GENERATING FUNCTIONS FOR THE NUMBER OF CURVES ON ABELIAN SURFACES

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Abstract. Let X be an Abelian surface and C a holomorphic curve in X representing a primitive homology class. The space of genus g curves in the class of C is g dimensional. We count the number of such curves that pass through g generic points and we also count the number of curves in the fixed linear system $|C|$ passing through $g-2$ generic points. These two numbers, (defined appropriately) only depend on n and g where $n = \frac{C \cdot C}{2} + 1 - g$ and not on the particular X or C (n is the number of nodes when a curve is nodal and reduced).

Generalizing a formula of Yau and Zaslow [11], Göttsche conjectured that certain quasi-modular forms are the generating functions for the number of curves in a fixed linear system [5]. Our theorem proves this formula and shows that (a different) modular form also arises in the problem of counting curves without fixing a linear system. We use techniques that were developed in [1] for similar questions on $K3$ surfaces. The techniques include Gromov-Witten invariants for families and a degeneration to an elliptic fibration. One new feature of the Abelian surface case is the presence of non-trivial $\mathrm{Pic}^0(X)$. We show that for any surface S the cycle in the moduli space of stable maps defined by requiring that the image of the map lies in a fixed linear system is homologous to the cycle defined by requiring the image of the map meets b_1 generic loops in S representing the generators of $H_1(S; \mathbf{Z})/\operatorname{Tor}$.

Contents

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1. Introduction

Let X be an Abelian surface and let C be a holomorphic curve in X representing a primitive homology class. For n and g satisfying $C\cdot C=2g-2+2n$, there is a g dimensional space of curves of genus g in the class of C. To define an enumerative problem, one must impose g constraints on the curves. There are two natural ways to do this. One way is to count the number of curves passing through g generic points which we denote $N_{g,n}(X,C)$. The second way is to count the number of curves in the fixed linear system $|C|$ passing through $g-2$ generic points which we denote $N^{FLS}_{g,n}(X,C)$. We define (modified) Gromov-Witten invariants that compute the numbers $N_{g,n}(X,C)$ and $N^{FLS}_{g,n}(X,C)$ and we prove that they do not depend on X or C but are universal numbers henceforth denoted $N_{g,n}$ and $N^{FLS}_{g,n}$. Our main theorem computes these numbers as the Fourier coefficients of quasi-modular forms. Note that X does not contain any genus 0 curves so implicitly $g>0$ throughout.

Theorem 1.1 (Main theorem). *The universal numbers $N^{FLS}_{g,n}$ and $N_{g,n}$ are given by the following generating functions:*

$$\begin{aligned}\sum_{n=0}^{\infty} N_{g,n}q^{n+g-1} &= g(DG_2)^{g-1},\\ \sum_{n=0}^{\infty} N^{FLS}_{g,n}q^{n+g-1} &= (DG_2)^{g-2}D^2G_2\\ &= (g-1)^{-1}D((DG_2)^{g-1})\end{aligned}$$

where D is the operator $q\frac{d}{dq}$ and G_2 is the Eisenstein series, i.e.

$$G_2(q)=-\frac{1}{24}+\sum_{k=1}^{\infty}(\sum_{d|k}d)q^k.$$

Note that the right hand sides are quasi-modular forms (quasi-modular forms are an algebra generated by modular forms, G_2, and D, c.f. [5]) and the left hand sides are reminiscent of theta series since the power of q is $\frac{C\cdot C}{2}=n+g-1$.

The surprising fact that quasi-modular forms appear in generating functions for counting curves on surfaces was first discovered by Yau and Zaslow [11] who gave a formula for the number of rational curves on $K3$ surfaces. This formula was subsequently generalized by Göttsche to a conjectural formula that applies for all genus and all surfaces [5]. The formula for $N^{FLS}_{g,n}$ in Theorem 1.1 is a special case of the general Göttsche-Yau-Zaslow formula thus proving it (given our definition of $N^{FLS}_{g,n}$) for Abelian surfaces. Göttsche was able to prove the genus 2 case for Abelian surfaces (see also [2]).

In general, the Göttsche-Yau-Zaslow formula provides a conjectural generating functions for the number of curves with n nodes in any n-dimensional sublinear system of $|C|$ for any sufficiently ample divisor C on any surface S. The formulas involve only $c_2(S)$, $C\cdot C$, $C\cdot K$, $K\cdot K$ and universal functions (K is the canonical class). For surfaces with numerically trivial canonical class, the formula reduces to quasi-modular forms.

For concreteness, we expand the equations of the theorem to write:

$$\sum_{n=0}^{\infty} N_{g,n} q^{n+g-1} = g\left(\sum_{k=1}^{\infty} k(\sum_{d|k} d)q^k\right)^{g-1},$$

$$\sum_{n=0}^{\infty} N_{g,n}^{FLS} q^{n+g-1} = \left(\sum_{k=1}^{\infty} k(\sum_{d|k} d)q^k\right)^{g-2} \left(\sum_{k=1}^{\infty} k^2(\sum_{d|k} d)q^k\right).$$

So for example, for small values of n and g, $N_{g,n}^{FLS}$ and $N_{g,n}$ are given in the following tables:

$N_{g,n}^{FLS}$	n= 0	n=1	n=2	n=3	n=4	n=5	n=6	n=7
g=2	1	12	36	112	150	432	392	960
g=3	1	18	120	500	1620	4116	9920	19440
g=4	1	24	240	1464	6594	23808	73008	198480
g=5	1	30	396	3220	18960	88452	344960	1169520

$N_{g,n}$	n= 0	n=1	n=2	n=3	n=4	n=5	n=6	n=7
g=2	2	12	24	56	60	144	112	240
g=3	3	36	180	600	1620	3528	7440	12960
g=4	4	72	576	2928	11304	35712	97344	238176
g=5	5	120	1320	9200	47400	196560	689920	2126400

Gromov-Witten invariants have been remarkably effective in answering many questions in enumerative geometry for certain varieties such as $\mathbf{P}^n$; however, the ordinary Gromov-Witten invariants are not very effective for counting curves on most surfaces. One basic reason is that the moduli space of stable maps often fails to be a good model for a linear system (and the corresponding Severi varieties) for dimensional reasons. For an bundle L such that $L - K$ is ample, the dimension of the Severi variety $V_g(L)$ (the closure of the set of geometric genus g curves in the complete linear system $|L|$) is

$$\dim_{\mathbf{C}} V_g(L) = -K \cdot L + g - 1 + p_g - q$$

where K is the canonical class, $p_g = \dim H^0(X, K)$, and $q = \dim H^1(X, \mathcal{O})$. On the other hand, the virtual dimension of the moduli space $\mathcal{M}_{g,L}(X)$ of stable maps of genus g in the class dual to $c_1(L)$ is

$$\operatorname{virdim}_{\mathbf{C}} \mathcal{M}_{g,L}(X) = -K \cdot L + g - 1.$$

The discrepancy $p_g - q$ arises from two sources. Since the image of maps in $\mathcal{M}_{g,L}(X)$ are divisors not only in $|L|$ but also potentially in every linear system in $\operatorname{Pic}^{c_1(L)}(X)$, one would expect $\dim \mathcal{M}_{g,L}(X)$ to exceed $\dim V_g(L)$ by $q = \dim \operatorname{Pic}^{c_1(L)}(X)$ (we use $\operatorname{Pic}^c(X)$ to denote the component of $\operatorname{Pic}(X)$ with Chern class c). We show that this discrepancy can be accounted for within the framework of the usual Gromov-Witten invariants (see Theorem 2.1).

