This Examination paper consists of 12 pages (including this one). Make sure you have all 12.

**INSTRUCTIONS:**

No extra materials allowed. No calculator allowed. No communication devices allowed.

Rules:

- Each candidate must be prepared to produce, upon request, a UBC card for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
  - having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
  - speaking or communicating with other candidates; and
  - purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

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**Names of Instructors:** Bryan, Mac Lean
Q1 [10 marks]

Consider the motion of a thumbtack stuck in the tread of a tire which is on a bicycle moving at constant speed. This motion is given by the parametrized curve \( \vec{r}(t) = (t - \sin t, 1 - \cos t) \) with \( t > 0 \).

(a) Sketch the curve in the xy-plane for \( 0 < t < 4\pi \).

(b) Find and simplify the formula for the curvature \( \kappa(t) \).

(c) Find the radius of curvature of the osculating circle to \( \vec{r}(t) \) at \( t = \pi \).

(d) Find the equation of the osculating circle to \( \vec{r}(t) \) at \( t = \pi \).

\[
\begin{align*}
(a) & \\
& \text{Sketch the curve in the xy-plane for } 0 < t < 4\pi. \\
(b) & \\
& \vec{r}'(t) = \langle 1 - \cos t, \sin t \rangle \\
& |\vec{r}'| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{1 - 2\cos t + \cos^2 t} \\
& \vec{r}'' = \langle \sin t, \cos t \rangle \\
& \vec{r}' \times \vec{r}'' = \\
& K(t) = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{1 - \cos t}{2 \sqrt[2]{1 - \cos t}} \\
(c) & \\
& \text{rad} = \frac{1}{K(\pi)} = \frac{1}{2\sqrt[2]{1 - (1)}} = \frac{1}{4} \\
(d) & \\
& \text{At } t = \pi \quad \vec{r}(\pi) = \langle \pi, 2 \rangle \quad \text{and} \quad \vec{r}'(\pi) = \langle 2, 0 \rangle \\
& \text{so normal } \vec{n} \text{ is in the } -\hat{j} \text{ direction. Thus center is } \langle \pi, 2 \rangle - 4\langle 0, 1 \rangle = \langle \pi, -2 \rangle \\
& (x - \pi)^2 + (y + 2)^2 = 4^2.
\end{align*}
\]
Q2  [10 marks]

A particle moves so that its position vector is given by \( \mathbf{r}(t) = (\cos t, \sin t, c \sin t) \), where \( t > 0 \) and \( c \) is a constant.

(a) Find the velocity \( \mathbf{v}(t) \) and the acceleration \( \mathbf{a}(t) \) of the particle.

(b) Find the speed \( v(t) \) of the particle.

(c) Find the tangential component of the acceleration of the particle.

(d) Show that the trajectory of this particle lies in a plane.

\[
\begin{align*}
\mathbf{v}(t) &= \mathbf{r}'(t) = \langle -\sin t, \cos t, c \cos t \rangle \\
\mathbf{a}(t) &= \mathbf{r}''(t) = \langle -\cos t, -\sin t, -c \sin t \rangle \\

\mathbf{v}(t) &= \sqrt{\sin^2 t + \cos^2 t + c^2 \cos^2 t} \\
&= \sqrt{1 + c^2 \cos^2 t}
\end{align*}
\]

(c) From the formula \( \mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N} = \frac{d}{dt} \mathbf{T} + \kappa \mathbf{N} \),

we get
\[
\begin{align*}
a_T &= \frac{d}{dt} \sqrt{1 + c^2 \cos^2 t} = \frac{-2c^2 \cos t \sin t}{2\sqrt{1 + c^2 \cos^2 t}} = \frac{-c^2 \cos t \sin t}{\sqrt{1 + c^2 \cos^2 t}}
\end{align*}
\]

(d) Since \( x = \cos t \), \( y = \sin t \), \( z = c \sin t \) for \( t > 0 \)

this curve lies on the plane \( z = cy \).
Q3  [10 marks]

Find the arc length of the curve \( \mathbf{r}(t) = (t^m, t^n, t^{3m/2}) \) for \( 0 \leq a \leq t \leq b \), and where \( m > 0 \). Express your result in terms of \( m, a, \) and \( b \). (Hint: The integral you get can be evaluated with a simple substitution. You may want to factor the integrand first.)

