mukai (c.f. [40] Proposition 2.18 and Corollary 2.19).

Theorem 5.1. *Let $\mathcal{E}$ be a holomorphic bundle on A with $ch(\mathcal{E}) = (n, 0, 0)$, let $\tau : \mathcal{E} \to \mathcal{O}_{p_0}$ be a surjective sheaf map, and let $\mathcal{F} = \mathrm{Ker}(\tau)$. Then the following are equivalent:*

1. *$\mathbf{R}(\Delta S)(\mathcal{F})$ is a sheaf;*
2. *$\widehat{\mathcal{E}}$, the Fourier-Mukai transform of $\mathcal{E}$, is of the form $\mathcal{O}_{\widehat{Z}} = \mathcal{O}/\mathcal{I}_{\widehat{Z}}$ where $\widehat{Z} \subset \widehat{A}$ is a length n, 0 dimensional subscheme of $\widehat{A}$.*
3. *$\mathcal{F}$ is simple, i.e. $\mathrm{End}(\mathcal{F}) = \mathbf{C}$;*

In this case $\mathbf{R}(\Delta S)(\mathcal{F}) = \mathcal{I}_{\widehat{Z}}[-1]$.

Note that the theorem implies that $\mathcal{F}$ is independent of the choice of τ (as long as $\mathrm{Ker}(\tau)$ is simple).

Proof: First we note that $\mathbf{R}(\Delta S)(\mathcal{E}) = \mathbf{R}\Delta(\mathbf{R}S(\mathcal{E})) = \mathbf{R}\Delta(\widehat{\mathcal{E}}[2]) = \widehat{\mathcal{E}}$ and $\mathbf{R}(\Delta S)(\mathcal{O}_{p_0}) = \mathbf{R}\Delta(\mathcal{O}) = \mathcal{O}$. We apply the functor $\mathbf{R}(\Delta S)$ to the exact sequence $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O}_{p_0} \to 0$ to get the exact triangle

$$\mathbf{R}(\Delta S)(\mathcal{F}) \leftarrow \widehat{\mathcal{E}} \leftarrow \mathcal{O} \leftarrow \mathbf{R}(\Delta S)(\mathcal{F})[-1]$$

Suppose that $\mathbf{R}(\Delta S)(\mathcal{F})$ is a sheaf. Since $\widehat{\mathcal{E}}$ is a sheaf supported on a finite number of points, $R^{-1}(\Delta S)(\mathcal{F}) = \mathrm{Ker}(\mathcal{O} \to \widehat{\mathcal{E}}) \neq 0$. Thus $\mathbf{R}(\Delta S)(\mathcal{F})$ is a sheaf implies that $R^0(\Delta S)(\mathcal{F}) = \mathrm{Coker}(\mathcal{O} \to \widehat{\mathcal{E}}) = 0$. Thus $\mathcal{O} \to \widehat{\mathcal{E}}$ is surjective and since $\widehat{\mathcal{E}}$ is supported on points the kernel of $\mathcal{O} \to \widehat{\mathcal{E}}$ must be an ideal sheaf of a zero dimensional subscheme. Thus (1) implies (2) and $\mathbf{R}(\Delta S)(\mathcal{F}) = \mathcal{I}_{\widehat{Z}}[-1]$. It follows then that $\mathcal{F}$ is simple since $\mathcal{I}_{\widehat{Z}}$ is a simple sheaf and $\mathbf{R}(\Delta S)$ is an equivalence of categories. It remain to be seen that (3) implies (1).

Assume that $\mathcal{F}$ is simple and suppose that $\mathbf{R}(\Delta S)(\mathcal{F})$ is not a sheaf. Then $R^0(\Delta S)(\mathcal{F}) = \mathrm{Coker}(\mathcal{O} \to \widehat{\mathcal{E}}) \neq 0$. Since $\widehat{\mathcal{E}}$ is supported on a finite number of points, so is $R^0(\Delta S)(\mathcal{F})$

and so there must exist some point $\widehat{x} \in \widehat{A}$ so that

$$\mathrm{Hom}_{\widehat{A}}(R^0(\Delta S)(\mathcal{F}), \mathcal{O}_{\widehat{x}}) \neq 0.$$

Since $R^i(\Delta S)(\mathcal{F}) = 0$ for $i > 0$ we have

$$\mathrm{Hom}_{\widehat{A}}(R^0(\Delta S)(\mathcal{F}), \mathcal{O}_{\widehat{x}}) = \mathrm{Hom}_{D(\widehat{A})}(\mathbf{R}(\Delta S)(\mathcal{F}), \mathcal{O}_{\widehat{x}})$$

where $\mathcal{O}_{\widehat{x}}$ is regarded as a complex concentrated in degree 0. Now $\mathcal{O}_{\widehat{x}} = \mathbf{R}(\Delta S)(L_{\widehat{x}})$ since $\mathbf{R}S(L_{\widehat{x}}) = \mathcal{O}_{\widehat{x}}[2]$ and $\mathbf{R}\Delta(\mathcal{O}_{\widehat{x}}[2]) = \mathcal{O}_{\widehat{x}}$. So