However, even if we consider $\mathcal{M}_{g,L}$ as a model for the parametrized Severi varieties

$$V_g(c_1(L)) \equiv \bigcup_{L' \in \mathrm{Pic}^{c_1(L)}(X)} V_g(L'),$$

there is still a p_g dimensional discrepancy (see also [3]).

The reason is the following. The virtual dimension of $\mathcal{M}_{g,L}(X)$ is the dimension of the space of curves that persist as *pseudo-holomorphic* curves when we perturb the Kähler structure to a generic almost Kähler structure. The difference of p_g in the dimensions of $\mathcal{M}_{g,L}$ and $V_g(c_1(L))$ means that only a codimension p_g subspace of $V_g(c_1(L))$ persists as pseudo-holomorphic curves when we perturb the Kähler structure. One way to rectify this situation is to find a compact p_g-dimensional[1] *family* of almost Kähler structures that has the property that the only almost Kähler structure in the family that supports pseudo-holomorphic curves in the class $c_1(L)$ is the original Kähler structure. If T is such a family, then the moduli space $\mathcal{M}_{g,L}(X,T)$ of stable maps for the family T is a better model for the space $V_g(c_1(L))$ in the sense that its dimension is stable under generic perturbations of the family $T \mapsto T'$.

It is straight-forward to extend the notion of the ordinary Gromov-Witten invariants to invariants for families of almost Kähler structures. The invariants will only depend on the deformation class of the underlying family of symplectic structures. Given the existence of a p_g-dimensional family as described above, these invariants can be used to answer enumerative geometry questions for the corresponding surface and linear system.

In general, it is not clear when such a family will exist; however, if X has a hyperkähler metric g (*i.e.* X is an Abelian or $K3$ surface), then there is a natural candidate for T, namely the hyperkähler family of Kähler structures. We call this family the *twistor family* associated to the metric g and we denote it T_g. It is parameterized by a 2-sphere and so $\dim_{\mathbf{R}} T_g = 2 = 2p_g$ as it should. Furthermore, the property that all the curves in $\mathcal{M}_{g,L}(X,T_g)$ are holomorphic for the original complex structure can be proved with Hodge theory (of course this need no longer be the case for a perturbation of T_g to a generic family of almost Kähler structures).

We will define the numbers $N^{FLS}_{g,n}$ and $N_{g,n}$ as certain Gromov-Witten invariants for the twistor family T_g associated to a hyperkähler metric on an Abelian surface X (see our previous paper [1] for the $K3$ case). We show that the invariants only depend on g and n (not X or C) and they count each irreducible geometric genus g curve with positive integral multiplicity. Furthermore, the multiplicity is 1 if additionally the curve is nodal. There are additional Gromov-Witten invariants for the twistor family that we also compute. These invariants are easier to describe with the notation of the Gromov-Witten invariants so we will postpone the statement of the result until section 3 (Definition 3.3 and Theorem 4.1). The enumerative problem these additional invariants correspond to is counting curves that pass through $g-1$ points and lie in a certain one dimensional family of linear systems.

Section 2 reviews Gromov-Witten invariants for families and formulates our result that equates the subset of $\mathcal{M}_{g,L}(X)$ consisting of maps whose image have fixed divisor class with a cycle more familiar in ordinary Gromov-Witten theory (see Theorem 2.1). In section 3 we discuss properties of the twistor family, define $N^{FLS}_{g,n}$

[1] By this we mean a real $2p_g$ dimensional family. Note that the parameter space for the family need not have an almost complex structure.

and $N_{g,n}$, and prove they have the enumerative properties discussed above. In section 4 we compute the invariants to complete the proof of our main theorem and its generalization. We conclude with an appendix containing the proof of Theorem 2.1 that was postponed in the main exposition.

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2. Gromov-Witten invariants for families

We review Gromov-Witten invariants for families and we refer the reader to [1] for details.

Let (X, ω) be any compact symplectic manifold with an almost Kähler structure. Recall that an n-marked, genus g *stable map* of degree $C \in H_2(X, \mathbf{Z})$ is a (pseudo)-holomorphic map $f : \Sigma \to X$ from an n-marked nodal curve $(\Sigma, x_1, \ldots, x_n)$ of geometric genus g to X with $f_*([\Sigma]) = C$ that has no infinitesimal automorphisms. Two stable maps $f : \Sigma \to X$ and $f' : \Sigma' \to X$ are equivalent if there is a biholomorphism $h : \Sigma \to \Sigma'$ such that $f = f' \circ h$. We write $\mathcal{M}_{g,n,C}(X, \omega)$ for the moduli space of equivalence classes of genus g, n-marked, stable maps of degree C to X. We will often drop the ω or X from the notation if they are understood and we sometimes will drop the n from the notation when it is 0. If B is a family of almost Kähler structures, we denote parameterized version of the moduli space:

$$\mathcal{M}_{g,n,C}(X, B) = \bigcup_{t \in B} \mathcal{M}_{g,n,C}(X, \omega_t).$$

If B is a compact, connected, oriented manifold then $\mathcal{M}_{g,n,C}(X, B)$ has a fiduciary cycle $[\mathcal{M}_{g,n,C}(X, B)]^{vir}$ called the virtual fundamental cycle (see [1] and the fundamental papers of Li and Tian [7][8][9]). The dimension of the cycle is

$$\dim_{\mathbf{R}} [\mathcal{M}_{g,n,C}(X, B)]^{vir} = 2c_1(X)(C) + (6 - \dim_{\mathbf{R}} X)(g - 1) + 2n + \dim_{\mathbf{R}} B.$$

The invariants are defined by evaluating cohomology classes of $\mathcal{M}_{g,n,C}$ on the virtual fundamental cycle. The cohomology classes are defined via incidence relations of the maps with cycles in X. The framework is as follows. There are maps

$$\begin{array}{ccc} \mathcal{M}_{g,1,C} & \xrightarrow{ev} & X \\ \Big\downarrow{\scriptstyle ft} & & \\ \mathcal{M}_{g,C} & & \end{array}$$

called the *evaluation* and *forgetful* maps defined by $ev(\{f : (\Sigma, x_1) \to X\}) = f(x_1)$ and $ft(\{f : (\Sigma, x_1) \to X\}) = \{f : \Sigma \to X\}$.[2] The diagram should be regarded as the universal map over $\mathcal{M}_{g,C}$.