\[
\text{Length} = \int_a^b |\mathbf{r}'(t)| \, dt
\]

\[
\mathbf{r}'(t) = \left< mt^{-1}, nt^{-1}, \frac{3m}{2} t^{3m/2 - 1} \right>
\]

\[
= mt^{-1} \left< 1, 1, \frac{3}{2} t^{m/2} \right>
\]

\[
|\mathbf{r}'(t)| = mt^{-1} \sqrt{1^2 + 1^2 + \frac{3}{2} \left( \frac{3}{2} t^{m/2} \right)^2}
\]

\[
= mt^{-1} \sqrt{2 + \frac{9}{4} t^m}
\]

\[
\text{Length} = \int_a^b mt^{-1} \sqrt{2 + \frac{9}{4} t^m} \, dt
\]

\[
u = 2 + \frac{9}{4} t^m
\]

\[
du = \frac{9}{4} m t^{-1} dt
\]

\[
\text{Length} = \int_{2 + \frac{9}{4} a^m}^{2 + \frac{9}{4} b^m} \frac{4}{9} \sqrt{u} \, du
\]

\[
= \left[ \frac{2}{3} \frac{4}{9} u^{3/2} \right]_{2 + \frac{9}{4} a^m}^{2 + \frac{9}{4} b^m}
\]

\[
= \frac{8}{27} \left[ \left(2 + \frac{9}{4} b^m\right)^{3/2} - \left(2 + \frac{9}{4} a^m\right)^{3/2} \right]
\]
Q4  [10 marks]

Find the value of \( \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S} \), where \( \mathbf{F} = (z - y, x, -x) \) and \( S \) is the hemisphere \( \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4, \ z \geq 0\} \) oriented so the surface normals point away from the centre of the hemisphere.

\[
\iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}
\]

\( C : \begin{align*}
\mathbf{r}(t) &= \langle 2\cos t, 2\sin t, 0 \rangle \quad 0 \leq t \leq 2\pi \\
\mathbf{r}'(t) &= \langle -2\sin t, 2\cos t, 0 \rangle
\end{align*} \)

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle -2\sin t, 2\cos t, -2\cos t \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle dt
\]

\[
= \int_0^{2\pi} (4\sin^2 t + 4\cos^2 t) dt = 4 \int_0^{2\pi} dt = 8\pi
\]
Q5  [10 marks]

Compute the net outward flux of the vector field

$$\mathbf{F} = \mathbf{F} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}$$

across the boundary of the region \textit{between} the spheres of radius 1 and radius 2 centred at the origin.

\[ \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathbf{N}_1 dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{N}_2 dS \]

\[ S_1 \text{ is sphere of radius 1 } \mathbf{N}_1 = -\frac{\mathbf{r}}{1^2} \text{ points inward} \]

\[ S_2 \text{ is sphere of radius 2 } \mathbf{N}_2 = \frac{\mathbf{r}}{2^2} \text{ points outward} \]

\[ \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \frac{\mathbf{F}}{1^2} \cdot \frac{\mathbf{r}}{1^2} dS + \iint_{S_2} \frac{\mathbf{F}}{2^2} \cdot \frac{\mathbf{r}}{2^2} dS \]

\[ = -\iint_{S_1} dS + \iint_{S_2} dS \]

\[ = -\text{area}(S_1) + \text{area}(S_2) \]

\[ = -4\pi + 4\pi (2^2) \]

\[ = 12\pi \]
Q5 [10 marks]

Compute the net outward flux of the vector field

\[ \mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} \]

across the boundary of the region between the spheres of radius 1 and radius 2 centred at the origin.