$$\begin{aligned}\mathrm{Hom}_{D(\widehat{A})}(\mathbf{R}(\Delta S)(\mathcal{F}), \mathcal{O}_{\widehat{x}}) &= \mathrm{Hom}_{D(\widehat{A})}(\mathbf{R}(\Delta S)(\mathcal{F}), \mathbf{R}(\Delta S)(L_{\widehat{x}}))\\ &= \mathrm{Hom}_{D(A)}(L_{\widehat{x}}, \mathcal{F})\\ &= \mathrm{Hom}_{A}(L_{\widehat{x}}, \mathcal{F})\end{aligned}$$

using the fact that $\mathbf{R}(\Delta S)$ is an anti-equivalence of categories. This gives us $\mathrm{Hom}(L_{\widehat{x}}, \mathcal{F}) \neq 0$ which implies $\mathrm{Hom}(L_{\widehat{x}}, \mathcal{E}) = H^0(\mathcal{E} \otimes L_{\widehat{x}}^{\vee}) \neq 0$. This in turn implies (by Proposition 4.18 of [38]) that $H^0(\mathcal{E}^{\vee} \otimes L_{\widehat{x}}) = \mathrm{Hom}(\mathcal{E}, L_{\widehat{x}}) \neq 0$ from which we get $\mathrm{Hom}(\mathcal{F}, L_{\widehat{x}}) \neq 0$. Therefore we get a (necessarily non-constant) endomorphism $\mathcal{F} \to L_{\widehat{x}} \to \mathcal{F}$ which contradicts the simplicity of $\mathcal{F}$. $\square$

5.4. Mukai stability

We can regard the three equivalent conditions in Theorem 5.1 as giving a different stability condition for bundles. We call this condition Mukai stability and we extend it to principal $G^{\mathbf{C}}$ bundles:

Definition 5.2. We say a semi-stable holomorphic bundle $\mathcal{E}$ on A is *Mukai-stable* if there exists a framing $\tau : \mathcal{E} \to \mathcal{O}_{p_0}$ such that $\mathrm{Ker}(\tau)$ is simple, i.e. $\mathrm{End}(\mathrm{Ker}(\tau)) = \mathbf{C}$. We say that a holomorphic $GL(n, \mathbf{C})$, $SL(n, \mathbf{C})$, or $Sp(n, \mathbf{C})$ bundle is Mukai stable if the associated vector bundle (induced by the standard representation) is Mukai stable.

Using the facts that $\mathrm{Hilb}^n(\widehat{A})$ is a fine moduli space and $\mathbf{R}(\Delta S)$ is an equivalence of categories, and applying Theorem 5.1, we get the following (c.f. Theorem 2.20 of [40]):

Theorem 5.2. *Let $\widetilde{M}(n,0,0)$ be the space of Mukai-stable bundles $\mathcal{E}$ on A with $ch(\mathcal{E}) = (n,0,0)$, then $\widetilde{M}(n,0,0)$ is a fine moduli space and the functor $\mathbf{R}(\Delta S)$ applied to* $\mathrm{Ker}(\tau)$ *induces an isomorphism $\widetilde{M}(n,0,0) \cong \mathrm{Hilb}^n(\widehat{A})$. Moreover the map $\widetilde{M}(n,0,0) \to M(n,0,0)$ induced by sending $\mathcal{E}$ to its s-equivalence class fits into the following commutative diagram*

$$\begin{tikzcd} \widetilde{M}(n,0,0) \arrow[r] \arrow[d] & M(n,0,0) \arrow[d] \\ \mathrm{Hilb}^n(\widehat{A}) \arrow[r] & \mathrm{Sym}^n(\widehat{A}) \end{tikzcd}$$

where the vertical arrows are isomorphisms induced by $\mathbf{R}(\Delta S)$ *applied to* $\mathrm{Ker}(\tau)$ *and* $\mathcal{E}$ *respectively. In particular,* $\widetilde{M}(n,0,0)$ *is a resolution of singularities of* $M(n,0,0)$.

This theorem provides an answer to Question 1.2 in the case when G is the (non-semi-simple) group $U(n)$. That is, it shows that the moduli space $M_{U(n)}(A) = M(n,0,0)$ of flat $U(n)$ connections on A has a hyperkähler resolution given by the moduli space $\widetilde{M}(n,0,0)$ of Mukai stable $U(n)^{\mathbf{C}}$-bundles (*i.e.* $Gl(n,\mathbf{C})$-bundles).

We now use this theorem to analyze the Mukai-stable moduli spaces in the $Sp(n)$ and $SU(n)$ cases.

Definition 5.3. Let $\widetilde{M}_{SU(n)}(A) \subset \widetilde{M}(n,0,0)$ be the subset of Mukai stable bundles that arise as the associated vector bundles of principal holomorphic $SL(n,\mathbf{C}) = SU(n)^{\mathbf{C}}$ bundles. For the case of $Sp(n)$, let $\widetilde{M}_{Sp(n)}(A) \subset \widetilde{M}(2n,0,0)$ be the *closure* of the subset of Mukai stable bundles that arise as the associated vector bundles of principal holomorphic $Sp(n,\mathbf{C}) = Sp(n)^{\mathbf{C}}$ bundles.