Given geometric cycles $\alpha_1, \ldots, \alpha_l$ in X representing classes $[\alpha_1], \ldots, [\alpha_l] \in H_*(X, \mathbf{Z})$ with Poincaré duals $[\alpha_1]^\vee, \ldots, [\alpha_l]^\vee$, we can define the Gromov-Witten invariant

$$\Phi^{(X,B)}_{g,C}(\alpha_1, \ldots, \alpha_l) = \int_{[\mathcal{M}_{g,C}(X,B)]^{vir}} ft_* ev^*([\alpha_1]^\vee) \cup \cdots \cup ft_* ev^*([\alpha_l]^\vee).$$

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[2] There is some subtlety to making this definition rigorous since forgetting the point may make a stable map unstable, but it can be done.

$\Phi_{g,C}^{(X,B)}(\alpha_1,\dots,\alpha_l)$ counts the number of genus g, degree C maps which are pseudo-holomorphic with respect to some almost Kähler structure in B and such that the image of the map intersects each of the cycles $\alpha_1,\dots,\alpha_l$ (the integral is defined to be 0 if the integrand is not a class of the correct degree.). The Gromov-Witten invariants are multi-linear in the α's and they are symmetric for α's of even degree and skew symmetric for α's of odd degree. If $p_1,\dots,p_k$ are points in a path-connected X, we will use the shorthand

$$\Phi_{g,C}^{(X,B)}(\mathrm{pt.}^k,\alpha_{k+1},\dots,\alpha_l) := \Phi_{g,C}^{(X,B)}(p_1,\dots,p_k,\alpha_{k+1},\dots,\alpha_l).$$

Now suppose that X is a Kähler surface. In order to count curves in a fixed linear system $|L|$ with $c_1(L)=[C]^\vee$ one would like to restrict the above integral to the cycle defined by $\Psi_{\Sigma_0}^{-1}(0)$ where Ψ_{Σ_0} is the map

$$\Psi_{\Sigma_0} : \mathcal{M}_{g,C}(X,\omega)\to \mathrm{Pic}^0(X)$$

given by $f\mapsto \mathcal{O}(\mathrm{Im}(f)-\Sigma_0)$ where $\Sigma_0\in|L|$ is a fixed divisor. Dually, one can add the pullback by Ψ_{Σ_0} of the volume form on $\mathrm{Pic}^0(X)$ to the integrand defining the invariant:

$$\int_{[\mathcal{M}_{g,C}(X)]^{vir}} \Psi_{\Sigma_0}^*([\mathrm{pt.}]^\vee)\cup ft_*ev^*([\alpha_1]^\vee)\cup\dots\cup ft_*ev^*([\alpha_l]^\vee).$$

We show that $\Psi_{\Sigma_0}^*([\mathrm{pt.}]^\vee)$ can be expressed in the usual Gromov-Witten framework.

Theorem 2.1. *Let X be a Kähler surface and let $[\gamma]\in H_1(X,\mathbf{Z})$ and let $\tilde\gamma$ be the corresponding class in $H^1(\mathrm{Pic}^0(X),\mathbf{Z})$ induced by the identification $\mathrm{Pic}^0(X)\cong H^1(X,\mathbf{R})/H^1(X,\mathbf{Z})$. Then*

$$\Psi_{\Sigma_0}^*(\tilde\gamma) = ft_*ev^*([\gamma]^\vee).$$

Corollary 2.2. *Let $[\gamma_1],\dots,[\gamma_{b_1}]$ be an oriented integral basis for $H_1(X;\mathbf{Z})$. Then $\Psi_{\Sigma_0}^*([pt.]^\vee) = ft_*ev^*([\gamma_1]^\vee)\cup\dots\cup ft_*ev^*([\gamma_{b_1}]^\vee)$.*

Proof: We defer the proof of Theorem 2.1 to the appendix. The corollary follows immediately.

The upshot is that we count curves in a fixed linear system using the usual constraints from Gromov-Witten theory. Namely, the invariant:

$$\Phi_{g,C}(\gamma_1,\dots,\gamma_{b_1},\alpha_1,\dots,\alpha_l)$$

counts the number of genus g maps whose image lie in a fixed linear system $|L|$ with $c_1(L)=[C]^\vee$ and hit the cycles $\alpha_1,\dots,\alpha_l$.

3. The twistor family and the definition of $N_{g,n}$ and $N_{g,n}^{FLS}$

The discussion in this section is very similar to the corresponding discussion for $K3$ surfaces given in section 3 of [1]. We show that there is a unique family T (up to deformation) corresponding to the twistor family. We define $N_{g,n}^{FLS}$ and $N_{g,n}$ using a suitable set of Gromov-Witten invariants for the family T. We show that the invariants solve the enumerative problems we are interested in. We use this framework to prove that the invariants $N_{g,n}^{FLS}$ and $N_{g,n}$ are universal numbers independent of the choice of the Abelian surface X and the linear system $|L|$.

Let (X,ω) be an Abelian surface and let g be the unique hyperkähler metric given by Yau's theorem [10]. Define T_g to be the family of Kähler structures given by the unit sphere in $\mathcal{H}^2_{+,g}$, the space of self-dual, harmonic 2-forms.

Proposition 3.1. *For any two hyperkähler metrics g and g', the corresponding twistor families T_g and $T_{g'}$ are deformation equivalent. There is therefore a well-defined deformation class which we denote by T.*

PROOF: The moduli space of complex structures on the 4-torus is connected and the space of hyperkähler structures for a fixed complex torus is contractible (it is the Kähler cone). Therefore the space parametrizing hyperkähler 4-tori is connected (in fact it is just the space of flat metrics). We can thus find a path of hyperkähler metrics connecting g to g' and by associating the twistor family to each metric, we obtain a continuous deformation of T_g to $T_{g'}$. □

An important observation concerning the twistor family is the following corollary.