\[ \iiint_{R} \nabla \cdot \mathbf{F} \, dV = \iint_{S} \nabla \cdot \mathbf{F} \, dS. \]

\[ S = \partial R \]

\[ R = \{(x, y, z) : \ 1 \leq x^2 + y^2 + z^2 \leq 4 \} \]

\[ \nabla \cdot \mathbf{F} = \frac{2}{r^3} \left( \frac{x}{r} \right) + \frac{2}{r} \left( \frac{y}{r} \right) + \frac{2}{r} \left( \frac{z}{r} \right) \]

\[ \frac{2}{r^2} \left( \frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{x^2}{x^2 + y^2 + z^2} - \frac{x^2}{x^2 + y^2 + z^2} \]

\[ = \frac{1}{r} - \frac{x^2}{r^3}, \text{ so} \]

\[ \nabla \cdot \mathbf{F} = \left( \frac{1}{r} - \frac{x^2}{r^3} \right) + \left( \frac{1}{r} - \frac{y^2}{r^3} \right) + \left( \frac{1}{r} - \frac{z^2}{r^3} \right) = \frac{3}{r} - \frac{x^2 + y^2 + z^2}{r^3} \]

\[ = \frac{3}{r} - \frac{r^2}{r^3} = \frac{2}{r} \]

\[ \iint_{S} \mathbf{F} \cdot d\mathbf{S} = \int_{\rho=1}^{2} \int_{\theta=0}^{\pi} \int_{\phi=0}^{\pi} \frac{2}{\rho} \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho \]

\[ = 2\pi \int_{\rho=1}^{2} 2\rho d\rho \int_{\phi=0}^{\pi} \sin \phi d\phi = 2\pi \left[ \rho^2 \right]_{1}^{2} \left[ -\cos \phi \right]_{0}^{\pi} = 2\pi (4-1) \cdot \frac{1}{2} = 12\pi \]
Q6 [15 marks] \( P \quad Q \quad R \)

Let \( \vec{F} = (y^2 e^{3z} + Axy^3) \hat{i} + (2xye^{3z} + 3x^2y^2) \hat{j} + Bxy^2e^{3z} \hat{k} \).

(a) Find all values of \( A \) and \( B \) for which the vector field \( \vec{F} \) is conservative.

(b) If \( A \) and \( B \) have values found in (a), find a scalar potential function for \( \vec{F} \).

(c) Let \( C \) be the curve with parametrization \( \vec{r}(t) = e^{2t} \hat{i} + e^{-t} \hat{j} + \ln(1 + t) \hat{k} \) from \((1, 1, 0)\) to \((e^2, \frac{1}{e}, \ln 2)\). Evaluate \( \int_C (y^2 e^{3z} + xy^3) \, dx + (2xye^{3z} + 3x^2y^2) \, dy + 3xy^2e^{3z} \, dz \).

(Hint: This vector field is not conservative, but it is almost to the conservative vector field \( \vec{F} \).)

\[
\text{\( \vec{F} \) conservative } \Rightarrow \text{ curl} \vec{F} = 0 \quad \text{curl} \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y^2 e^{3z} + Axy^3 & 2xye^{3z} + 3x^2y^2 & Bxy^2e^{3z}
\end{vmatrix}
\]

\[
= \begin{pmatrix}
Bxy^2e^{3z} - 6xye^{3z} \\
y^2e^{3z} - Bye^{3z} \\
2y^3e^{3z} - 2ye^{3z} \cdot 3Axy^2
\end{pmatrix} + 6xy^2
\]

So to be zero we need \( B = 3 \), \( A = 2 \)

\[
f_x = y^2 e^{3z} + 2xy^3 \Rightarrow f = xy^2 e^{3z} + x^2y^3 + g_1(y, z)
\]

\[
f_y = 2xye^{3z} + 3x^2y^2 \Rightarrow f = xy^2 e^{3z} + x^2y^3 + g_2(x, z)
\]

\[
f_z = 3xy^2 e^{3z} \Rightarrow f = xy^2 e^{3z} + g_3(x, y)
\]

so \( f(x, y, z) = xy^2 e^{3z} + x^2y^3 \)

(c) The integral is \( \int_C P \, dx + Q \, dy + R \, dz = \int_C xy^3 \, dx \)

\[
f(e^2, \frac{1}{e}, \ln 2) - f(1, 1, 0) = \int_0^1 e^{2t} \cdot e^{-3t} \cdot 2e^{2t} \, dt
\]

\[
e^{2(\frac{1}{2})} e^{3\ln 2} + e^4(\frac{1}{e})^3 - 1 - 1 = 2\int_0^1 e^t \, dt
\]

\[
e^{\ln(2^3)} + e - 7 - 2[2e] = 8 + e - 2(e - 1) = 10 - e
\]
Q7 [10 marks]