We now wish to show (as the notation suggests) that $\widetilde{M}_{SU(n)}(A)$ and $\widetilde{M}_{Sp(n)}(A)$ are hyperkähler resolutions of $M_{SU(n)}(A)$ and $M_{Sp(n)}(A)$.

We first treat the $SU(n)$ case. A holomorphic vector bundle $\mathcal{E}$ is the associated bundle of a holomorphic principal $SL(n,\mathbf{C})$ bundle if and only if $\det \mathcal{E} \cong \mathcal{O}$. There fore $\widetilde{M}_{SU(n)}(A) \subset \widetilde{M}(n,0,0)$ is the subset of bundles with trivial determinant. Since the determinant of a bundle is constant in its s-equivalence class, $\widetilde{M}_{SU(n)}(A)$ is the preimage of the subset of bundles in $M(n,0,0)$ which have trivial determinant. This set is just $M_{SU(n)}(A)$ which under the isomorphism $M(n,0,0) \cong \mathrm{Sym}^n(\widehat{A})$ is the fiber over p_0 of the sum map $\mathrm{Sym}^n(\widehat{A}) \to \widehat{A}$. Thus we recover the *ad hoc* hyperkähler resolution constructed in Section 2:

Theorem 5.3. *The moduli space of Mukai stable* $SL(n,\mathbf{C})$ *bundles* $\widetilde{M}_{SU(n)}(A)$ *is smooth, holomorphic symplectic, and isomorphic to the generalized Kummer variety* KA_{n-1}. *The natural map* $\widetilde{M}_{SU(n)}(A) \to M_{SU(n)}(A)$ *is a hyperkähler resolution.*

We now turn to the $Sp(n)$ case. A holomorphic bundle $\mathcal{E}$ is the associated bundle of a holomorphic principal $Sp(n,\mathbf{C})$ bundle if and only if there is an isomorphism

$$\phi : \mathcal{E} \to \mathcal{E}^{\vee}$$

such that $\phi^{\vee} = -\phi$ (*i.e.* a symplectic form). We apply the Fourier-Mukai transform to translate this into a condition for $\widetilde{M}_{Sp(n)}(A) \subset \widetilde{M}(2n,0,0) \cong \mathrm{Hilb}^{2n}(\widehat{A})$. The result is the following.

Theorem 5.4. *The moduli space of Mukai stable* $Sp(n,\mathbf{C})$ *bundles* $\widetilde{M}_{Sp(n)}(A)$ *(Definition 5.3) is smooth, holomorphic symplectic, and birationally equivalent to* $\mathrm{Hilb}^n(X)$ *where* X *is the Kummer* $K3$ *surface associated to* A. *The natural map* $\widetilde{M}_{Sp(n)}(A) \to M_{Sp(n)}(A)$ *is a hyperkähler resolution.*

Remark 5.1. Note that strictly speaking, Definition 5.3 is not consistent with our earlier *ad hoc* definition of $\widetilde{M}_{Sp(n)}(A)$ as $\mathrm{Hilb}^n(X)$ since Theorem 5.4 only asserts that they are birationally equivalent. However, from the point of view of this paper, the distinction is not very important—we have compared the Hodge numbers of the resolution to the stringy Hodge numbers and so we can work with either of the desingularizations since the Hodge numbers of birationally equivalent hyperkähler manifolds are the same. As we remarked earlier, it is believed (but has not been proved) that birationally equivalent hyperkähler manifolds are actually deformation equivalent and hence diffeomorphic, c.f. [31].

We also note that the subset of Mukai stable bundles that admit a symplectic form is not closed in $\widetilde{M}(2n,0,0)$. Thus the points of $\widetilde{M}_{Sp(n)}(A)$ parameterize not only symplectic Mukai stable bundles but also Mukai stable bundles with degenerate symplectic forms that occur as the limits of symplectic Mukai stable bundles. We will give examples of such bundles in the course of the proof of the theorem (see Remark A.2).

Proof of Theorem 5.4: Since $\mathcal{E}$ is Mukai stable, by Theorem 5.1, $\mathbf{R}S(\mathcal{E}) = \widehat{\mathcal{E}}[2] = \mathcal{O}_{\widehat{Z}}[2]$ where $\widehat{Z} \subset \widehat{A}$ is a length $2n$, 0 dimensional subscheme of $\widehat{A}$. In the sequel, we drop the hats and just write $Z \subset A$. Let $(-1) : A \to A$ be the involution $x \mapsto -x$ and note that $(-1)^*$ induces an involution on $\mathrm{Hilb}^{2n}(A)$. We use $-Z$ to denote the image of Z under $(-1)_*$ so that we also have $(-1)^*\mathcal{O}_Z = \mathcal{O}_{-Z}$. There is a natural equivalence of functors [39]

$$\mathbf{R}(S\Delta) = (-1)^*\mathbf{R}(\Delta S)[2]$$

and so

$$\begin{aligned}\mathbf{R}S(\mathcal{E}^\vee) &= \mathbf{R}(S\Delta)(\mathcal{E})\\ &= (-1)^*\mathbf{R}\Delta(\mathcal{O}_Z[2])\\ &= (-1)^*\,\mathcal{E}xt^2(\mathcal{O}_Z,\mathcal{O})\\ &= \mathcal{E}xt^2(\mathcal{O}_{-Z},\mathcal{O}).\end{aligned}$$