Corollary 3.2. *Let $f : X \to X$ be an orientation preserving diffeomorphism, then $f^*(T)$ is deformation equivalent to T.*

PROOF: Let T_g be the twistor family for a hyperkähler metric g. Then $f^*(T_g) = T_{f^*(g)}$ where $f^*(g)$ is the pullback metric which is also hyperkähler. □

The corollary has the consequence that for any orientation preserving diffeomorphism f the Gromov-Witten invariants for the twistor family satisfy

$$\Phi^{X,T}_{g,C}(\alpha_1, \dots, \alpha_l) = \Phi^{X,T}_{g,f_*(C)}(f_*\alpha_1, \dots, f_*\alpha_l).$$

There is an orientation preserving diffeomorphism of the 4-torus for every element of $Sl_4(\mathbf{Z})$ given by the descent of the linear action on the universal cover $\mathbf{R}^4$. It follows from the elementary divisor theorem that for any two classes C and C' in $H_2(X, \mathbf{Z})$ with the same divisibility and square there is a linear diffeomorphism f such that $f_*(C) = C'$ ([6] pg. 47). This means that there is a lot of symmetry among the Gromov-Witten invariants for the twistor family so that (for primitive classes) they can be encompassed by $N_{g,n}$ and $N^{FLS}_{g,n}$ and the other invariants which we define below.

Definition 3.3. *Let $\gamma_1, \dots, \gamma_4$ be loops in X representing an oriented basis for $H_1(X, \mathbf{Z})$ and let $C \in H_2(X, \mathbf{Z})$ be a primitive class with $C \cdot C = 2g - 2 + 2n$. We define:*

$$\begin{aligned} N_{g,n} &= \Phi^{(X,T)}_{g,C}(pt.^g) \\ N^{FLS}_{g,n} &= \Phi^{(X,T)}_{g,C}(\gamma_1, \gamma_2, \gamma_3, \gamma_4, pt.^{g-2}). \end{aligned}$$

Now let $C = [\gamma_1] \wedge [\gamma_2] + (n + g - 1)[\gamma_3] \wedge [\gamma_4]$ and define[3]

$$N^{ij}_{g,n} = \Phi^{(X,T)}_{g,C}(\gamma_i, \gamma_j, pt.^{g-1})$$

for $i < j$.

The invariants $N_{g,n}$, $N^{FLS}_{g,n}$, and $N^{ij}_{g,n}$ encompass all possible Gromov-Witten invariants for the twistor family and primitive homology classes. Since X has real dimension four, the only non-trivial constraints come from the point class and $H_1(X; \mathbf{Z})$. Since the point class is invariant under orientation preserving diffeomorphisms, $\Phi^{(X,T)}_{g,C}(\text{pt.}^g)$ only depends on the square (and divisibility) of C. Similarly, since Φ is skew-symmetric in the classes γ_i, the only possibility with

[3]Here we use the fact that on any torus there is a natural identification

$$\Lambda^2 H_1(X; \mathbf{Z}) \cong H_2(X, \mathbf{Z}).$$

four γ-constraints are the invariants $\Phi^{(X,T)}_{g,C}(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\mathrm{pt.}^{g-2})$ which also only depends on the square (and divisibility) of C. [4]

For dimensional reasons, the invariants with one or three γ's are zero. The invariants $N^{ij}_{g,n}$ encompass the remaining invariants since for any primitive C with $C^2 = 2g-2+2n$ we can first move C to $[\gamma_1]\wedge[\gamma_2]+(n+g-1)[\gamma_3]\wedge[\gamma_4]$ by an orientation preserving diffeomorphism.

We now wish to show that $N_{g,n}$, $N^{FLS}_{g,n}$, and $N^{ij}_{g,n}$ enumerate holomorphic curves as we want.

Lemma 3.4. *Suppose that X is an Abelian surface with a hyperkähler metric g and suppose that $C\subset X$ is a holomorphic curve. Then the only Kähler structure in T_g that has a holomorphic curve in the class $[C]\in H_2(X;\mathbf{Z})$ is the original Kähler structure for which C is holomorphic.*

PROOF: A necessary condition for the class $[C]$ to contain holomorphic curves is that $[C]^\vee \in H^{1,1}(X,\mathbf{R})$ and $[C]$ pairs positively with the Kähler form. Since the class orthogonal to $\mathcal{H}^2_{+,g}$ are always of type $(1,1)$ we just need to see when the projection of $[C]^\vee$ to $\mathcal{H}^2_{+,g}$ is type $(1,1)$.

We may assume $[C]^2$ is non-negative since X has no rational curves and so the projection of $[C]^\vee$ onto $\mathcal{H}^2_{+,g}$ is non-zero. By definition, the complex structure associated to a form $\omega\in T_g\subset\mathcal{H}^2_{+,g}$ defines the orthogonal splitting $\mathcal{H}^2_{+,g}\cong \omega\mathbf{R}\oplus(H^{2,0}\oplus H^{0,2})_{\mathbf{R}}$. Therefore, the only $\omega\in T_g$ for which $[C]^\vee$ is type $(1,1)$ and pairs positively with ω is when ω is a positive multiple of the projection of $[C]^\vee$ to $\mathcal{H}^2_{+,g}$. This is unique and so must be the original Kähler structure for which $C\subset X$ is holomorphic. $\square$

In general, Gromov-Witten type invariants may give a different count from the corresponding enumerative problem because Gromov-Witten invariants count maps instead of curves and the image of a map may have geometric genus smaller than its domain. We show that in the case at hand this does not happen.

Lemma 3.5. *Suppose X is generic among those Abelian surfaces admitting a curve in the class of C. Then the invariants $N_{g,n}$, $N^{FLS}_{g,n}$, and $N^{ij}_{g,n}$ count only maps whose image has geometric genus g and they are counted with positive integral multiplicity.*

PROOF: The assumption guarantees that C generates $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ so that all the curves in the class $[C]$ are reduced and irreducible. Then a dimension count shows that maps with contracting components of genus one or greater are not counted. The details are the same as in the proof of Theorem 3.5 in [1]. The argument shows that all the maps are birational onto their image and the corresponding moduli space is zero-dimensional. Furthermore, the contribution from maps that have a fixed image is determined solely by the singularities of the image curve. This contribution is shown in [4] to be a positive, integral number which is one if all the singularities are nodal. $\square$

[4]This invariant only depends on the square and divisibility of C since for any orientation preserving diffeomorphism f we have:

$$\begin{aligned}\Phi^{(X,T)}_{g,C}(\gamma_1,\dots,\gamma_4,\mathrm{pt.}^{g-2}) &= \Phi^{(X,T)}_{g,f_*(C)}(f_*\gamma_1,\dots,f_*\gamma_4,\mathrm{pt.}^{g-2})\\ &= \det(f_*:H_1\to H_1)\Phi^{(X,T)}_{g,f_*(C)}(\gamma_1,\dots,\gamma_4,\mathrm{pt.}^{g-2})\\ &= \Phi^{(X,T)}_{g,f_*(C)}(\gamma_1,\dots,\gamma_4,\mathrm{pt.}^{g-2}).\end{aligned}$$

We summarize the results of this section in the following theorem.