(a) (6 marks) Give parametric descriptions of the form \( \vec{F}(u,v) = (x(u,v), y(u,v), z(u,v)) \) for the following surfaces. Be sure to state the domains of your parametrizations.

i) The part of the plane \( 2x + 4y + 3z = 16 \) in the first octant \( \{(x,y,z) : x \geq 0, y \geq 0, z \geq 0 \} \).

ii) The cap of the sphere \( x^2 + y^2 + z^2 = 16 \) for \( 4/\sqrt{2} \leq z \leq 4 \).

iii) The hyperboloid \( z^2 = 1 + x^2 + y^2 \) for \( 1 \leq z \leq 10 \).

\[ \begin{align*}
(1) & \quad \text{use } \vec{r}(u,v) = \vec{r}_0 + u\vec{A} + v\vec{B} \text{ where} \\
& \quad \vec{r}_0 = \langle 0, 0, 0 \rangle, \quad \vec{A} = \langle -8, 4, 0 \rangle, \quad \vec{B} = \langle -8, 0, \frac{16}{3} \rangle \\
& \quad \vec{F}(u,v) = \langle 8 - 8u, 8v, \frac{16}{3}v \rangle \\
& \quad x, y, z \geq 0 \Rightarrow u \geq 0, v \geq 0, 8 - 8u - 8v \geq 0 \Rightarrow 12u + v \\
& \quad \vec{r}(u,v) = \langle 8 - 8u - 8v, 4u, \frac{16}{3}v \rangle \quad 0 \leq u, 0 \leq v, u + v \leq 1 \\
& \quad \text{or para. as the graph of a function: } z = \frac{1}{3}(16 - 2x - 4y) \\
& \quad \vec{F}(u,v) = \langle u, v, \frac{1}{3}(16 - 2u - 4v) \rangle \quad u > 0, v > 0 \quad \frac{1}{3}(16 - 2u - 4v) > 0 \\
& \quad \Leftrightarrow \quad 8 > u + 2v \\
& \quad \text{lots of other valid solutions.} \\
(ii) & \quad \text{so triangle is a } 45^\circ \\
& \quad \vec{F}(\theta, \phi) = \langle 4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi \rangle \\
& \quad 0 \leq \theta \leq 2\pi \quad 0 \leq \phi \leq \frac{\pi}{4}
\end{align*} \]
(ii) Alternate: 
\[ z = \sqrt{16 - x^2 - y^2} \quad \frac{4}{\sqrt{2}} \leq \sqrt{16 - x^2 - y^2} \]

\[ \Rightarrow 8 \leq 16 - x^2 - y^2 \Rightarrow x^2 + y^2 \leq 8 \]

\[ \vec{r}(x, y) = \langle x, y, \sqrt{16 - x^2 - y^2} \rangle \quad x^2 + y^2 \leq 8 \]

(iii) \[ 1 \leq z \leq 10 \]

\[ z = \sqrt{1 + x^2 + y^2} \]

\[ z \leq 10 \Rightarrow 1 + x^2 + y^2 \leq 100 \]

\[ \Rightarrow x^2 + y^2 \leq 99 \]

\[ 1 \leq z \text{ automatic} \]

\[ \vec{r}(x, y) = \langle x, y, \sqrt{1 + x^2 + y^2} \rangle \quad x^2 + y^2 \leq 99 \]

Alternative: \[ \vec{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, \sqrt{1 + r^2} \rangle \quad r \leq \sqrt{99} \]
(b) (4 marks) Use your parametrization from part (a) to compute the surface area of the cap of the sphere \( x^2 + y^2 + z^2 = 16 \) for \( 4/\sqrt{2} \leq z \leq 4 \).

From class, we know that \( \left| \frac{\partial}{\partial \theta} \vec{r} \right| = \sin \phi \) for the standard parametrization of the unit sphere. Since our parametrization is scaled by \( 4 \), we have \( \left| \frac{\partial}{\partial \theta} \vec{r} \right| = 16 \sin \phi \), so

\[
\text{Area} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} 16 \sin \phi \, d\phi \, d\theta = 2\pi \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} 16 \cos \phi \, d\phi \, d\theta
\]

\[
= 2\pi \left[ -16 \left[ \frac{\sqrt{z}}{2} - 1 \right] \right] = -32\pi \left( 1 - \frac{\sqrt{z}}{2} \right)
\]