For an arbitrary 0-dimensional subscheme $W \subset A$, the sheaf $\mathcal{E}xt^2(\mathcal{O}_W,\mathcal{O})$ is not necessarily isomorphic to $\mathcal{O}_W$ because it may fail to be the structure sheaf of a subscheme. However, if $\mathcal{E}xt^2(\mathcal{O}_W,\mathcal{O})$ *is* the structure sheaf of a subscheme, then $\mathcal{E}xt^2(\mathcal{O}_W,\mathcal{O})$ is isomorphic to $\mathcal{O}_W$ (although not canonically!); this assertion will be proved in the course of the proof of Lemma 5.5. In the case at hand, since $\mathcal{E} \cong \mathcal{E}^\vee$ we have $\mathcal{O}_Z \cong \mathcal{E}xt^2(\mathcal{O}_{-Z},\mathcal{O}) \cong \mathcal{O}_{-Z}$ and so $Z = -Z$. In particular, $\widetilde{M}_{Sp(n)}(A)$ is contained in the fixed point set of the action of $(-1)^*$ on $\mathrm{Hilb}^{2n}(A)$. More precisely, we have the following Lemma:

Lemma 5.5. *$\widetilde{M}_{Sp(n)}(A)$ is a connected component of the fixed locus of $(-1)^*$ acting on $\mathrm{Hilb}^{2n}(A)$.*

We defer the proof of the lemma to the appendix. The theorem follows easily from the lemma: First, since $\mathrm{Hilb}^{2n}(A)$ is smooth, the components of the fixed point set of the involution $(-1)^*$ are smooth. Furthermore, since $(-1)^*$ preserves the holomorphic symplectic form on $\mathrm{Hilb}^{2n}(A)$, the fixed components are also holomorphic symplectic. Finally, the fixed component that we claim is $\widetilde{M}_{Sp(n)}(A)$ lies over the subset of $\mathrm{Sym}^{2n}(A)$ consisting of n pairs of points of the form $\{x,-x\}$. This set is naturally identified with $\mathrm{Sym}^n(A/\pm 1)$. Thus $\widetilde{M}_{Sp(n)}(A)$ is birational to $\mathrm{Sym}^n(A/\pm 1)$ which is birational to $\mathrm{Hilb}^n(X)$. □

Appendix A. Miscellaneous details

Here we provide the details that were suppressed in the main discourse.

A.1. The basic example for D_4.

Theorem A.1. *Let W and Λ be the Weyl group and coroot lattice for $Spin(8)$. There exists a point of $(A\otimes\Lambda)/W$ locally modeled on $\mathbf{C}^8/\pm 1$.*

PROOF: We first derive a useful description of $A\otimes\Lambda$. The coroot lattice Λ is the sublattice of $\mathbf{Z}^4$ generated by the simple coroots e_1-e_2, e_2-e_3, e_3-e_4, and e_3+e_4. W is generated by the reflections through the planes perpendicular to the simple coroots.

Thus one can easily see that

$$\Lambda=\{\sum_{i=1}^{4}a_ie_i\in\mathbf{Z}^4:\ \sum a_i\equiv 0 \bmod 2\}$$

and

$$W=S_4\ltimes\{\pm1\}^3\subset S_4\ltimes\{\pm1\}^4$$

where the action of W on Λ is the restriction of the action on $\mathbf{Z}^4$ given by permuting the factors and multiplying by -1 on some even number of factors. The elements $\sum a_ie_i$ with $a_i\equiv 0 \bmod 2$ form a sublattice of Λ giving us the exact sequence:

$$0\to(2\mathbf{Z})^4\to\Lambda\to(\mathbf{Z}/2)^3\to 0$$

where $(\mathbf{Z}/2)^3\subset(\mathbf{Z}/2)^4$ is the kernel of the sum map $(\mathbf{Z}/2)^4\to\mathbf{Z}/2$. Noting that $A\otimes(2\mathbf{Z})^4\cong A\otimes\mathbf{Z}^4=A^4$ we apply $A\otimes(\cdot)$ to the sequence and examine the Tor sequence to arrive at:

$$0\to\mathrm{Tor}_1(A,(\mathbf{Z}/2)^3)\to A^4\to A\otimes\Lambda\to 0.$$

The subgroup $\mathrm{Tor}_1(A,(\mathbf{Z}/2)^3)\subset A^4$ is concretely given as

$$\mathrm{Tor}_1(A,(\mathbf{Z}/2)^3)=\{(\tau_1,\dots,\tau_4)\in A^4:\ 2\tau_i=0,\ \sum\tau_i=0\}.$$

Thus we can regard elements of $A\otimes\Lambda$ as orbits of points in A^4 by translation by the finite number of elements in $\mathrm{Tor}_1(A,(\mathbf{Z}/2)^3)$. The action of $W=S_4\ltimes\{\pm1\}^3$ on $A\otimes\Lambda$ is induced from the natural action on A^4.