Theorem 3.6. *Let $C \subset X$ be a holomorphic curve C in an Abelians surface X with $C^2 = 2g-2+2n$. Then the invariants $N_{g,n}$, $N^{FLS}_{g,n}$, and $N^{ij}_{g,n}$ defined in Definition 3.3 count the number of genus g curves in the class $[C]$ that:*

1. *pass through g generic points ($N_{g,n}$);*
2. *pass through $g-2$ generic points and are confined to the linear system $|C|$ ($N^{FLS}_{g,n}$).*
3. *pass though $g-1$ generic points and are confined to lie in a certain 1-dimensional family of linear systems determined by $ij \in \{12, 13, 14, 23, 24, 34\}$ ($N^{ij}_{g,n}$).*

Furthermore, the invariants do not depend on the particular Abelian surface X or the curve C and the invariants count all irreducible, reduced curves with positive multiplicity that is one for nodal curves.

4. Computing the invariants

In this section, we compute the invariants $N_{g,n}$, $N^{FLS}_{g,n}$ and $N^{ij}_{g,n}$ by using a particular choice for X and C. Namely, we will choose X to be a product of elliptic curves and C to be a section together with a multiple of the fiber. The computations give our main theorem as well as the generalization to the $N^{ij}_{g,n}$ case. For convienence we state the results below:

Theorem 4.1. *The invariants $N_{g,n}$, $N^{FLS}_{g,n}$ and $N^{ij}_{g,n}$ (defined in Definiton 3.3) are given by the following generating functions:*

$$\sum_{n=0}^{\infty} N_{g,n} q^{n+g-1} = g\,(DG_2)^{g-1} \tag{1}$$

$$\sum_{n=0}^{\infty} N^{FLS}_{g,n} q^{n+g-1} = (DG_2)^{g-2} D^2 G_2 = (g-1)^{-1} D((DG_2)^{g-1}) \tag{2}$$

$$\sum_{n=0}^{\infty} N^{12}_{g,n} q^{n+g-1} = D\left((DG_2)^{g-1}\right) \tag{3}$$

$$\sum_{n=0}^{\infty} N^{34}_{g,n} q^{n+g-1} = (DG_2)^{g-1} \tag{4}$$

$$\sum_{n=0}^{\infty} N^{13}_{g,n} q^{n+g-1} = 0 \tag{5}$$

$$\sum_{n=0}^{\infty} N^{14}_{g,n} q^{n+g-1} = 0$$

$$\sum_{n=0}^{\infty} N^{23}_{g,n} q^{n+g-1} = 0$$

$$\sum_{n=0}^{\infty} N^{24}_{g,n} q^{n+g-1} = 0.$$

where $D = q\frac{d}{dq}$ and $G_2 = -\frac{1}{24} + \sum_{k=1}^{\infty} (\sum_{d|k} d) q^k$.

From Theorem 3.6 we are free to compute the invariants for any choice of X and C. Let X be the product of two generic elliptic curves $S\times F$ and let C be the primitive homology class $S+(g+n-1)F\in H_2(X,\mathbf{Z})$. We write $X=S^1\times S^1\times S^1\times S^1$ by choosing a diffeomorphism between $S^1\times S^1$ with S and also one between $S^1\times S^1$ with F . Next we choose representatives for the loops and points on X. We consider following four loops generating $H_1(X,\mathbf{Z})$:

$$\begin{array}{ll}\gamma_1 : S^1 \to X, & \gamma_1\left(e^{it}\right)=\left(e^{it},b_1,c_1,d_1\right),\\ \gamma_2 : S^1 \to X, & \gamma_2\left(e^{it}\right)=\left(a_2,e^{it},c_2,d_2\right),\\ \gamma_3 : S^1 \to X, & \gamma_3\left(e^{it}\right)=\left(a_3,b_3,e^{it},d_3\right),\\ \gamma_4 : S^1 \to X, & \gamma_4\left(e^{it}\right)=\left(a_4,b_4,c_4,e^{it}\right).\end{array}$$

where the a_i's, b_i's, c_i's, and d_i's are distinct points on S^1. We also choose g generic points $p_1=(s_1,f_1),...,p_g=(s_g,f_g)$ on $X=S\times F$. For any $f\in F$ and $s\in S$ we call $S_f=S\times\{f\}\subset X$ a section curve and $F_s=\{s\}\times F$ a fiber curve of X.

Proof of Equation 1:

We suppose that $\phi : D\to X$ is a stable map from a genus g curve D with g marked points to X representing the class $C=S+(g+n-1)F$ and sending corresponding marked points to the p_i's. First we observe that the image of ϕ consists of one section curve and some fiber curves. This is because the projection of any irreducible component of $\mathrm{Im}(\phi)$ to S is of degree zero except for one which has degree one. On the other hand, the projection of the degree one component to F must have degree zero because there is no nontrivial morphism between two generic elliptic curves S and F. Therefore $\mathrm{Im}(\phi)$ consists of a single section curve and a number of fibers.

In order for ϕ to pass through the g generic points $p_1,\ldots,p_g$ on X, we need at least $(g-1)$ fiber curves in the image of ϕ. On the other hand the geometric genus of D is g and covering each section or fiber curve will take up at least one genus of D. Therefore $\mathrm{Im}(\phi)$ consists of exactly $g-1$ fiber curves (possibly with multiplicity) and one section curve.

In fact, D has to have precisely g irreducible components $D_1,\ldots,D_g$ and each component is a genus one curve and contains one marked point. (Since an Abelian surface has no non-constant rational curves, stability implies that every rational component of D must meet at least three other components. It is easy to rule this possibility out.) The restriction of ϕ to each D_i is either (i) a covering of some fiber curve containing one of the p_i's or (ii) an isomorphism to a section curve containing one of the p_i's.

We can assume that ϕ restricted to D_g is an isomorphism onto one of the section curves containing some $p_i=(s_i,f_i)$. There are g choices of such p_i's. Without loss of generality we assume $\phi(D_g)$ contains p_g or equivalently $\phi(D_g)=S_{f_g}$. In this case the marked point on D_g must be the unique point $\phi^{-1}(p_g)$.

For $i<g$, we label the component of D covering F_{s_i} as D_i. Then $\phi:D_i\to F_{s_i}$ is an unbranched cover by the Hurwitz theorem because both D_i and F_{s_i} are genus one curves. We denote $k_i=\deg(\phi:D_i\to F_{s_i})>0$ then we have $\sum_{i=1}^{g-1}k_i=g+n-1$ since ϕ represents the homology class of $S+(g+n-1)F$. The number of elliptic curves that admit a degree k homomorphism to a fixed elliptic curve is classically known to be $\sum_{d|k}d$. We fix the origin of the elliptic curve D_i (and F_{s_i}) to be its intersection point with D_g (and S_{f_g} respectively), so that $\phi : D_i\to F_{s_i}$ is a homomorphism and thus the number of choices of $(\phi:D_i\to F_{s_i})$ is given by

$\sum_{d|k_i} d$. Since the marked point on D_i could be any one of the k_i points in $\phi^{-1}(p_i)$, there are a total of $k_i \sum_{d|k_i} d$ choices for each marked curve D_i.