Or using \( \vec{r}(x, y) = (x, y, \sqrt{16-x^2-y^2}) \) for \( x^2+y^2 \leq 8 \)

\[
\left| \frac{\partial}{\partial x} \vec{r} \times \frac{\partial}{\partial y} \vec{r} \right| = \begin{vmatrix} i & j & k \\ 1 & 0 & -x \\ 0 & 1 & -y \end{vmatrix} = \left\langle \frac{x}{\sqrt{16-x^2-y^2}}, \frac{y}{\sqrt{16-x^2-y^2}}, 1 \right\rangle
\]

\[
= \sqrt{\frac{x^2}{16-x^2-y^2} + \frac{y^2}{16-x^2-y^2} + \frac{16-x^2-y^2}{16-x^2-y^2}} = \frac{4}{\sqrt{16-x^2-y^2}}
\]

\[
\text{Area} = \iint_{x^2+y^2 \leq 8} \frac{4 \sin \theta}{\sqrt{16-r^2}} \, dr \, d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{8}} \frac{4r \, dr \, d\theta}{\sqrt{16-r^2}} = 2\pi \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{8}} \frac{-2(-2r \, dr)}{\sqrt{16-r^2}}
\]

\[
= 2\pi \int_{u=16}^{2\sqrt{8}} -2 \, du = -4\pi \left[ 2\sqrt{u} \right]_{16}^{2\sqrt{8}} = -8\pi \left[ \sqrt{8} - 4 \right] = -32\pi \left( 1 - \frac{\sqrt{z}}{2} \right)
\]
Q8  [15 marks]

Determine if the given statements are True or False. Provide a reason or a counterexample. (Grading: 1 mark for a correct T or F, 2 marks for a correct reason or counterexample. 1 mark for a reasonable attempt at a reason or counterexample.)

(a) A constant vector field is conservative on $\mathbb{R}^3$.

\[
\begin{align*}
\text{True} & \quad \text{if} \quad \vec{F} = \langle a, b, c \rangle \quad \text{then} \\
\vec{F} &= \nabla f \quad \text{where} \quad f = ax + by + cz
\end{align*}
\]

(b) If $\text{div} \vec{F} = 0$ for all points in the domain of $\vec{F}$ then $\vec{F}$ is a constant vector field.

\[
\begin{align*}
\text{False} & \quad \vec{F} = \langle x, -y, 0 \rangle \quad \text{is not constant} \\
\text{but} \quad \text{div} \vec{F} = 0
\end{align*}
\]
(c) Let \( \mathbf{r}(t) \) be a parametrization of a curve \( C \) in \( \mathbb{R}^3 \). If \( \mathbf{r}(t) \) and \( \frac{d\mathbf{r}}{dt} \) are orthogonal at all points of the curve \( C \), then \( C \) lies on the surface of a sphere \( x^2 + y^2 + z^2 = a^2 \) for some \( a > 0 \).

True \[ \mathbf{r}(t) \text{ is orthogonal to } \frac{d\mathbf{r}}{dt} \] Then

\[
\frac{d}{dt} (\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}) = \frac{d}{dt} \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} + \mathbf{r} \cdot \frac{d^2\mathbf{r}}{dt^2} = 0
\]

and so \( \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = \text{const.} = a^2 \)
thus \( |\mathbf{r}(t)| = a \) and so \( \mathbf{r}(t) \) lies on the sphere.

(d) The curvature \( \kappa \) at a point on a curve depends on the orientation of the curve.

False \[ \text{The definition of curvature is } \]

\[
\kappa = \left| \frac{d\mathbf{T}}{ds} \right|
\]
changing the orientation changes the sign of \( \mathbf{T} \) which does not effect \( \kappa \) since the absolute value makes the sign go away.

(e) The domain of a conservative vector field must be simply connected.

False \[ \text{Let } \mathbf{f}(x,y) = \frac{1}{x^2 + y^2} \text{ Then} \]

\[
\nabla \mathbf{f} = \left\langle \frac{-2x}{(x^2 + y^2)^2}, \frac{-2y}{(x^2 + y^2)^2} \right\rangle
\]
is conservative
but its domain is \( \mathbb{R}^2 - \{0,0\} \) which is not simply connected.