Now we choose 3 distinct, non-zero 2-torsion points of A that sum to 0; that is, let $\tau_1,\tau_2,\tau_3\in A$ be such that $2\tau_i=0$, $\tau_i\neq\tau_j\neq 0$ for all $i\neq j$, and $\tau_1+\tau_2+\tau_3=0$. Note that $(0,\tau_1,\tau_2,\tau_3)\in\mathrm{Tor}_1(A,(\mathbf{Z}/2)^3)$.

We then choose "square-roots" of the τ_i's; that is, elements $\tau_i/2 \in A$, such that $2(\tau_i/2) = \tau_i$.

Let

$$p = (0, \tau_1/2, \tau_2/2, \tau_3/2) \in A^4.$$

In $A \otimes \Lambda$, $-p$ is equivalent to p since

$$\begin{aligned}(0, -\tau_1/2, -\tau_2/2, -\tau_3/2) &= (0, \tau_1 - \tau_1/2, \tau_2 - \tau_2/2, \tau_3 - \tau_3/2) \\ &= (0, \tau_1/2, \tau_2/2, \tau_3/2).\end{aligned}$$

Thus the subgroup $\{\pm 1\} \subset S_4 \ltimes \{\pm 1\}^3 \subset S_4 \ltimes \{\pm 1\}^4$ with generator $Id \times (-1, -1, -1, -1)$ fixes p in $A \otimes \Lambda$. In fact, the stabilizer of p in $A \otimes \Lambda$ is exactly $\{\pm 1\}$: The stabilizer does not contain non-trivial permutations because 0, $\tau_1/2$, $\tau_2/2$, and $\tau_3/2$ are distinct. No other subgroup of $\{\pm 1\}^3$ stabilizes p since $\tau_i/2 \neq -\tau_i/2$ and we must have at least two -1's acting. Thus, the local model of the quotient of $A \otimes \Lambda$ by W near p is $\mathbf{C}^8/\pm 1$ (where ± 1 acts non-trivially on all factors) as asserted by Theorem A.1. □

Remark A.1. If we replace A with E in the above discussions, we see that in $M_{Spin(8)}(E) = (E \otimes \Lambda)/W$ there is a point modeled on $\mathbf{C}^4/\pm 1$. Looijenga's theorem (Theorem 2.3) tells us that $M_{Spin(8)}(E)$ is in fact $\mathbf{CP}(1,1,1,1,2)$ which has a unique singular point (modeled on $\mathbf{C}^4/\pm 1$). In an elliptic curve E, there are exactly 3 non-zero 2-torsion points and so the choice of the τ_i is unique (up to permutation) and the choice of the square roots $\tau_i/2$ is unique up to addition by a 2-torsion point. It is then easily checked that the W orbit of the point $(0, \tau_1/2, \tau_2/2, \tau_3/2) \in A \otimes \Lambda$ is unique and corresponds to the predicted singular point in $\mathbf{CP}(1,1,1,1,2)$. In A, there are many choices for the τ_i's and so there are multiple points in $M^0_{Spin(8)}(A)$ where a crepant resolution does not exist locally.

A.2. The basic example for B_3.

Theorem A.2. *Let W and Λ be the Weyl group and coroot lattice for $Spin(7)$. There exists a point of $(A \otimes \Lambda)/W$ locally modeled on $\mathbf{C}^6/\pm 1$.*

Proof: The coroot lattice of B_n in general is given as the sublattice of $\mathbf{Z}^n$ defined by the condition that the sum of the coordinates should be even. That is, there is an exact sequence

$$0 \longrightarrow \Lambda_{B_n} \longrightarrow \mathbf{Z}^n \xrightarrow{sum} \mathbf{Z}/2 \longrightarrow 0. \qquad (6)$$

The Weyl group $W = W_{B_n}$ is the same as the Weyl group of the C_n coroot system: $W_{C_n} \cong S_n \ltimes \{\pm 1\}^n$. The action of W on Λ_{B_n} is induced by the action on $\mathbf{Z}^n \cong \Lambda_{C_n}$ which is given by permuting the factors and multiplying by ± 1 on each coordinate. We remark that for the *root* lattices, the situation is exactly reversed, $\Lambda^*_{C_n} \subset \Lambda^*_{B_n} \cong \mathbf{Z}^n$. These facts are easily seen by examining the simple roots (see for example the tables in Appendix C of [34]). The simple roots of the B_n root system are given by

$$\{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n\}$$

which span the full lattice $\mathbf{Z}^n = \sum_i e_i\mathbf{Z}$, whereas the simple roots of C_n are given by

$$\{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, 2e_n\}$$

which spans the kernel of the (mod 2) sum $\mathbf{Z}^n \to \mathbf{Z}/2$. To obtain the coroot system from the root system, one replaces each root α with its coroot (see for example page 496 of [20])

$$\alpha' = \frac{2}{||\alpha||^2}\alpha.$$

It is then clear that the coroot system of B_n is isomorphic to the root system of C_n and vice versa since $(e_i - e_{i+1})' = e_i - e_{i+1}$, $e_n' = 2e_n$, and $(2e_n)' = e_n$.