We denote the $(g-1)$-tuple $k_1, \dots, k_{g-1}$ by $\mathbf{k}$ and we write $|\mathbf{k}|$ for $\sum_i k_i$. From the preceding discussion, the number of stable maps ϕ of geometric genus g and g marked points to X in the class of $S + (g+n-1)F$ is given by

$$g \sum_{\mathbf{k}:|\mathbf{k}|=n+g-1} \prod_{i=1}^{g-1} k_i \left(\sum_{d|k_i} d \right). \tag{6}$$

It is not difficult to see that each such stable map ϕ contributes one to the family Gromov-Witten invariant: The analysis is essentially the same as in Section 5 of [1] where the analogous results are proved in the $K3$ case and we refer the reader there for full details. We remark however that the situation here is much more elementary than in the $K3$ case: each map is an isolated smooth point as can be see by examining the infinitesimal deformations of maps (c.f. subsection 5.2 of [1]). As in the $K3$ case, the infinitesimal obstruction to deforming the map f in the direction of the twistor family cancels with $H^1(S, NS)$ where NS is the normal bundle of S in X. The space $H^1(S, NS)$ is essentially the obstruction to extending a deformation of the complex structure of the section component of the domain to a deformation of the map f.

The formula is now proved by summing Equation 6 over n and rearranging:

$$\begin{aligned}
\sum_{n=0}^{\infty} N_{g,n} q^{n+g-1} &= \sum_{n=0}^{\infty} \left(g \sum_{|\mathbf{k}|=n+g-1} \prod_{i=1}^{g-1} k_i \left(\sum_{d|k_i} d \right) \right) q^{n+g-1} \\
&= g \sum_{n=0}^{\infty} \left(\sum_{|\mathbf{k}|=n+g-1} \prod_{i=1}^{g-1} \left(k_i \sum_{d|k_i} d q^{k_i} \right) \right) \\
&= g \left(\sum_{k=1}^{\infty} k \sum_{d|k} d q^k \right)^{g-1} \\
&= g (DG_2)^{g-1}.
\end{aligned}$$

$\square$

Proof of Equations 3, 4, and 5:

For these computations we count the number of stable maps $\phi : D \to X$ of genus g with $g+1$ marked points that represent the homology class $C = S + (g+n-1)F$. The maps are constrained by requiring that the first $g-1$ marked points are mapped to the points $p_1, \dots, p_{g-1}$ and the image of the remaining two points must lie on γ_i and γ_j respectively. We argue as before that D has g irreducible components $D_1, ..., D_g$ where each D_i is a genus one curve and it contains one marked point except D_g which contains two marked points. Moreover, the image of the two marked points on D_g must lie on the two loops γ_i and γ_j. It is easy to check that when the points $p_i, a_i, \dots, d_i$ are in general positions, then no single section or fiber curve can pass through both one of the loops γ_i and one of the points p_i. We can also verify directly the following lemma.

Lemma 4.2. *No fiber or section curve can pass through two different γ_i's unless they are (γ_1, γ_2) or (γ_3, γ_4).*

No section curve can pass through both γ_1 and γ_2. The only fiber curve passing through γ_1 and γ_2 is $F_{12} = F_{(a_2,b_1)}$. Moreover, $F_{12} \cap \gamma_1 = (a_2, b_1, c_1, d_1)$ and $F_{12} \cap \gamma_2 = (a_2, b_1, c_2, d_2)$.

Similarly, no fiber curve can pass through both γ_3 and γ_4. The only section curve passing through γ_3 and γ_4 is $S_{34} = S_{(c_4,d_3)}$. Moreover, $S_{34} \cap \gamma_3 = (a_3, b_3, c_4, d_3)$ and $S_{34} \cap \gamma_4 = (a_4, b_4, c_4, d_3)$.

From the lemma the moduli space of stable maps is empty in all cases except $(\gamma_i, \gamma_j) = (\gamma_1, \gamma_2)$ or (γ_3, γ_4) up to permutations. Therefore the corresponding family Gromov-Witten invariants vanish (proving Equation 5)

$$N^{13}_{g,n} = N^{14}_{g,n} = N^{23}_{g,n} = N^{24}_{g,n} = 0.$$

Next we compute $N^{34}_{g,n}$. Let ϕ be any stable map in the corresponding moduli space, we have $\phi(D_g) = S_{34}$ by the above lemma. Moreover the restriction of ϕ to D_g is an isomorphism and ϕ must send the two marked points on D_g to (a_3, b_3, c_4, d_3) and (a_4, b_4, c_4, d_3). It is not difficult to check that the orientation of the moduli space is compatible with the one induced from (γ_3, γ_4). The calculations for the contribution from the other D_i's are identical with the earlier computation and we obtain

$$\sum_{n=0}^{\infty} N^{34}_{g,n} q^{n+g-1} = \left(\sum_{k=1}^{\infty} k \sum_{d|k} dq^k \right)^{g-1} = (DG_2)^{g-1}$$

thus proving Equation 4.

Now we compute $N^{12}_{g,n}$. From the lemma again we have $\phi(D_g) = F_{12}$ for any stable map ϕ in the corresponding moduli space. Let us denote the degree of ϕ restricted to D_g by k_0. Then the number of possible $(\phi : D_g \to F_{12})$ is given by $\sum_{d|k_0} d$ as before. The image of the two marked points on D_g are (a_2, b_1, c_1, d_1) and (a_2, b_1, c_2, d_2). Since ϕ is an unbranched covering map, there are k_0 choices for the location of each marked point. There are then a total of $k_0^2 \sum_{d|k_0} d$ different choices associated the component D_g.