As in the D_4 case (indeed, D_n in general) we can regard Λ_{B_n} as a quotient of $(2\mathbf{Z})^n$:

$$0 \to (2\mathbf{Z})^n \to \Lambda_{B_n} \to (\mathbf{Z}/2)^{n-1} \to 0$$

where $(\mathbf{Z}/2)^{n-1}$ is the kernel of the sum map $(\mathbf{Z}/2)^n \to \mathbf{Z}/2$. Thus, as in the D_4 case, we have a sequence

$$0 \to \operatorname{Tor}_1(A, (\mathbf{Z}/2)^{n-1}) \to A^n \to A \otimes \Lambda_{B_n} \to 0.$$

Specializing to $n = 3$, we can express the subgroup $\operatorname{Tor}_1(A, (\mathbf{Z}/2)^{n-1}) \subset A^3$ as

$$\{(\tau_1, \tau_2, \tau_3) \in A^3 : 2\tau_i = 0 \text{ and } \sum \tau_i = 0\}.$$

As in the D_4 case, choose $(\tau_1, \tau_2, \tau_3) \in \operatorname{Tor}_1(A, (\mathbf{Z}/2)^{n-1})$ such that $\tau_i \neq \tau_j \neq 0$ for all $i \neq j$ and choose elements $\tau_i/2 \in A$ such that $2(\tau_i/2) = \tau_i$. Define

$$p = (\tau_1/2, \tau_2/2, \tau_3/2) \in A^3/\operatorname{Tor}_1(A, (\mathbf{Z}/2)^{n-1}).$$

By essentially the same argument as in the D_4 case, the stabilizer of p is $\{\pm 1\}$. Therefore the local model of the image of p in $(A \otimes \Lambda_{B_3})/W$ is $\mathbf{C}^6/\{\pm 1\}$. □

A.3. The proof of Lemma 5.5

Let $F^0 \subset \operatorname{Hilb}^{2n}(A)$ be the locus of subschemes consisting of $2n$ distinct points of the form $\{p_1, -p_1, \ldots, p_n, -p_n\}$. Let F be the closure of F^0. Note that F has dimension $2n$ an is a connected component of the fixed locus of $(-1)^*$ and is hence smooth. Let $S = S_n \subset \operatorname{Hilb}^{2n}(A)$ be the locus of subschemes whose Fourier-Mukai transforms admit a symplectic form so by definition $\widetilde{M}_{Sp(n)}(A) = \overline{S}$. We will prove that $\overline{S} = F$.

Let $H_k \subset \operatorname{Hilb}^{2k}(A)$ be the locus of subschemes supported at the origin and let $H_k' \subset \operatorname{Hilb}^{2k}(A)$ be the locus of subschemes supported at the two-torsion points. By Theorem 1.13 of [41], $\dim H_k \leq 2k-1$ for $k \geq 1$. It follows that $\dim H_k' \leq 2k-1$. In fact, a component of H_k' parameterizing subschemes supported at precisely l of the two-torsion points has dimension $2k - l$.

Now S has a stratification

$$S = \bigcup_{k \geq 0} S^0_{n-k} \times (H_k' \cap S_k),$$

where $S^0_{n-k} \subset S_{n-k} \subset \mathrm{Hilb}^{2(n-k)}(A)$ parameterizes subschemes whose support is *disjoint* from the two-torsion points. Since $\dim(S^0_{n-k}) = 2(n-k)$, we see that all strata of S have dimension less than or equal to $2n-1$, except for S^0_n which is irreducible of dimension $2n$: In fact, there is a quasi-finite map from an open subset of $\mathrm{Hilb}^n(A)$ onto S^0_n. Note that F^0 is a dense open subset of S^0_n. Indeed, if $Z = \{p_1, -p_1, \ldots, p_n, -p_n\}$ is a subscheme corresponding to a point in F^0, then the Fourier-Mukai transform of $\mathcal{O}_Z$ is the direct sum $L_{p_1} \oplus L_{-p_1} \oplus \cdots \oplus L_{p_n} \oplus L_{-p_n}$ of degree zero line bundles which has an obvious symplectic form.

We claim that all the strata of S are contained in the closure $\overline{S^0_n}$ and hence $S = \overline{F^0} = F$ which proves the lemma. This claim is equivalent to showing that $\dim_z(S) = 2n$ at every point $z \in S$. By factoring S locally near z, this reduces to showing that $\dim_z(S_k) = 2k$ at each $z \in S_k \cap H_k$. Since n is arbitrary, we prove this for $k = n$.

To prove this claim, we begin by examining the Fourier-Mukai transform of the condition $-\phi = \phi^\vee$. We get $-\mathbf{R}S(\phi) = \mathbf{R}S(\phi^\vee) = \mathbf{R}(S\Delta)(\phi) = (-1)^*\mathbf{R}(\Delta S)(\phi)[2]$ and so the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}_Z \arrow[r, "-R^2S(\phi)"] \arrow[d, "(-1)^*"'] & \mathcal{E}xt^2(\mathcal{O}_{-Z}, \mathcal{O}) \arrow[d, "(-1)^*"] \\ \mathcal{O}_{-Z} \arrow[r, "R^0(\Delta S)(\phi)"] & \mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O}) \end{tikzcd} \tag{7}$$

Let $V_Z = H^0(\mathcal{O}_Z)$ so by Serre duality $V_Z^\vee = \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O})$. Applying the global sections functor Γ to the above diagram and writing Φ for $\mathbf{R}(\Gamma S)(\phi)$ we get

$$\begin{tikzcd} V_Z \arrow[r, "-\Phi"] \arrow[d, "(-1)^*"'] & V_{-Z}^\vee \arrow[d, "(-1)^*"] \\ V_{-Z} \arrow[r, "\Phi^\vee"] & V_Z^\vee \end{tikzcd}$$

so that in particular, $(-1)^* \circ \Phi$ is a symplectic form on V_Z.

Suppose now that $z \in S_n \cap H_n$. That is, z corresponds to a subscheme $Z \subset A$ with $\mathrm{Supp}(Z) = p_0$ such that we have the Diagram (7). The sheaf $\mathcal{O}_Z$ is then determined by the corresponding module over the local ring $\widehat{\mathcal{O}}_{p_0} \cong \mathbf{C}[[x,y]]$. This has a concrete description in terms of linear algebra (c.f. Nakajima [41] section 1.2), namely $\mathcal{O}_Z$ is determined by the actions of x and y on V_Z. That is, $\mathcal{O}_Z$ is determined by a pair of nilpotent endomorphisms $M_x, M_y \in \mathrm{End}(V_Z)$ that commute. Conversely, suppose M_x and M_y are any pair of commuting, nilpotent, $2n \times 2n$ dimensional complex matrices. Then the action of M_x and M_y give $\mathbf{C}^{2n}$ the structure of a finite length module over $\mathbf{C}[[x,y]]$; this module will be of the form $\mathbf{C}[[x,y]]/\mathcal{I}_Z$ (and hence correspond to a point in H_0) if and only if there exists a vector $v \in \mathbf{C}^{2n}$ such that the vectors $\{M_x^i M_y^j(v)\}_{i,j\geq 0}$ span $\mathbf{C}^{2n}$. In this case, $\mathcal{I}_Z = \{f \in \mathbf{C}[[x,y]] : f(M_x, M_y) = 0\}$ and the matrices are unique up to simultaneous conjugation.

More generally, if (M_x, M_y) are a pair of (not necessarily nilpotent) commuting matrices satisfying the above spanning condition, then they determine (uniquely up to simultaneous conjugation) an ideal $\mathcal{I} \subset \mathbf{C}[x,y]$ of finite length and hence a 0 dimensional subscheme of $\mathbf{C}^2$.

Note that $H_n \subset \mathrm{Hilb}^{2n}(A)$ has a neighborhood $\nu(H_n)$, open in the analytic topology, parameterizing subschemes whose support is contained in an open ϵ-polydisc about p_0. $\nu(H_n)$ is isomorphic to the corresponding neighborhood of $H_n \subset \mathrm{Hilb}^{2n}(\mathbf{C}^2)$ whose points are given by (equivalence classes of) commuting matrices (M_x, M_y) whose eigenvalues have modulus less than ϵ. Under this identification, Diagram (7) for A corresponds to the same diagram for $\mathbf{C}^2$. We may therefore replace A and the subschemes S_n and H_n of $\mathrm{Hilb}^{2n}(A)$ by $\mathbf{C}^2$ and the corresponding subschemes of $\mathrm{Hilb}^{2n}(\mathbf{C}^2)$.

If the sheaf $\mathcal{O}_Z$ is given by the pair (M_x, M_y), then the sheaf $\mathcal{O}_{-Z}$ is then clearly given by the pair $(-M_x, -M_y)$. We can also determine the matrices corresponding to the sheaf $\mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O})$. This sheaf must be given by the pair (M_x^t, M_y^t) where M_x^t, M_y^t denote the transpose matrices. This follows from the uniqueness of the dualizing functor for modules over a local ring ([27] pg. 275): both $\mathcal{O}_Z \mapsto \mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O})$ and $(M_x, M_y) \mapsto (M_x^t, M_y^t)$ satisfy the conditions of a dualizing functor and so they must coincide[7]. In this language, the existence of Diagram (7) means that there is a skew-symmetric, invertible matrix $\Phi = -\Phi^t$ such that $\Phi M_\bullet \Phi^{-1} = -M_\bullet^t$ for $\bullet = x, y$.