There are $g-1$ choices for which curve $D_1, \ldots, D_{g-1}$ is mapped to a section. After making such a choice we may assume without loss of generality that D_{g-1} is mapped to the section curve passing through p_{g-1} (since we can relabel the points). Then since ϕ is an isomorphism on D_{g-1}, the location of the marked point on it is determined. The analysis for the other fibers is the same as before and the issue of the orientation is the same as above. In conclusion we have

$$N^{12}_{g,n} = (g-1) \sum_{\mathbf{k}:|\mathbf{k}|=n+g-1} \left(k_0^2 \sum_{d|k_0} d \prod_{i=1}^{g-2} k_i \sum_{d|k_i} d \right),$$

where the summation is over $(g-1)$-tuples $\mathbf{k} = \{k_0, \ldots, k_{g-2}\}$ with $k_i > 0$ and $|\mathbf{k}| = \sum_{i=0}^{g-2} k_i = g+n-1$. Summing over n, we prove Equation 3:

$$\begin{aligned} \sum_{n=0}^{\infty} N^{12}_{g,n} q^{n+g-1} &= (g-1) \left(\sum_{k=1}^{\infty} k^2 \sum_{d|k} dq^k \right) \left(\sum_{k=1}^{\infty} k \sum_{d|k} dq^k \right)^{g-2} \\ &= (g-1) \left(D^2 G_2\right) (DG_2)^{g-2} \\ &= D\left((DG_2)^{g-1}\right). \end{aligned}$$

□

PROOF OF EQUATION 2:

Suppose that $\phi : D \to X$ is a stable map of genus g with $g+2$ marked points and represents the homology class $C = S + (g+n-1)F$. We argue as before that D has g irreducible components $D_1, ..., D_g$. The curves $D_1, \ldots, D_{g-2}$ each have one marked point mapping to $p_1, \ldots, p_{g-2}$ respectively. The curves D_{g-1} and D_g contain the remaining four marked points which must lie on the four loops $\gamma_1, ..., \gamma_4$. The image of each D_i must either be a section curve or a fiber curve. By the previous lemma the only fiber curve that intersects two γ_i's must be F_{12} and it intersects γ_1 and γ_2 at (a_2, b_1, c_1, d_1) and (a_2, b_1, c_2, d_2) respectively. Similarly, the only section that intersects two γ's is S_{34} and it must intersect γ_3 and γ_4 at (a_3, b_3, c_4, d_3) and (a_4, b_4, c_4, d_3) respectively. Hence the image of D_{g-1} and D_g must be F_{12} and S_{34} respectively. For $i \leq g-2$, we have $\phi(D_i) = F_{s_i}$. Let the degree of ϕ on D_i be k_i where $1 \leq i \leq g-1$. Then $\sum k_i = g+n-1$. The number of choices of $(\phi : D_i \to F_{s_i})$ is again given by $\sum_{d|k_i} d$. For $i < g-1$ there are k_i choices of the location of the marked point on D_i. For $i = g-1$, there are two marked points on D_{g-1} and each has k_{g-1} possible locations. Again it is not difficult to check that such maps each contribute $+1$ to the family Gromov-Witten invariants. In conclusion we have

$$N_{g,n}^{FLS} = \sum_{\mathbf{k}:|\mathbf{k}|=n+g-1} \left(k_{g-1}^2 \sum_{d|k_{g-1}} d \right) \prod_{i=1}^{g-2} \left(k_i \sum_{d|k_i} d \right)$$

and so summing over n we have

$$\begin{aligned} \sum_{n=0}^{\infty} N_{g,n}^{FLS} q^{n+g-1} &= \left(\sum_{k=1}^{\infty} k \sum_{d|k} dq^k \right)^{g-2} \left(\sum_{k=1}^{\infty} k^2 \sum_{d|k} dq^k \right) \\ &= (DG_2)^{g-1} \left(D^2 G_2 \right) \end{aligned}$$

proving Equation 2. □

APPENDIX A. PROOF OF THEOREM 2.1

First we determine a more explicit formulation of the map

$$\Psi_{\Sigma_0} : \mathcal{M}_{g,C} \to \mathrm{Pic}^0(X) = H^1(X, \mathbf{R})/H^1(X, \mathbf{Z}).$$

Throughout this section we will abuse notation and refer to a stable map $f : \Sigma \to X$ by its image so that when we say $\Sigma \in \mathcal{M}_{g,C}$ we mean the curve $\Sigma \subset X$ given by the image of the map $f : \Sigma \to X$. By definition, $\Psi_{\Sigma_0}(\Sigma) = \mathcal{O}(\Sigma - \Sigma_0)$. We wish to get a 1-form representative for $\mathcal{O}(\Sigma - \Sigma_0)$. First, choose a C^∞ trivialization of the bundle $\mathcal{O}(\Sigma - \Sigma_0)$. Let s_0 and s be defining sections of $\mathcal{O}(\Sigma_0)$ and $\mathcal{O}(\Sigma)$. Then via the trivialization, $h = s_0/s$ is a non-vanishing C^∞ function on $X^o = X - \{\Sigma_0 \cup \Sigma\}$. Define an integrable real 1-form a_h by

$$a_h = \frac{1}{2\pi i} \left(\frac{\bar{\partial} h}{h} - \frac{\partial \overline{h}}{\overline{h}} \right).$$

The form a_h defines a real 1-current and let $a_{\Sigma_0}^{\Sigma}$ be the 1-form representing the harmonic projection of a_h. We wish to define a map to $H^1(X, \mathbf{R})/H^1(X, \mathbf{Z})$ using $a_{\Sigma_0}^{\Sigma}$ and we need the following lemma:

Lemma A.1. *The harmonic 1-form $a^{\Sigma}_{\Sigma_0}$ is independent of the choices of s_0, s, and the trivialization up to translation by an integral harmonic 1-form.*

Proof: Let $h' = s'_0/s'$ be defined using a possibly different trivialization. Then $h' = gh$ for some C^∞ map $g : X \to \mathbf{C}^*$ and so

$$a_h - a_{h'} = \frac{1}{2\pi i}\left(\frac{\bar\partial g}{g} - \frac{\partial \overline{g}}{\overline{g}}\right) = a_g$$

is a smooth 1-form. We need to show that the harmonic projection of a_g is integral. Write $g = e^t\theta$ where $t : X \to \mathbf{R}$ and $\theta : X \to S^1 \subset \mathbf{C}$ are C^∞. A direct computation shows that

$$a_g = d^c t + \frac{1}{2\pi i}\theta^{-1} d\theta = d^*(t\omega) + \frac{1}{2\pi i}\theta^{-1} d\theta$$

where the last equality uses the Kähler identities and so we see that the harmonic projection of a_g is the integral class $\frac{1}{2\pi i}\theta^{-1} d\theta$. $\square$

The lemma shows that

$$\Sigma \mapsto a^{\Sigma}_{\Sigma_0}$$

determines a well-defined map

$$\mathcal{M}_{g,C} \to H^1(X, \mathbf{R})/H^1(X, \mathbf{Z}).$$

This is the same map as Ψ_{Σ_0} since

1. if Σ is linearly equivalent to Σ_0 then $a^{\Sigma}_{\Sigma_0} \equiv 0$ and
2. the sum of divisors corresponds to the sum of forms, *i.e.* $a^{\Sigma+\Sigma'}_{2\Sigma_0} = a^{\Sigma}_{\Sigma_0} + a^{\Sigma'}_{\Sigma_0}$.

The first property holds since if $\Sigma \sim \Sigma_0$ then h can be chosen to be a meromorphic function and so $a_h \equiv 0$. The second property follows from $a_{hh'} = a_h + a_{h'}$.

To prove Theorem 2.1 we need to show that $\Psi^*_{\Sigma_0}(\tilde\gamma) = ft_* ev^*([\gamma]^\vee)$. Recall that γ is a loop in X, $[\gamma]^\vee$ is the Poincare dual of the 1-cycle $[\gamma]$, and $\tilde\gamma \in H^1(\mathrm{Pic}^0(X), \mathbf{Z})$ is the natural class associated to $[\gamma]$ arising from the identification $\mathrm{Pic}^0(X) \cong H^1(X, \mathbf{R})/H^1(X, \mathbf{Z})$.

We use the general correspondence between elements of $H^1(M, \mathbf{Z})$ and homotopy classes of maps $M \to S^1$. One can get a circle valued function on M from a class $\phi \in H^1(M; \mathbf{Z})$ by choosing a base point $x_0 \in M$ and defining $f_\phi : M \to \mathbf{R}/\mathbf{Z}$ by

$$f_\phi(x) = \int_{\Gamma^x_{x_0}} \phi \mod \mathbf{Z}$$

where $\Gamma^x_{x_0}$ is a path from x_0 to x. Since ϕ is an integral class, f_ϕ does not depend on the choice of the path (mod $\mathbf{Z}$).

Using Σ_0 as the base point for $\mathcal{M}_{g,C}$, the class $ft_* ev^*([\gamma]^\vee) \in H^1(\mathcal{M}_{g,C}, \mathbf{Z})$ is given by the S^1-valued map

$$\begin{aligned}
\Sigma \quad \mapsto \quad & \int_{\Gamma^{\Sigma}_{\Sigma_0}} ft_* ev^*([\gamma]^\vee) \mod \mathbf{Z} \\
= \quad & \int_{ev(ft^{-1}(\Gamma^{\Sigma}_{\Sigma_0}))} [\gamma]^\vee \mod \mathbf{Z} \\
= \quad & \int_{W^{\Sigma}_{\Sigma_0}} [\gamma]^\vee \mod \mathbf{Z}
\end{aligned}$$

where $W^{\Sigma}_{\Sigma_0}$ is the 3-chain in X swept out by the curves in the path $\Gamma^{\Sigma}_{\Sigma_0}$. Note that $\partial W^{\Sigma}_{\Sigma_0} = \Sigma_0 - \Sigma$ and that the map

$$\Sigma \mapsto \int_W [\gamma]^\vee \mod \mathbf{Z}$$

is the same for any 3-chain W such that $\partial W = \Sigma_0 - \Sigma$ (since the difference $W^{\Sigma}_{\Sigma_0} - W$ is a 3-cycle and $[\gamma]^\vee$ is an integral class).

On the other hand, the class $\tilde{\gamma} \in H^1(\mathrm{Pic}^0, \mathbf{Z})$ is by definition given by the S^1-valued function on Pic^0 defined by

$$[a] \mapsto \int_X a \wedge [\gamma]^\vee \mod \mathbf{Z}$$

where $a \in H^1(X, \mathbf{R})$ and $[a]$ is the corresponding equivalence class in

$$H^1(X, \mathbf{R})/H^1(X, \mathbf{Z}).$$

The class $\Phi^*_{\Sigma_0}(\tilde{\gamma})$ is therefore represented by the S^1-valued function

$$\Sigma \mapsto \int_X a^{\Sigma}_{\Sigma_0} \wedge [\gamma]^\vee \mod \mathbf{Z}$$

and to prove theorem 2.1 then we need to show that

$$\int_X a^{\Sigma}_{\Sigma_0} \wedge [\gamma]^\vee = \int_{W^{\Sigma}_{\Sigma_0}} [\gamma]^\vee \mod \mathbf{Z}.$$

Recall that $a^{\Sigma}_{\Sigma_0}$ is the harmonic projection of a_h. Let ζ be the harmonic representative for $[\gamma]^\vee$. Then

$$\int_X a^{\Sigma}_{\Sigma_0} \wedge [\gamma]^\vee = \int_X a_h \wedge \zeta$$

so we need to show that

$$\int_{X^o} \frac{1}{2\pi i}\left(\frac{\bar\partial h}{h} - \frac{\partial \overline{h}}{\overline{h}}\right) \wedge \zeta = \int_{W^{\Sigma}_{\Sigma_0}} \zeta \mod \mathbf{Z}. \tag{7}$$

Recall that h is a $\mathbf{C}^*$-valued function on $X^o = X - \{\Sigma_0 \cup \Sigma\}$ and so writing $h = e^t\theta$ for smooth functions $t : X^o \to \mathbf{R}$ and $\theta : X^o \to S^1$ we can rewrite the left hand side of Equation 7 as

$$LHS = \int_{X^o} d^*(t\omega) \wedge \zeta + \int_{X^o} \frac{1}{2\pi i}\theta^{-1} d\theta \wedge \zeta.$$

We can do the first of these integrals by first integrating along the fibers of $t : X^o \to \mathbf{R}$:

$$\begin{aligned} t_*(d^*(t\omega) \wedge \zeta) &= -t_*(*d(*(t\omega)) \wedge \zeta) \\ &= t_*(d(t\omega) \wedge *\zeta) \\ &= t_*(dt \wedge \omega \wedge *\zeta) \\ &= dt \wedge t_*(\omega \wedge *\zeta) \end{aligned}$$

but $t_*(\omega \wedge *\zeta) = 0$ because $\omega \wedge *\zeta$ is a closed form defined on *all* of X and $t^{-1}(\mathrm{pt.})$ is a boundary 3-chain in X.

We perform the remaining integral by integrating first along the fibers of $\theta : X^o \to S^1$. Since $\frac{1}{2\pi i}\theta^{-1} d\theta = \theta^*(dvol_{S^1})$, we can write the remaining integral as

$$\int_{X^o} \theta^*(dvol_{S^1}) \wedge \zeta = \int_{S^1} \theta_*(\zeta) dvol_{S^1}.$$

Now $\theta_*(\zeta) = \int_{\theta^{-1}(c)} \zeta$ is independent of c (mod $\mathbf{Z}$) since $\theta^{-1}(c)$ is a 3-chain in X with boundary $\Sigma_0 - \Sigma$ and so (mod $\mathbf{Z}$) we have

$$\begin{aligned}\text{LHS of Eqn 7} &= \left(\int_W \zeta\right) \int_{S^1} dvol_{S^1} \\ &= \int_W \zeta \\ &= \text{RHS of Eqn 7}\end{aligned}$$

which proves the theorem. □

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