Note that whenever $(M_\bullet, \Phi)$ satisfy $\Phi M_\bullet \Phi^{-1} = -M_\bullet^t$, then $(P^{-1}M_\bullet P, P^t\Phi P)$ satisfy the same equation. Since Φ is skew-symmetric and invertible, there exists a P so that $P^t \Phi P = J$ where $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ is the standard symplectic form. Thus we may assume that (M_x, M_y) satisfy $JM_\bullet + M_\bullet^t J = 0$, *i.e.* $(M_x, M_y) \in \mathfrak{sp}(n) \oplus \mathfrak{sp}(n)$. This equation is preserved under $M_\bullet \mapsto g^{-1} M_\bullet g$ if and only if $g^t J g = J$, *i.e.* $g \in Sp(n, \mathbf{C})$.

We conclude that $S \subset \mathrm{Hilb}^{2n}(\mathbf{C}^2)$ is the image of an open set $\mathcal{C}^0$ of the "commuting variety"

$$\mathcal{C}^0 \subset \mathcal{C} = \{(g_1, g_2) \in \mathfrak{sp} \oplus \mathfrak{sp} : [g_1, g_2] = 0\}$$

under a morphism whose fibers are the simultaneous $Sp(n, \mathbf{C})$ conjugacy classes in $\mathcal{C}^0$. The open set $\mathcal{C}^0$ is the set of $(g_1, g_2) \in \mathcal{C}$ satisfying the spanning condition; that is, there exists a vector $v \in \mathbf{C}^{2n}$ such that $\{g_1^i g_2^j v\}_{i,j\geq 0}$ spans $\mathbf{C}^{2n}$. By Theorem A of Richardson [45], $\mathcal{C}$ is irreducible. It follows that $S \subset \mathrm{Hilb}^{2n}(\mathbf{C}^2)$, the image of $\mathcal{C}^0$, is also irreducible,

[7] One sees from this description that the sheaf $\mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O})$ is isomorphic (non-canonically) to $\mathcal{O}_Z$ as long as $\mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O})$ is of the form $\mathcal{O}/\mathcal{I}$. This follows since the ideals $\{f \in \mathbf{C}[[x,y]] : f(M_x, M_y) = 0\}$ and $\{f \in \mathbf{C}[[x,y]] : f(M_x^t, M_y^t) = 0\}$ coincide. However, it can happen that (M_x^t, M_y^t) fails the spanning condition even if it is met by (M_x, M_y). For example, $\mathcal{E}xt^2(\mathcal{O}/(x^2, xy, y^2), \mathcal{O})$ is not isomorphic to the structure sheaf of a subscheme. In this example, the pair $(M_x, M_y) = \left(\left(\begin{smallmatrix} 0&0&0 \\ 1&0&0 \\ 0&0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{smallmatrix}\right)\right)$ satisfies the spanning condition, but the transpose pair (M_x^t, M_y^t) does not.

hence everywhere of dimension at least $2n$, and so the same must hold for the original $S \subset \mathrm{Hilb}^{2n}(A)$. $\square$

Remark A.2. We have shown that $\overline{S} = F$ but in general $S \neq F$—the symplectic form can degenerate in a family of Mukai stable symplectic bundles. For example, let $\mathcal{E}$ be the Fourier-Mukai transform of $\mathcal{O}_Z$ where Z is a subscheme supported at $p_0 \in A$ defined by the ideal $\mathcal{I} = (x,y)^3 \cup (y^2 - xy, x^2 - xy)$. A matrix representation of this subscheme is given by the pair

$$M_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

One can check by hand that there does not exist an *invertible* matrix Φ such that $\Phi = -\Phi^t$ and $\Phi M_\bullet = -M_\bullet^t \Phi$ for $\bullet = x, y$. Thus $\mathcal{E}$ is not symplectic.

Another example with a slightly different flavor is as follows. Let $\mathcal{E}$ be the Fourier-Mukai transform of $\mathcal{O}/(\mathcal{I}_{p_0})^2$. As we showed earlier in the footnote, $\mathcal{O}/(\mathcal{I}_{p_0})^2$ is invariant under $(-1)^*$ but it is not isomorphic to $\mathcal{E}xt^2(\mathcal{O}/(\mathcal{I}_{p_0})^2, \mathcal{O})$. This implies that $\mathcal{E} \not\cong \mathcal{E}^\vee$. Consequently, any bundle of the form $(L \otimes \mathcal{E}) \oplus (L^{-1} \otimes \mathcal{E})$ is Mukai stable but cannot have a non-degenerate symplectic form (here L is a degree zero line bundle that is *not* two-torsion). Examples of this type occur in codimension 4; in fact, one can prove that in general, the components of the locus of bundles in $\widetilde{M}_{Sp(n)}(A)$ with degenerate symplectic forms have codimension at least 4.

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